Brownian Motion on $\mathbb{GL}_N$: Large-N Limit and Fluctuations

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Giving Credit where Credit is Due

Based partly on joint work with Bruce Driver (UC San Diego), Brian Hall (Notre Dame), and Guillaume Cébron (Université Paris 6).

- Driver; Hall; K: *The large-N limit of the Segal–Bargmann transform on $\mathbb{U}_N$*. J. Funct. Anal. 265, 2585–2644 (2013)
- K: *Heat kernel empirical laws on $\mathbb{U}_N$ and $\mathbb{GL}_N$*. arXiv:1306.2140, June 2013.
- K: *The Large-N Limits of Brownian Motions on $\mathbb{GL}_N$*. arXiv:1306.6033, July 2013.
- Cébron; K: *Fluctuations of Brownian Motions on $\mathbb{GL}_N$*. Preprint in preparation.

Heat Kernels and Brownian Motion on Lie Groups

The Laplacian Δ on a Riemannian Manifold

Let M be a Riemannian manifold, with metric $\langle\cdot,\cdot\rangle$ and resulting volume form dV. If $f \in C^\infty(M)$, the **gradient** $\nabla f = \nabla_M f$ is the vector field defined by

$$\langle \nabla f, X\rangle = df(X) = X(f), \quad X \in \mathrm{Vec}(M).$$

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- Citations

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If U is an isometry of M, then $(\Delta f)\circ U=\Delta(f\circ U)$. This means Δ can be computed by the same expression in any orthonormal basis. If $M=\mathbb{R}^n$ with its usual Euclidean metric, $\Delta=\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.

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$$e^{\frac{t}{2}\Delta}f(x) = \int_M f(y)\rho(t,x,y)\,dV(y), \quad f \in L^1(M).$$

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$$\rho(t,x,y) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2t}}.$$

This Gaussian tail behavior is universal; but in general there is no formula for the heat kernel on any non-Euclidean manifold.

Left-Invariant Laplacian on a Lie Group

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. If $\langle\cdot,\cdot\rangle$ is a real inner product on $\mathfrak{g}$, by (right-)translation it gives rise to a *left*-invariant Riemannian metric on G (which has positive curvature).

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Fix any orthonormal basis β of $\mathfrak{g}$; then define

$$\Delta_\beta = \sum_{\xi \in \beta} \partial_\xi^2.$$

In fact, this does not depend on the choice of basis β; it is equal to the Laplace operator on the Riemannian manifold G.

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In fact, this does not depend on the choice of basis β; it is equal to the Laplace operator on the Riemannian manifold G. *And we can compute with it!*

Left-Invariant Heat Kernel on a Lie Group

Because of the left-invariance of Δ_G, the heat kernel $\rho(t,x,y)$ takes the form of a convolution kernel: letting $\rho_t(x) = \rho(t,x,1_G)$,

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$$\rho(t, x, y) = \rho_t(y^{-1}x) \qquad \text{that is to say}$$

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- Since $e^{\frac{t}{2}\Delta_G}(1) = 1$, ρ_t is a probability density.
- Since $e^{\frac{s+t}{2}\Delta_G} = e^{\frac{s}{2}\Delta_G} e^{\frac{t}{2}\Delta_G}$, $\rho_{s+t} = \rho_s * \rho_t$.

We will also denote by $d\rho_t$ the **heat kernel measure** (with density ρ_t). This measure is determined (by definition) by

$$\int_G f\,d\rho_t = \left(e^{\frac{t}{2}\Delta_G} f\right)(1_G), \qquad f \in C_c(G).$$

Brownian Motion on Riemannian Manifolds, and Lie Groups

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- $t \mapsto B_t$ is a continuous map from $\mathbb{R}_+$ into G a.s.
- For $0 \le s < t < \infty$, $B_s^{-1}B_t$ has distribution ρ_{t-s}, and is independent from $(B_r)_{0\le r\le s}$.

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There is an even more explicit representation, as a kind of projection of the Brownian motion on the Lie algebra. Let β be an o.n. basis of $\mathfrak{g}$, and

$$W_t = \sum_{\xi\in\beta} W_t^{(\xi)}\xi, \quad \{W_t^{(\xi)}\}_{\xi\in\beta} \text{ i.i.d. Brownian motions on } \mathbb{R}.$$

Then, in Stratonovich form, $\quad dB_t = B_t \circ dW_t.$

Brownian Motions on $\mathfrak{u}_N$ and $\mathfrak{gl}_N$

Fix the inner product $\langle \xi, \eta \rangle_N = N\Re\mathrm{Tr}(\xi^* \eta)$ on $\mathfrak{gl}_N = \mathbb{M}_N$ (and therefore on $\mathfrak{u}_N \subset \mathbb{M}_N$).

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As a (real) orthonormal basis of $\mathfrak{gl}_N$, we can take the matrix units $\{\frac{1}{\sqrt{N}} E_{jk}\}_{1 \le j,k \le n} \cup \{\frac{i}{\sqrt{N}} E_{jk}\}_{1 \le j,k \le n}$, and so the Brownian motion $Z^N(t)$ can be written as

$$[Z^N(t)]_{jk} = \frac{1}{\sqrt{N}} [W_{jk}(t) + i W'_{jk}(t)]$$

where $\{W_{jk}, W'_{jk}\}_{1 \le j,k \le N}$ are independent Brownian motions.

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It is a routine exercise to find an o.n. basis for $\mathfrak{u}_N$, and find that the Brownian motion there has the form $-iX^N(t)$, where

$$X^N(t) = \frac{1}{2}[Z^N(t) + Z^N(t)^*].$$

Brownian Motions on $\mathbb{U}_N$ and $\mathbb{GL}_N$

There is a general procedure for converting Stratonovich integrals to Itô integrals. In the case of $\mathfrak{g}$-valued Brownian motion W_t, this gives the Itô SDE

$$dB_t = B_t \circ dW_t = B_t\, dW_t + \frac{1}{2} B_t \sum_{\xi \in \beta} \xi^2\, dt.$$

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There is a "magic formula": if β is an o.n. basis of $\mathfrak{u}_N$, then for any matrix A,

$$\sum_{\xi\in\beta} \xi A \xi = -\mathrm{tr}(A) I = -\frac{1}{N}\mathrm{Tr}(A).$$

In particular, $\sum_{\xi\in\beta}\xi^2 = -I$. Similarly, $\beta' = \beta\cup i\beta$ is an o.n. basis for $\mathfrak{gl}_N$, and so it follows that $\sum_{\xi\in\beta'}\xi^2 = -I - (i^2)I = 0$.

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$$dU_t = iU_t\, dX_t - \frac{1}{2} U_t\, dt, \qquad dB_t = B_t\, dZ_t.$$

Large-N Limits of Brownian Motions on $\mathfrak{u}_N$, $\mathfrak{gl}_N$, $\mathbb{U}_N$, and $\mathbb{GL}_N$

Free Additive Brownian Motions

Let $(\mathscr{A},\tau)$ be a W^*-probability space sufficiently rich to contain an infinite sequence of freely independent semicircular elements (e.g. any free group factor). Then $\mathscr{A}$ contains **free additive Brownian motions**: *free semicircular Brownian motion* $(x_t)_{t\geq 0}$ and *free circular Brownian motion* $(z_t)_{t\geq 0}$. These are defined by

- $x_0 = z_0 = 1$.
- For $0 < s < t < \infty$, $x_t - x_s$ is semicircular with variance $t-s$; $z_t - z_s$ is circular with variance $t-s$.
- For $0 < s < t < \infty$, $x_t - x_s$ is freely independent from $(x_r)_{0\leq r\leq s}$; $z_t - z_s$ is freely independent from $(z_r)_{0\leq r\leq s}$.

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Note: if $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ are two freely independent free semicircular Brownian motions, then $z_t = \frac{1}{\sqrt{2}}(x_t + iy_t)$ is a free circular Brownian motion. Vice versa: if $(z_t)_{t\geq 0}$ is a free circular Brownian motion then $\sqrt{2}\mathrm{Re}(z_t)$ and $\sqrt{2}\mathrm{Im}(z_t)$ are free semicircular Brownian motions.

Large-N Limits of Free Additive Brownian Motion

Theorem. [Voiculescu, 1991] Let $X^N(t)$ and $Z^N(t)$ be the Brownian motions on $\mathfrak{u}_N$ and $\mathfrak{gl}_N$. Then, for any times $t_1, \ldots, t_n \ge 0$,

$$(X^N_{t_1}, \ldots, X^N_{t_n}) \xrightarrow{\mathscr{D}} (x_{t_1}, \ldots, x_{t_n}), \quad \text{and}$$

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$$(Z^N_{t_1}, \ldots, Z^N_{t_n}) \xrightarrow{\mathscr{D}} (z_{t_1}, \ldots, z_{t_n}).$$

Note: this is convergence in noncommutative distribution, meaning that if p is any fixed noncommutative polynomial in $2n$ variables,

$$\mathbb{E}\mathrm{tr}[p(Z^N_{t_1}, Z^{N*}_{t_1}, \ldots, Z^N_{t_n}, Z^{N*}_{t_n})] \to \tau[p(z_{t_1}, z^*_{t_1}, \ldots, z_{t_n}, z^*_{t_n})].$$

It is also true that the *random* moments converge *almost surely* to their means. This highlights the fact that these are really **strong laws of large numbers** for these "flat" Brownian motions.

Free Stochastic Differential Equations

In the mid 1990s, Roland Speicher and Philippe Biane (and others) showed that the technology of stochastic integrals and stochastic differential equations can be made sense of for the “stochastic processes” x_t and z_t, as in the classical setting. That is, one can solve equations like

$$da_t = \sigma(t, a_t)\, dx_t + \mu(t, a_t)\, dt$$

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subject to regularity constraints on the functions σ and μ. In particular, one can solve the precise analogs of the SDEs that define the Brownian motions U_t and B_t on $\mathbb{U}_N$ and $\mathbb{GL}_N$:

$$du_t = iu_t\, dx_t - \frac{1}{2}u_t\, dt, \qquad db_t = b_t\, dz_t.$$

It is now natural to ask whether the same kind of convergence of processes $U_t \to u_t$ and $B_t \to b_t$ holds true.

Free Unitary Brownian Motion

Theorem. [Biane, 1997] Let U_t^N be the Brownian motion on $\mathbb{U}_N$, and let u_t be a free unitary Brownian motion, defined by $du_t = iu_t\,dx_t - \frac{1}{2}u_t\,dt$. Then for any times $t_1, \ldots, t_n \geq 0$,

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For the proof, Biane used an explicit characterization of the irreducible representations of $\mathbb{U}_N$, and also made use of the spectral theorem, both of which are unavailable for generic matrices in $\mathbb{GL}_N$.

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Theorem. [K, 2013] Let B_t^N be Brownian motion on $\mathbb{GL}_N$, and let b_t be a free multiplicative Brownian motion, defined by $db_t = b_t\,dz_t$. Then for any times $t_1,\ldots,t_n \geq 0$,

$$(B_{t_1}^N,\ldots,B_{t_n}^N) \xrightarrow{\mathscr{D}} (b_{t_1},\ldots,b_{t_n}).$$

Trace Polynomials and their Intertwining Space

An Example Calculation

Example. Consider the function $f(A) = \mathrm{tr}(A^2A^*)$ on $\mathbb{GL}_N$. We use the "magic formulas"

$$\sum_{\xi\in\beta_N} \xi A\xi = -\mathrm{tr}(A)I_N, \quad \sum_{\xi\in\beta_N} \mathrm{tr}(A\xi)\xi = -\frac{1}{N^2}A.$$

Let $g(A) = \mathrm{tr}(A)\mathrm{tr}(AA^*)$. We can readily compute that

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$$\Delta_{\mathbb{GL}_N} f = 4f + 4g$$
$$\Delta_{\mathbb{GL}_N} g = \frac{4}{N^2} f + 4g.$$

This 2×2 system can be exponentiated by a (good) freshman, and we see that

$$e^{\frac{t}{2}\Delta_{\mathbb{GL}_N}} f = e^{2t}\cosh(2t/N) f + e^{2t} N \sinh(2t/N) g$$

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$$\begin{aligned}\Delta_{\mathbb{GL}_N} f &= 4f + 4g \\ \Delta_{\mathbb{GL}_N} g &= \frac{4}{N^2}f + 4g.\end{aligned}$$

This 2×2 system can be exponentiated by a (good) freshman, and we see that

$$\begin{aligned}e^{\frac{t}{2}\Delta_{\mathbb{GL}_N}} f &= e^{2t}\cosh(2t/N)f + e^{2t}N\sinh(2t/N)g \\ &= e^{2t}f + 2te^{2t}g + O(1/N^2).\end{aligned}$$

(Abstract) Trace Polynomial Space

Let $\mathscr{P}$ denote the commutative $\mathbb{C}$-algebra generated by the set of finite words $\varepsilon \in \bigcup_{n=0}^{\infty}\{1,*\}^n$. For convenience, label the basis elements v_ε. For example

$$P = v_1 - 2v_{1*1} + v_{*1}v_1.$$

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$$P = v_1 - 2v_{1*1} + v_{*1}v_1.$$

Call such elements **abstract trace polynomials**. The reason is the following. For any $N \in \mathbb{N}$ and any $P \in \mathscr{P}$, define a function $P_N \colon \mathbb{M}_N \to \mathbb{C}$ as follows: for any word $\varepsilon = \varepsilon_1 \cdots \varepsilon_n$, let

$$[v_\varepsilon]_N(A) = \mathrm{tr}(A^{\varepsilon_1} \cdots A^{\varepsilon_n})$$

then extend the map $P \mapsto P_N$ to be an algebra homomorphism. For example, with the above P,

$$P_N(A) = \mathrm{tr}(A) - 2\mathrm{tr}(AA^*A) + \mathrm{tr}(A^*A)\mathrm{tr}(A).$$

Such functions are called **trace polynomials**.

Intertwining Formula for the Laplacian

Theorem. [Driver, Hall, K.] The space $[\mathscr{P}]_N$ of trace polynomials is a reducing subspace for $\Delta_{\mathbb{GL}_N}$. There exist first- and second-order differential operators $\mathcal{D}$ and $\mathcal{L}$ on $\mathscr{P}$ so that

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The point is that $e^{\frac{t}{2}\left(\mathcal{D}+\frac{1}{N^2}\mathcal{L}\right)} = e^{\frac{t}{2}\mathcal{D}} + O\left(\frac{1}{N^2}\right)$. Since $e^{\frac{t}{2}\mathcal{D}}$ is an algebra homomorphism, this leads to the following core estimate.

Corollary. For any trace polynomials P, Q,

$$\mathrm{Cov}(P_N(B_t^N), Q_N(B_t^N)) = O\left(\frac{1}{N^2}\right) .$$

Idea of the Proof of Convergence $(B_t^N)_{t\geq 0} \to (b_t)_{t\geq 0}$

Fix $t > 0$. We have $dB_t = B_t\, dZ_t$ and $db_t = b_t\, dz_t$. Because the diffusion terms are linear, we can proceed by induction on the degree of the moment. Using stochastic calculus, the difference can be expressed as an integral of terms consisting of the difference between lower-order moments (which $\to 0$ by inductive hypothesis), plus the covariance of the involved terms (which $\to 0$ as above).

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Fluctuations of Matrix Brownian Motions

Fluctuations of Flat Brownian Motions

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For the "flat" Brownian motions $X^N(t)$ and $Z^N(t)$, this was answered by Speicher and Mingo and simultaneously by Chatterji in the mid 2000s, for the case of "linear statistics".

Theorem. [Mingo, Speicher] Let $p_1, \ldots, p_n$ be polynomials in one variable. Let $t_1, \ldots, t_n \geq 0$. Then the random variables

$$N[\mathrm{tr}(p_j(X^N(t_j))) - \mathbb{E}\mathrm{tr}(p_j(X^N(t_j)))], \qquad j = 1 \ldots n$$

are, in the limit as $N \to \infty$, jointly Gaussian (with a covariance that is determined by $p_1, \ldots, p_n$).

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A similar result (involving polynomials in the variables and their adjoints) holds for $Z^N(t)$.

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Note, since $U^N(t)$ is a normal matrix, $f(U^N(t))$ can be made sense of for any measurable function f on the unit circle, via functional calculus. But this doesn't allow for multiple times, since that introduces real noncommutativity. Using a different approach, we can handle the more general $\mathbb{U}_N$ case, and the $\mathbb{GL}_N$ case, simultaneously.

Fluctuations of Brownian Motion on $\mathbb{GL}_N$

Theorem. [Cébron, K, 2014] Let $P_1, \ldots, P_n$ be trace polynomials. Let $t_1, \ldots, t_n \geq 0$. Let $\Xi^N(t)$ denote either $U^N(t)$ or $B^N(t)$. Then the random variables

$$X_j = N[P_j(\Xi^N(t_1), \ldots, \Xi^N(t_n)) - \mathbb{E}P_j(\Xi^N(t_1), \ldots, \Xi^N(t_n))]$$

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Recall that $\Delta_{\mathbb{GL}_N} \sim \mathcal{D} + \frac{1}{N^2}\mathcal{L}$. The fluctuations are therefore controlled by the second-order operator $\mathcal{L}$; in fact, by its *carré du champ* operator

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Indeed, we can express the covariance of the asymptotically Gaussian random vector $(X_1, \ldots, X_n)$ as follows:

The Covariance of the Fluctuations of B_t^N (for a fixed t)

Theorem. [Cébron, K, 2014] The asymptotic covariance matrix of $(X_1, \ldots, X_n)$ has (i,j)-entry $\sigma(P_i, P_j)$, where the function σ is determined as follows: given $P, Q \in \mathscr{P}$, there is a trace polynomial $\tilde{\Gamma}(P,Q)$ in three variables such that, if a_t, b_t, c_t are three freely independent multiplicative Brownian motions,

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E.g. Suppose p, q are single-variable polynomials. Then

$$\sigma(\mathrm{tr}(p), \mathrm{tr}(q^*)) = \int_0^t \tau\left[p'(b_{t-s}a_s)q'(c_{t-s}a_s)^*\right]\, ds.$$

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In the unitary case, we can compute that this converges (as $t \to \infty$) to $\langle p, q\rangle_{H_{1/2}}$, agreeing with [Diaconis, Evans, 2001] in the Haar unitary case.

Second Order Distribution and Freeness

In alternate terminology (due to Collins, Mingo, Śniady, Speicher, 2006-2007), the fact that the fluctuations of $(B_t^N)_{t\geq 0}$ are Gaussian says that any collection $(B_{t_1}^N, \ldots, B_{t_n}^N)$ possesses a **second-order limit distribution**.

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