Playing with binomial coefficients

Recall that we define the binomial coefficients $C(n, k)$ as follows:

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

If A is a set of size n, then $\binom{n}{k}$ counts the number of subsets of size k. For example, the set $\{1, 2, 3, 4, 5\}$ has $\binom{5}{2} = 10$ subsets of size 2:

$$\begin{array}{ccccc} \{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} \\ \{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\} \end{array}$$

Prove the following identities of binomial coefficients, using combinatorial arguments if possible. A direct or inductive proof is often possible, but less satisfying.

1. For any n, k, both ≥ 0,
$$\binom{n}{k} = \binom{n}{n-k}$$

2. For any $n > 1, 0 < k < n$,
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

3. For any $n \geq 0$,
$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

4. For any $n \geq 0$,
$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$
For instance, when $n = 6$ we get $1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$.

5. (Challenge) For any $n \geq 0$,
$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

For combinatorial arguments, only the idea will be given, since that is most of the content of the proof.

1

Any set of size k has a complement of size $n - k$. This process is a bijection. (Can also be proven directly from the definition).

2

Fix some element $a \in A$ (say, the first one). Then there are two kinds of sets of size k; either those which do not contain a, and thus have $k-1$ of the remaining $n-1$, and those which do not, and thus have k of the remaining $n-1$. (Can also be proven directly from the definition).

3

If we add together the subsets of size 0, the subsets of size 1, and so on, then we'll get all the subsets. (Can also be proven by induction on i, or with the binomial theorem:

$$\begin{aligned}\sum_{i=0}^{n} \binom{n}{i} &= \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} \\ &= (1+1)^n\end{aligned}$$

but the binomial theorem relies on a proof that is at least as difficult as the induction proof).

4

We want there to be the same number of even and odd subsets. Fix some element $a \in A$. Then given an even subset we can construct an odd one by adding or removing a. This process is a bijection. (Could also be proven using the formula in (2), or with the same binomial-theorem trick).

5

Split $2n$ elements into two halves. Then to pick n elements, we pick i from the first half and $n - i$ from the second half, so

$$\sum_{i=0}^{n} \binom{n}{i} \cdot \binom{n}{n-i} = \binom{2n}{n}$$

Using the identity in (2) we get what we want. (Or, we pick i from the first half and we exclude i from the second half).