Free monotone transport without a trace

Brent Nelson

UCLA

March 27, 2014

- $(\mathcal{H}_\mathbb{R}, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and $\{U_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H}_\mathbb{R})$ a strongly continuous one-parameter group of orthogonal transformations.

- $(\mathcal{H}_{\mathbb{R}}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $\{U_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H}_{\mathbb{R}})$ a strongly continuous one-parameter group of orthogonal transformations.
- $\mathcal{H}_{\mathbb{C}} := \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$, extend inner product to be $\mathbb{C}$-linear in second coordinate, and extend $\{U_t\}_{t\in\mathbb{R}}$ to unitary transformations.

- $(\mathcal{H}_\mathbb{R}, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and $\{U_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H}_\mathbb{R})$ a strongly continuous one-parameter group of orthogonal transformations.
- $\mathcal{H}_\mathbb{C} := \mathcal{H}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$, extend inner product to be $\mathbb{C}$-linear in second coordinate, and extend $\{U_t\}_{t\in\mathbb{R}}$ to unitary transformations.
- $\{U_t\}_{t\in\mathbb{R}}$ has a generator A: $A \geq 0$, non-singular, and $A^{it} = U_t$ for all $t \in \mathbb{R}$.

- $(\mathcal{H}_\mathbb{R}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $\{U_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H}_\mathbb{R})$ a strongly continuous one-parameter group of orthogonal transformations.
- $\mathcal{H}_\mathbb{C} := \mathcal{H}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$, extend inner product to be $\mathbb{C}$-linear in second coordinate, and extend $\{U_t\}_{t\in\mathbb{R}}$ to unitary transformations.
- $\{U_t\}_{t\in\mathbb{R}}$ has a generator A: $A \geq 0$, non-singular, and $A^{it} = U_t$ for all $t \in \mathbb{R}$.
- Examples. (i) $U_t = 1$ and $A = 1$. (ii):

$$U_t = \begin{pmatrix} \cos(t\log\lambda) & -\sin(t\log\lambda) \\ \sin(t\log\lambda) & \cos(t\log\lambda) \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\lambda+\lambda^{-1}}{2} & -i\frac{\lambda-\lambda^{-1}}{2} \\ i\frac{\lambda-\lambda^{-1}}{2} & \frac{\lambda+\lambda^{-1}}{2} \end{pmatrix}.$$

- $(\mathcal{H}_\mathbb{R}, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and $\{U_t\}_{t\in\mathbb{R}} \subset \mathcal{B}(\mathcal{H}_\mathbb{R})$ a strongly continuous one-parameter group of orthogonal transformations.
- $\mathcal{H}_\mathbb{C} := \mathcal{H}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \mathcal{H}_\mathbb{R} + i\mathcal{H}_\mathbb{R}$, extend inner product to be $\mathbb{C}$-linear in second coordinate, and extend $\{U_t\}_{t\in\mathbb{R}}$ to unitary transformations.
- $\{U_t\}_{t\in\mathbb{R}}$ has a generator A: $A \geq 0$, non-singular, and $A^{it} = U_t$ for all $t \in \mathbb{R}$.
- Examples. (i) $U_t = 1$ and $A = 1$. (ii):

$$U_t = \begin{pmatrix} \cos(t \log \lambda) & -\sin(t \log \lambda) \\ \sin(t \log \lambda) & \cos(t \log \lambda) \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{\lambda+\lambda^{-1}}{2} & -i\frac{\lambda-\lambda^{-1}}{2} \\ i\frac{\lambda-\lambda^{-1}}{2} & \frac{\lambda+\lambda^{-1}}{2} \end{pmatrix}.$$

- Define a new inner product

$$\langle f, g \rangle_U = \left\langle \frac{2}{1+A^{-1}} f, g \right\rangle, \qquad f, g \in \mathcal{H}_\mathbb{C},$$

and let $\mathcal{H} = \overline{\mathcal{H}_\mathbb{C}}^{\|\cdot\|_U}$.

- For $-1 < q < 1$, the *q-Fock space* $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ with respect to the inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_{U,q} := \delta_{n=m} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{k=1}^{n} \left\langle f_k, g_{\pi(k)} \right\rangle_U .$$

- For $-1 < q < 1$, the *q-Fock space* $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathbb{C}\Omega \oplus \bigoplus_{n=1}^\infty \mathcal{H}^{\otimes n}$ with respect to the inner product

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_{U,q} := \delta_{n=m} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{k=1}^n \left\langle f_k, g_{\pi(k)} \right\rangle_U .$$

- For each $f \in \mathcal{H}$, define the *left q-creation operator* $l_q(f) \in \mathcal{B}(\mathcal{F}_q(\mathcal{H}))$ densely by

$$\begin{aligned} l_q(f)\Omega &= f, \qquad \text{and} \\ l_q(f) g_1 \otimes \cdots \otimes g_n &= f \otimes g_1 \otimes \cdots \otimes g_n. \end{aligned}$$

- Its adjoint, the *left q-annihilation operator* $l_q(f)^*$ is densely defined by

$$l_q(f)^*\Omega = 0, \qquad \text{and}$$

$$l_q(f)^* g_1 \otimes \cdots \otimes g_n = \sum_{k=1}^n q^{k-1} \left\langle f, g_k \right\rangle_U g_1 \otimes \cdots \otimes \hat{g}_k \otimes \cdots \otimes g_n.$$

- Denote $s_q(f) = l_q(f) + l_q(f)^*$, $f \in \mathcal{H}$. When $q = 0$, this is semicircular.

- Denote $s_q(f) = l_q(f) + l_q(f)^*$, $f \in \mathcal{H}$. When $q = 0$, this is semicircular.
- The *q-deformed Araki-Woods algebra* is then defined as

$$\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)'' = W^*(s_q(f) \colon f \in \mathcal{H}_{\mathbb{R}}) \subset \mathcal{B}(\mathcal{F}_q(\mathcal{H})).$$

For $q = 0$ we simply write $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ and call it the free Araki-Woods factor.

- Denote $s_q(f) = l_q(f) + l_q(f)^*$, $f \in \mathcal{H}$. When $q = 0$, this is semicircular.
- The *q-deformed Araki-Woods algebra* is then defined as

$$\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' = W^*(s_q(f) \colon f \in \mathcal{H}_\mathbb{R}) \subset \mathcal{B}(\mathcal{F}_q(\mathcal{H})).$$

  For $q = 0$ we simply write $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ and call it the free Araki-Woods factor.
- The vector-state corresponding to Ω, $\varphi(= \varphi_{q,U})$, is faithful and non-degenerate on $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$, and is called a *q-quasi-free state*, or a *free quasi-free state* when $q = 0$.

- Denote $s_q(f) = l_q(f) + l_q(f)^*$, $f \in \mathcal{H}$. When $q = 0$, this is semicircular.
- The *q-deformed Araki-Woods algebra* is then defined as

$$\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' = W^*(s_q(f) \colon f \in \mathcal{H}_\mathbb{R}) \subset \mathcal{B}(\mathcal{F}_q(\mathcal{H})).$$

For $q = 0$ we simply write $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ and call it the free Araki-Woods factor.

- The vector-state corresponding to Ω, $\varphi(= \varphi_{q,U})$, is faithful and non-degenerate on $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$, and is called a *q-quasi-free state*, or a *free quasi-free state* when $q = 0$.
- The modular automorphism group $\{\sigma_t^\varphi\}$ of φ is well known:

$$\sigma_z^\varphi(s_q(e_j)) = \sum_{k=1}^N [A^{iz}]_{jk} s_q(e_k), \qquad z \in \mathbb{C},$$

where $\{e_j\} \subset \mathcal{H}_\mathbb{R}$ is an o.n. basis.

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$		
$U_t \neq 1$		

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$	
$U_t \neq 1$		

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	
$U_t \neq 1$		

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$
$U_t \neq 1$		

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$		

Let $N = \dim \mathcal{H}_\mathbb{R}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	[Guionnet, Shlyakhtenko 2013]: for $N < \infty$ there is a constant $C(N) > 0$ so that for $\lvert q\rvert < C(N)$ this is isomorphic to $L(\mathbb{F}_N)$ via transport arguments
$U_t \neq 1$		

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$		

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$	$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$	

Let $N = \dim \mathcal{H}_\mathbb{R}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_\mathbb{R})$ the q-deformed free group factor
$U_t \neq 1$	$\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ free Araki-Woods factor	

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$	[Shlyakhtenko 1997]: factoriality and type classification are obtained, latter determined by spectrum(A)	

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$	$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ free Araki-Woods factor	

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$	$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ free Araki-Woods factor	$\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$	$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ free Araki-Woods factor	$\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ q-deformed Araki-Woods algebra

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$	$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ free Araki-Woods factor	[Hiai 2003]: factoriality and type classification are obtained for A with either infinitely many mutually orthogonal eigenvectors or no eigenvectors (i.e. $N = \infty$)

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$	$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ free Araki-Woods factor	[N. 2013]: for $N < \infty$ there is a constant $C(N,U) > 0$ so that for $\lvert q\rvert < C(N,U)$ $\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ is isomorphic to $\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ via transport arguments

Let $N = \dim \mathcal{H}_{\mathbb{R}}$.

	$q = 0$	$q \neq 0$
$U_t = 1$	$L(\mathbb{F}_N)$ the free group factor with N generators	$\Gamma_q(\mathcal{H}_{\mathbb{R}})$ the q-deformed free group factor
$U_t \neq 1$	$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)''$ free Araki-Woods factor	$\Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$ q-deformed Araki-Woods algebra

- Fix an orthonormal basis $\{e_n\}$ of $\mathcal{H}_{\mathbb{R}}$, it will still be a normalized basis for $\mathcal{H}$ but not orthogonal unless $U_t = 1$.
- Denote $s_q(e_n) = X_n (= X_n^{(q)})$ and want to compute φ on monomials in $\mathbb{C}\langle\{X_n\}\rangle$.
- The combinatorics associated to this task makes interesting transistions as we vary U_t and q.

Example 1.1 (Free group factor; top-left corner)

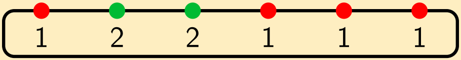
Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q = 0$.

Example 1.1 (Free group factor; top-left corner)

Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q = 0$. Wish to compute $\varphi(X_1 X_2^2 X_1^3)$.

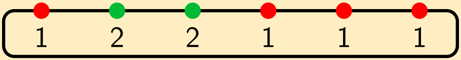
Example 1.1 (Free group factor; top-left corner)

Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q = 0$. Wish to compute $\varphi(X_1 X_2^2 X_1^3)$. Draw



Example 1.1 (Free group factor; top-left corner)

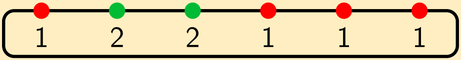
Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q = 0$. Wish to compute $\varphi(X_1 X_2^2 X_1^3)$. Draw



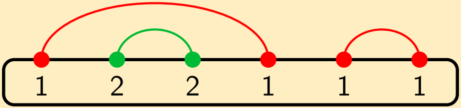
Pair up nodes of the same color and connect them with strings so that strings do not cross. In this case there are two such diagrams:

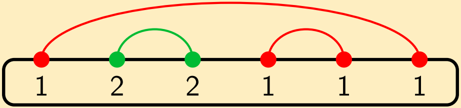
Example 1.1 (Free group factor; top-left corner)

Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q = 0$. Wish to compute $\varphi(X_1 X_2^2 X_1^3)$. Draw



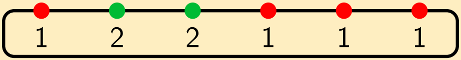
Pair up nodes of the same color and connect them with strings so that strings do not cross. In this case there are two such diagrams:



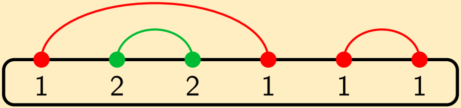


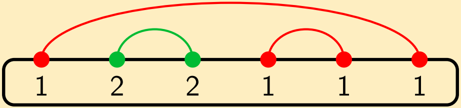
Example 1.1 (Free group factor; top-left corner)

Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q = 0$. Wish to compute $\varphi(X_1 X_2^2 X_1^3)$. Draw



Pair up nodes of the same color and connect them with strings so that strings do not cross. In this case there are two such diagrams:





Each such diagram contributes a term of 1 so that $\varphi(X_1 X_2^2 X_1^3) = 2$.

Example 1.2 (q-deformed free group factor; top-right corner)

Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q \neq 0$.

Example 1.2 (q-deformed free group factor; top-right corner)

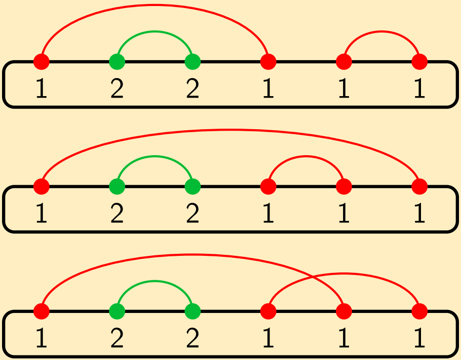
Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q \neq 0$. Again we compute $\varphi(X_1 X_2^2 X_1^3)$.

Example 1.2 (q-deformed free group factor; top-right corner)

Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q \neq 0$. Again we compute $\varphi(X_1 X_2^2 X_1^3)$. Same setup as before, but now strings may cross:

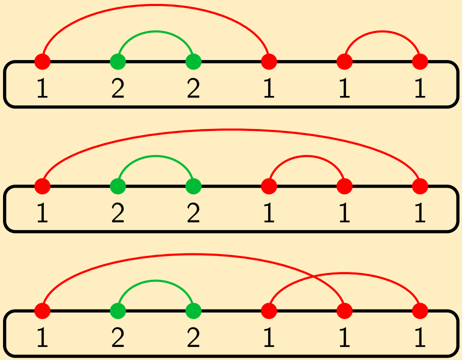
Example 1.2 (q-deformed free group factor; top-right corner)

Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q \neq 0$. Again we compute $\varphi(X_1 X_2^2 X_1^3)$. Same setup as before, but now strings may cross:



Example 1.2 (q-deformed free group factor; top-right corner)

Suppose $U_t = 1$ for all $t \in \mathbb{R}$ and $q \neq 0$. Again we compute $\varphi(X_1 X_2^2 X_1^3)$. Same setup as before, but now strings may cross:



A crossing adds a factor of q to the weight of a diagram, so here $\varphi(X_1 X_2 X_2 X_1^3) = 2 + q$.

Example 1.3 (Free Araki-Woods factor; bottom-left corner)

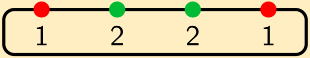
Now let U_t be non-trivial and $q = 0$.

Example 1.3 (Free Araki-Woods factor; bottom-left corner)

Now let U_t be non-trivial and $q = 0$. We will compute $\varphi(X_1 X_2 X_2 X_1)$.

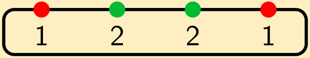
Example 1.3 (Free Araki-Woods factor; bottom-left corner)

Now let U_t be non-trivial and $q = 0$. We will compute $\varphi(X_1 X_2 X_2 X_1)$. Setup is again the same:



Example 1.3 (Free Araki-Woods factor; bottom-left corner)

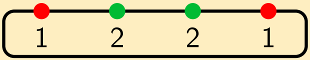
Now let U_t be non-trivial and $q = 0$. We will compute $\varphi(X_1 X_2 X_2 X_1)$. Setup is again the same:



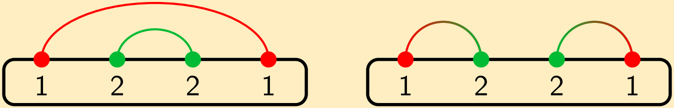
However, now pairings can be made regardless of color:

Example 1.3 (Free Araki-Woods factor; bottom-left corner)

Now let U_t be non-trivial and $q = 0$. We will compute $\varphi(X_1 X_2 X_2 X_1)$. Setup is again the same:

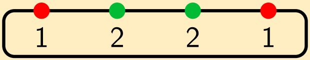


However, now pairings can be made regardless of color:

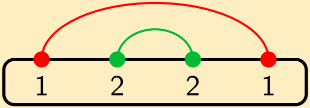


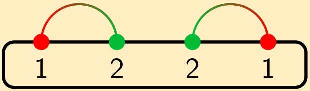
Example 1.3 (Free Araki-Woods factor; bottom-left corner)

Now let U_t be non-trivial and $q = 0$. We will compute $\varphi(X_1X_2X_2X_1)$. Setup is again the same:



However, now pairings can be made regardless of color:

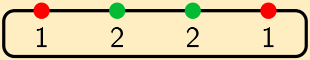




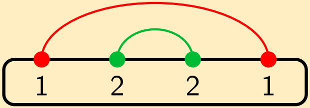
When nodes corresponding to the vectors e_j and e_k are connected (form left to right), the diagram gains a factor of $\langle e_j, e_k \rangle_U$.

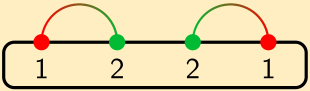
Example 1.3 (Free Araki-Woods factor; bottom-left corner)

Now let U_t be non-trivial and $q = 0$. We will compute $\varphi(X_1X_2X_2X_1)$. Setup is again the same:



However, now pairings can be made regardless of color:





When nodes corresponding to the vectors e_j and e_k are connected (form left to right), the diagram gains a factor of $\langle e_j, e_k\rangle_U$. So here

$$\varphi(X_1X_2X_2X_1) = 1 + \langle e_1, e_2\rangle_U \langle e_2, e_1\rangle_U .$$

Example 1.4 (q-deformed Araki-Woods algebra; bottom-right corner)

Finally let U_t be non-trivial and $q \neq 0$.

Example 1.4 (q-deformed Araki-Woods algebra; bottom-right corner)

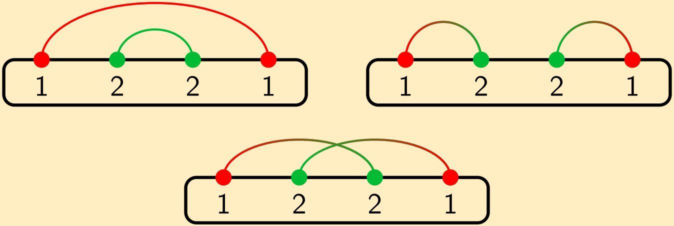
Finally let U_t be non-trivial and $q \neq 0$. We again compute $\varphi(X_1 X_2 X_2 X_1)$.

Example 1.4 (q-deformed Araki-Woods algebra; bottom-right corner)

Finally let U_t be non-trivial and $q \neq 0$. We again compute $\varphi(X_1X_2X_2X_1)$. The transition to non-zero q is the same as before: strings may cross and each contributes a factor of q.

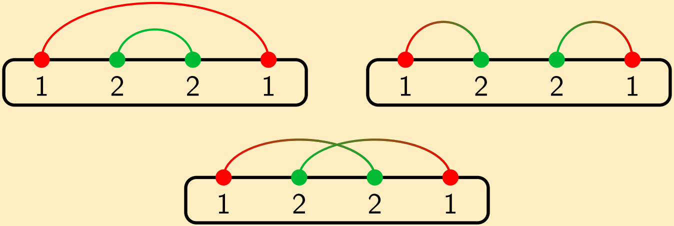
Example 1.4 (q-deformed Araki-Woods algebra; bottom-right corner)

Finally let U_t be non-trivial and $q \neq 0$. We again compute $\varphi(X_1X_2X_2X_1)$. The transition to non-zero q is the same as before: strings may cross and each contributes a factor of q.



Example 1.4 (q-deformed Araki-Woods algebra; bottom-right corner)

Finally let U_t be non-trivial and $q \neq 0$. We again compute $\varphi(X_1X_2X_2X_1)$. The transition to non-zero q is the same as before: strings may cross and each contributes a factor of q.



So here

$$\varphi(X_1X_2X_2X_1) = 1 + \langle e_1, e_2\rangle_U \langle e_2, e_1\rangle_U + q\langle e_1, e_2\rangle_U \langle e_2, e_1\rangle_U .$$

Let $N < \infty$ and $\{U_t : t \in \mathbb{R}\}$ with generator A.
Fix $-1 < q < 1$, and write $X_j = X_j^{(q)}$ and $\varphi = \varphi_{q,U}$. Denote $\mathscr{P} = \mathbb{C}\langle X_1, \ldots, X_N \rangle$.

Let $N < \infty$ and $\{U_t : t \in \mathbb{R}\}$ with generator A.
Fix $-1 < q < 1$, and write $X_j = X_j^{(q)}$ and $\varphi = \varphi_{q,U}$. Denote $\mathscr{P} = \mathbb{C}\langle X_1, \ldots, X_N\rangle$.
For each $j = 1, \ldots, N$ we define the *σ-difference quotient* $\partial_j : \mathscr{P} \to \mathscr{P} \otimes \mathscr{P}^{op}$ and the *σ-cyclic derivative* $\mathscr{D}_j : \mathscr{P} \to \mathscr{P}$ by

$$\partial_j(X_{k_1} \cdots X_{k_n}) = \sum_{l=1}^{n} \left[\frac{2}{1+A}\right]_{k_l j} X_{k_1} \cdots X_{k_{l-1}} \otimes X_{k_{l+1}} \cdots X_{k_n},$$

and

$$\mathscr{D}_j(X_{k_1} \cdots X_{k_n}) = \sum_{l=1}^{n} \left[\frac{2}{1+A}\right]_{j k_l} \sigma^{\varphi}_{-i}(X_{k_{l+1}} \cdots X_{k_n}) X_{k_1} \cdots X_{k_{l-1}},$$

respectively.

Example 2.1

Let $\alpha_{jk} = \left[\frac{2}{1+A}\right]_{jk}$

Example 2.1

Let $\alpha_{jk} = \left[\frac{2}{1+A}\right]_{jk}$ then

$$\partial_1(X_1X_2X_1) = \alpha_{11} 1 \otimes X_2X_1 + \alpha_{21} X_1 \otimes X_1 + \alpha_{11} X_1X_2 \otimes 1,$$

and

$$\mathscr{D}_1(X_1X_2X_1) = \alpha_{11}\sigma_{-i}^{\varphi}(X_2X_1) + \alpha_{12}\sigma_{-i}^{\varphi}(X_1)X_1 + \alpha_{11}X_1X_2.$$

Example 2.1

Let $\alpha_{jk} = \left[\frac{2}{1+A}\right]_{jk}$ then

$$\partial_1(X_1X_2X_1) = \alpha_{11} 1 \otimes X_2X_1 + \alpha_{21}X_1 \otimes X_1 + \alpha_{11}X_1X_2 \otimes 1,$$

and

$$\mathscr{D}_1(X_1X_2X_1) = \alpha_{11}\sigma^{\varphi}_{-i}(X_2X_1) + \alpha_{12}\sigma^{\varphi}_{-i}(X_1)X_1 + \alpha_{11}X_1X_2.$$

Special example:

$$V_0 = \frac{1}{2}\sum_{j,k=1}^{N}\left[\frac{1+A}{2}\right]_{jk} X_kX_j.$$

An easy computation shows $\mathscr{D}_j V_0 = X_j$.

Let ψ be a state defined on $\mathscr{P}$ and let $V \in \mathscr{P}$. We say that ψ satisfies the *Schwinger-Dyson equation with potential* V if for each $j = 1, \ldots, N$

$$\psi(\mathscr{D}_j V \cdot P) = \psi \otimes \psi^{op}(\partial_j P)$$

for all $P \in \mathscr{P}$, and we call ψ the *free Gibbs state with potential* V, $\psi = \varphi_V$.

Let ψ be a state defined on $\mathscr{P}$ and let $V \in \mathscr{P}$. We say that ψ satisfies the *Schwinger-Dyson equation with potential* V if for each $j = 1, \ldots, N$

$$\psi(\mathscr{D}_j V \cdot P) = \psi \otimes \psi^{op}(\partial_j P)$$

for all $P \in \mathscr{P}$, and we call ψ the *free Gibbs state with potential* V, $\psi = \varphi_V$.
In particular, $\varphi_{0,U}$ is the free Gibbs state with potential V_0, where V_0 is as above.
Since $\mathscr{D}_j V_0 = X_j$, this is immediate from considering the diagrams in our earlier examples.

We collect everything in vector and matrix notation:

$$X := (X_1, \ldots, X_N) \in \mathscr{P}^N;$$

for $P \in \mathscr{P}$

$$\mathscr{D}P := (\mathscr{D}_1 P, \ldots, \mathscr{D}_N P) \in \mathscr{P}^N;$$

and for $G \in \mathscr{P}^N$ we define $\mathscr{J}_\sigma G \in M_{N\times N}(\mathscr{P} \otimes \mathscr{P}^{op})$ by

$$[\mathscr{J}_\sigma G]_{jk} = \partial_k(G_j)$$

for each $j, k \in \{1, \ldots, N\}$.
So the Schwinger-Dyson equation with potential V reads:

$$\psi(\mathscr{D}V \cdot G) = \psi \otimes \psi^{op} \otimes \mathrm{Tr}(\mathscr{J}_\sigma G)$$

Theorem 2.2 (N. 2013)

For $|q|$ sufficiently small (depending on N and $\|A\|$) $\varphi_{q,U}$ satisfies the Schwinger-Dyson equation with a potential $V_q \in \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''$:

$$\varphi_{q,U}(\mathscr{D}V_q \# P) = \varphi_{q,U} \otimes \varphi_{q,U}^{op} \otimes Tr(\mathscr{J}_\sigma P) \tag{1}$$

for all $P \in \mathbb{C}\left\langle X_1^{(q)}, \ldots, X_N^{(q)} \right\rangle^N$.
Moreover, $V_q \to V_0$ as $|q| \to 0$ (with respect to a particular Banach norm).

Theorem 2.2 (N. 2013)

For $|q|$ sufficiently small (depending on N and $\|A\|$) $\varphi_{q,U}$ satisfies the Schwinger-Dyson equation with a potential $V_q \in \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$:

$$\varphi_{q,U}(\mathscr{D}V_q \# P) = \varphi_{q,U} \otimes \varphi_{q,U}^{op} \otimes Tr(\mathscr{J}_\sigma P) \tag{1}$$

for all $P \in \mathbb{C}\left\langle X_1^{(q)}, \ldots, X_N^{(q)} \right\rangle^N$.
Moreover, $V_q \to V_0$ as $|q| \to 0$ (with respect to a particular Banach norm).

Theorem 2.3 (Guionnet, Maurel-Segala 2006)

For potentials V sufficiently close to V_0 (with respect to particular Banach norm), the free Gibbs state with potential V is unique.

For $P \in \mathscr{P}$ we write

$$P = \sum_{n=0}^{\deg P} \sum_{j_1,\ldots,j_n=1}^{N} c(j_1,\ldots,j_n) X_{j_1} \cdots X_{j_n} = \sum_{n=0}^{\deg P} \pi_n(P),$$

and define

$$\|P\|_R := \sum_{n=0}^{\deg P} \sum_{j_1,\ldots,j_n=1}^{N} |c(j_1,\ldots,j_n)| R^n$$

and $\mathscr{P}^{(R)} = \overline{\mathscr{P}}^{\|\cdot\|_R}$.

For $P \in \mathscr{P}$ we write

$$P = \sum_{n=0}^{\deg P} \sum_{j_1,\ldots,j_n=1}^{N} c(j_1,\ldots,j_n) X_{j_1} \cdots X_{j_n} = \sum_{n=0}^{\deg P} \pi_n(P),$$

and define

$$\|P\|_R := \sum_{n=0}^{\deg P} \sum_{j_1,\ldots,j_n=1}^{N} |c(j_1,\ldots,j_n)| R^n$$

and $\mathscr{P}^{(R)} = \overline{\mathscr{P}}^{\|\cdot\|_R}$.

Define σ-*cyclic rearrangements* by

$$\rho(X_{j_1} \cdots X_{j_n}) := \sigma_{-i}^{\varphi}(X_{j_n}) X_{j_1} \cdots X_{j_{n-1}},$$

and define

$$\|P\|_{R,\sigma} := \sum_{n=0}^{\deg P} \sup_{k_n \in \mathbb{Z}} \|\rho^{k_n}(\pi_n(P))\|_R$$

and $\mathscr{P}^{(R,\sigma)} = \overline{\mathscr{P}\mathit{finite}}^{\|\cdot\|_{R,\sigma}}$.

Denote

$$\mathscr{P}_{\varphi}^{(R)} := \mathscr{P}^{(\varphi)} \cap \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''_{\varphi} \qquad \mathscr{P}_{\varphi}^{(R,\sigma)} = \mathscr{P}^{(R,\sigma)} \cap \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''_{\varphi},$$

Denote

$$\mathscr{P}_\varphi^{(R)} := \mathscr{P}^{(\varphi)} \cap \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)_\varphi'' \qquad \mathscr{P}_\varphi^{(R,\sigma)} = \mathscr{P}^{(R,\sigma)} \cap \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)_\varphi'',$$

and

$$\mathscr{P}_{c.s.}^{(R,\sigma)} := \{P \in \mathscr{P}^{(R,\sigma)} \colon \rho(P) = P\}.$$

$V_q \in \mathscr{P}_{c.s.}^{(R,\sigma)}$ for sufficiently small $|q|$.

Denote

$$\mathscr{P}_{\varphi}^{(R)} := \mathscr{P}^{(\varphi)} \cap \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''_{\varphi} \qquad \mathscr{P}_{\varphi}^{(R,\sigma)} = \mathscr{P}^{(R,\sigma)} \cap \Gamma_q(\mathcal{H}_{\mathbb{R}}, U_t)''_{\varphi},$$

and

$$\mathscr{P}_{c.s.}^{(R,\sigma)} := \{P \in \mathscr{P}^{(R,\sigma)} \colon \rho(P) = P\}.$$

$V_q \in \mathscr{P}_{c.s.}^{(R,\sigma)}$ for sufficiently small $|q|$.
The map

$$\mathscr{S} := \sum_{n\geq 1} \frac{1}{n} \sum_{k=0}^{n-1} \rho^k \circ \pi_n,$$

is a contraction

$$\mathscr{P}_{\varphi}^{(R,\sigma)} \xrightarrow{\mathscr{S}} \mathscr{P}_{c.s.}^{(R,\sigma)}$$

- Classically, transport is a map $T\colon (X,\mu) \to (Z,\nu)$ so that $T_*(\mu) = \nu$. Yields a measure-preserving embedding $L^\infty(Z,\nu) \hookrightarrow L^\infty(X,\mu)$ via $f \mapsto f \circ T$.

- Classically, transport is a map $T\colon (X,\mu) \to (Z,\nu)$ so that $T_*(\mu) = \nu$. Yields a measure-preserving embedding $L^\infty(Z,\nu) \hookrightarrow L^\infty(X,\mu)$ via $f \mapsto f \circ T$.
- Given N-tuples $X_1, \ldots, X_N \in (\mathcal{A}, \varphi)$ and $Z_1, \ldots, Z_N \in (\mathcal{B}, \psi)$ write $X = (X_1, \ldots, X_N)$ and $Z = (Z_1, \ldots, Z_N)$ and let φ_X and ψ_Z be their respective joint laws.
- Transport from φ_X to ψ_Z is an N-tuple $Y_1, \ldots, Y_N \in W^*(X_1, \ldots, X_N)$ whose joint law with respect to φ, φ_Y, is the same as ψ_Z.
- Its existence implies $W^*(Z_1, \ldots, Z_N) \hookrightarrow W^*(X_1, \ldots, X_N)$ via $Z_j \mapsto Y_j$, and the embedding is state preserving.

Theorem 2.4 (N. 2013)

For $\|V - V_0\|_{R,\sigma}$ *sufficiently small, there exists* $Y_1, \ldots, Y_N \in \mathscr{P}^{(R)} \subset \Gamma(\mathcal{H}_\mathbb{R}, U_t)''$ *whose law with respect to* $\varphi_{0,U}$ *is the free Gibbs state with potential* V*.*
Moreover, each $Y_j = G_j(X_1^{(0)}, \ldots, X_N^{(0)})$ *and*

$$G = (G_1, \ldots, G_N) \colon (\mathscr{P}^{(R)})^N \to (\mathscr{P}^{(R)})^N$$

is invertible.
Consequently, for $|q|$ *sufficiently small (depending on* N *and* $\|A\|$*) there is a state preserving isomorphism*

$$\begin{array}{rcl} \Gamma_q(\mathcal{H}_\mathbb{R}, U_t)'' & \cong & \Gamma(\mathcal{H}_\mathbb{R}, U_t)'' \\ X_j^{(q)} & \longmapsto & G_j(X_1^{(0)}, \ldots, X_N^{(0)}) \\ (G^{-1})_k(X_1^{(q)}, \ldots, X_N^{(q)}) & \longleftarrow & X_k^{(0)} \end{array}$$

Define $F\colon \mathscr{P}_{c.s.}^{(R,\sigma)} \to \mathscr{P}_{\varphi}^{(R,\sigma)}$ by

$$\begin{aligned} F(g) = &- W(X + \mathscr{D}g) - \frac{1}{4}\left\{(1+A)\#\mathscr{D}g\right\}\#\mathscr{D}g \\ &+ \sum_{m\geq 1}\frac{(-1)^{m+1}}{m}(1\otimes\varphi^{op})\circ \mathrm{Tr}\left(A\#\left[\mathscr{J}_\sigma\mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right) \\ &+ \sum_{m\geq 1}\frac{(-1)^{m+1}}{m}(\varphi\otimes 1)\circ \mathrm{Tr}\left(A^{-1}\#\left[\mathscr{J}_\sigma\mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right), \end{aligned}$$

where $W = V - V_0$.

Define $F\colon \mathscr{P}^{(R,\sigma)}_{c.s.} \to \mathscr{P}^{(R,\sigma)}_{\varphi}$ by

$$F(g) = - W(X + \mathscr{D}g) - \frac{1}{4}\left\{(1+A)\#\mathscr{D}g\right\}\#\mathscr{D}g$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(1\otimes\varphi^{op})\circ \mathrm{Tr}\left(A\#\left[\mathscr{J}_\sigma\mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right)$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(\varphi\otimes 1)\circ \mathrm{Tr}\left(A^{-1}\#\left[\mathscr{J}_\sigma\mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right),$$

where $W = V - V_0$.
Then $\mathscr{S}\pi_{n\geq 1}F$ has a fixed point $g \in \mathscr{P}^{(R,\sigma)}_{c.s.}$ and setting

$$Y = X + \mathscr{D}g$$

yields $Y_1, \ldots, Y_N$ whose law with respect to φ is the free Gibbs state with potential V.

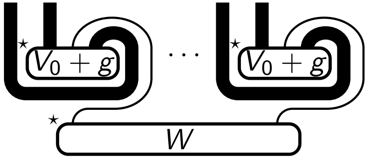
F is equivalent to a planar tangle:

$$F(g) = -W(X + \mathscr{D}g) - \frac{1}{4}\left\{(1+A)\#\mathscr{D}g\right\}\#\mathscr{D}g$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(1\otimes\varphi^{op})\circ \mathrm{Tr}\left(A\#\left[\mathscr{J}_\sigma\mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right)$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(\varphi\otimes 1)\circ \mathrm{Tr}\left(A^{-1}\#\left[\mathscr{J}_\sigma\mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right),$$

F is equivalent to a planar tangle:

$$F(g) = -W(X + \mathscr{D}g) - \frac{1}{4}\{(1+A)\#\mathscr{D}g\}\#\mathscr{D}g$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(1 \otimes \varphi^{op}) \circ \mathrm{Tr}\left(A\#\left[\mathscr{J}_\sigma \mathscr{D}g \# \mathscr{J}_\sigma X^{-1}\right]^m\right)$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(\varphi \otimes 1) \circ \mathrm{Tr}\left(A^{-1}\#\left[\mathscr{J}_\sigma \mathscr{D}g \# \mathscr{J}_\sigma X^{-1}\right]^m\right),$$

where the term in red corresponds to:



F is equivalent to a planar tangle:

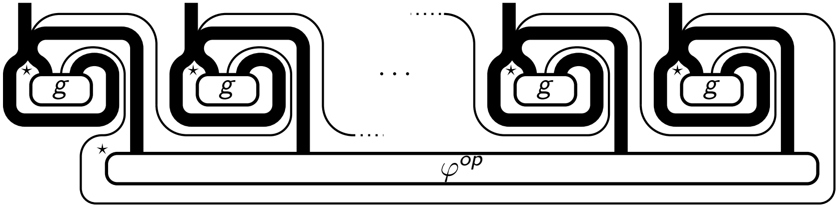
$$F(g) = -W(X + \mathscr{D}g) - \frac{1}{4}\{(1 + A)\#\mathscr{D}g\}\#\mathscr{D}g$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(1 \otimes \varphi^{op}) \circ \mathrm{Tr}\left(A\#\left[\mathscr{J}_\sigma \mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right)$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(\varphi \otimes 1) \circ \mathrm{Tr}\left(A^{-1}\#\left[\mathscr{J}_\sigma \mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right),$$

where the term in red corresponds to:

F is equivalent to a planar tangle:

$$F(g) = -W(X + \mathscr{D}g) - \frac{1}{4}\{(1+A)\#\mathscr{D}g\}\#\mathscr{D}g$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m} (1 \otimes \varphi^{op}) \circ \mathrm{Tr}\left(A\#\left[\mathscr{J}_\sigma \mathscr{D}g \# \mathscr{J}_\sigma X^{-1}\right]^m\right)$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m} (\varphi \otimes 1) \circ \mathrm{Tr}\left(A^{-1}\#\left[\mathscr{J}_\sigma \mathscr{D}g \# \mathscr{J}_\sigma X^{-1}\right]^m\right),$$

where the term in red corresponds to:



F is equivalent to a planar tangle:

$$F(g) = -W(X + \mathscr{D}g) - \frac{1}{4}\left\{(1+A)\#\mathscr{D}g\right\}\#\mathscr{D}g$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(1 \otimes \varphi^{op}) \circ \mathrm{Tr}\left(A\#\left[\mathscr{J}_\sigma\mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right)$$
$$+ \sum_{m\geq 1} \frac{(-1)^{m+1}}{m}(\varphi \otimes 1) \circ \mathrm{Tr}\left(A^{-1}\#\left[\mathscr{J}_\sigma\mathscr{D}g\#\mathscr{J}_\sigma X^{-1}\right]^m\right),$$

where the term in red corresponds to: