Counting

1. How many integers between 1 and 100 are divisible by
   (a) 3?
   (b) 2, 3, and 5?
   (c) 2, 3, *or* 5?
   (d) Exactly one of 2, 3, and 5?
2. (Set up, but do not try to evaluate) How many five-card poker hands (from a standard deck of 52 cards) have
   (a) exactly two clubs?
   (b) at least two clubs?
   (c) more black cards than red cards?
   (d) three of a kind? (That is, three of one rank, with the other two ranks distinct. So 33345 counts, but 777AA and QQQQK do not).
3. Consider the string of letters 'DISCRETE'.
   (a) How many ways are there to rearrange the letters to form another string (such as TCREIEDS)?
   (b) In how many of those are the two 'E's nonadjacent?
   (c) In how many is the 'T' still between the 'E's (but not necessarily directly between: EDISCRTE counts)?
   (d) How many distinct substrings (such as SCREE) does this string have?

1a

These are the numbers $\{3, 6, 9, \ldots, 99\}$, of which there are 33. (Note that this is exactly $\lfloor 100/3 \rfloor$.

1b

These are the numbers that are multiples of 30; there are 3, namely 30, 60, and 90. This is $\lfloor 100/30 \rfloor$.

1c

Here we need to use inclusion/exlcusion; we get

$$\begin{aligned}
\text{(what we want)} &= \text{(multiples of 2)} + \text{(multiples of 3)} + \text{(multiples of 5)} \\
&\quad - \text{(multiples of 2 and 3)} - \text{(multiples of 3 and 5)} - \text{(multiples of 2 and 5)} \\
&\quad + \text{(mutiples of 2, 3, and 5)} \\
&= \lfloor 100/2 \rfloor + \lfloor 100/3 \rfloor + \lfloor 100/5 \rfloor \\
&\quad - \lfloor 100/6 \rfloor - \lfloor 100/15 \rfloor - \lfloor 100/10 \rfloor + \lfloor 100/30 \rfloor \\
&= 50 + 33 + 20 - 16 - 6 - 10 + 3 &= 74
\end{aligned}$$

1d

$$\begin{aligned}
\text{(what we want)} &= \text{(multiples of 2, 3, or 5)} - \text{(multiples of at least two of 2, 3, and 5)} \\
&= 74 - \text{(multiples of 2 and 3)} - \text{(multiples of 3 and 5)} - \text{(multiples of 2 and 5)} + 2 \cdot \text{(multiples} \\
&= 74 - 16 - 6 - 10 + 2 \cdot 3 \\
&= 48
\end{aligned}$$

2a

This is

$$\underbrace{\binom{13}{2}}_{\text{ways to pick two clubs}} \cdot \underbrace{\binom{39}{3}}_{\text{ways to pick three non-clubs}}$$

2b

To get this, we add the numbers of hands with two, three, four, and five clubs to get

$$\binom{13}{2} \cdot \binom{39}{3} + \binom{13}{3} \cdot \binom{39}{2} + \binom{13}{4} \cdot \binom{39}{1} + \binom{13}{5} \cdot \binom{39}{0}$$

Note that this is *not* the same thing as

$$\binom{13}{2} \cdot \binom{50}{3}$$

2c

Half of the hands will have more black than red, so it's

$$\frac{1}{2}\binom{52}{5}$$

2d

$$\left[\binom{13}{1}\cdot\binom{4}{3}\right]\cdot\left[\binom{12}{2}\cdot\binom{4}{1}^2\right]$$

3a

If we pretend the two 'E's are distinct, then there would be 8! ways of doing it. But we're double-counting as the two 'E's can be switched; so there's really only

$$\frac{8!}{2} = 20\ 160$$

different ways.

3b

There are $\binom{8}{2}$ different places for the 'E's to go, but 7 of them are a pair of adjacent places. So, once we pick the $\binom{8}{2} - 7 = 21$ possible places for the 'E's, there are 6! ways of arranging the other letters. Thus there are

$$(6!)\cdot(21) = 15\ 120$$

of these.

3c

This will be true for 1/3 of the permutations (since we can divide the permutations in groups of six as follows

$$\{\alpha E\beta E\gamma, \beta E\alpha E\gamma, \ldots, \gamma E\beta E\alpha\}$$

and in each of those groups, two of them will have the T between the Es). So there are

$$\frac{8!}{2}\cdot\frac{1}{3} = 6\ 720$$

of them.

3d

Ordinarily there would be 2^8 of them; but some are repeats because of the Es. But the only repeats occur when you have exactly one E and do not have T (otherwise we can tell which E we are looking at). And the number of such repeats is 2^5. So there are

$$2^8 - 2^5 = 224$$

such substrings.