Counting

1. How many integers between 1 and 100 are divisible by
   (a) 3?
   (b) 2, 3, and 5?
   (c) 2, 3, *or* 5?
   (d) Exactly one of 2, 3, and 5?

2. (Set up, but do not try to evaluate) How many five-card poker hands (from a standard deck of 52 cards) have
   (a) exactly two clubs?
   (b) at least two clubs?
   (c) more black cards than red cards?
   (d) three of a kind? (That is, three of one rank, with the other two ranks distinct. So 33345 counts, but 777AA and QQQQK do not).

1a

These are the numbers $\{3, 6, 9, \ldots, 99\}$, of which there are 33. (Note that this is exactly $\lfloor 100/3 \rfloor$.

1b

These are the numbers that are multiples of 30; there are 3, namely 30, 60, and 90. This is $\lfloor 100/30 \rfloor$.

1c

Here we need to use inclusion/exlcusion; we get

$$\begin{aligned}
\text{(what we want)} &= \text{(multiples of 2)} + \text{(multiples of 3)} + \text{(multiples of 5)} \\
&\quad - \text{(multiples of 2 and 3)} - \text{(multiples of 3 and 5)} - \text{(multiples of 2 and 5)} \\
&\quad + \text{(mutiples of 2, 3, and 5)} \\
&= \lfloor 100/2 \rfloor + \lfloor 100/3 \rfloor + \lfloor 100/5 \rfloor \\
&\quad - \lfloor 100/6 \rfloor - \lfloor 100/15 \rfloor - \lfloor 100/10 \rfloor + \lfloor 100/30 \rfloor \\
&= 50 + 33 + 20 - 16 - 6 - 10 + 3 \qquad = 74
\end{aligned}$$

1d

$$\begin{aligned}
\text{(what we want)} &= \text{(multiples of 2, 3, or 5)} - \text{(multiples of at least two of 2, 3, and 5)} \\
&= 74 - \text{(multiples of 2 and 3)} - \text{(multiples of 3 and 5)} - \text{(multiples of 2 and 5)} + 2 \cdot \text{(multiples} \\
&= 74 - 16 - 6 - 10 + 2 \cdot 3 \\
&= 48
\end{aligned}$$

2a

This is

$$\underbrace{\binom{13}{2}}_{\text{ways to pick two clubs}} \cdot \underbrace{\binom{39}{3}}_{\text{ways to pick three non-clubs}}$$

2b

To get this, we add the numbers of hands with two, three, four, and five clubs to get

$$\binom{13}{2} \cdot \binom{39}{3} + \binom{13}{3} \cdot \binom{39}{2} + \binom{13}{4} \cdot \binom{39}{1} + \binom{13}{5} \cdot \binom{39}{0}$$

Note that this is *not* the same thing as

$$\binom{13}{2} \cdot \binom{50}{3}$$

2c

Half of the hands will have more black than red, so it's

$$\frac{1}{2}\binom{52}{5}$$

2d

$$\left[\binom{13}{1}\cdot\binom{4}{3}\right]\cdot\left[\binom{12}{2}\cdot\binom{4}{1}\right]$$