Induction

1. In the famous song "The Twelve Days of Christmas," the singer's lover gives them 1 gift on the first day, then 1+2 gifts on the second day, 1+2+3 gifts on the third day, and so on. The total number of gifts received up to and including day n is

$$\frac{n(n+1)(n+2)}{6}, \text{ or } \binom{n+2}{3}$$

(That is, by the first day we have 1 gift, by the second we have 4, then 10, and so on).

Functions

1. Define a function f from the set of natural numbers $\{1, 2, 3, \dots\}$ to the set of even natural numbers $\{2, 4, 6, \dots\}$ by letting $f(x) = 2x$. Show carefully that f is a bijection and conclude that these two sets have the same cardinality, even though one is a subset of the other.

2. Suppose we have a function $f : A \to B$ and a function $g : B \to C$. Then we can define the composition function $g \circ f : A \to C$ which is defined by $(f \circ g)(a) = f(g(a))$. (Note that it is 'backwards': $g \circ f$ means first f, then g).

   (a) If $g \circ f$ is surjective, what can we say about f and g?

   (b) If $g \circ f$ is injective, what can we say about f and g?

   (c) If we want to show $g \circ f$ is surjective, what would we need to know about f and g (in terms of injectivity and surjectivity)?

   (d) If we want to show $g \circ f$ is injective, what would we need to know about f and g?

(Remember that 'surjective' or 'onto' means that the function hits everything in the target space; 'injective' or 'one-to-one' means that everything that is hit is hit only once; and 'bijective' means that both of these happen).

Solutions

1

We induct on n. The base case occurs when $n = 1$; on the first day, the singer receives one gift, which is

$$\frac{1 \cdot (1+1) \cdot (1+2)}{6} = \frac{6}{6} = 1.$$

So suppose we've received the correct number of gifts up to day n. Then we receive $1 + \cdots + (n+1)$ gifts on the $n+1$st day. That's equal to $\frac{(n+1)(n+2)}{2}$, so we have gotten

$$\begin{aligned} &\frac{n(n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2} \\ &= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{6} \\ &= \frac{(n+3)(n+1)(n+2)}{6} \end{aligned}$$

by the $n+1$st day, as required. Note that we can get a slightly nicer argument using binomial coefficients: by the nth day, we've received $\binom{n+2}{3}$, and on the $n+1$st day we get $\frac{(n+2)(n+1)}{2} = \binom{n+2}{2}$. And we know

$$\binom{n+2}{3} + \binom{n+2}{2} = \binom{n+3}{3}$$

by binomial identities. The numbers we computed here are called *tetrahedral numbers* because the nth one is the number of spheres needed to make a regular tetrahedron with side length n.

1

We break up the proof into two parts. First, we show that f is injective. That means we need to prove

$$\text{If } 2x = 2y, \text{ then } x = y.$$

But if $2x = 2y$, then we can just divide both sides by 2 to conclude $x = y$. So f is in fact injective.

Next we show that f is surjective. That means we need to prove

$$\text{For any even natural number } y, \text{ there is a natural number } x \text{ such that } y = 2x.$$

But there is such a natural number, namely $y/2$. So the function is also surjective.

Thus f is a bijection between the two sets, and if there is a bijection the sets must have the same cardinality.

2a

All we can say is that g is surjective. For if we have some $c \in C$, then the surjectivity of $g \circ f$ implies that $(g \circ f)(a) = c$ for some $a \in A$; then $g(f(a)) = c$, which means that $f(a)$ does what we need. But neither has to be injective, and f does not have to be surjective; consider the functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \{1\}$ which are both the constant function 1.

2b

All we can say is that f is injective. For if $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$; so if $f(a_1) = f(a_2)$, we get $g(f(a_1)) = g(f(a_2))$ and so $a_1 = a_2$. We can't say that g is injective, or that either is surjective; consider the functions $f : \{1\} \to \mathbb{R}$, $g : \mathbb{R} \to \{1\}$ where f is the identity function and g is the constant function 1.

2c

We need *both* f and g to be surjective to conclude that the composition is. (Just proving that g is surj. is not sufficient[1]). If both are surjective, that means for any $c \in C$, we can find $b \in B$ such that $g(b) = c$, and then $a \in A$ such that $f(a) = b$. So $(g \circ f)(a) = g(f(a)) = g(b) = c$.

2d

We need *both* f and g to be injective to conclude that the composition is. (Just proving that f is inj. is not sufficient[2]). If both are injective, that means whenever $a_1 \neq a_2$, we have $f(a_1) \neq f(a_2)$, so $g(f(a_1)) \neq g(f(a_2))$.

[1] Consider the functions $f : \mathbb{Z} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ where f is the identity function and $g(x) = x^2$. g is surjective but $g \circ f$ is not, as nothing maps to 2.

[2] Same example as before: f is injective but $(g \circ f)(2) = (g \circ f)(-2)$