Free convolution semigroups and coefficient stripping

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Notation.

Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} \, d\mu(x),$$

F-transform

$$F_\mu(z) = \frac{1}{G_\mu(z)}.$$

Definition. (Belinschi, Nica)

$$\Phi : \mathcal{P} \to \mathcal{P}_2, \qquad F_{\Phi[\nu]}(z) = z - G_\nu(z).$$

Note

$$m(\Phi[\nu]) = 0, \quad \mathrm{Var}(\Phi[\nu]) = 1.$$

Left inverse of Φ.

Definition.

Coefficient stripping is the map

$$J : \{\mu \in \mathcal{P}_2 \mid \operatorname{Var}(\mu) > 0\} \to \mathcal{P}, \qquad F_\mu(z) = z - \beta - \gamma G_{J[\mu]}(z),$$

where $\beta = m(\mu)$, $\gamma = \operatorname{Var}(\mu)$.

J strips off the first pair of Jacobi parameters.

Warm-up: Boolean convolution semigroups.

A Boolean convolution semigroup

$$\{\mu^{\uplus t} : t \geqslant 0\}$$

is determined by

$$z - F_{\mu^{\uplus t}} = t(z - F_\mu).$$

Then

$$(t\beta + t\gamma G_{J[\mu^{\uplus t}]}(z)) = z - F_{\mu^{\uplus t}}(z) = t(z - F_\mu(z)) = t(\beta + \gamma G_{J[\mu]}(z))$$

and so

$$J[\mu^{\uplus t}] = J[\mu].$$

Free convolution semigroups.

Denoting $\varphi_\mu(z) = F_\mu^{-1}(z) - z$, a free convolution semigroup

$$\{\mu_t : t \geqslant 0\}$$

is determined by

$$\varphi_{\mu_t}(z) = t\varphi_\mu(z).$$

Free convolution semigroups with finite variance are in bijection with canonical triples

$$\{(\beta, \gamma, \rho) : \beta \in \mathbb{R}, \gamma > 0, \rho \in \mathcal{P}\} \cup \{(\beta, 0, \cdot) : \beta \in \mathbb{R}\}$$

via (Maassen)

$$\varphi_{\mu_t}(z) = \beta t + \gamma t G_\rho(z).$$

For F^{-1}, not F.

Stripping a free convolution semigroup.

Proposition. (A)

If $\{\mu_t\}$ is a free convolution semigroup with finite positive variance,

$$J[\mu_t] = \rho \boxplus \sigma_{\beta,\gamma}^{\boxplus t},$$

where $\sigma_{\beta,\gamma}$ is the semicircle law with mean β and variance γ.

Proof. Will prove for mean zero, variance one.

Recall the Belinschi-Nica transformations

$$\mathbb{B}_t[\mu] = \left(\mu^{\boxplus(1+t)}\right)^{\uplus \frac{1}{1+t}},$$

for which $\mathbb{B}_1 = \mathbb{B}$, the Boolean-to-free Bercovici-Pata bijection.

Stripping a free convolution semigroup.

Belinschi and Nica proved that

$$\mathbb{B}_t[\Phi[\rho]] = \Phi[\rho \boxplus \sigma^{\boxplus t}].$$

Note that the Maassen representation $\varphi_\mu(z) = G_\rho(z)$ says

$$\mu = \mathbb{B}[\Phi[\rho]].$$

So

$$\mathbb{B}_{t-1}[\mu] = \Phi[\rho \boxplus \sigma^{\boxplus t}],$$

$$\mu^{\boxplus t} = \Phi[\rho \boxplus \sigma^{\boxplus t}]^{\uplus t},$$

and

$$J[\mu_t] = \rho \boxplus \sigma^{\boxplus t}.$$

Inverse coefficient stripping.

What about inverse coefficient stripping?

For what $\widetilde{\mu}_t$ is

$$J[\widetilde{\mu}_t] = \widetilde{\rho} \boxplus \tau^{\boxplus t}$$

for other $\tau \in \mathcal{ID}^{\boxplus}$?

More notation.

Definition. (Bożejko, Speicher)

Define $\varphi_{\tilde{\mu},\mu}$ via

$$(\varphi_{\tilde{\mu},\mu} \circ F_\mu)(z) + F_{\tilde{\mu}}(z) = z.$$

A *two-state free convolution semigroup* is a family $\{(\tilde{\mu}_t, \mu_t) : t \geqslant 0\}$ such that

$$\varphi_{\tilde{\mu}_t,\mu_t} = t\varphi_{\tilde{\mu},\mu}, \quad \varphi_{\mu_t} = t\varphi_\mu.$$

Definition. (Lenczewski, Nica)

The *subordination distribution* $\mu \boxplus\!\!\!\!\!\!\;\vdash \nu$ is the unique probability measure such that

$$G_{\mu \boxplus\!\!\!\!\!\!\;\vdash \nu}(z) = G_\nu(F_{\mu \boxplus\!\!\!\!\!\!\;\vdash \nu}(z)).$$

Inverse coefficient stripping I.

Proposition.

Let $\widetilde{\rho} \in \mathcal{P}$ and $\tau \in \mathcal{ID}^{\boxplus}$. Denote

$$\mu_t = \tau^{\boxplus t} \boxplus \widetilde{\rho},$$

$$\widetilde{\mu}_t = \Phi[\widetilde{\rho} \boxplus \tau^{\boxplus t}]^{\uplus t}.$$

Then $\{(\widetilde{\mu}_t, \mu_t) : t \geqslant 0\}$ form a two-state free convolution semi-group and

$$J[\widetilde{\mu}_t] = \widetilde{\rho} \boxplus \tau^{\boxplus t}.$$

The converse is not true.

Reason: $\widetilde{\rho} \boxplus \tau^{\boxplus t}$ may “make sense” even if $\tau \notin \mathcal{ID}^{\boxplus}$.

Examples.

Example.

If $\widetilde{\rho} = \nu \boxplus \tau$, then

$$\widetilde{\rho} \boxplus \tau^{\boxplus t} = \nu \boxplus \tau^{\boxplus (1+t)}.$$

Example. (A, Belinschi)

Suppose τ is supported in a small neighborhood of 0, so that φ_τ is small outside of the unit disk. Bercovici, Voiculescu ⇒

$$\sigma \boxplus \tau^{\boxplus t} \text{ positive for } t < 1.$$

But also for $t \geqslant 1$.

Examples.

τ need not even be positive.

Example.

Free Binomial distributions distributions: ν_γ, positive for $\gamma \geqslant 1$, defined as linear functionals for $0 \leqslant \gamma < 1$.

$$\nu_{\gamma_1} \boxplus \nu_{\gamma_2} = \nu_{\gamma_1 + \gamma_2}.$$

Let

$$\tilde{\rho} = \nu_c : \quad c \geqslant 1.$$

$$\tau = \nu_\gamma : 0 < \gamma < 1.$$

Then

$$\tilde{\rho} \boxplus \tau^{\boxplus t} = \nu_{c+\gamma t} : \quad c + \gamma t \geqslant 1.$$

This is positive for all $t \geqslant 0$, while τ is not.

Inverse coefficient stripping II.

Theorem. (A)

Let $\tilde{\rho}, \tau \in \mathcal{P}$ such that $\tau \boxplus \tilde{\rho} \in \mathcal{ID}^{\boxplus}$. Then

$$\tilde{\rho} \boxplus \tau^{\boxplus t}$$

is well-defined (positive) for all $t \geqslant 0$. Denote

$$\mu_t = (\tau \boxplus \tilde{\rho})^{\boxplus t},$$

$$\tilde{\mu}_t = \Phi[\tilde{\rho} \boxplus \tau^{\boxplus t}]^{\uplus t}.$$

Then $\{(\tilde{\mu}_t, \mu_t) : t \geqslant 0\}$ form a two-state free convolution semigroup and

$$J[\tilde{\mu}_t] = \tilde{\rho} \boxplus \tau^{\boxplus t}.$$

Subordination.

Question.

Suppose $\tilde{\rho} \boxplus \tau^{\boxplus t}$ is positive for all $t \geqslant 0$.

Does this imply that $\tau \boxplus \tilde{\rho} \in \mathcal{ID}^{\boxplus}$?

Equivalently, does this imply that $F_{\tilde{\rho} \boxplus \tau^{\boxplus t}}$ is subordinate to $F_{\tilde{\rho}}$ for all $t > 0$?

Partial converse.

Theorem. (A)

Let $\{(\widetilde{\mu}_t, \mu_t) : t \geqslant 0\}$ be a two-state free convolution semigroup of *compactly supported* measures. Then there exist $\widetilde{\rho}, \omega \in \mathcal{P}$ and $p > 0$ such that

$$J[\widetilde{\mu}_t] = \widetilde{\rho} \boxplus \omega^{\boxplus(t/p)}.$$

Here $\tau = \omega^{\boxplus(1/p)}$ need not be positive.

Partial converse.

Proof. Compactly supported measures ⇒ linear functionals.

Algebraic argument ⇒

$J[\tilde{\mu}_t] = \tilde{\rho} \boxplus \tau^{\boxplus t}$ for τ linear, unital, not necessarily positive, $\mathrm{Var}(\tau) = \mathrm{Var}(\mu) > 0$.

Then Bercovici, Voiculescu ⇒

there exists a $p > 0$ such that $D_{1/\sqrt{p}} \tau^{\boxplus p}$ is positive

⇒ $\tau^{\boxplus p}$ is positive.

Additional examples.

Example.

Let μ_t be a general free convolution semigroup. Then

$$J[\widetilde{\mu}] = \mu_t,$$

where $\{(\widetilde{\mu}_t, \mu_t)\}$ are the (distributions of) two-state free Brownian motions.

Example.

Recall that for $\gamma > 0$, $J[\mu_t] = \rho \boxplus \sigma_{\beta,\gamma}^{\boxplus t}$.

What if $\gamma = 0$, $\sigma_{\beta,0}^{\boxplus t} = \delta_{\beta t}$? Answer: $J[\widetilde{\mu}] = \widetilde{\rho} \boxplus \delta_{\beta t}$,

where $\{(\widetilde{\mu}_t, \delta_{\beta t})\}$ are the generalized Boolean semigroups from (A, Młotkowski).

Two-state free Meixner distributions.

A, Młotkowski ⇒ $\{(\widetilde{\mu}_t, \mu_t)\}$ form a two-state free convolution semigroup, where

$$\widetilde{\mu}_t \sim \begin{pmatrix} \widetilde{\beta} t, & \widetilde{b} + \beta t, & b + \beta t, & b + \beta t, & \ldots \\ \widetilde{\gamma} t, & \widetilde{c} + \gamma t, & c + \gamma t, & c + \gamma t, & \ldots \end{pmatrix}$$

and

$$\mu_t \sim \begin{pmatrix} \beta t, & b + \beta t, & b + \beta t, & \ldots \\ \gamma t, & c + \gamma t, & c + \gamma t, & \ldots \end{pmatrix}.$$

Then

$$J[\widetilde{\mu}_t] \sim \begin{pmatrix} \widetilde{b} + \beta t, & b + \beta t, & b + \beta t, & \ldots \\ \widetilde{c} + \gamma t, & c + \gamma t, & c + \gamma t, & \ldots \end{pmatrix}.$$

Two-state free Meixner distributions.

$$J[\tilde{\mu}_t] \sim \begin{pmatrix} \tilde{b} + \beta t, & b + \beta t, & b + \beta t, & \ldots \\ \tilde{c} + \gamma t, & c + \gamma t, & c + \gamma t, & \ldots \end{pmatrix},$$

so $J[\tilde{\mu}_t] = \tilde{\rho} \boxplus \tau^{\boxplus t}$, where

$$\tilde{\rho} \sim \begin{pmatrix} \tilde{b}, & b, & b, & b, & \ldots \\ \tilde{c}, & c, & c, & c, & \ldots \end{pmatrix}$$

and

$$\tau \sim \begin{pmatrix} \beta, & \beta + b - \tilde{b}, & \beta + b - \tilde{b}, & \beta + b - \tilde{b}, & \ldots \\ \gamma, & \gamma + c - \tilde{c}, & \gamma + c - \tilde{c}, & \gamma + c - \tilde{c}, & \ldots \end{pmatrix}.$$

Two-state free Meixner distributions.

$$\tau \sim \begin{pmatrix} \beta, & \beta + b - \widetilde{b}, & \beta + b - \widetilde{b}, & \beta + b - \widetilde{b}, & \ldots \\ \gamma, & \gamma + c - \widetilde{c}, & \gamma + c - \widetilde{c}, & \gamma + c - \widetilde{c}, & \ldots \end{pmatrix},$$

where

$$\widetilde{c} \geqslant 0, \quad \gamma > 0.$$

So:

- For $c \geqslant \widetilde{c}$, $\tau \in \mathcal{ID}^{\boxplus}$.
- For $c \geqslant \widetilde{c} - \gamma$, $\tau \in \mathcal{P}$.
- For $c < \widetilde{c} - \gamma$, τ is not a positive functional. But $\tau^{\boxplus p}$ is positive for $p \geqslant (\widetilde{c} - c)/\gamma$.