Pigeonhole principle and recurrences

1. A *blue moon* is the rare occurrence of having two full moons in the same calendar month (thus the phrase "once in a blue moon"). Show that there is at least one blue moon every year. You may assume that the lunar cycle is exactly 28 days.

2. Suppose we have a 4-by-82 grid of points, each of which is colored taupe, vermilion, or chartreuse. Show that there is a rectangle with side lines parallel to the grid axes which has all four corners the same color.

3. John loves drinking soda, and wants to spend all the money in his wallet on soda today. So, he'll keep making trips to the soda stand until all his money is exhausted. There are three kinds of soda to buy:

   - Cheapo Cola, which costs \$1.
   - Diet Sugar, which costs \$2.
   - Eau de Water, which costs \$2.

   (So there are five ways of spending \$3: CC-DS, DS-CC, CC-EdW, EdW-CC, and CC-CC-CC. Note that John will always spend all his money, and the order of the drinks does matter.)

   How many ways are there for John to spend \$$n$?

4. Consider a permutation P of the numbers $\{1, \ldots, n\}$. Call it *132-avoiding* if there are no a, b, c such that $a < b < c$ but

   $$P(a) < P(c) < P(b)$$

   That is, there are no three entries where the values of P go low-high-middle. For example, the permutation **3**6**7**5812**4** is not 132-avoiding because of the bolded entries, but 54673812 is.

   Show that the number of 132-avoiding permutations of $\{1, \ldots, n\}$ is the nth Catalan number. (Hint: where does n go?)

Only the idea will be given, since that is most of the content of the proof.

1

Since the lunar cycle is 28 days, and $28 \cdot 13 = 364$, there must be at least 13 full moons in a year. So there must be two in a month.

2

Since there are 81 possible patterns of four-element rows, at least one row must repeat. That repeated row must have a particular color show up twice (since there are four slots and only three colors). But that constitutes a rectangle (if we look at both of our rows).

3

We get the recurrence

$$a_n = a_{n-1} + 2a_{n-2}, a_0 = 1, a_1 = 1$$

which we can solve to get

$$a_n = \frac{1}{3}(2 \cdot 2^n + (-1)^n).$$

4

Write out the permutation and look at where n went. Then everything to the left of n needs to be bigger than everything to the right of n. And each of the sides must be 132-avoiding. So we have the recurrence

$$a_n = \sum_{i=0}^{n-1} a_i a_{n-1-i}$$

where $a_0 = 1$. This is exactly the Catalan numbers.