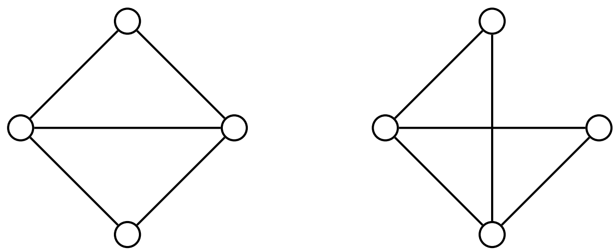
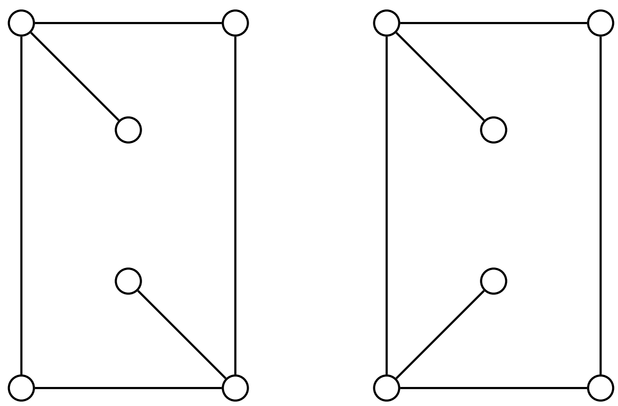
Graphs

1. For the following pairs of graphs, tell whether they are the same or different:

(a)



(b)



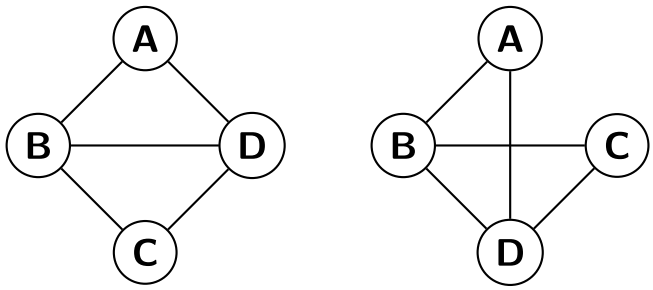
2. Let G be an arbitrary graph with vertex set V. Consider the relation $\sim$ on V defined by $v_1 \sim v_2$ if there is a (not necessarily simple) path from v_1 to v_2 in G. Show that this is an equivalence relation. The equivalence classes under this relation are called the *connected components* of G.

3. Under the same conditions as the previous problem, suppose our relation is that $v_1 \sim v_2$ if there is a *simple* path from v_1 to v_2 in G. Show that this is also an equivalence relation. How does it compare to the equivalence relation in the previous problem?

4. (Challenge - this is a nice thing to know and think about, but nailing down the details is tricky. See if you can get an intuitive understanding of why this is true). Show that a graph G is bipartite if and only if it has no odd-length cycles.

1a

Yes, these graphs are 'the same'. Suppose we label the vertices like so:



Then we can see that corresponding pairs of vertices are either both adjacent or both nonadjacent. So, they are the same.

1b

These graphs are not the same (even though they have the same numbers of vertices and edges and even the same degree sequence). Here's a few reasons why:

- The graph on the right has two adjacent degree-3 vertices, but the one on the left does not.
- The graph on the left has a length-4 path both of whose endpoints are leaves, but the one on the right does not.
- The graph on the right has a Hamiltonian path, but the one on the left does not.
- The graph on the left has four distinct automorphisms (isomorphisms to itself), but the one on the right only has two.

2

We quickly check the axioms for an equivalence relation:

- Reflexivity: given a vertex v the length-0 path $\{v\}$ represents a path from v to itself.
- Symmetry: If we have a path $\{v_1, w_1, w_2, \ldots, w_k, v_2\}$ from v_1 to v_2, then the reverse path $\{v_2, w_k, w_{k-1}, \ldots, w_1, v_1\}$ constitues a path from v_2 to v_1. (Here we assume the graph is undirected; if the graph is directed then in general we just get a preorder).
- Transitivity: If we have a path $\{v_1, w_1, w_2, \ldots, w_k, v_2\}$ from v_1 to v_2 and a path $\{v_2, u_1, u_2, \ldots, u_\ell, v_3\}$ from v_2 to v_3, then we can just concatenate them to get a path

$$\{v_1, w_1, \ldots, w_k, v_2, u_1, \ldots, u_\ell, v_3\}$$

from v_1 to v_3. Note that this may not be a simple path even if the two constituent paths are simple. In part (3) we show that this isn't really a problem.

3

We make the following claim: In a graph G, two vertices are connected by a path if and only if they are connected by a simple path. This will show that the relations in (2) and (3) are the same, and so the relation in (3) is an equivalence relation. Here is a proof:

One direction is quick: if two vertices are connected by a path they are connected by a simple path. For the other direction, suppose v_1 and v_2 are connected by a path. Let P be a path *of minimal length* connecting the two vertices. Then we claim P is simple.

If P is not simple, then it repeats a vertex w. So P looks like

$$P = \{v_1, x_1, \ldots, x_k, w, x_{k+2}, \ldots, x_\ell, w, x_{\ell+2}, \ldots, x_n, v_2\}$$

so we can make a shorter path by dropping everything between the two ws. This includes the second w:

$$P' = \{v_1, x_1, \ldots, x_k, w, x_{\ell+2}, \ldots, x_n, v_2\}$$

This is a shorter path, contradicting minimality of P. So P is a simple path.

4

To prove this 'if and only if' statement, we need to argue both directions.

First, suppose our graph is bipartite. Then we can write V (the vertex set) as $A \cup B$ for some disjoint A and B where there are no edges within A or within B. Let $\{v_1, v_2, \ldots, v_n, v_1\}$ be a cycle in V of length n. We want to show that n is not odd.

Without loss of generality, suppose $v_1 \in A$. Then since A has no interior edges, we have $v_2 \in B$. Similarly, $v_3 \in A$, and so on; v_i is in A if i is odd, and B if i is even. But we also know at the end of the cycle that v_n connects to v_1. That means $v_n \in B$, so n is even. Thus all cycles are of even length.

Conversely, suppose there are no cycles of odd length. As a graph is bipartitie if and only if all its connected components are bipartite (can be checked quickly), it suffices to look at the case where the graph is connected. The null-graph is vacuously bipartite, so take any vertex $v \in V$, and suppose we put it in A. Then we have to put all its neighbors in B, and all the neighbors of those in A, and so on. This won't work for all graphs though; if at some stage we give two neighboring vertices the same letter, there will be a problem. To show the lack of odd cycles prevents this, we make the following claim:

- Claim: if G is a connected graph with no odd cycles, then for any $v, w \in G$, either every path connecting them has odd length or every path connecting them has even length.

Note that this is exactly what we need, as for our v we can just let A be the set of vertices that are even distance from v and B the set of vertices that are odd distance from v. The claim tells us that no two vertices of A can be adjacent, because then we could take a path to one of them, followed by stepping to the other, to get an odd path.

To prove the claim, we have to work a little bit. Suppose paths P and Q go from v to w. Then P followed by Q is a path from v to itself[1], but it's not necessarily a cycle. But the conditions mean that there cannot be an odd length path from v to itself either. If there were, we could take a minimal odd-length path from v to itself. By assumption, this is not a cycle, so it repeats at least one vertex (besides the fact that v is at the beginning and end). It could repeat several vertices several times.

Consider a repetition where the two repeated vertices are minimal distance apart. Then the part of the path between them is a cycle, because any repetitions would have to be closer together. Hence we can delete this cycle to get a shorter path, which is still an odd-length path since the cycle was even. This contradicts minimality of the original path.

So P followed by Q can't have odd length, so it has even length meaning that P and Q have the same parity, as required. So this proves our claim, so the graph is bipartite.

[1] There is some ambiguity in notation here. The book lets paths repeat edges and vertices, and I am following that definition here.