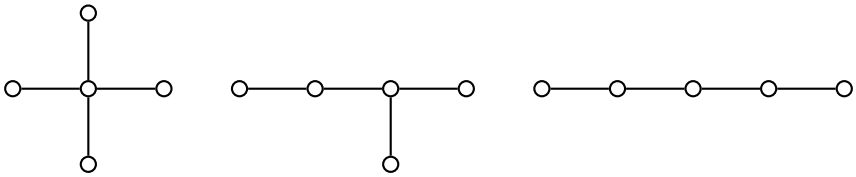
Planar Graphs and Trees

1. One of the key facts about trees is:

   Every tree T with at least 2 vertices has a *leaf* (a vertex of degree 1).

   Prove this statement. This allows us to prove facts about trees by induction, since we can reduce trees to smaller ones by trimming the leaves.

2. Here are the three (unrooted) trees with five vertices:



   Draw all the unrooted trees with 6 vertices (there are six of them). Hint: Question 1. For more, see `oeis.org/A000055`. Contrast the 156 total graphs on 6 vertices (`oeis.org/A000088`) of which 112 are connected (`oeis.org/A001349`). On 10 vertices, there are over 12 million possible graphs, of which 106 are trees.

3. Draw all the rooted trees with 5 vertices (there are nine of them). Why are there more rooted trees than unrooted ones?

(1)

Pick any vertex to start, and begin walking around the graph without repeating edges until we can't go anymore. Eventually, since we have finitely many edges, we need to stop. Either we stop at a leaf, in which case we win, or we stop at somewhere we've already been, which constitutes a cycle.

(2)

Here they are. Note that they can all be obtained from trees on 5 vertices by adding leaves: