On Two-Faced Families of Non-Commutative Random Variables

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Combinatorially-Bi-Free Families

Definition (Mastnak, Nica; 2013)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. There exists a family of multilinear functionals

$$(\kappa_\chi : \mathcal{A}^n \to \mathbb{C})_{n\geq 1, \chi:\{1,\ldots,n\}\to\{\ell,r\}}$$

which are uniquely determined by the requirement

$$\varphi(z_1 \cdots z_n) = \sum_{\pi \in \mathcal{P}^{(\chi)}(n)} \left(\prod_{V\in\pi} \kappa_{\chi|_V}((z_1, \ldots, z_n)|V) \right)$$

for every $n \geq 1$, $\chi \in \{\ell, r\}^n$, and $z_1, \ldots, z_n \in \mathcal{A}$. These κ_χ's will be called the (ℓ, r)-cumulant functionals of $(\mathcal{A}, \varphi)$.

Combinatorially-Bi-Free Families

Definition (Mastnak, Nica; 2013)

A pair of two-faced families

$$z' = ((z'_i)_{i \in I}, (z'_j)_{j \in J}) \qquad \text{and} \qquad z'' = ((z''_i)_{i \in I}, (z''_j)_{j \in J})$$

in $(\mathcal{A}, \varphi)$ are said to be combinatorially-bi-free if

$$\kappa_\chi\left(z^{\epsilon_1}_{\alpha(1)}, \ldots, z^{\epsilon_n}_{\alpha(n)}\right) = 0$$

whenever $\alpha : \{1, \ldots, n\} \to I \sqcup J$, $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ is such that $\alpha^{-1}(I) = \chi^{-1}(\{\ell\})$, and $\epsilon \in \{', ''\}^n$ is non-constant.

Combinatorially-Bi-Free Families

Remarks

If z' and z'' are combinatorially-bi-free, it is easy to see that

$$\kappa_\alpha(z' + z'') = \kappa_\alpha(z') + \kappa_\alpha(z'').$$

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Are z' and z'' bi-freely independent if and only if z' and z'' are combinatorially bi-freely independent?

The answer is yes!!!

Combinatorially-Bi-Free Families

For $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\mathcal{P}^{(\chi)}(n)$ is the set of all partitions π on $\{1, \ldots, n\}$ such that if

$$\chi^{-1}(\{\ell\}) = \{i_1 < \cdots < i_p\}, \qquad \chi^{-1}(\{r\}) = \{j_1 < \cdots < j_{n-p}\},$$

and $s_\chi \in S_n$ is defined by

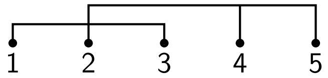
$$s_\chi(k) = \begin{cases} i_k & \text{if } k \leq p \\ j_{n+1-k} & \text{if } k > p \end{cases}.$$

then $s_\chi^{-1} \cdot \pi \in NC(n)$.

Bi-Non-Crossing Partitions

Let $\chi^{-1}(\{\ell\}) = \{1, 2, 4\}$, $\alpha^{-1}(\{r\}) = \{3, 5\}$, and

$$\pi = \left\{ \{1, 3\}, \{2, 4, 5\} \right\} = s_\chi \cdot \left\{ \{1, 5\}, \{2, 3, 4\} \right\}.$$



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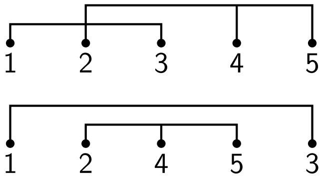

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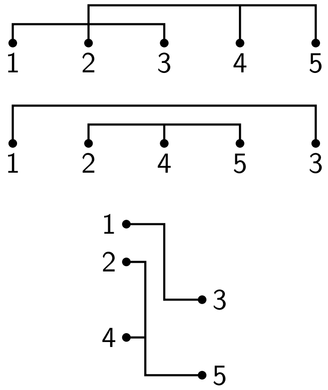

Bi-Non-Crossing Diagrams

Definition

Given $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, we say a partition π on $\{1, \ldots, n\}$ is bi-non-crossing if $\pi \in \mathcal{P}^{\chi}(n)$; that is, if one draws two vertical lines and places nodes ordered from the top down such that the k^{th} node is on the left if $\chi(k) = \ell$ and on the right if $\chi(k) = r$, then π is bi-non-crossing if π can be drawn to be a non-crossing diagram on these nodes.

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We will use $BNC(\chi)$ instead of $\mathcal{P}^{\chi}(n)$ to denote the bi-non-crossing diagrams on χ.

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We will use $BNC(\chi)$ instead of $\mathcal{P}^{\chi}(n)$ to denote the bi-non-crossing diagrams on χ.

In addition, if $\alpha : \{1, \ldots, n\} \to I \sqcup J$, we let $BNC(\alpha)$ denote the set of all bi-non-crossing diagrams in $BNC(\chi)$ where $\chi^{-1}(\{\ell\}) = \alpha^{-1}(I)$ where the k^{th} node is labelled $\alpha(k)$.

Incident Algebra on *BNC*

Note $BNC(\chi)$ inherits the lattice structure from $\mathcal{P}(n)$.

Proposition

Let $\pi, \sigma \in BNC(\chi)$ be such that $\pi \leq \sigma$. The interval

$$[\pi, \sigma] = \{\rho \in BNC(\chi) \mid \pi \leq \rho \leq \sigma\}$$

can be associated to a product of full lattices

$$\prod_{j=1}^{k} BNC(\beta_k)$$

for some $\beta_k : \{1, \ldots, m_k\} \to \{\ell, r\}$ so that the lattice structure is preserved.

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This is obtained by viewing π, σ as elements of $NC(n)$ and using the usual decomposition while maintaining a notion of left and right nodes.

Incident Algebra on BNC

Definition

A function $f \in IA(BNC)$ is said to be multiplicative if whenever $\pi, \sigma \in BNC(\chi)$ are such that

$$[\pi, \sigma] \leftrightarrow \prod_{j=1}^{k} BNC(\beta_k)$$

for some $\beta_k : \{1, \ldots, m_k\} \to \{\ell, r\}$, then

$$f(\pi, \sigma) = \prod_{j=1}^{k} f(0_{\beta_k}, 1_{\beta_k}).$$

For a multiplicative function $f \in IA(BNC)$, we will call the collection $\{f([0_\chi, 1_\chi]) \mid n \geq 1, \chi : \{1, \ldots, n\} \to \{\ell, r\}\} \subseteq \mathbb{C}$ the multiplicative net associated to f.

Incident Algebra on *BNC*

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The delta function on BNC (which is the identity element in $IA(BNC)$) and the zeta function on BNC are given by

$$\delta_{BNC}(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{otherwise} \end{cases} \qquad \zeta_{BNC}(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi \leq \sigma \\ 0 & \text{otherwise} \end{cases}$$

The Möbius function on BNC is defined such that

$$\mu_{BNC} * \zeta_{BNC} = \zeta_{BNC} * \mu_{BNC} = \delta_{BNC}.$$

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Note μ_{BNC} is multiplicative and for any $\pi, \sigma \in BNC(\chi)$

$$\mu_{BNC}(\pi, \sigma) = \mu(s_\chi^{-1} \cdot \pi, s_\chi^{-1} \cdot \sigma).$$

Incident Algebra on *BNC*

For $T_1, \ldots, T_n \in (\mathcal{A}, \varphi)$ and $\pi \in BNC(\chi)$ where $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ and $V_t = \{k_{t,1} < \cdots < k_{t,m_t}\}$ for $t \in \{1, \ldots, k\}$ being the blocks of π, let

$$\varphi_\pi(T_1, \ldots, T_n) := \prod_{t=1}^{k} \varphi(T_{k_{t,1}} \cdots T_{k_{t,m_t}}) \qquad \text{and}$$

$$\kappa_\pi(T_1, \ldots, T_n) := \sum_{\sigma \in BNC(\chi), \sigma \leq \pi} \varphi_\pi(T_1, \ldots, T_n) \mu_{BNC}(\sigma, \pi).$$

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Then one can show that

$$\kappa_\pi(T_1, \ldots, T_n) = \prod_{t=1}^{k} \kappa_{\pi|_{V_t}}(T_{k_{t,1}}, \ldots, T_{k_{t,m_t}}) \qquad \text{and}$$

$$\varphi(T_1 \ldots T_n) = \sum_{\pi \in BNC(\chi)} \kappa_\pi(T_1, \ldots, T_n).$$

In particular, $\kappa_{1_\chi} = \kappa_\chi$ are the (ℓ, r)-cumulant functions.

Bi-Free Families

Definition (Voiculescu; 2013)

A family

$$\pi = (((\mathfrak{B}_k, \beta_k), (\mathfrak{C}_k, \gamma_k)))_{k \in K}$$

of pairs of faces in a non-commutative probability space $(\mathcal{A}, \varphi)$ is said to be bi-freely independent if there exists a family of vector spaces with specified vector state $\left((\mathcal{X}_k, \mathcal{X}_k^\perp, \xi_k)\right)_{k \in K}$ and unital homomorphisms

$$l_k : \mathfrak{B}_k \to \mathcal{L}(\mathcal{X}_k) \text{ and } r_k : \mathfrak{C}_k \to \mathcal{L}(\mathcal{X}_k)$$

such that if

$$\tilde{\pi} = (((\mathfrak{B}_k, \beta_k \circ l_k), (\mathfrak{C}_k, \gamma_k \circ r_k)))_{k \in K}$$

is the family of faces of $(\mathcal{L}(\mathcal{X}), p)$ where $(\mathcal{X}, \mathcal{X}^\perp, \xi) = *_{k \in K} \left(\mathcal{X}_k, \mathcal{X}_k^\perp, \xi_k\right)$, then we have the equality of distributions $\mu_\pi = \mu_{\tilde{\pi}}$.

Bi-Free Families

Remarks

- If $z' = ((z'_i)_{i \in I}, (z'_j)_{j \in J})$ and $z'' = ((z''_i)_{i \in I}, (z''_j)_{j \in J})$ are bi-freely independent in $(\mathcal{A}, \varphi)$, then there exists universal polynomials such that any joint moment can be computed from moments of elements from individual families. These polynomials generalize the characterizing property of freely independent random variables.

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- There are universal bi-free cumulant polynomials that are additive on bi-free families. These polynomials generalize the free cumulants.
- "A pair of faces" was inspired by the Roman 'god of beginnings and transitions' Janus whom is depicted as having two faces - one looking to the future and one looking to the past.

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Definition

For $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ and $\epsilon \in \{', ''\}^n$, we define $LR(\chi, \epsilon)$ recursively. In the following, $'$ will be coloured black and $''$ will be coloured blue.

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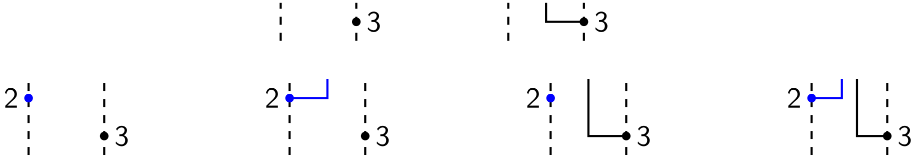

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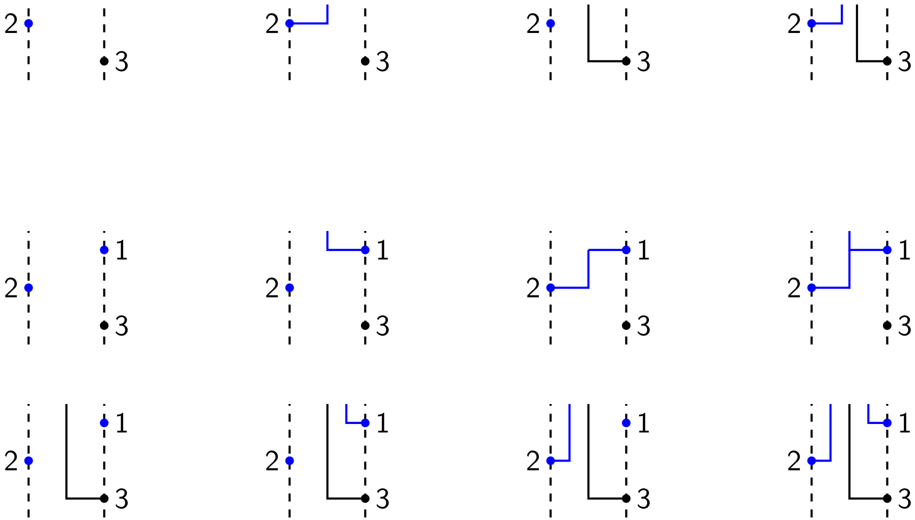
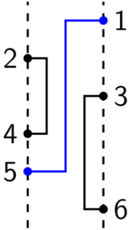


Universal Polynomials of Voiculescu

Definition

For $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ and $\epsilon \in \{', ''\}^n$, we define $BNC(\chi, \epsilon)$ to be the subset of all diagrams in $LR(\chi, \epsilon)$ where no strings reach the top.



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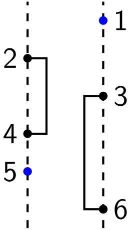

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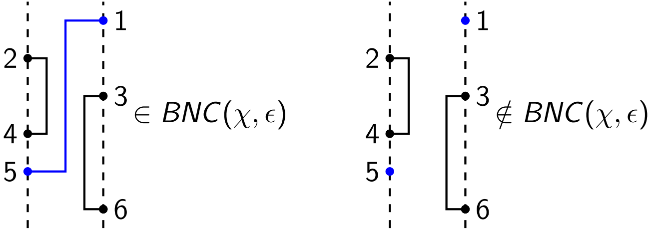


Definition

For $\alpha : \{1, \ldots, n\} \to I \sqcup J$, we define $BNC(\alpha, \epsilon)$ to be $BNC(\chi, \epsilon)$ for the unique χ where $\chi^{-1}(\{\ell\}) = \alpha^{-1}(I)$ where, in addition, we label the k^{th} node from the top $\alpha(k)$.

Universal Polynomials of Voiculescu

Theorem (CNS; 2014)

Let $z' = ((z'_i)_{i\in I}, (z'_j)_{j\in J})$ *and* $z'' = ((z''_i)_{i\in I}, (z''_j)_{j\in J})$ *be a pair of two-faced families in* $(\mathcal{A}, \varphi)$*. Then* z' *and* z'' *are bi-free if and only if for every map* $\alpha\colon \{1, \ldots, n\} \to I \sqcup J$ *and* $\epsilon \in \{', ''\}^n$ *we have*

$$\varphi_\alpha(z^\epsilon) = \sum_{\pi\in BNC(\alpha)} \left(\sum_{\substack{\sigma\in BNC(\alpha,\epsilon)\\ \sigma\geq_{\mathrm{lat}}\pi}} (-1)^{|\pi|-|\sigma|} \right) \varphi_\pi(z^\epsilon),$$

where $z^\epsilon = \left(z^{\epsilon_1}_{\alpha(1)}, \ldots, z^{\epsilon_n}_{\alpha(n)}\right)$.

The Tale of Two Sums

For bi-free independence, we need to verify

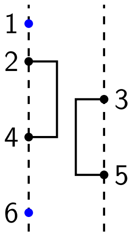
$$\varphi_\alpha(z^\epsilon) = \sum_{\pi \in BNC(\alpha)} \left(\sum_{\substack{\sigma \in BNC(\alpha,\epsilon) \\ \sigma \geq_{\mathrm{lat}} \pi}} (-1)^{|\pi|-|\sigma|} \right) \varphi_\pi(z^\epsilon),$$

whereas, using previous formulae, for combinatorially-bi-free independence, we need to verify

$$\varphi_\alpha(z^\epsilon) = \sum_{\pi \in BNC(\alpha)} \left(\sum_{\substack{\sigma \in BNC(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma) \right) \varphi_\pi(z^\epsilon).$$

The Tale of Two Sums

$$\sum_{\substack{\sigma \in BNC(\alpha,\epsilon) \\ \sigma \geq_{\mathrm{lat}} \pi}} (-1)^{|\pi|-|\sigma|} \qquad \sum_{\substack{\sigma \in BNC(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma)$$



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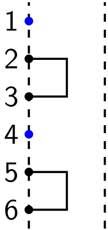

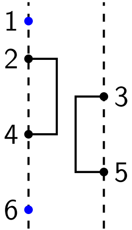

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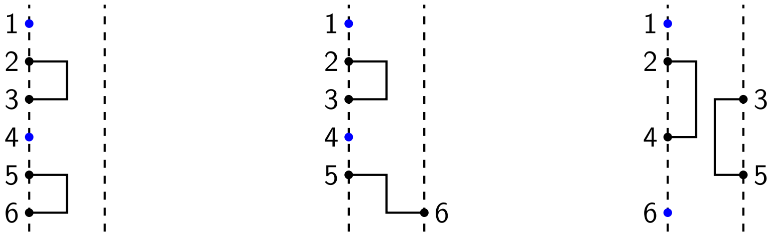
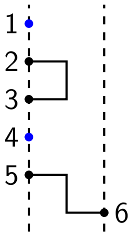
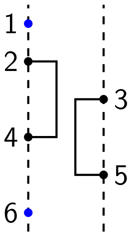


The Tale of Two Sums

$$\sum_{\substack{\sigma \in BNC(\alpha,\epsilon) \\ \sigma \geq_{\mathrm{lat}} \pi}} (-1)^{|\pi| - |\sigma|} \qquad \sum_{\substack{\sigma \in BNC(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma)$$





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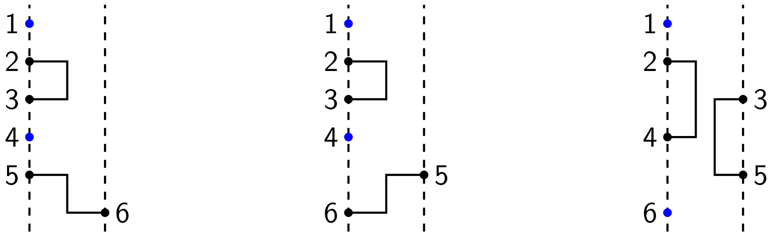

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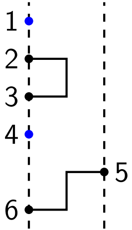

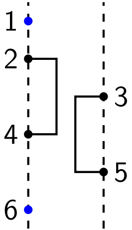

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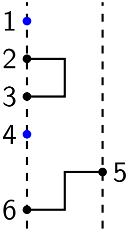
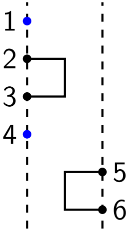
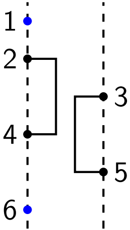
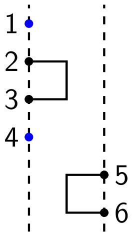


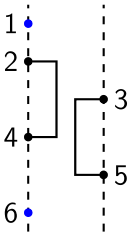




The Tale of Two Sums

$$\sum_{\substack{\sigma\in BNC(\alpha,\epsilon)\\ \sigma\geq_{\mathrm{lat}}\pi}} (-1)^{|\pi|-|\sigma|} \qquad\qquad \sum_{\substack{\sigma\in BNC(\alpha)\\ \pi\leq\sigma\leq\epsilon}} \mu_{BNC}(\pi,\sigma)$$





The Tale of Two Sums

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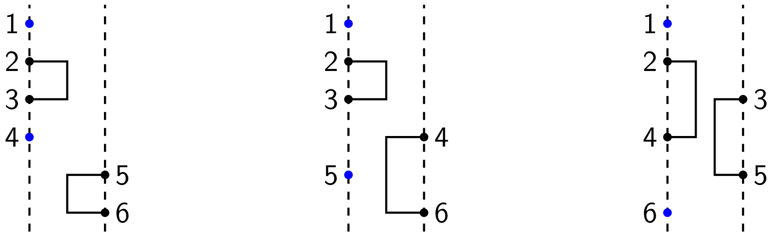
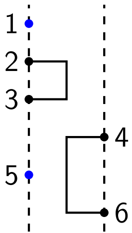
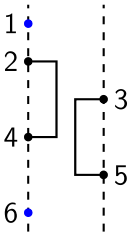


The Tale of Two Sums

$$\sum_{\substack{\sigma \in BNC(\alpha,\epsilon) \\ \sigma \geq_{\mathrm{lat}} \pi}} (-1)^{|\pi| - |\sigma|} \qquad \sum_{\substack{\sigma \in BNC(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma)$$





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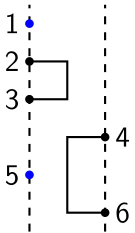
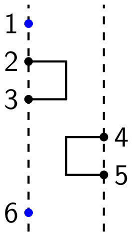
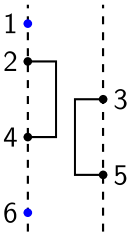






The Tale of Two Sums

For bi-free independence, we need to verify

$$\varphi_\alpha(z^\epsilon) = \sum_{\pi \in BNC(\alpha)} \left(\sum_{\substack{\sigma \in BNC(\alpha,\epsilon) \\ \sigma \geq_{\mathrm{lat}} \pi}} (-1)^{|\pi|-|\sigma|} \right) \varphi_\pi(z^\epsilon),$$

whereas, using previous formulae, for combinatorially-bi-free independence, we need to verify

$$\varphi_\alpha(z^\epsilon) = \sum_{\pi \in BNC(\alpha)} \left(\sum_{\substack{\sigma \in BNC(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi,\sigma) \right) \varphi_\pi(z^\epsilon).$$

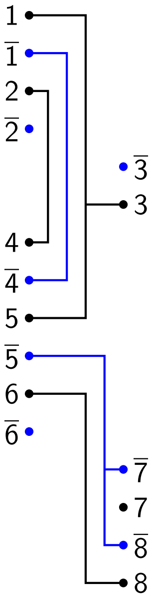
As these expressions are the same, the notions of bi-free independence and combinatorially-bi-free independence agree!!!

Multiplicative Bi-Free Convolution

Definition

For any $\chi : \{1, \ldots, n\} \to I \sqcup J$ and $\pi \in BNC(\chi)$, the Kreweras complement of π, denoted $K_{BNC}(\pi)$, is the element of $BNC(\chi)$ obtained by applying s_χ to the Kreweras complement in $NC(n)$ of $s_\chi^{-1} \cdot \pi$; explicitly

$$K_{BNC}(\pi) = s_\chi \cdot K_{NC}(s_\chi^{-1} \cdot \pi).$$



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Multiplicative Bi-Free Convolution

Theorem (CNS; 2014)

Let $z' = (\{z'_\ell\}, \{z'_r\})$ *and* $z'' = (\{z''_\ell\}, \{z''_r\})$ *be a bi-free family of pairs of faces and let* $z = (\{z'_\ell z''_\ell\}, \{z''_r z'_r\})$. *If* κ_z, $\kappa_{z'}$, *and* $\kappa_{z''}$ *are the multiplicative elements of* $IA(BNC)$ *with multiplicative nets corresponding to the* (ℓ, r)*-cumulants of* z, z', *and* z'' *respectively, then*

$$\kappa_z = \kappa_{z'} * \kappa_{z''}.$$

In particular

$$\kappa_\chi(z) = \sum_{\pi \in BNC(\chi)} \kappa_\pi(z')\kappa_{K_{BNC}(\pi)}(z'')$$

for all $\chi : \{1, \ldots, n\} \to \{\ell, r\}$.

Bi-Free Operator Model

Theorem (Nica; 1996)

Let $X = \{X_i\}_{i \in I}$ be a collection of random variables in a non-commutative probability space $(\mathcal{A}, \varphi)$. Let $\omega : \mathcal{L}\left(\mathcal{F}\left(\mathbb{C}^{|I|}\right)\right) \to \mathbb{C}$ be defined by $\omega(T) = \langle T\Omega, \Omega\rangle$ and consider the (unbounded) operator

$$\Theta_X := I_{\mathcal{F}(\mathbb{C}^{|I|})} + \sum_{k \geq 1} \sum_{i_1, \ldots, i_k \in I} \kappa(X_{i_1}, \ldots, X_{i_k}) L_{i_k} \cdots L_{i_1}.$$

The joint distribution of the operators

$$Z_i := L_i^* \Theta_X = L_i^* + \sum_{k \geq 0} \sum_{i_1, \ldots, i_k \in I} \kappa(X_{i_1}, \ldots, X_{i_k}, X_i) L_{i_k} \cdots L_{i_1}$$

with respect to ω is the same as the joint distribution of $\{X_i\}_{i \in I}$ with respect to φ.

Bi-Free Operator Model

Theorem (CNS; 2014)

Let $z = (\{z_i\}_{i\in I}, \{z_j\}_{j\in J})$ *be a pair of faces in a non-commutative probability space* $(\mathcal{A}, \varphi)$*. Consider the (unbounded) operator*

$$\Theta_z := I + \sum_{n\geq 1} \sum_{\alpha:\{1,\ldots,n\}\to I\sqcup J} \kappa_\alpha(z) T_\alpha.$$

For $k \in I \sqcup J$ *define*

$$Z_k := L_k^* \Theta_z = L_k^* + \sum_{n\geq 0} \sum_{\substack{\alpha:\{1,\ldots,n+1\}\to I\sqcup J \\ \alpha(n+1)=k}} \kappa_\alpha(z) S_\alpha.$$

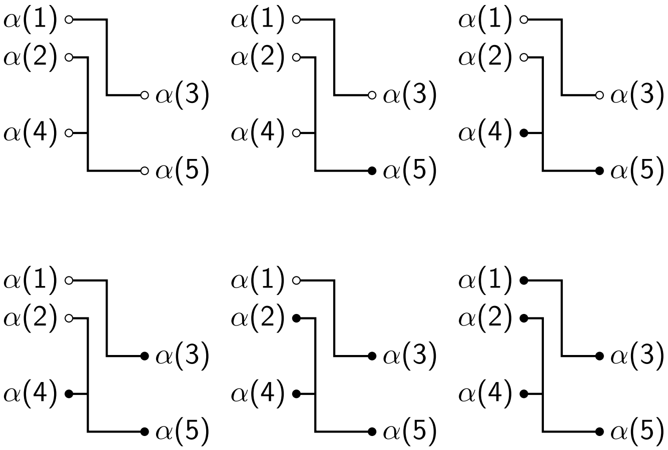
Then, if $T \in \mathrm{alg}(\{Z_k\}_{k\in I\sqcup J})$ *then* $\langle T\Omega, \Omega\rangle$ *is well-defined. Moreover, if* $\psi(T) = \langle T\Omega, \Omega\rangle$*, the joint distribution of* $\{Z_k\}_{k\in I\sqcup J}$ *with respect to* ψ *is the same as the joint distribution of* z *with respect to* φ*.*

Bi-Free Operator Model

Definition

Let $\alpha : \{1, \ldots, n\} \to I \sqcup J$. For a bi-non-crossing partition $\pi \in BNC(\alpha)$, the skeletons on π are the diagrams of π where the k^{th}-node from the top is labelled $\alpha(k)$, each node is either an open circle or a closed circle, and if node $\alpha(k)$ is open, then node $\alpha(k')$ is open for all $k' < k$.

Bi-Free Operator Model



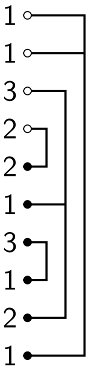
Bi-Free Operator Model

The idea behind this operator model is to show that products of L_i^* and $\kappa(X_{i_1}, \ldots, X_{i_k}, X_i) L_{i_k} \cdots L_{i_1}$ that have non-zero ω value correspond to non-crossing partitions on the generators and the value of ω on the product is the correct product of free cumulants. These operators can be viewed as operators on skeletons as follows.

Note there is not a bijection between skeletons and basis vectors on the Fock space as the skeleton retains information on the order and type of operators applied.

For the vector $e_3 \otimes e_1 \otimes e_2$

Bi-Free Operator Model



is constructed by first applying $L_1^* L_1 L_1 L_1$ (weighted by $\kappa(X_1, X_1, X_1)$), then $L_2^* L_2 L_1 L_3$ (weighted by $\kappa(X_3, X_1, X_2)$), then $L_1^* L_1 L_3$ (weighted by $\kappa(X_3, X_1)$), then L_3^*, then L_1^*, and then $L_2^* L_2 L_2$ (weighted by $\kappa(X_2, X_2)$).

Bi-Free Operator Model

- The operator model acts on $\mathcal{F}(\mathbb{C}^{|I|+|J|})$ with $\{e_i\}_{i\in I} \cup \{e_j\}_{j\in J}$ an orthonormal basis for $\mathbb{C}^{|I|+|J|}$. The vectors $\{e_i\}_{i\in I}$ are 'left basis vectors' and $\{e_j\}_{j\in J}$ are 'right basis vectors'.

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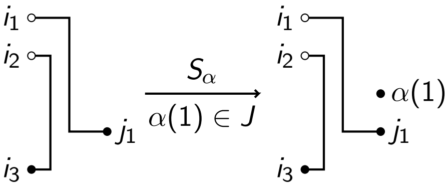
- In particular, S_α acts on certain vectors differently but ultimately behave like creation operators where the creation may occur in multiple places on the left, right, or in the middle of a tensor.
- If $\alpha : \{1, \ldots, n\} \to I$, S_α acts on tensor products of the 'left basis vectors' as the appropriate left creation operators to recover the usual operator model.

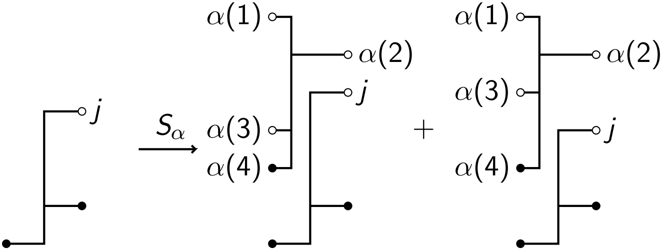
Bi-Free Operator Model

For $|\alpha| = 1$,



Bi-Free Operator Model

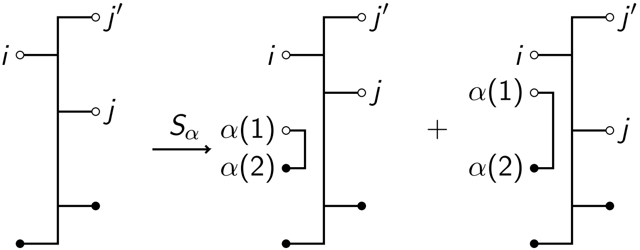
For $\alpha(n) \in I$,



$$S_\alpha e_j = e_{\alpha(3)} \otimes e_j \otimes e_{\alpha(2)} \otimes e_{\alpha(1)} + e_j \otimes e_{\alpha(3)} \otimes e_{\alpha(2)} \otimes e_{\alpha(1)}.$$

Bi-Free Operator Model

For $\alpha(n) \in I$, if $\alpha(k) \in I$ for all $k \in \{1, \ldots, n-1\}$,



$$S_\alpha(e_j \otimes e_i \otimes e_{j'}) = e_{\alpha(1)} \otimes e_j \otimes e_i \otimes e_{j'} + e_j \otimes e_{\alpha(1)} \otimes e_i \otimes e_{j'}$$

Otherwise S_α applied to this vector is zero.

Thanks for Listening!

arXiv:1403.4907