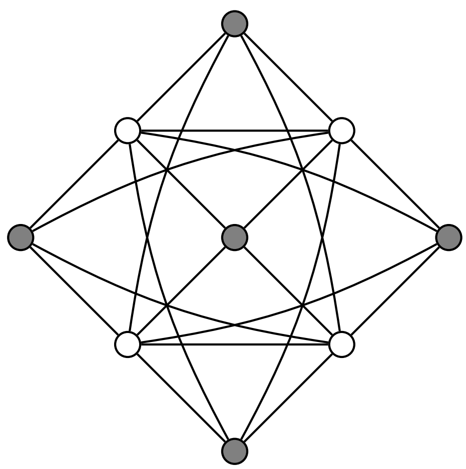
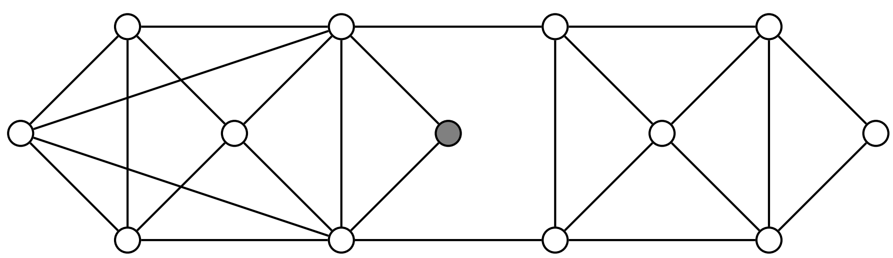
Graphs

1. (To be done on the board) Find a closed-form solution for the following recurrence relation:
$$a_n = 4a_{n-1} - 4a_{n-2} - n$$
with the initial conditions $a_0 = 7$, $a_1 = 2$.

2. For the following graphs, show that they are Eulerian but not Hamiltonian. (The highlighted vertices are meant to give a hint as to where to look. If you are still stuck, try using (3)).



(a)



(b)

3. For a graph G, we say that a subset W of its vertex set is an *independent set* if it has no internal edges (that is, there are no edges w_1w_2 with $w_1, w_2 \in W$). For instance, a graph is bipartitie if and only if its vertex set is the union of two independent sets. Suppose G has n vertices. Show that if G has an independent set of size $> n/2$, then G is not Hamiltonian.

1

Since this is an inhomogeneous equation, we need to find a particular solution first. As our inhomogeneity is $-n$, we could guess that our particular solution is of the form $Bn + C$ for some constants B and C. Plugging this into the recurrence we get

$$\begin{aligned}(Bn + C) &= 4(B(n-1) + C) - 4(B(n-2) + C) - n \\ Bn + C &= (4B - 4B - 1)n + (-4B + 4C + 8B - 4C)\end{aligned}$$

Since this is supposed to hold true for all n, we can equate the coefficients of n and the constant terms to get

$$\begin{aligned}B &= 4B - 4B - 1 = -1 \\ C &= -4B + 8B + 4C - 4C = 4B = -4\end{aligned}$$

so our particular solution is $-4 - n$. We can check this as follows:

$$\begin{aligned}(-4-n) &\stackrel{?}{=} 4(-4-n+1) - 4(-4-n+2) - n \\ -4-n &\stackrel{?}{=} -12 - 4n + 8 + 4n - n \\ -4-n &= -4-n\end{aligned}$$

so this is indeed a solution.

Now we just need to solve the homogeneous recurrence: it has characteristic polynomial $t^2 - 4t + 4$, which has a double root at 2. So a basis for the solution set is $\{2^n, n \cdot 2^n\}$. Thus our solution looks like

$$a_n = K_1 2^n + K_2 n 2^n - n - 4$$

Now we need to make use of the initial conditions to find K_1 and K_2. We have

$$\begin{aligned}7 = a_0 &= K_1 \cdot 2^0 + K_2 \cdot 0 \cdot 2^0 - 0 - 4 = K_1 - 4 \\ 2 = a_1 &= K_1 \cdot 2^1 + K_2 \cdot 1 \cdot 2^1 - 1 - 4 = 2K_1 + 2K_2 - 5\end{aligned}$$

from which we can conclude $K_1 = 3$ and $K_2 = 1/2$. So our final answer is

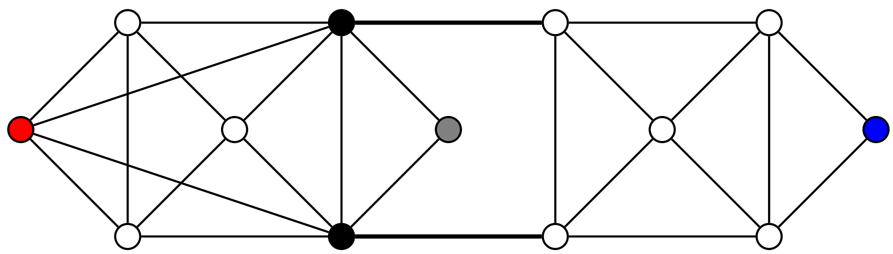
$$a_n = 3 \cdot 2^n + (1/2)n2^n - n - 4$$

2(a)

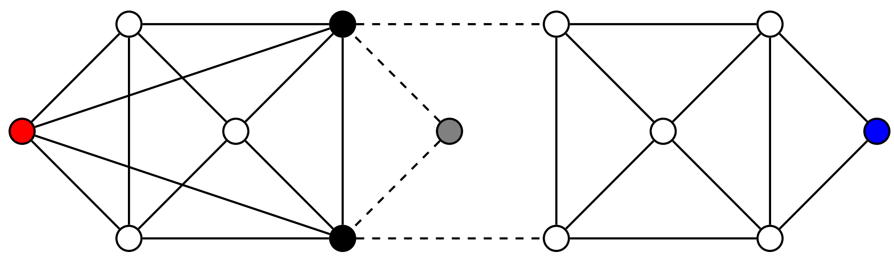
This graph has 9 vertices, and an independent set of size 5 (as indicated by the gray vertices). So by (3) it cannot be Hamiltonian. It's Eulerian since it's connected and every vertex has degree 4 or 6.

(We can also argue as follows: the graph has 18 edges. In order to get a Hamiltonian cycle we would need to delete two edges incident to each of the gray points; these are all distinct, though, so we end up with at most 18-10 = 8 edges. That's not enough for a Hamiltonian cycle on 9 vertices).

2(b)



Note since we have a degree-2 vertex in gray which we need to pass through, we must have its two edges as part of our graph. In order to have a Hamiltonian cycle, we need two disjoint paths from the leftmost (red) vertex to the rightmost (blue) vertex. As any such path needs to use one of the two thick edges incident to the black points, we have to have the following dashed path as part of our cycle:



But now we've cut the graph into two pieces, so is no way we can contain both the blue and red vertices. So there's no Hamiltonian path.

3

Suppose we have an independent set of size $> n/2$, and call it W. Now if our graph is Hamiltonian, then we can define a path

$$\{v_1, v_2, \ldots, v_n\}$$

containing each vertex exactly once, such that v_i and v_{i+1} are adjacent for every i, and also v_n and v_1 are adjacent. Some of these vertices are in W; let A be the set of indices such that $v_i \in W$. Now let's rotate the path by one step:

$$\{v_1', v_2', \ldots, v_{n-1}', v_n'\} = \{v_2, v_3, \ldots, v_n, v_1\}.$$

This is another Hamiltonian path, and we can let B be the set of indices such that $v_i' \in W$. Now $|A| = |B| = |W| > n/2$, so $|A| + |B| > n$. And since $A, B \subseteq \{1, 2, \ldots, n\}$, which has size n, the pigeonhole principle tells us that there is an $i \in A \cap B$. But that means $v_i \in W$ and $v_{i-1} \in W$, so we have two adjacent vertices in W, which can't happen as we assumed W to be an independent set.