Freeness and the Transpose

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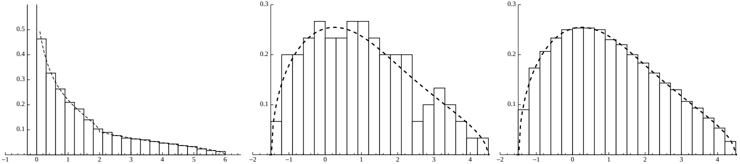
Eigenvalue distributions and the transpose

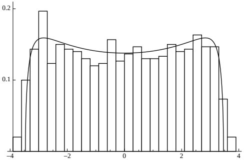
- Let X_N be the $N \times N$ GUE. (dotted curves show limit distributions)



$X_{1000} + X_{1000}^2$ $\qquad$ $X_{100} + (X_{100}^2)^t$ $\qquad$ $X_{1000} + (X_{1000}^2)^t$

- if we let Y_N be the $N \times N$ GOE then $Y_N + (Y_N^2)^t = Y_N + Y_N^2$; so we would ***not*** get different pictures
- let U_N be the $N \times N$ Haar distributed unitary matrix

$U_{10} + U_{10}^* + (U_{10} + U_{10}^*)^t$
sampled 100 times



Unitary Case

For a unitary matrix let $U^{(1,1)} = U$, $U^{(-1,1)} = U^{\mathrm{T}}$, $U^{(1,-1)} = \overline{U}$, $U^{(-1,-1)} = U^*$.

If whenever we have a reduced word $U^{(\epsilon_1,\eta_1)}U^{(\epsilon_2,\eta_2)}\cdots U^{(\epsilon_n,\eta_n)}$ (i.e. $(\epsilon_k,\eta_k) \neq -(\epsilon_{k+1},\eta_{k+1})$ for all k), we can show that

$$\mathrm{E}(\mathrm{Tr}(U^{(\epsilon_1,\eta_1)}U^{(\epsilon_2,\eta_2)}\cdots U^{(\epsilon_n,\eta_n)})) = O(1)$$

then we will have shown that $\{U, U^*\}$ is free from $\{U^{\mathrm{T}}, \overline{U}\}$.

For a $N \times N$ Haar distributed unitary matrix

$$\mathrm{E}(\mathrm{Tr}(U^{(\epsilon_1,\eta_1)}U^{(\epsilon_2,\eta_2)}\cdots U^{(\epsilon_n,\eta_n)}))$$

$$= \sum_{p,q\in\mathcal{P}_2^{\eta}(n)} \mathrm{Wg}(p,q)\, |\{j : [\pm n] \to [N] \mid p\delta q\delta \vee \epsilon\gamma\delta\gamma^{-1}\epsilon \leqslant \ker(j)\}|$$

$$= \sum_{p,q\in\mathcal{P}_2^{\eta}(n)} \mathrm{Wg}(p,q)\, N^{\#(p\delta q\delta\vee\epsilon\gamma\delta\gamma^{-1}\epsilon)}$$

$$E(\mathrm{Tr}(U^{(\epsilon_1,\eta_1)}U^{(\epsilon_2,\eta_2)}\dots U^{(\epsilon_n,\eta_n)}))$$
$$= \sum_{p,q\in\mathcal{P}_2^{\eta}(n)} \mathrm{Wg}(p,q)\, N^{\#(p\delta q\delta\vee\epsilon\gamma\delta\gamma^{-1}\epsilon)}$$

$$\mathrm{Wg}(p,q) = O(N^{-n+\#(p\vee q)})$$

so we must show that

$$\#(p\vee q) + \#(p\delta q\delta \vee \epsilon\gamma\delta\gamma^{-1}\epsilon) \leqslant n$$

or, since $\#(p\vee q) \leqslant n/2$, that

$$\#(p\delta q\delta \vee \epsilon\gamma\delta\gamma^{-1}\epsilon) \leqslant n/2$$

or that the permutation $\delta\gamma^{-1}\epsilon(p\delta q\delta)\epsilon\gamma\delta$ has no fixed points, which can be easily checked since $(\epsilon_k,\eta_k) \neq -(\epsilon_{k+1},\eta_{k+1})$

Wigner Matrices (with R. Speicher)

Let $X = X^* = \frac{1}{\sqrt{N}}(x_{ij})_{ij}$ be a $N \times N$ complex Wigner matrix:

- $\mathrm{E}(x_{ij}) = 0$
- $\mathrm{E}(|x_{ij}|^2) = 1$
- $\mathrm{E}(x_{ij}^2) = 0$
- $\{x_{ii}\}_i \cup \{x_{ij}\}_{i<j}$ independent
- $\{x_{ii}\}_i$ identically distributed
- $\{x_{ij}\}_{i<j}$ identically distributed

then X and X^{T} are asymptotically free

the proof is very different from the unitary case as there is no equivalent to the Weingarten calculus

Wishart Random Matrices

- Suppose $G_1, \ldots, G_{d_1}$ are $d_2 \times p$ random matrices where $G_i = (g_{jk}^{(i)})_{jk}$ and $g_{jk}^{(i)}$ are complex Gaussian random variables with mean 0 and (complex) variance 1, i.e. $\mathrm{E}(|g_{jk}^{(i)}|^2) = 1$. Moreover suppose that the random variables $\{g_{jk}^{(i)}\}_{i,j,k}$ are independent.
- $$W = \frac{1}{p} \begin{pmatrix} G_1 \\ \hline \vdots \\ \hline G_{d_1} \end{pmatrix} \left(\begin{array}{c|c|c} G_1^* & \cdots & G_{d_1}^* \end{array} \right) = (G_i G_j^*)_{ij}$$

  is a $d_1 d_2 \times d_1 d_2$ Wishart matrix. We write $W = (W_{ij})_{ij}$ as $d_1 \times d_1$ block matrix with each entry the $d_2 \times d_2$ matrix $G_i G_j^*$.

Partial Transposes

- G_i a $d_2 \times p$ matrix
- $W_{ij} = \frac{1}{p} G_i G_j^*$, a $d_2 \times d_2$ matrix,
- $W = (W_{ij})_{ij}$ is a $d_1 \times d_1$ block matrix with entries W_{ij}
- $W^{\mathrm{T}} = (W_{ji}^{\mathrm{T}})_{ij}$ is the "full" transpose
- $W^{\urcorner} = (W_{ji})_{ij}$ is the "left" partial transpose
- $W^{\Gamma} = (W_{ij}^{\mathrm{T}})_{ij}$ is the "right" partial tarnspose
- we **assume** that $\dfrac{p}{d_1 d_2} \to \alpha$ and $0 < \alpha < \infty$
- eigenvalue distributions of W and W^{T} converge to Marchenko-Pastur with parameter α
- eigenvalues of $W^{\urcorner}$ and W^{Γ} converge to a shifted semi-circular with mean 1 and variance $1/\alpha$ (Aubrun)
- W and W^{T} are asymptotically free (M. and Popa)
- what about W^{Γ} and $W^{\urcorner}$?

Semi-circle and Marchenko-Pastur Distributions

Suppose $\frac{d_1}{\sqrt{p}} \to \frac{1}{\alpha_1}$ and $\frac{d_2}{\sqrt{p}} \to \frac{1}{\alpha_2}$ and $\alpha = \alpha_1\alpha_2$ ($c = 1/\alpha$.)

- limit eigenvalue distribution of W (Marchenko-Pastur)

$$\lim \mathrm{E}(\mathrm{tr}(W^n)) = \sum_{\sigma \in NC(n)} \left(\frac{1}{\alpha}\right)^{\#(\sigma)-1} = \sum_{\sigma \in NC(n)} \left(\frac{1}{\alpha}\right)^{\#(\gamma\sigma^{-1})-1}$$

(here $\#(\sigma)$ is the number of blocks of σ, $\gamma = (1, \ldots, n)$ and $\gamma\sigma^{-1}$ is the "other" Kreweras complement)

- limit eigenvalue distribution of W^Γ (semi-circle)

$$\lim \mathrm{E}(\mathrm{tr}((W^\Gamma)^n)) = \sum_{\sigma \in NC_{1,2}(n)} \left(\frac{1}{\alpha}\right)^{\#(\gamma\sigma^{-1})-1}$$

$NC_{1,2}(n)$ is the set of non-crossing partitions with only blocks of size 1 and 2. (*c.f.* Fukuda and Śniady (2013) and Banica and Nechita (2013))

main theorem

- THM: The matrices $\{W, W^{\daleth}, W^{\Gamma}, W^{\mathrm{T}}\}$ form an asymptotically free family
- let $(\epsilon, \eta) \in \{-1, 1\}^2 = \mathbb{Z}_2^2$.
- let $W^{(\epsilon,\eta)} = \begin{cases} W & \text{if } (\epsilon,\eta) = (1,1) \\ W^{\daleth} & \text{if } (\epsilon,\eta) = (-1,1) \\ W^{\Gamma} & \text{if } (\epsilon,\eta) = (1,-1) \\ W^{\mathrm{T}} & \text{if } (\epsilon,\eta) = (-1,-1) \end{cases}$
- let $(\epsilon_1, \eta_1), \ldots, (\epsilon_n, \eta_n) \in \mathbb{Z}_2^n$

$$\mathrm{E}(\mathrm{Tr}(W^{(\epsilon_1,\eta_1)} \cdots W^{(\epsilon_n,\eta_n)}))$$

$$= \sum_{\sigma \in S_n} \left(\frac{d_1}{\sqrt{p}}\right)^{f_\epsilon(\sigma)} \left(\frac{d_2}{\sqrt{p}}\right)^{f_\eta(\sigma)} p^{\#(\sigma) + \frac{1}{2}(f_\epsilon(\sigma) + f_\eta(\sigma)) - n}.$$

where $f_\epsilon(\sigma) = \#(\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \vee \sigma\delta\sigma^{-1})$ (*"$\vee$" means the sup of partitions and # means the number of blocks or cycles*)

Computing Moments via Permutations, I

- $[d_1] = \{1, 2, \ldots, d_1\}$,
- given $i_1, \ldots, i_n \in [d_1]$ we think of this n-tuple as a function $i : [n] \to [d_1]$
- $\ker(i) \in \mathcal{P}(n)$ is the partition of $[n]$ such that i is constant on the blocks of $\ker(i)$ and assumes different values on different blocks
- if $\sigma \in S_n$ we also think of the cycles of σ as a partition and write $\sigma \leqslant \ker(i)$ to mean that i is constant on the cycles of σ
- given $\sigma \in S_n$ we extend σ to a permutation on $[\pm n] = \{-n, \ldots, -1, 1, \ldots, n\}$ by setting $\sigma(-k) = -k$ for $k > 0$
- $\gamma = (1, 2, \ldots, n)$, $\delta(k) = -k$
- $\delta\gamma^{-1}\delta\gamma\delta = (1, -n)(2, -1) \cdots (n, -(n-1))$

Computing Moments via Permutations, II

- $\delta\gamma^{-1}\delta\gamma\delta = (1,-n)(2,-1)\cdots(n,-(n-1))$
- if $A_k = (a^{(k)}_{ij})_{ij}$ then

$$\mathrm{Tr}(A_1\cdots A_n) = \sum_{i_1,\ldots,i_n=1}^{N} a^{(1)}_{i_1i_2}a^{(2)}_{i_2i_3}\cdots a^{(n)}_{i_ni_1} = \sum_{\substack{i_{\pm1},\ldots,i_{\pm n}\\ \delta\gamma^{-1}\delta\gamma\delta\leqslant\ker(i)}} a^{(1)}_{i_1i_{-1}}\cdots a^{(n)}_{i_ni_{-n}}$$

$$\begin{aligned}
&\mathrm{Tr}\big(W^{(\epsilon_1,\eta_1)}\cdots W^{(\epsilon_n,\eta_n)}\big)\\
&= \sum_{i_1,\ldots,i_n}\mathrm{Tr}\Big(\big(W^{(\epsilon_1,\eta_1)}\big)_{i_1i_2}\cdots\big(W^{(\epsilon_n,\eta_n)}\big)_{i_ni_1}\Big)\\
&= \sum_{i_{\pm1},\ldots,i_{\pm n}}\mathrm{Tr}\Big(\big(W^{(\epsilon_1,\eta_1)}\big)_{i_1i_{-1}}\cdots\big(W^{(\epsilon_n,\eta_n)}\big)_{i_ni_{-n}}\Big)\\
&= \sum_{j_{\pm1},\ldots,j_{\pm n}}\mathrm{Tr}\Big(W^{(\eta_1)}_{j_1j_{-1}}\cdots W^{(\eta_n)}_{j_nj_{-n}}\Big)
\end{aligned}$$

where $\delta\gamma^{-1}\delta\gamma\delta\leqslant\ker(i)$, $\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon\leqslant\ker(j)$ and $j = i\circ\epsilon$

Computing Moments via Permutations, III

$$\mathrm{Tr}\big(W^{(\epsilon_1,\eta_1)}\cdots W^{(\epsilon_n,\eta_n)}\big) = \sum_{j_{\pm 1},\dots,j_{\pm n}} \mathrm{Tr}\Big(W^{(\eta_1)}_{j_1 j_{-1}}\cdots W^{(\eta_n)}_{j_n j_{-n}}\Big)$$

with $\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \leqslant \ker(j)$. Let $s = r\circ\eta$ then for $\delta\gamma^{-1}\delta\gamma\delta \leqslant \ker(r)$

$$\begin{aligned}
&\mathrm{Tr}\big(W^{(\eta_1)}_{j_1 j_{-1}}\cdots W^{(\eta_n)}_{j_n j_{-n}}\big)\\
&= \sum_{r_{\pm 1},\dots,r_{\pm n}} \big(W^{(\eta_1)}_{j_1 j_{-1}}\big)_{r_1 r_{-1}}\cdots\big(W^{(\eta_n)}_{j_n j_{-n}}\big)_{r_n r_{-n}}\\
&= \sum_{s_{\pm 1},\dots,s_{\pm n}} \big(W_{j_1 j_{-1}}\big)_{s_1 s_{-1}}\cdots\big(W_{j_n j_{-n}}\big)_{s_n s_{-n}}\\
&= p^{-n}\sum_{s_{\pm 1},\dots,s_{\pm n}} \big(G_{j_1}G^*_{j_{-1}}\big)_{s_1 s_{-1}}\cdots\big(G_{j_n}G^*_{j_{-n}}\big)_{s_n s_{-n}}\\
&= p^{-n}\sum_{s_{\pm 1},\dots,s_{\pm n}}\sum_{t_1,\dots,t_n} g^{(j_1)}_{s_1 t_1}\overline{g^{(j_{-1})}_{s_{-1} t_1}}\cdots g^{(j_n)}_{s_n t_n}\overline{g^{(j_{-n})}_{s_{-n} t_n}}
\end{aligned}$$

Gaussian entries

$$\mathrm{E}(\mathrm{Tr}(W^{(\epsilon_1,\eta_1)} \dots W^{(\epsilon_1,\eta_1)}))$$

$$= p^{-n} \sum_{j_{\pm 1},\dots,j_{\pm n}} \sum_{s_{\pm 1},\dots,s_{\pm n}} \sum_{t_1,\dots,t_n} \mathrm{E}(g_{s_1 t_1}^{(j_1)} \overline{g_{s_{-1} t_1}^{(j_{-1})}} \cdots g_{s_n t_n}^{(j_n)} \overline{g_{s_{-n} t_n}^{(j_{-n})}})$$

$$= p^{-n} \sum_{j_{\pm 1},\dots,j_{\pm n}} \sum_{s_{\pm 1},\dots,s_{\pm n}} \sum_{t_1,\dots,t_n} \mathrm{E}(g_{s_1 t_1}^{(j_1)} \cdots g_{s_n t_n}^{(j_n)} \overline{g_{s_{-1} t_1}^{(j_{-1})}} \cdots \overline{g_{s_{-n} t_n}^{(j_{-n})}})$$

[subject to the condition that $\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \leqslant \ker(j)$ and $\eta\delta\gamma^{-1}\delta\gamma\delta\eta \leqslant \ker(s)$]

$$= p^{-n} \sum_{j_{\pm 1},\dots,j_{\pm n}} \sum_{s_{\pm 1},\dots,s_{\pm n}} \sum_{t_1,\dots,t_n} \mathrm{E}(g_{\alpha(1)} \cdots g_{\alpha(n)} \overline{g_{\beta(1)}} \cdots \overline{g_{\beta(n)}})$$

where $g_{\alpha(k)} = g_{s_k t_k}^{(j_k)}$ and $g_{\beta(k)} = g_{s_{-k} t_k}^{(j_{-k})}$. Using $\mathrm{E}(g_{\alpha(1)} \cdots g_{\alpha(n)} \overline{g_{\beta(1)}} \cdots \overline{g_{\beta(n)}}) = |\{\sigma \in S_n \mid \beta = \alpha \circ \sigma\}|$

Thus
$E(\mathrm{Tr}(W^{(\epsilon_1,\eta_1)}\ldots W^{(\epsilon_1,\eta_1)}))$

$$= p^{-n} \sum_{j_{\pm 1},\ldots,j_{\pm n}} \sum_{s_{\pm 1},\ldots,s_{\pm n}} \sum_{t_1,\ldots,t_n} |\{\sigma \in S_n \mid \text{“various conditions”}\}|$$

where “various conditions” means

- $\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \leqslant \ker(j)$
- $\eta\delta\gamma^{-1}\delta\gamma\delta\eta \leqslant \ker(s)$
- $j_{-k} = j_{\sigma(k)}$ which is equivalent to $\sigma\delta\sigma^{-1} \leqslant \ker(j)$
- $s_{-k} = s_{\sigma(k)}$ which is equivalent to $\sigma\delta\sigma^{-1} \leqslant \ker(s)$
- $t_k = t_{\sigma(k)}$ which is equivalent to $\sigma \leqslant \ker(t)$

$$E(\mathrm{Tr}(W^{(\epsilon_1,\eta_1)}\ldots W^{(\epsilon_n,\eta_n)}))$$
$$= \sum_{\sigma\in S_n} \left(\frac{d_1}{\sqrt{p}}\right)^{f_\epsilon(\sigma)} \left(\frac{d_2}{\sqrt{p}}\right)^{f_\eta(\sigma)} p^{\#(\sigma)+\frac{1}{2}(f_\epsilon(\sigma)+f_\eta(\sigma))-n}.$$

where $f_\epsilon(\sigma) = \#(\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \vee \sigma\delta\sigma^{-1})$ (*“$\vee$” means the sup of partitions*)

finding the highest order terms

- general fact: if p and q are pairings then $\#(p \vee q) = \frac{1}{2}\#(pq)$. In fact we can write the permutation pq as a product of cycles $c_1 c_1' \cdots c_k c_k'$ where $c_i' = q c_i^{-1} q$ and the blocks of $p \vee q$ are $c_i \cup c_i'$
- $\#(\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \vee \sigma\delta\sigma^{-1}) = \frac{1}{2}\#(\delta\gamma^{-1}\delta\gamma \cdot \epsilon\delta\sigma\delta\sigma^{-1}\epsilon)$
- if $\pi, \sigma \in S_n$ and $\langle \pi, \sigma \rangle$ (the subgroup generated by π and σ) has only one orbit then there is an integer g (the "genus") such that

$$\#(\pi) + \#(\pi^{-1}\sigma) + \#(\sigma) = n + 2(1 - g)$$

  and $g = 0$ only when π is planar or non-crossing with respect to σ.
- $\delta\gamma^{-1}\delta\gamma$ has two cycles so $\langle \delta\gamma^{-1}\delta\gamma, \epsilon\delta\sigma\delta\sigma^{-1}\epsilon \rangle$ can have either 1 or 2 orbits
- if $\langle \delta\gamma^{-1}\delta\gamma, \epsilon\delta\sigma\delta\sigma^{-1}\epsilon \rangle$ has one orbit then $\#(\epsilon\delta\gamma^{-1}\delta\gamma\delta\epsilon \vee \sigma\delta\sigma^{-1}) + \#(\sigma) \leqslant n$

$$\begin{aligned} &\mathrm{E}(\mathrm{tr}(W^{(\epsilon_1,\eta_1)}\cdots W^{(\epsilon_n,\eta_n)})) \\ &= \sum_{\sigma\in S_n}\left(\frac{d_1}{\sqrt{p}}\right)^{f_\epsilon(\sigma)-1}\left(\frac{d_2}{\sqrt{p}}\right)^{f_\eta(\sigma)-1} p^{\#(\sigma)+\frac{1}{2}(f_\epsilon(\sigma)+f_\eta(\sigma))-(n+1)}. \end{aligned}$$

- σ will not contribute to the limit unless $\langle\delta\gamma^{-1}\delta\gamma, \epsilon\delta\sigma\delta\sigma^{-1}\epsilon\rangle$ has two orbits, i.e. ϵ is constant on the cycles of σ (write $\epsilon\delta\sigma\delta\sigma^{-1}\epsilon = \delta\epsilon\sigma\epsilon\delta(\epsilon\sigma\epsilon)^{-1}$)
- if ϵ is constant on the cycles of σ there is $\sigma_\epsilon \in S_n$ such that $\epsilon\delta\sigma\delta\sigma^{-1}\epsilon = \delta\sigma_\epsilon\delta\sigma_\epsilon^{-1}$ (if $\sigma = c_1c_2\cdots c_k$ then $\sigma_\epsilon = c_1^{\lambda_1}\cdots c_k^{\lambda_k}$ where λ_i is the sign of ϵ on c_i)
- then $\frac{1}{2}\#(\delta\gamma^{-1}\delta\gamma\cdot\epsilon\delta\sigma\delta\sigma^{-1}\epsilon) = \#(\gamma\sigma_\epsilon^{-1})$
- $\#(\sigma) + f_\epsilon(\sigma) = \#(\sigma_\epsilon) + \#(\gamma\sigma_\epsilon^{-1}) \leqslant n+1$ with equality only if σ_ϵ is non-crossing
- $\#(\sigma) + f_\eta(\sigma) = \#(\sigma_\eta) + \#(\gamma\sigma_\eta^{-1}) \leqslant n+1$ with equality only if σ_η is non-crossing

$$\begin{aligned}&\mathrm{E}(\mathrm{tr}(W^{(\epsilon_1,\eta_1)}\cdots W^{(\epsilon_n,\eta_n)}))\\&=\sum_{\sigma\in S_n}\left(\frac{d_1}{\sqrt{p}}\right)^{f_\epsilon(\sigma)-1}\left(\frac{d_2}{\sqrt{p}}\right)^{f_\eta(\sigma)-1}+O\left(\frac{1}{p^2}\right).\end{aligned}$$

where the sum runs over σ such that

- ϵ and η are constant on the cycles of σ and
- both σ_ϵ and σ_η are non-crossing.
- if $\epsilon \neq \eta$ on a cycle of σ then this cycle must be either a fixed point or a pair; $\sigma_\epsilon = \sigma_\eta$ and so $f_\epsilon(\sigma) = f_\eta(\sigma)$
- σ can only connect $W^{(1,1)}$ to another $W^{(1,1)}$, a $W^{(-1,1)}$ to another $W^{(-1,1)}$, a $W^{(1,-1)}$ to another $W^{(1,-1)}$, and a $W^{(-1,-1)}$ to another $W^{(-1,-1)}$