POINT-SET TOPOLOGY

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Introduction

Topology might be called the theory of space, not just the usual Euclidean space $\mathbb{R}^3$ in which we appear to live, but those far more bizarre and abstract. A space starts with a set on which you wish to place some notion of points being "near" to each other, without necessary actually having a notion of distance like Euclidean space. This notion is a called a topology, and a set together with a topology is called a topological space.

The definition of a topology is rather simple: it is a collection of subsets known as its open sets satisfying certain axioms. In Euclidean space, the open sets consist of the union of open balls about points. The axioms required of a topology aim to make the definition of a continuous function between topological spaces reasonable. In topology, a function is said to be continuous if the inverse image of every open set is open. This matches with the usual definition of continuity of maps between Euclidean spaces.

The key objects in the study of topology, or more specifically its foundational branch known as point-set topology that is our focus, are topological spaces and continuous functions. Among the topological spaces considered are the spaces found in multivariable calculus: $\mathbb{R}^n$, spheres, tori, and other real manifolds (which are spaces that look locally like $\mathbb{R}^n$). Some might say that topology aims to study geometric objects while essentially forgetting about geometry: that is, in topology, a circle and a square are in some sense viewed as the same thing, because there exists a bijection from the square to the circle that is both continuous and has a continuous inverse. To a topologist, any collection of points, all separated from each other, is a perfectly good space too.

Consider for instance the topological space $\mathbb{R}^2$, i.e., the plane. Around every point in the plane are centered discs without their boundaries: these are open sets. The open sets in $\mathbb{R}^2$ are arbitrary unions of these open disks. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ sends x to y. To say that f is continuous at x is to say that given an open disk B centered at y, the inverse image $f^{-1}(B)$ contains an open ball centered at x. Then f is continuous if it is continuous at every point. But looking closely at the definition of a continuous function between topological spaces, one sees that this is in fact the same definition: f is continuous if and only if $f^{-1}(U)$ is open for every open set U in $\mathbb{R}^2$. The key point here is that any open set U contains an open disk about any point that it contains.

There are several key notions that will arise in our study of point-set topology, among them separability, closure, connectedness, and compactness.

- Spaces can have rather strange topologies. For instance, on any set X, one could put the "trivial" topology that the only open sets are $\varnothing$ and X. The only continuous functions from this space X to any other space are constant. In topological terms, this space is highly inseparable, and so we often put requirements on our spaces that prevent this, e.g., that the space be what is known as Hausdorff. A space is Hausdorff if given any two distinct points, one can find open sets containing each point that do not intersect each other. In some sense, this means we can separate the two points from each other. Any space with the trivial topology and at least two elements cannot be Hausdorff.

- As for closure, we should define what it means for a set to be closed, which is exactly to say that its complement will be open. In $\mathbb{R}^2$, a closed disk, i.e., a disk including its boundary, is closed. And in fact, that brings us quickly to the notion of closure of a set, which is to say the smallest closed set containing a given set. In $\mathbb{R}^2$, the closure of an open disk is the closed disk that is the open disk and its boundary.

- A space is connected if it is not a disjoint union of two nonempty open sets, which is to say it has no nonzero subsets which are both open and closed. Then $\mathbb{R}^2$ is connected, but inside it, the union of open disks of radius 1 about 0 and 2 is not. Some spaces are highly disconnected: for instance, for any set X, we may put on it the "discrete" topology, whereby every subset is open. With this topology, every open set is also closed, and so every point is what is known as a connected component. The whole space itself is then called totally disconnected.

- In $\mathbb{R}^2$, a set is compact if it is closed and bounded, the latter meaning that the whole set is contained within a large enough disk about the origin. In fact, the abstract definition is rather different: it says that a space is compact if whenever it is a union of some collection of open sets, only finitely many of these open sets are needed. To see why these might be the same, try thinking about a closed interval $[a,b]$ in $\mathbb{R}$. Cover it by a union of open intervals, and since each has some finite length, only finitely many of them are needed to cover.

Among the most important of topological spaces are metric spaces, which are sets with a distance function satisfying the triangle inequality. We define a topology on a metric space by again considering the open balls of varying radius about a point and defining an arbitrary open set to be a union of these. When we consider this underlying topological space, we may then forget about the metric. So, one should consider a metric space as being more than just a topological space: i.e., it has a notion of distance that a topological space itself does not. This notion of distance is crucial for many applications in analysis, but it is also useful to have the flexibility to work with spaces that have no natural metric. A topological space that has a metric that gives rise to the original topology is called metrizable.

Toward the end of these notes, we will turn from point-set topology to the very beginnings of algebraic topology, which motivated many of the latter definitions. In particular, we shall explain the notion of homotopy, which allows one to transform one space into another. From the perspective of homotopy theory, two spaces are homotopic if one can be continuously transformed into another. From the perspective of homotopy, a closed disk in $\mathbb{R}^2$ is just a point, in that it its radius can be shrunk at a continuous rate until it becomes a point. This perspective is, for instance, quite useful for classifying compact surfaces by how many "holes" they have: a sphere and a torus are topologically different as they are not homotopic. The first object of abstract algebra that can be used to measure this difference is known as the fundamental group, and it consists of "homotopy classes" of loops on a space beginning and ending at a fixed point. For instance, the fundamental group of a circle is just the integers, with the integer determined by how many times a loop wraps around the circle and in what direction.

CHAPTER 1

Topological spaces

1.1. Topologies

We begin by defining topological spaces.

DEFINITION 1.1.1. A *topology* on a set X is a set $\mathscr{T}$ of subsets of X such that

i. the empty set $\varnothing$ and the set X are contained in $\mathscr{T}$,

ii. if $\mathscr{U}$ is a subset of $\mathscr{T}$, then $\bigcup_{U \in \mathscr{U}} U \in \mathscr{T}$, and

iii. if $U_1, \ldots, U_n \in \mathscr{T}$ for some $n \geq 1$, then $U_1 \cap \cdots \cap U_n \in \mathscr{T}$.

In other words, a topology $\mathscr{T}$ is a collection of subsets of X containing $\varnothing, X$ and which is closed under arbitrary unions and finite intersections.

DEFINITION 1.1.2. A *topological space* is a pair $(X, \mathscr{T})$ consisting of a set X and a topology $\mathscr{T}$ on X.

DEFINITION 1.1.3. Let X be a set.

a. The *discrete topology* on X is the topology equal to the set of all subsets (i.e., the power set) of X. We say that a topological space X is *discrete* if its topology is the discrete topology on X.

b. The *trivial topology* on X is the topology $\{\varnothing, X\}$.

DEFINITION 1.1.4. Let $(X, \mathscr{T})$ be a topological space. A subset U of a topological space X is called *open* if $U \in \mathscr{T}$.

TERMINOLOGY 1.1.5. We often omit the notation of a topology $\mathscr{T}$ on a topological space X and simply refer to X as a topological space when its topology $\mathscr{T}$ is understood. At times, we say that a topological space X is *endowed with* (or *has*) a topology $\mathscr{T}$. We sometimes refer to a topological space more simply as a *space*.

DEFINITION 1.1.6. An element of a topological space X is called a *point* of X.

EXAMPLE 1.1.7. A topological space X has the discrete topology if and only if every subset of X is open.

EXAMPLES 1.1.8.

a. Let $X = \{a, b\}$ be a two-point set. Then there are 4 distinct topologies on X, all equal to $\{\varnothing, X\} \cup S$, where S is some subset of $\{\{a\}, \{b\}\}$.

b. Let $X = \{a,b,c\}$ be a three-point set. Then $\{\varnothing, \{a\}, \{b\}, X\}$ is not a topology on X, as it is not closed under unions, while $\{\varnothing, \{a,b\}, \{a,c\}, X\}$ is not a topology on X, as it is not closed under finite intersections.

DEFINITION 1.1.9. A set X has a topology under which a nonempty subset U is open if and only if its complement $X - U = \{x \in X \mid x \notin U\}$ is finite. This topology is known as the *finite complement topology* on X.

EXAMPLE 1.1.10. The Euclidean topology on $\mathbb{R}$ is the unique topology under which a set is open if it is a union of open intervals. This is a topology as both $\varnothing$ and $\mathbb{R}$ are open intervals, any union of unions of open intervals is a union of open intervals, and any finite intersection of open intervals is an open interval (which, as the reader will check, implies that any finite intersection of unions of open intervals is a union of open intervals).

DEFINITION 1.1.11. If $\mathscr{T}$ and $\mathscr{T}'$ are topologies on a set X with $\mathscr{T} \subseteq \mathscr{T}'$, we say that $\mathscr{T}'$ is *finer* (or *stronger*) than $\mathscr{T}$, and $\mathscr{T}$ is *coarser* (or *weaker*) than $\mathscr{T}'$. If, in addition, $\mathscr{T} \neq \mathscr{T}'$, we say that $\mathscr{T}'$ is *strictly finer* (or *strictly stronger*) than $\mathscr{T}$, and $\mathscr{T}$ is *strictly coarser* (or *strictly weaker*) than $\mathscr{T}'$.

REMARK 1.1.12. We think of a topology with more open sets as being finer in that we think of open sets as separating points from each other, so a topology with more open sets is more "fine-grained", in a sense. The discrete topology is the finest topology on any set, while the trivial topology is the coarsest.

REMARK 1.1.13. The terminology of a "finer" topology including one that may be the same is in some sense unfortunate, but it is the most standard usage.

EXAMPLE 1.1.14. Consider the three-point set $X = \{a,b,c\}$ with topologies $\mathscr{T}_1 = \{\varnothing, X\}$, $\mathscr{T}_2 = \{\varnothing, \{a\}, X\}$, $\mathscr{T}_3 = \{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$, and $\mathscr{T}_4$ the discrete topology. Then $\mathscr{T}_{i+1}$ is strictly finer than $\mathscr{T}_i$ for each $1 \leq i \leq 3$. If we set $\mathscr{T}_3' = \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, X\}$, then $\mathscr{T}_3'$ is strictly finer than $\mathscr{T}_1$ and $\mathscr{T}_2$ and strictly coarser than $\mathscr{T}_4$ but has no such relation with $\mathscr{T}_3$.

DEFINITION 1.1.15. An *open neighborhood* of a point x in a topological space X is an open set containing x. We say that an open neighborhood V of x is an *open neighborhood of V in a subset* A if V is contained in A.

LEMMA 1.1.16. *A subset U of a topological space X is open if and only if every point of U has an open neighborhood in U.*

PROOF. If U is open and $x \in U$, then U is an open neighborhood of x in U. Conversely, if every point x of a subset U of X has an open neighborhood V_x in U, then $U = \bigcup_{x \in U} V_x$, so U is open as a union of open sets in X. □

At times, we may wish to speak of closed neighborhoods, in which case the following definition is useful.

DEFINITION 1.1.17. A *neighborhood* of a point x in a topological space X is any subset of X containing an open neighborhood of x.

PROPOSITION 1.1.18. *A set of subsets $\mathscr{T}$ of X is a topology on X if and only if $\varnothing, X \in \mathscr{T}$, the set $\mathscr{T}$ is closed under arbitrary unions, and for all $U_1, \dots, U_n \in \mathscr{T}$ with $n \geq 1$ and $x \in \bigcap_{i=1}^n U_i$, there exists an open neighborhood of x contained in $\bigcap_{i=1}^n U_i$.*

PROOF. Suppose that $U_1, \dots, U_n \in \mathscr{T}$ for some $n \geq 1$, and set $W = \bigcap_{i=1}^n U_i$. If $\mathscr{T}$ is a topology, then W is open, so we may take W to be the open neighborhood of the proposition. If on the other hand we have that for each $x \in W$, there exists an open neighborhood V_x of x in W, then $W = \bigcup_{x \in W} V_x$ is an element of $\mathscr{T}$ if $\mathscr{T}$ is closed under unions. Thus, $\mathscr{T}$ is a topology under the conditions of the proposition. $\square$

NOTATION 1.1.19. Given a set X and a subset A, we write A^c for the complement $X - A$ of A when X is understood.

DEFINITION 1.1.20. A subset A of a topological space X is *closed*, or a *closed subset* of X, if its complement $X - A$ is a open.

EXAMPLES 1.1.21.

a. Every subset of a discrete space is closed.

b. The only closed subsets of a space X with the trivial topology are $\varnothing$ and X.

c. In $\mathbb{R}$ with its Euclidean topology, closed intervals are closed subsets, as are their finite unions. Some but not all infinite unions of closed intervals are also closed: e.g., the set $\mathbb{Z}$ of integers inside $\mathbb{R}$ is an infinite union of closed intervals of length zero that is closed.

PROPOSITION 1.1.22. *Let X be a set. A set $\mathscr{T}$ of subsets of X forms topology on X if and only if the set $\mathscr{C} = \{U^c \mid U \in \mathscr{T}\}$ has the properties that it contains $\varnothing$ and X, intersections of elements of $\mathscr{C}$ are contained in $\mathscr{C}$, and finite unions of elements of $\mathscr{C}$ are contained in $\mathscr{C}$.*

PROOF. Let $\mathscr{T}$ be a set of subsets of X and $\mathscr{C}$ be the set of complements of elements of $\mathscr{T}$. We have $\varnothing = X^c$ and $X = X^c$, so $\varnothing, X \in \mathscr{C}$ if and only if $\varnothing, X \in \mathscr{T}$. We have

$$\left(\bigcap_{A \in \mathscr{C}} A \right)^c = \bigcup_{A \in \mathscr{C}} A^c,$$

so $\mathscr{C}$ is closed under intersections if and only if $\mathscr{T}$ is closed under unions. If $A_1, \dots, A_n \in \mathscr{C}$ for some $n \geq 1$, then

$$\left(\bigcup_{i=1}^n A_i \right)^c = \bigcap_{i=1}^n A_i^c,$$

so $\mathscr{C}$ is closed under finite unions if and only if $\mathscr{T}$ is closed under finite intersections. Thus, $\mathscr{T}$ is a topology if and only if $\mathscr{C}$ satisfies the conditions of the proposition. $\square$

1.2. Subspaces

DEFINITION 1.2.1. Let $(X,\mathscr{T})$ be a topological space and A be a subset of X. The *subspace topology* on A is the set $\mathscr{T}_A = \{U \cap A \mid U \in \mathscr{T}\}$ of subsets of A.

DEFINITION 1.2.2. A topological space A with underlying set a subset of a topological space X is called a *subspace* if its topology is the subspace topology from X.

We verify that the subspace topology on a topological space is in fact a topology.

PROPOSITION 1.2.3. *Let X be a topological space and A be a subset of X. Then the subspace topology on A is a topology on A.*

PROOF. Let $\mathscr{T}$ denote the topology on X and $\mathscr{T}_A$ the subspace topology on A. We have $\varnothing = \varnothing \cap A \in \mathscr{T}_A$ and $A = X \cap A \in \mathscr{T}_A$, so $\mathscr{T}_A$ satisfies property (i) of a topology.

If $\mathscr{V} \subseteq \mathscr{T}_A$, then for each $V \in \mathscr{V}$, there exists $U_V \in \mathscr{T}$ with $V = U_V \cap A$. We then have

$$\bigcup_{V\in\mathscr{V}} V = \bigcup_{V\in\mathscr{V}} (U_V \cap A) = \left(\bigcup_{V\in\mathscr{V}} U_V\right) \cap A \in \mathscr{T}_A$$

since $\bigcup_{V\in\mathscr{V}} U_V \in \mathscr{T}$ as $\mathscr{T}$ is a topology. Thus, $\mathscr{T}_A$ satisfies property (ii) of a topology.

If $V_1,\dots,V_n \in \mathscr{T}_A$ for some $n \geq 1$, then $V_i = U_i \cap A$ for some $U_i \in \mathscr{T}$ for $1 \leq i \leq n$. We then have

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (U_i \cap A) = \left(\bigcap_{i=1}^{n} U_i\right) \cap A \in \mathscr{T}_A,$$

since $\bigcap_{i=1}^{n} U_i \in \mathscr{T}$ as $\mathscr{T}$ is a topology. Thus, $\mathscr{T}_A$ satisfies property (iii) of a topology. □

The subspace topology on an open subset of a topological space is a subset of the topology on the space.

PROPOSITION 1.2.4. *If X is a topological space and U is an open subset of X, then the subspace topology on U is the set of open subsets of X that are contained in U.*

PROOF. If V is an open subset of U, then $V = W \cap U$ for some open subset W of X, so V is open in X as an intersection of two of its open subsets. Conversely, if V is an open subset of X contained in U, then $V = V \cap U$, so V is open in U as well. □

EXAMPLES 1.2.5.

a. Every subspace of a discrete space X is discrete.

b. The subspace topology on an open interval $(a,b) \in \mathbb{R}$ with $a < b$ consists of $\varnothing$ and all unions of open intervals (a',b') with $a \leq a' \leq b' \leq b$.

c. Consider the four-point set $X = \{a,b,c,d\}$ with the topology $\{\varnothing,\{a\},\{b\},\{a,b\},X\}$. Then the subspace topology on $\{a,b\}$ is the discrete topology, while the subspace topology on $\{c,d\}$ is the trivial topology.

The reader may check the following lemma.

LEMMA 1.2.6. *Let A be a closed subset of a topological space X. Then the closed sets of A under the subspace topology are exactly the intersections of closed subsets of X with A.*

TERMINOLOGY 1.2.7. An open (resp., closed) subset of a topological space X, when endowed with the subspace topology, is called an open (resp., closed) subspace of X.

1.3. Bases

DEFINITION 1.3.1. A subset $\mathscr{B}$ of a topology $\mathscr{T}$ on a topological space X is said to be a *base*, or *basis*, for the topology $\mathscr{T}$ on X if every open set $U \in \mathscr{T}$ is a (possibly empty) union of elements of $\mathscr{B}$.

EXAMPLE 1.3.2. The set of open intervals in $\mathbb{R}$ is a base for the Euclidean topology on $\mathbb{R}$.

EXAMPLE 1.3.3. If X is a set with the discrete topology, then the set $\{\{x\} \mid x \in X\}$ is a base of the topology on X.

DEFINITION 1.3.4. We say that a set $\mathscr{S}$ of subsets of X *covers* a subset A of X if $A \subseteq \bigcup_{S \in \mathscr{S}} S$, and $\mathscr{S}$ is a *cover* of A if $\mathscr{S}$ covers A.

THEOREM 1.3.5. *Let X be a set, and let $\mathscr{B}$ be a set of subsets of X such that*

i. $\mathscr{B}$ covers X and

ii. for every $U,V \in \mathscr{B}$ and $x \in U \cap V$, there exists $W \in \mathscr{B}$ with $x \in W$ and $W \subseteq U \cap V$.

Then the collection

$$\mathscr{T} = \left\{ \bigcup_{U \in \mathscr{C}} U \mid \mathscr{C} \subseteq \mathscr{B} \right\}$$

of arbitrary unions of elements of $\mathscr{B}$ is a topology on X, and $\mathscr{B}$ is a base for the topology $\mathscr{T}$. Moreover, if X is a topological space and $\mathscr{B}$ is a base of open sets in X, then $\mathscr{T}$ is the topology on X.

PROOF. If $\mathscr{T}$ is a topology, then $\mathscr{B}$ will by definition be a base, so we need only verify that $\mathscr{T}$ is a topology. Note that $\varnothing$ is the empty union of elements of $\mathscr{B}$ and $X \in \mathscr{T}$ by property (i) of $\mathscr{B}$. We also have that $\mathscr{T}$ is closed under arbitrary unions, as its elements are just the arbitrary unions of elements of $\mathscr{B}$. To check that $\mathscr{T}$ is closed under finite intersections, it suffices by recursion to show that it is closed under intersections of two elements, so let $U_1, U_2 \in \mathscr{T}$. Any $x \in U_1 \cap U_2$ lies in the intersection of some $V_1 \in \mathscr{B}$ contained in U_1 and some $V_2 \in \mathscr{B}$ contained in U_2. So, by property (ii) of $\mathscr{B}$, there exists $W \in \mathscr{B}$ with

$$x \in W \subseteq V_1 \cap V_2 \subseteq U_1 \cap U_2.$$

Since $\mathscr{B} \subset \mathscr{T}$, we have $W \in \mathscr{T}$ as well. By Proposition 1.1.18, the set $\mathscr{T}$ is a topology on X.

If X is a topological space with topology $\mathscr{T}'$ and $\mathscr{B}$ is a base of open sets in X, then both $\mathscr{T}$ above and the topology $\mathscr{T}'$ are the set of unions of elements of $\mathscr{B}$, the first by definition of $\mathscr{T}$, and the second by definition of a base, so $\mathscr{T} = \mathscr{T}'$. □

DEFINITION 1.3.6. The topology $\mathscr{T}$ on a set X given by arbitrary unions of elements of a set of subsets $\mathscr{B}$ of X satisfying the two conditions of Theorem 1.3.5 is called the *topology generated by the base $\mathscr{B}$*. We say that $\mathscr{B}$ *generates* the topology $\mathscr{T}$.

We may now easily define the Euclidean topology on $\mathbb{R}^n$.

EXAMPLE 1.3.7. The Euclidean topology on $\mathbb{R}^n$ is the topology generated by the base consisting of open balls in $\mathbb{R}^n$ of finite radius. To see this is a topology, one need only note that around any point inside any nonempty intersection of two open balls, there exists another open ball.

DEFINITION 1.3.8. Let X be a topological space. A *base of open neighborhoods* of a point x is a set $\mathscr{B}_x$ of open neighborhoods of x such that for every open neighborhood U of x in X, there exists $V \in \mathscr{B}_x$ with $V \subseteq U$. An element of $\mathscr{B}_x$ is said to be a *basic open neighborhood* of x.

LEMMA 1.3.9. *Let X be a topological space, and for each $x \in X$, let $\mathscr{B}_x$ be a base of open neighborhoods of x. Then $\mathscr{B} = \bigcup_{x \in X} \mathscr{B}_x$ is a base for the topology on X.*

PROOF. Since each $\mathscr{B}_x$ is nonempty, we have $X = \bigcup_{U \in \mathscr{B}} U$. If $U \in \mathscr{B}_x$ and $V \in \mathscr{B}_y$ for some $x, y \in X$ and $z \in U \cap V$, then there exists $W \in \mathscr{B}_z$ with $W \subseteq U \cap V$. By Theorem 1.3.5, $\mathscr{B}$ is a base for a topology $\mathscr{T}'$ on X. We have $\mathscr{T}' \subseteq \mathscr{T}$ since every set in $\mathscr{B}$ is open and $\mathscr{T}$ is closed under unions. We claim that $\mathscr{T}'$ is in fact the original topology $\mathscr{T}$.

Let $W \in \mathscr{T}$. For $x \in W$, since $\mathscr{B}_x$ is a base of open neighborhoods of x, there exists $U_x \in \mathscr{B}_x$ with $U_x \subseteq W$. Then $W \in \mathscr{T}'$ by Lemma 1.1.16. Thus $\mathscr{T} = \mathscr{T}'$. □

EXAMPLE 1.3.10. In $\mathbb{R}^n$, the set of open balls centered at a point x forms a base of open neighborhoods of x.

In fact, any set of subsets of X with union X gives rise to a topology on X.

DEFINITION 1.3.11. A set $\mathscr{S}$ of open subsets of a topological space X is called a *subbase*, or *subbasis*, for the topology on X if every proper open set in X is a union of finite intersections of elements of $\mathscr{S}$.

The following is an simple consequence of Theorem 1.3.5.

PROPOSITION 1.3.12. *Let $\mathscr{S}$ be a set of subsets of X. Let $\mathscr{B}$ be the set consisting of X and all finite intersections of elements of $\mathscr{S}$. Then $\mathscr{B}$ is a base for a topology on X for which $\mathscr{S}$ is a subbase.*

EXAMPLE 1.3.13. The set of all intervals (a, ∞) and $(-\infty, b)$ with $a, b \in \mathbb{R}$ is a subbase for the Euclidean topology on $\mathbb{R}$.

1.4. Closure

There is a smallest closed set containing a given subset of a topological space, known as its closure.

DEFINITION 1.4.1. The *closure* $\overline{A}$ of a subset A of a topological space X is the intersection of all closed subsets of X containing A.

LEMMA 1.4.2. *The closure $\overline{A}$ of a subset A of X is the smallest closed subset of X containing A in the sense that $\overline{A}$ is closed and contains A and, if B is a closed subset of X with $A \subseteq B$, then $\overline{A} \subseteq B$.*

PROOF. First, we remark that $\overline{A}$ is closed as an intersection of closed sets and contains A, as all of these closed sets contain A. Moreover, if B is closed and contains A, then B contains the intersection $\overline{A}$ of all closed sets containing A, since B is one of the sets over which the intersection is taken. □

EXAMPLE 1.4.3. The closure of an open interval (a,b) in $\mathbb{R}$ is the closed interval $[a,b]$, as $[a,b]$ is a closed set containing (a,b), and none of (a,b), $[a,b)$, and $(a,b]$ is closed.

We have the following alternative characterization of the closure.

PROPOSITION 1.4.4. *Let A be a subset of a topological space X. Then $x \in \overline{A}$ if and only if every (open) neighborhood of x in X has nonempty intersection with A.*

PROOF. We have $x \in \overline{A}$ if and only if $x \in B$ for all closed sets B containing A. The latter holds if and only if $x \notin U$ for all open sets $U \subseteq A^c$. And this holds if and only if every open set U containing x is not contained in A^c, hence has nonempty intersection with A. □

EXAMPLE 1.4.5. If $x \in (a,b)$, then clearly every open neighborhood of x intersects (a,b), in particular in x. Any open interval of the form $(a-\varepsilon, a+\varepsilon)$ intersects (a,b) in $\{a+\delta \mid 0 < \delta < \varepsilon\}$. Similarly, any interval $(b-\varepsilon, b+\varepsilon)$ intersects (a,b) in $\{b-\delta \mid 0 < \delta < \varepsilon\}$. On the other hand, if $x > b$ or $x < a$, then there exists a sufficiently small interval centered at x that does not intersect (a,b). Since every open neighborhood of a point in $\mathbb{R}$ contains an open interval (centered at the point), Proposition 1.4.4 again tells us that the closure of (a,b) is $[a,b]$.

DEFINITION 1.4.6. A subset A of a topological space X is *dense* in X if $\overline{A} = X$.

EXAMPLE 1.4.7. The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$ with its Euclidean topology. That is, every open interval containing a real number contains a rational number, being that the interval has finite nonzero length.

We also have the notion of an interior of a set.

DEFINITION 1.4.8. The *interior* A° of a subset A of a topological space X is the union of all open sets of X contained in A.

Note that $A^\circ = (\overline{A^c})^c$ by its definition. We then have the following as a consequence of Lemma 1.4.2.

LEMMA 1.4.9. *The interior of a subset A of a topological space X is the largest open subset of X contained in A.*

We also have a notion of boundary.

DEFINITION 1.4.10. The boundary ∂A of a subset A of X is the complement of the interior A° of A in the closure $\overline{A}$.

EXAMPLE 1.4.11. In $\mathbb{R}$, the closure of an open interval (a,b) with $a,b \in \mathbb{R}$ and $a < b$ is the closed interval $[a,b]$, and the interior of $[a,b]$ is (a,b). The boundary of (a,b), or $[a,b]$, is $\{a\} \cup \{b\}$.

EXAMPLE 1.4.12. In $\mathbb{R}^n$, the closure of the open ball of radius ε about a point x is the closed ball of radius ε about the point x, while the boundary is the sphere of radius ε centered at x.

EXAMPLE 1.4.13. If X has the trivial topology, then the closure of any nonempty subset A is X, while the interior of A is empty unless $A = X$. So, any nonempty, proper subset A of X has boundary X, whereas the boundary of X is empty.

EXAMPLE 1.4.14. Consider the three-point set $X = \{a,b,c\}$ with topology

$$\mathscr{T} = \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, X\}.$$

The closed sets of $\mathscr{T}$ are $\varnothing$, $\{b\}$, $\{c\}$, $\{b,c\}$, and X. Thus, the closure of $\{a\}$ is X, while $\{b\}$ and $\{c\}$ are closed.

REMARK 1.4.15. The taking of subspaces can change interiors and closures. For instance, the interior of the closed interval $[a,b]$ with $a < b$ in $\mathbb{R}$ is (a,b), but its interior in $[a,b]$ is $[a,b]$, since $[a,b]$ is open in $[a,b]$.

1.5. Limit points

DEFINITION 1.5.1. A point x in a topological space X is called a *limit point* of a subset A of X if every (open) neighborhood of x intersects A in a point other than x.

The following is a corollary of Proposition 1.4.4.

LEMMA 1.5.2. *The closure of a subset A of a topological space X is the union of A and the set of its limit points.*

PROOF. By Proposition 1.4.4, if $x \in \overline{A}$, then every open neighborhood of x intersects A. If $x \notin A$, these intersections cannot contain x, so x is a limit point. □

Since closed sets are their own closures, we have the following.

COROLLARY 1.5.3. *Any closed subset A of a topological space contains all of its limit points.*

REMARK 1.5.4. In Proposition 1.4.4, we may replace the condition on every open neighborhood with the same condition restricted to open neighborhoods in any base.

One might ask how the notion of a limit point compares to the notion of points to which sequences converge. For this, we need the following definition.

DEFINITION 1.5.5. A sequence $(x_n)_{n\geq 1}$ of points of a topological space X *converges* to a point $x \in X$ if for every open neighborhood U of x, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$. The point x is said to be a *limit* of the *sequence* x_n.

REMARK 1.5.6. By definition, a limit of a convergent sequence $(x_n)_{n\geq 1}$ that is not eventually constant is a limit point of the set $\{x_n \mid n \geq 1\}$. However, the latter set may have more then one limit point even if $(x_n)_{n\geq 1}$ converges.

EXAMPLE 1.5.7. If X has the trivial topology, then every sequence in X converges to every point of X.

To avoid such pathologies as in the previous example, it is useful to put the following condition on a space.

DEFINITION 1.5.8. A topological space X is called is *Hausdorff* is for every two distinct points $a, b \in X$, there exist open neighborhoods U of a and V of b such that $U \cap V = \varnothing$.

More briefly, X is Hausdorff if every two distinct points of X have disjoint neighborhoods. In Hausdorff spaces, points are closed.

LEMMA 1.5.9. *In a Hausdorff space X, every singleton set $\{x\}$ for $x \in X$ is closed.*

PROOF. If $x \in X$ and $a \in X$ with $x \neq a$, then there exists an open neighborhood U_a of a not containing x. As the union of all such open sets U_a is the complement of $\{x\}$, the set $\{x\}$ is closed. □

The property of being Hausdorff is stronger than that of points being closed, however.

EXAMPLE 1.5.10. In an infinite set X with the finite complement topology, points are closed as the complement of open sets. However, any two nonempty open sets in X intersect in all but finitely many elements of X, so X is not Hausdorff. Moreover, every non-repeating sequence $(x_n)_{n\geq 1}$ in X converges to every point of X, as the reader should check using the fact that open sets have finite complements.

Even better, in Hausdorff spaces, every convergent sequence has a unique limit.

PROPOSITION 1.5.11. *Every convergent sequence in a Hausdorff space has a unique limit.*

PROOF. If $x \in X$ is a limit of a convergent sequence $(x_n)_{n\geq 1}$ in a Hausdorff space X, then for any $y \in X - \{x\}$, we have have disjoint open neighborhoods U of x and V of y. For sufficiently large n, the point x_n are all in U, hence not in V, and therefore y is not a limit point of the sequence. □

The following is immediate from the definitions.

LEMMA 1.5.12. *Every subspace of a Hausdorff space is Hausdorff.*

The following characterization of the Hausdorff property is often useful.

LEMMA 1.5.13. *A topological space X is Hausdorff if and only if for every two distinct points $x, y \in X$, there exists an open neighborhood U of x with $y \notin \overline{U}$.*

PROOF. Let x and y be distinct points of X. We have disjoint open neighborhoods U and V of x and y, respectively, if and only if we have an open neighborhood U of x and a closed neighborhood A containing U and not containing y. That is, we simply take A and V to be complements of each other. But such an A exists if and only if we can take it to be the closure $\overline{U}$, which is contained in any such A, being the smallest closed set containing U. □

1.6. Metric spaces

Metric spaces, and the open balls inside of them, provide fundamental examples of topological spaces. We review the definition here.

DEFINITION 1.6.1. A *metric* on a set X is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that for all $a, b, c \in X$, one has

i. $d(a,b) = 0$ if and only if $a = b$,

ii. $d(a,b) = d(b,a)$, and

iii. $d(a,c) \leq d(a,b) + d(b,c)$.

TERMINOLOGY 1.6.2. For a set X, the condition $d(a,c) \leq d(a,b) + d(b,c)$ on $d\colon X \times X \to \mathbb{R}_{\geq 0}$ for all $a, b, c \in X$ is called the *triangle inequality*.

DEFINITION 1.6.3. A pair (X,d) consisting of a set X and a metric d on X is called a *metric space*.

NOTATION 1.6.4. When the metric d on a metric space (X,d) is understood, we often write X for the metric space.

EXAMPLE 1.6.5. The set $\mathbb{R}^n$ is a metric space for the distance function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by the Euclidean metric

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$. We remark that any three points x, y, and z form the vertices of a triangle, and the distances between them are the lengths of the sides, so the triangle inequality reduces to the usual triangle inequality of Euclidean geometry.

DEFINITION 1.6.6. In a metric space (X,d), the *open ball* of *radius* $\varepsilon > 0$ about a point (or with *center*) $x \in X$ is the set

$$B(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}.$$

The *closed ball* of radius ε is

$$\overline{B}(x,\varepsilon) = \{y \in X \mid d(x,y) \leq \varepsilon\}.$$

PROPOSITION 1.6.7. *The set of open balls in a metric space X forms a base of a topology on X, and the set of open balls with center x is a base of open neighborhoods of x in this topology.*

PROOF. Clearly, the union of open balls in a metric space is X, so by Theorem 1.3.5, it suffices to show that for any two open balls $U = B(x,\varepsilon)$ and $V = B(y,\delta)$ in X and any point $z \in U \cap V$, there exists an open ball containing z and contained in $U \cap V$. For this, choose any positive real number

$$\rho < \min\{\varepsilon - d(x,z), \delta - d(y,z)\}.$$

Then $B(z,\rho) \subseteq U \cap V$ by the triangle inequality. That is, if $d(z,w) < \rho$, then

$$d(w,x) \leq d(w,z) + d(z,x) < \rho + d(x,z) < \varepsilon,$$

and similarly $d(w,y) \leq d(w,z) + d(z,y) < \delta$. Thus the set of open balls in X form a base.

If $x \in X$ and U is any open neighborhood of x, then it contains some basic open neighborhood $B(y,\varepsilon)$ of x, i.e., with $d(x,y) < \varepsilon$. Then $B(x,\varepsilon - d(x,y)) \subseteq U$ as in the above argument, so the balls centered at x form a base of open neighborhoods of x. □

DEFINITION 1.6.8. The *metric topology* on a set X induced by a metric d on X is the topology on X generated by the set of open balls under d.

LEMMA 1.6.9. *Let X be a metric space. Then every closed ball $\overline{B}(x,\varepsilon)$ is closed in the metric topology.*

PROOF. It suffices to show that the complement of $\overline{B}(x,\varepsilon)$ is open. If $y \notin \overline{B}(x,\varepsilon)$, then $\delta = d(x,y) - \varepsilon > 0$, and $B(y,\delta)$ and $\overline{B}(x,\varepsilon)$ are disjoint by the triangle inequality. That is, if $z \in B(y,\delta)$, then $d(x,z) + \delta > d(x,z) + d(z,y) \geq d(x,y)$, so $d(x,z) > \varepsilon$. Thus, $z \notin \overline{B}(x,\varepsilon)$. □

Two different metrics on a set X can have the same metric topology. Take the following example.

EXAMPLE 1.6.10. Consider the Euclidean metric d on $\mathbb{R}^n$ and the box metric d' on $\mathbb{R}^n$ defined by

$$d'(x,y) = \max\{|x_i - y_i| \mid 1 \leq i \leq n\}$$

for $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in \mathbb{R}^n$. The reader should check that $d'(x,y)$ is in fact a metric.

We have bases $B(x,\varepsilon)$ and $B'(x,\varepsilon)$ of open balls about a point x with respect to these respective metrics. For any $y \in \mathbb{R}^n$, we have

$$\max\{|x_i - y_i|^2 \mid 1 \leq i \leq n\} \leq \sum_{i=1}^n (x_i - y_i)^2 \leq n \max\{|x_i - y_i|^2 \mid 1 \leq i \leq n\},$$

so

$$B(x,\varepsilon) \subseteq B'(x,\varepsilon) \subseteq B(x,\sqrt{n}\varepsilon)$$

for all $\varepsilon > 0$. Thus, the two metric topologies coincide.

We will often consider a metric space as a topological space endowed with the metric topology. We can then examine the topological properties of metric spaces.

PROPOSITION 1.6.11. *Metric spaces are Hausdorff.*

PROOF. Let X be a metric space, and suppose that $x, y \in X$ are distinct points. Let $\varepsilon = \frac{1}{2}d(x,y)$. Then $B(x,\varepsilon)$ and $B(y,\varepsilon)$ are disjoint by the *triangle inequality*. □

DEFINITION 1.6.12. A topological space $(X,\mathscr{T})$ is *metrizable* if there exists a metric d on X such that the metric topology induced by d is the topology $\mathscr{T}$ on X.

The following is a direct corollary of Proposition 1.6.11.

COROLLARY 1.6.13. *If X is a metrizable topological space, then X is Hausdorff.*

Discrete spaces are metric spaces as well.

DEFINITION 1.6.14. Let X be a set. The *discrete metric d* on X is defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

We have the following.

LEMMA 1.6.15. *The discrete metric on a set X is a metric, and the topology induced by this metric is the discrete topology.*

PROOF. That d is a metric is straightforward. Since $B(x,\frac{1}{2}) = \{x\}$ for all $x \in X$, singleton sets are open in this topology, so X is discrete. □

The following example shows that even metric spaces can defy our intuition from Euclidean geometry.

EXAMPLE 1.6.16. While in any metric space, the closure of the open ball $B(x,\varepsilon)$ is contained in the closed ball $\overline{B}(x,\varepsilon)$, the closure of $B(x,\varepsilon)$ can in fact be smaller. For instance, if (X,d) is a metric space with the discrete metric d, then the set $B(x,1) = \{x\}$ is both open and closed (as are all subsets of X), while $\overline{B}(x,1) = X$.

DEFINITION 1.6.17. A subset A of a metric space X is *bounded* if there exists $N > 0$ such that $d(x,y) \leq N$ for all $x, y \in A$.

This notion of boundedness is not a topological one.

LEMMA 1.6.18. *If (X,d) is a metric space, then the function $d'\colon X\times X\to\mathbb{R}_{\geq 0}$ given by*

$$d'(x,y)=\min\{d(x,y),1\}$$

is a metric on X, and the metric topologies on X from d and d' are the same.

PROOF. Let $x,y,z\in X$. If $d(x,y)\leq 1$, then

$$d'(x,y)=d(x,y)\leq\min\{d(x,z)+d(z,y),1+d(z,y),d(x,z)+1,1+1\}=d'(x,z)+d'(z,y),$$

by the triangle inequality for the first of the terms in the set and the fact that $d(x,y)\leq 1$ for the others. If $d(x,y)\geq 1$, then

$$d'(x,y)=1\leq\min\{d(x,z)+d(z,y),1+d(z,y),d(x,z)+1,1+1\}=d'(x,z)+d'(z,y)$$

by the triangle inequality for the first term in the set (in that $1\leq d(x,y)\leq d(x,z)+d(z,y)$) and the fact that the other terms are clearly at least one. Thus, d' satisfies the triangle inequality, and it clearly satisfies the other two conditions for being a metric.

The open balls of radius less than 1 form a base of any metric topology, as any open ball contains one of these. Since these sets of open balls coincide for d and d', these metrics induce the same topology on X. □

EXAMPLE 1.6.19. Under the Euclidean metric, $\mathbb{R}$ is not bounded, but it is with respect to the metric $d'(x,y)=\min\{|x-y|,1\}$ on $\mathbb{R}$. Nevertheless, both of these metrics induces the Euclidean topology.

Sequences in metric spaces behave as one might expect.

PROPOSITION 1.6.20. *Let (X,d) be a metric space. A sequence $(x_n)_{n\geq 1}$ in X converges to $a\in X$ if and only if $\lim_{n\to\infty}d(a,x_n)=0$.*

PROOF. If U is an open neighborhood of a, then U contains some open ball $B(a,\varepsilon)$. If $\lim_{n\to\infty}d(a,x_n)=0$, then there exists $N\geq 1$ such that $d(a,x_n)<\varepsilon$ for all $n\geq N$, so $x_n\in U$. Conversely, if $(x_n)_{n\geq 1}$ converges to a, then for any ε, there exists $N\geq 1$ such that $x_n\in B(a,\varepsilon)$ for $n\geq N$, which is to say $d(a,x_n)<\varepsilon$. Thus, the limit $\lim_{n\to\infty}d(a,x_n)$ is 0. □

PROPOSITION 1.6.21. *Let X be a metrizable space, let $A\subseteq X$, and let $x\in X$. Then $x\in\overline{A}$ if and only if it is the limit of a convergent sequence of elements of A.*

PROOF. We need only see that any $x\in\overline{A}$ is such a limit. Fix a metric d on X so that we may consider open balls in X. If $x\in\overline{A}$, then for any $n\geq 1$, there exists $x_n\in B(x,\frac{1}{n})\cap A$ by definition of $\overline{A}$. Then the x_n converge to x by Proposition 1.6.20. □

CHAPTER 2

Continuous functions

2.1. Continuous functions

In this section, X and Y will denote topological spaces.

DEFINITION 2.1.1. A function $f\colon X \to Y$ is said to be *continuous* if $f^{-1}(V)$ is open for every open subset V of Y.

DEFINITION 2.1.2. We say that $f\colon X \to Y$ between topological spaces is *continuous at a point* $x \in X$ (or *continuous at* x) if for every open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(U) \subseteq V$.

REMARK 2.1.3. As every open subset is a union of open neighborhoods of its points, a function is continuous if and only if it is continuous at every point.

We can check continuity on basis elements of Y.

PROPOSITION 2.1.4. *Let $\mathscr{B}_X$ and $\mathscr{B}_Y$ be fixed bases of the topologies on X and Y, respectively. Let $f\colon X \to Y$ be a function.*

a. A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for all $V \in \mathscr{B}_Y$.

b. A function $f\colon X \to Y$ is continuous at $x \in X$ if and only if for every basic open neighborhood $V \in \mathscr{B}_Y$ of $f(x)$, there exists a basic open neighborhood $U \in \mathscr{B}_X$ of x with $f(U) \subseteq V$,

PROOF. If f is continuous, then $f^{-1}(V)$ is open for all $V \in \mathscr{B}_Y$ by definition. Conversely, suppose $f^{-1}(V)$ is open for all $V \in \mathscr{B}_Y$. As $\mathscr{B}_Y$ is a base, for any open subset W of Y, there exists $\mathscr{C} \subseteq \mathscr{B}_Y$ such that $W = \bigcup_{V\in\mathscr{C}} V$. Then $f^{-1}(W) = \bigcup_{V\in\mathscr{C}} f^{-1}(V)$, so $f^{-1}(W)$ is open as a union of open sets. Thus, we have proven part (a).

If f is continuous at $x \in X$ and V is a basic open neighborhood of $f(x)$, then $f^{-1}(V)$ contains an open neighborhood of x, and such an open neighborhood contains a basic open neighborhood of x. Conversely, if for every $V \in \mathscr{B}_Y$ with $f(x) \in V$, the set $f^{-1}(V)$ contains a (basic) open neighborhood of x, then for any open neighborhood W of $f(x)$, we have that W contains some such V. Thus, f is continuous at x. $\square$

Here is the fundamental example, which states that a map between metric spaces is continuous with respect to their metric topologies if and only if it is continuous in the usual sense.

PROPOSITION 2.1.5. *Let X and Y be metric spaces, which we endow with their metric topologies. A function $f\colon X \to Y$ is continuous if and only if for every $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that $f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$.*

PROOF. The sets $\mathscr{B}_X$ and $\mathscr{B}_Y$ of open balls of positive radius in X and Y, respectively, are bases for the metric topologies on X and Y. The result is therefore a direct consequence of the equivalence of Proposition 2.1.4(b). $\square$

We give a few more examples.

EXAMPLES 2.1.6.

a. If X is discrete, then any map $f\colon X \to Y$ is continuous.

b. If Y has the trivial topology, then any map $f\colon X \to Y$ is continuous.

c. If the set X has two topologies $\mathscr{T}_X$ and $\mathscr{T}_X'$ with $\mathscr{T}_X'$ finer than $\mathscr{T}_X$ and f is continuous for $\mathscr{T}_X$, then f is continuous for $\mathscr{T}_X'$.

d. If Y has two topologies $\mathscr{T}_Y$ and $\mathscr{T}_Y'$ with $\mathscr{T}_Y'$ finer than $\mathscr{T}_Y$ and f is continuous for $\mathscr{T}_Y'$, then it is continuous for $\mathscr{T}_Y$.

e. As a special case of the two latter examples, the identity map $\mathrm{id}_X : X \to X$ given by $\mathrm{id}_X(x) = x$ is continuous with potentially different topologies on the domain and codomain if and only if the topology of the domain is finer than the topology on the codomain.

We can also express continuity in terms of closed sets and closures.

LEMMA 2.1.7. *A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(B)$ is closed for every closed subset B of Y.*

PROOF. We have $f^{-1}(B) = X - f^{-1}(B^c)$, so $f^{-1}(B)$ is closed if and only if $f^{-1}(B^c)$ is open. The result follows since B^c runs over the open sets of Y as B runs over the closed sets. $\square$

PROPOSITION 2.1.8. *A function $f\colon X \to Y$ is continuous if and only if for every $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.*

PROOF. If f is continuous and $A' = f^{-1}(\overline{f(A)})$, then A' is closed and $A \subseteq f^{-1}(f(A)) \subseteq A'$, so $\overline{A} \subseteq A'$. Thus

$$f(\overline{A}) \subseteq f(A') = f(f^{-1}(\overline{f(A)})) \subseteq \overline{f(A)}.$$

Conversely, suppose that $f(\overline{A}) \subseteq \overline{f(A)}$ for all A. Take B to be a closed subset of Y, and set $A = f^{-1}(B)$. Then $f(A) \subseteq B$, and if $x \in \overline{A}$, then $f(x) \in \overline{f(A)} \subseteq B$. In other words, $\overline{A} \subseteq f^{-1}(B) = A$, so $f^{-1}(B)$ is closed. Therefore, f is continuous. $\square$

EXAMPLES 2.1.9.

a. Constant functions are continuous.

b. The inclusion map $\iota_A : A \to X$ of a subspace A of a space X is continuous. That is, if V is open in X, then $\iota^{-1}(V) = A \cap V$ is open in A in the subspace topology.

c. A composition of continuous maps is continuous.

d. If $f : X \to Y$ is a continuous function, then its restriction $f|_A$ to any subset A of X given by $f|_A(a) = f(a)$ for all $a \in A$ is continuous for the subspace topology on A, since $f|_A = f \circ \iota_A$. (We say that f restricts to $g = f|_A$ on A, that f extends g to X, and that f is an extension of g from A to X.)

e. If $f : X \to Y$ is a continuous function, then the map $g : X \to f(X)$ is continuous for $g(x) = f(x)$ for all $x \in X$, where $f(X)$ is endowed with the subspace topology from Y. If $V \subseteq f(X)$ is open, then $V = U \cap f(X)$ for some open subset U of Y, so $g^{-1}(V) = f^{-1}(U)$ is open.

DEFINITION 2.1.10. An *open cover* of a subset A of a topological space X is a collection $\mathscr{U}$ of open subsets of X that covers A.

LEMMA 2.1.11. *A function $f : X \to Y$ is continuous if and only if there is an open cover $\mathscr{U}$ of X such that $f|_U$ is continuous for all $U \in \mathscr{U}$.*

PROOF. We have seen that each $f|_U$ is continuous if f is. On the other hand, suppose $f|_U$ is continuous for all $U \in \mathscr{U}$. If $V \subset Y$ is an open subset, then $f^{-1}(V) \cap U = f|_U^{-1}(V)$ is open. Thus

$$f^{-1}(V) = \bigcup_{U \in \mathscr{U}} (f^{-1}(V) \cap U)$$

is open, so f is continuous. □

LEMMA 2.1.12. *If $f : X \to Y$ is a function, A and B are closed subspaces of X such that $A \cup B = X$, and $f|_A$ and $f|_B$ are continuous, then f is continuous.*

PROOF. Let $C \subseteq Y$ be closed. Then $f^{-1}(C) = f|_A^{-1}(C) \cup f|_B^{-1}(C)$, and $(f|_A)^{-1}(C)$ is closed in A and $(f|_B)^{-1}(C)$ is closed in B. Since A and B are closed in X, the latter two inverse images are closed in X, so $f^{-1}(C)$ is closed as a union of two closed sets. □

DEFINITION 2.1.13. A function $f : X \to Y$ is a *homeomorphism* if it is a continuous bijection and its inverse is continuous as well.

DEFINITION 2.1.14. Two spaces X and Y are *homeomorphic* if there exists a homeomorphism $f : X \to Y$.

We leave the following for the reader to verify.

LEMMA 2.1.15. *The relation $\simeq$ on any set of topological spaces given by $X \simeq Y$ if X is homeomorphic to Y is an equivalence relation.*

EXAMPLES 2.1.16.

a. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism, as it is continuous with continuous inverse $f^{-1}(x) = x^{1/3}$.

b. The function $f\colon (0,1) \to \mathbb{R}_{>0}$ given by $f(x) = \frac{x}{1-x}$ is a homeomorphism (where $(0,1)$ has the subspace topology from $\mathbb{R}$).

DEFINITION 2.1.17. A function $f\colon X \to Y$ is an *embedding* if it is a homeomorphism onto its image $f(X)$ with the subspace topology from Y.

DEFINITION 2.1.18. A function $f\colon X \to Y$ is an *open map*, or *open*, if $f(U)$ is open for every open subset U of X. A function $f\colon X \to Y$ is a *closed map*, or *closed*, if $f(U)$ is closed for every closed subset U of X.

REMARK 2.1.19. If $f\colon X \to Y$ is a bijection, then f is open if and only if f is closed.

By definition, homeomorphisms are open maps; that is, they are the continuous, open bijections. However, in general, continuous bijections may not be open.

EXAMPLES 2.1.20.

a. Let $f\colon S^1 \to \mathbb{R}^2$ be the natural embedding, where

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Then S^1 is open in itself, but $f(S^1) = S^1$ is not open in $\mathbb{R}^2$.

b. Let X be a set with two topologies $\mathscr{T}$ and $\mathscr{T}'$ with $\mathscr{T}'$ strictly finer than $\mathscr{T}$. Then the identity map $f\colon X \to X$ with the domain having the topology $\mathscr{T}'$ and the codomain having the topology $\mathscr{T}$ is continuous, but its inverse f^{-1} is not, so f is not an open map. On the other hand, f^{-1} is an open map.

We next consider the behavior of continuous functions and sequences.

PROPOSITION 2.1.21. *Let $f\colon X \to Y$ be a function. If f is continuous, then for every sequence $(x_n)_{n\geq 1}$ in X that converges to a point $x \in X$, the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x)$. If X is a metrizable space such that for every sequence $(x_n)_{n\geq 1}$ in X that converges to some $x \in X$, then $(f(x_n))_{n\geq 1}$ converges to $f(x)$, then f is continuous.*

PROOF. Suppose that f is continuous. Let $(x_n)_{n\geq 1}$ be a sequence in X converging to x. Let V be an open neighborhood of $f(x)$ in Y. Then $f^{-1}(V)$ is an open neighborhood of $x \in X$, so contains all x_n for $n \geq N$ for some $N \geq 1$. Thus $f(x_n) \in V$ for all such $n \geq N$, and therefore the sequence of $f(x_n)$ converges to $f(x)$.

Now let be X metrizable, and suppose that whenever $(x_n)_{n\geq 1}$ converges to $x \in X$, the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x)$. Let A be a subset of X and $x \in \overline{A}$. It suffices by Proposition 2.1.8 to show that $f(x) \in \overline{f(A)}$. For this, we need only exhibit a sequence $(x_n)_{n\geq 1}$ in A converging to x. Then $f(x)$ is the limit of the $f(x_n)$. But such a sequence exists by the metrizability of X and Proposition 1.6.21. $\square$

DEFINITION 2.1.22. Let $(f_n)_{n\geq 1}$ be a sequence of functions from a set X to a metric space Y with metric d_Y. The sequence $(f_n)_{n\geq 1}$ *converges uniformly* to $f\colon X\to Y$ if for every $\varepsilon>0$ there exists an integer $N\geq 1$ such that $d_Y(f(x),f_n(x))<\varepsilon$ for all $n\geq N$ and $x\in X$.

REMARK 2.1.23. If Y is a metrizable topological space, then the notion of uniform convergence of a sequence is independent of the choice of metric on d_Y yielding the topology.

PROPOSITION 2.1.24. *Let X be a topological space and Y be a metrizable space. Let $(f_n)_{n\geq 1}$ be a sequence of continuous functions $f_n\colon X\to Y$ that converges uniformly to a function $f\colon X\to Y$. Then f is continuous.*

PROOF. Let $\varepsilon>0$ and $a\in X$. Choose $N\geq 1$ such that $d_Y(f_n(x),f(x))<\frac{\varepsilon}{3}$ for all $n\geq N$ and $x\in X$. By the continuity of f_N, we can find an open neighborhood U of a such that if $x\in U$, then $d_Y(f_N(a),f_N(x))<\frac{\varepsilon}{3}$. For such $x\in X$, we then have

$$d_Y(f(a),f(x))\leq d_Y(f(a),f_N(a))+d_Y(f_N(a),f_N(x))+d_Y(f_N(x),f(x))<\varepsilon.$$

□

2.2. Product spaces

PROPOSITION 2.2.1. *Let I be an indexing set, and for each $i\in I$, let X_i be a topological space. Then the set $\mathscr{B}$ of product sets $\prod_{i=1}^n U_i$ with each U_i open in X_i and $U_i=X_i$ for all but finitely many $i\in I$ forms a base of a topology on $X=\prod_{i\in I}X_i$.*

PROOF. Let $U=\prod_{i\in I}U_i$ and $V=\prod_{i\in I}V_i$ be two sets in $\mathscr{B}$. Then

$$\left(\prod_{i\in I}U_i\right)\cap\left(\prod_{i\in I}V_i\right)=\prod_{i\in I}(U_i\cap V_i),$$

each $U_i\cap V_i$ is open in X_i, and all but finitely many $U_i\cap V_i$ equal X_i. Thus $U\cap V\in\mathscr{B}$. Since $X\in\mathscr{B}$ as well, it follows that $\mathscr{B}$ is a base. □

DEFINITION 2.2.2. Let I be an indexing set, and for each $i\in I$, let X_i be a topological space. The *product topology* on X is the topology generated by the base of sets $\prod_{i\in I}U_i$ with each U_i open in X_i and all but finitely many $U_i=X_i$.

REMARK 2.2.3. The product topology on a finite product $\prod_{i=1}^n X_i$ of topological spaces $X_1,\dots,X_n$ with $n\geq 1$ has a base $\prod_{i=1}^n U_i$ of open sets with U_i open in X_i for $1\leq i\leq n$. I.e., the condition that all but finitely many U_i be X_i is vacuous, since $\{1,\dots,n\}$ is a finite set.

DEFINITION 2.2.4. Let I be an indexing set, and for each $i\in I$, let X_i be a topological space. The jth *projection map* $\pi_j\colon\prod_{i\in I}X_i\to X_j$ for $j\in I$ is the function defined by $\pi_j((x_i)_{i\in I})=x_j$ for $(x_i)_{i\in I}\in\prod_{i\in I}X_i$.

The following proposition explains the seemingly strange choice of the product topology on X.

PROPOSITION 2.2.5. *Let I be an indexing set, and for each $i \in I$, let X_i be a topological space. The product topology on $X = \prod_{i\in I} X_i$ is the coarsest topology on X such that each projection map $\pi_j \colon X \to X_j$ with $j \in I$ is continuous.*

PROOF. Let $i_1, \dots, i_n \in I$ be elements for some $n \geq 1$, and for $1 \leq k \leq n$, let $U_{i_k} \subset X_{i_k}$ be an open subset. Then

$$\bigcap_{k=1}^{n} \pi_k^{-1}(U_{i_k}) = \prod_{i\in I} V_i$$

with $V_i = X_i$ unless $i \in \{i_1, \dots, i_n\}$ and V_i open in X_i if so. The collection of sets $\pi_j^{-1}(U_j)$ with $j \in I$ and $U_j \subset X_j$ therefore forms a subbase for the product topology on X. In other words, the product topology is the coarsest topology such that these sets are open in X, which is to say such that every π_j is continuous. □

PROPOSITION 2.2.6. *Let X be a topological space, let Y_i be topological spaces for each i in an indexing set I, and endow $Y = \prod_{i\in I} Y_i$ with the product topology. A function $f = (f_i)_{i\in I} \colon X \to Y$ is continuous if and only if each $f_i \colon X \to Y_i$ is continuous.*

PROOF. If f is continuous, then each $f_i = \pi_i \circ f$ is continuous, where $\pi_i \colon Y \to Y_i$ is the projection map in the i-coordinate. Conversely, suppose that each f_i is continuous. For J be a finite subset of I, and for each $j \in J$, let V_j be an open subset of Y_j. Let $V = \prod_{i\in I} V_i$ where we set $V_i = Y_i$ for $i \in I - J$. Then

$$f^{-1}(V) = \bigcap_{i\in I} f_i^{-1}(V_i) = \bigcap_{j\in J} f_j^{-1}(V_j),$$

which is open as a finite intersection of open sets. Therefore f is continuous. □

PROPOSITION 2.2.7. *Let I be an indexing set, and let X_i be a Hausdorff topological space for each $i \in I$. Then $X = \prod_{i\in I} X_i$ is Hausdorff in the product topology.*

PROOF. Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be distinct elements of X, and let $j \in I$ be such that $x_j \neq y_j$. Let U_j and V_j be disjoint open neighborhoods of x_j and y_j. Then $U = \pi_j^{-1}(U_j)$ and $V = \pi_j^{-1}(V_j)$ are open in X in the product topology, and $U \cap V = \varnothing$. □

REMARK 2.2.8. From our definition of the product topology on a direct product $X = \prod_{i\in I} X_i$ of topological spaces, it is not hard to see that if J is any finite subset of I and U is any open subset of $\prod_{j\in J} X_j$ in the product topology, then $W = U \times \prod_{i\in I-J} X_i$ is open in X. However, not every open subset need have this form.

For instance, consider $\mathbb{R}^I = \prod_{i\in I} \mathbb{R}$ for any infinite set I. Consider its open subsets

$$V_j = \{(x_i)_{i\in I} \mid x_j \in (0,1)\}$$

for $j \in I$. Then the union $V = \bigcup_{j\in I} V_j$ is open in X but is not a product of the stated form, since it consists of all tuples with at least one coordinate in $(0,1)$. To see this, consider an open set $W = U \times \mathbb{R}^{I-J}$ of the above form, where U is an open set of $\mathbb{R}^J$ (in the coordinates corresponding to J, and $\mathbb{R}^{I-J}$ is the product of $\mathbb{R}$ in all coordinates in $I-J$). If $W \subseteq V$, then since no condition is imposed on the ith coordinates of elements of W for $i \notin J$, we must have $W \subseteq \bigcup_{j\in J} V_j$. Take any $i \in I-J$, and note that the element of V which is 0 in all coordinates but the ith and $\frac{1}{2}$ in the ith coordinate is not in W. Thus $W \neq V$.

REMARK 2.2.9. Just to speak of an element of an arbitrary product $A = \prod_{i\in I} A_i$ for an uncountable set I and nonempty sets A_i, we run into the issue of needing to make uncountably many choices. If we cannot, we may not be able to say that that A contains a single element! That one can in fact do this is equivalent to an axiom of set theory, called the axiom of choice. It states that, given a collection of disjoint sets $\{A_i \mid i \in I\}$ for some indexing set I, there exists a function $f\colon I \to \coprod_{i\in I} A_i$ to the disjoint union of the sets A_i such that $f(i) \in A_i$ for each $i \in I$. In other words, we can pick one element from each set A_i.

There is a more simply described topology on a product that agrees with the product topology for finite products but is rather finer for infinite products. It might constitute a first guess at a natural topology on the product.

DEFINITION 2.2.10. Let X_i be a topological space for each i in an indexing set I, and set $X = \prod_{i\in I} X_i$. The *box topology* on X is the topology generated by the base consisting of products $\prod_{i\in I} U_i$ of open sets $U_i \subseteq X_i$ for each $i \in I$.

REMARK 2.2.11. Suppose that X is an infinite product of spaces that do not have the trivial topology. Then the box topology on X is strictly finer than the product topology.

PROPOSITION 2.2.12. *Let I be an indexing set. Let X_i be a topological space and let A_i be a subset of X_i for each $i \in I$. The closure of $A = \prod_{i\in I} A_i$ in the box or product topology on $\prod_{i\in I} X_i$ is $\prod_{i\in I} \overline{A_i}$.*

PROOF. If $a = (a_i)_{i\in I} \in \prod_{i\in I} X_i$ and $U = \prod_{i\in I} U_i$ is an open neighborhood of a in the product (i.e., all but finitely many $U_i = X_i$) or box topologies, then $U \cap A = \prod_{i\in I}(U_i \cap A_i)$ is nonempty if and only if each $U_i \cap A_i$ is nonempty. So, if $a \in \overline{A_i}$ for all $i \in I$, then a lies in the closure of $\prod_{i\in I} A_i$. Conversely, if a lies in the latter closure for all possible choices of U, then $a \in \overline{A_i}$ for each $i \in I$. □

The following tells us when a product of maps is continuous.

PROPOSITION 2.2.13. *For each i in an indexing set I, let $f_i\colon X_i \to Y_i$ be a function between topological spaces. Set $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$ and $f = (f_i)_{i\in I}\colon X \to Y$. Then f is continuous with respect to the product (resp., box) topology on X and Y if and only if each f_i is continuous.*

PROOF. Let J be a finite subset of I, and let V_j be open in Y_j for each $j \in J$. Let $V = \prod_{i\in I} V_i$ where $V_i = X_i$ for $i \in I-J$. Then $f^{-1}(V) = \prod_{i\in I} f_i^{-1}(V_i)$ is open if and only if $f_j^{-1}(V_j)$ is open for every $j \in J$. Thus in the product topology, f is continuous if and only if each f_j is continuous. In the box topology, we may simply replace J by I in the above argument. □

EXAMPLE 2.2.14. Consider $\mathbb{R}^I = \prod_{i\in I}\mathbb{R}$. The metric topology for the uniform metric given by

$$d(x,y) = \sup\{\min\{|x_i - y_i|, 1\} \mid i \in I\},$$

is called the uniform topology. The reason for the name is as follows: the space of functions $f\colon I \to \mathbb{R}$ from a set I to $\mathbb{R}$ is in bijection with $\mathbb{R}^I$ via the map which takes a function f to $(f(i))_{i\in I}$. A sequence of functions $(f_n)_{n\geq 1}$ with $f_n\colon I \to \mathbb{R}$ converges uniformly to some $f\colon I \to \mathbb{R}$ if and only if the corresponding sequence in $\mathbb{R}^I$ converges to $(f(i))_{i\in I}$ in the uniform topology.

EXAMPLE 2.2.15. If I is finite, the product, uniform, and box topologies on $\mathbb{R}^I$ are all simply the Euclidean topology. If I is infinite, then the box topology is strictly finer than the uniform topology, which is strictly finer than the product topology, as we next explain.

The product topology on $\mathbb{R}^I$ has a basis of open neighborhoods of 0 consisting of a product P_ε of open intervals $(-\varepsilon,\varepsilon)$ in finitely many coordinates and $\mathbb{R}$ in the others. The uniform topology has a basis of open balls $B_\varepsilon = B(\varepsilon, 0)$ of radius $\varepsilon < 1$. We have $B_\varepsilon \subset P_\varepsilon$, but B_ε contains no P_δ. The box topology has a basis of open neighborhoods consisting of products of open intervals centered at 0 of lengths depending on the coordinates. Inside B_ε, we have the product of open intervals $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ in every coordinate, which is open in the box topology. On the other hand, a product of open intervals centered at 0 the infimum of the lengths of which is 0 contains no B_ε.

PROPOSITION 2.2.16. *Let J be an indexing set. If J is infinite, then $\mathbb{R}^J$ with the box topology is not metrizable. The space $\mathbb{R}^J$ with the product topology is metrizable if and only if J is countable.*

PROOF. For the first statement, it suffices to consider the case that J is countable, which we can then take to be the set of positive integers. Consider the subset $U = \mathbb{R}^J_{>0}$ of $\mathbb{R}^J$ in the box topology. The $0 \in \overline{U}$, as the reader can check. However, if $(a_n)_{n\geq 1}$ is a sequence in U and $a_{n,m}$ is its mth coordinate, then the product $\prod_{n=1}^\infty (-a_{n,n}, a_{n,n})$ is an open neighborhood of 0 that does not contain any a_n. By Proposition 1.6.21, the space $\mathbb{R}^J$ with the box topology is not metrizable.

Now, take the product topology on $\mathbb{R}^J$ with J uncountable, and consider the set

$$A = \{(x_j)_{j\in J} \mid x_j = 1 \text{ for all but finitely many } j \in J\}.$$

Then $0 \in \overline{A}$, since any product of open intervals $(-\varepsilon,\varepsilon)$ in finitely many coordinates and $\mathbb{R}$ in the others clearly intersects A. On other other hand, if $(a_n)_{n\geq 1}$ is a sequence in A, let I be the subset of J consisting of those $j \in J$ such that the j-coordinate of some a_n is not 1. This is a countable set, so $J \neq I$. But then there exists $j \in J$ such that the j-coordinate of a_n is 1 for all $n \geq 1$, which means in particular the neighborhood of 0 that is $\mathbb{R}$ in all coordinates but the j-coordinate and $(-\frac{1}{2}, \frac{1}{2})$ in the j-coordinate does not contain any a_n.

For the product topology on $\mathbb{R}^J$ with $J = \mathbb{Z}_{\geq 1}$, the reader may check that we have a metric d defined by

$$d(x,y) = \sup\{\tfrac{1}{n}\min\{|x_n - y_n|, 1\} \mid n \geq 1\}.$$

$\square$

2.3. Quotient spaces

Recall that we say that a function $f\colon X \to Y$ is surjective (or a surjection) if $f(X) = Y$.

DEFINITION 2.3.1. A *quotient map* $\pi\colon X \to Y$ of topological spaces X and Y is a surjection such that a subset V of Y is open if and only if $\pi^{-1}(V)$ is open in X.

The following is easily verified.

LEMMA 2.3.2. *A surjective map $\pi\colon X \to Y$ is a quotient map if and only if a subset B of Y is closed if and only if $\pi^{-1}(B)$ is closed in X.*

EXAMPLES 2.3.3. Let X be a topological space.

a. The identity map $\mathrm{id}_X : X \to X$ is a quotient map.

b. Any constant map from a nonempty X is a quotient map onto its singleton image.

c. Let $X = \prod_{i \in I} X_i$ be a product of topological spaces. Then the projection maps $\pi_i\colon X \to X_i$ are quotient maps, as the reader can check.

EXAMPLE 2.3.4. Let S^1 denote the unit open circle in $\mathbb{C}$. Consider the map $f\colon \mathbb{R} \to S^1$ given by $f(x) = e^{2\pi i x}$. Then f is a quotient map.

The product map $F\colon \mathbb{R}^2 \to S^1 \times S^1$ given by $F(x,y) = (f(x), f(y))$ is a quotient map as well, realizing a torus (which has the shape of the surface of a donut) as a quotient of the plane. In fact, if we restrict this function to the square $X = [0,1]^2$, the resulting map $F\colon X \to S^1 \times S^1$ is also a quotient map. It satisfies $F(0,y) = F(1,y)$ and $F(x,0) = F(x,1)$ for all $x,y \in [0,1]$. In particular,

$$F(0,0) = F(1,0) = F(0,1) = F(1,1) = (1,1).$$

One may think of the quotient map as identifying the left and right sides of the square with each other and the top and bottom sides of the square with each other, which in the process identifies the four corners with each other.

The following example illustrates that quotient maps need not be open maps.

EXAMPLE 2.3.5. Let X be the subspace of $\mathbb{R}^2$ consisting of points with y-coordinate 0 or 1. A subset of X is open if and only if its intersection with each of the two lines in X is open in the Euclidean topology.

Define $f\colon X \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} x & \text{if } y = 0 \\ |x| & \text{if } y = 1. \end{cases}$$

Then f is a quotient map. Let U be the open subset of X consisting of those (x,y) with $y = 1$, or with $y = 0$ and $x = 0$. Then $f(U) = [0,\infty)$, so f is not open.

Now consider the restriction $f|_U : U \to \mathbb{R}$. It is a continuous surjection. However, $f|_U^{-1}([0,\infty)) = \{(x,1) \mid x \in \mathbb{R}\}$, but $[0,\infty)$ is not open, so $f|_U$ is not a quotient map.

LEMMA 2.3.6. *Let X be a topological space, and let $\pi\colon X \to Y$ be a surjective function onto a set Y. There exists a unique topology on Y such that π is a quotient map.*

PROOF. The set of subsets of Y such that $\pi^{-1}(V)$ is open is easily seen to be a topology on Y, since π^{-1} commutes with the taking of intersections and unions. Moreover, this is the only topology such that π is a quotient map, since π being a quotient map means exactly that the $V \subseteq Y$ such that $\pi^{-1}(V)$ is open are the open sets. □

Given Lemma 2.3.6, we may make the following definition.

DEFINITION 2.3.7. Let X be a topological space and $\pi\colon X \to Y$ a surjective function to a set Y. The *quotient topology* on Y is the unique topology on Y such that π is a quotient map.

DEFINITION 2.3.8. If $\pi\colon X \to Y$ is a quotient map between topological spaces, then Y is said to be a *quotient space* of X.

EXAMPLE 2.3.9. Let $Y = \{a,b\}$ be the two-point set. Consider the function $f\colon \mathbb{R} \to Y$ given by $f(x) = a$ if $x < 0$ and $f(x) = b$ if $x \geq 0$. The quotient topology on Y is contains the sets $\varnothing$, $\{a\}$ and $\{a,b\}$ but not $\{b\}$, as $f^{-1}(b)$ is not open. Then Y is not Hausdorff, though $\mathbb{R}$ is.

PROPOSITION 2.3.10. *Let $\pi\colon X \to Y$ be a quotient map. Let B be a subset of Y and $A = \pi^{-1}(B)$. Consider the map $p\colon A \to \pi(A)$ given by $p(a) = \pi(a)$ for $a \in A$.*

a. If A is open or closed in X, then p is a quotient map.

b. If π is open or closed, then p is a quotient map.

PROOF. By definition, p is continuous and surjective. If V is a subset of $\pi(A)$, then $p^{-1}(V) = \pi^{-1}(V)$ is open (resp., closed) in X if and only if V is open (resp., closed) in X. If A is open (resp., closed) in X, then $p^{-1}(V)$ is open (resp., closed) in X if and only if $p^{-1}(V)$ is open (resp., closed) in A, and therefore, p is a quotient map. Thus, we have part a.

As for part b, suppose that π is open. If $V \subseteq B$ is such that $p^{-1}(V)$ is open in A, then $\pi^{-1}(V) = p^{-1}(V) = U \cap A$ for some open U in X, and

$$V = \pi(\pi^{-1}(V)) = \pi(U \cap A) = \pi(U) \cap B,$$

the last step as $\pi^{-1}(A) = B$. Since π is open, $\pi(U)$ is open in Y, and therefore $\pi(U) \cap B$ is open in B under the subspace topology. That is, V is open in B, so p is a quotient map. The argument in the case that π is closed proceeds in the same manner, replacing “open” by “closed” everywhere. □

The quotient map has a certain universality property, as expressed in the following theorem.

THEOREM 2.3.11. *Let $\pi\colon X \to Y$ be a quotient map, and let $f\colon X \to Z$ be any continuous map such that f is constant on $\pi^{-1}(\{y\})$ for all $y \in Y$. Then there exists a unique function $g\colon Y \to Z$ such that $f = g \circ \pi$, and g is continuous.*

PROOF. The function g is determined by $g(y) = f(x)$ where $\pi(x) = y$, which is well-defined by the constancy of f on the nonempty set $\pi^{-1}(\{y\})$. Let W be open in Z, and note that $\pi^{-1}(g^{-1}(W)) = f^{-1}(W)$. The set $f^{-1}(W)$ is open in X, so as π is a quotient map, $g^{-1}(W)$ is open in Y. Thus, g is continuous. $\square$

EXAMPLE 2.3.12. Let X be a topological space, and let A be a subset. Let $X/A = (X-A) \amalg \{*\}$ as sets, and define $\pi\colon X \to X/A$ by

$$\pi(x) = \begin{cases} x & \text{if } x \in X - A \\ * & \text{if } a \in A. \end{cases}$$

We give Y the quotient topology. This is the space given by collapsing A to a point. It may not be Hausdorff even if X is: for instance, if we take $X = \mathbb{R}$ and $A = (0,\infty)$, then 0 is in every open neighborhood of $*$ in Y. However, Y will be Hausdorff if for every $x \notin A$, there exist disjoint open sets U containing x and V containing A.

EXAMPLE 2.3.13. Let X be a topological space. The cone $C(X)$ on X is the quotient space of (the cylinder) $X \times [0,1]$ in which we collapse $X \times \{0\}$ to a point. If $X = S^1$, this is homeomorphic to a cone in $\mathbb{R}^3$, e.g.,

$$D = \{(rx, ry, r) \mid r \in [0,1], (x,y) \in S^1\},$$

where S^1 is the unit circle in $\mathbb{R}^2$. Here, the homeomorphism $D \to C(S^1)$ takes (rx, ry, r) to the image of $((x,y),r) \in X \times [0,1]$ in $C(S^1)$.

2.4. Disjoint unions

To give maps to a direct product of sets is to give a collection of maps to the individual sets, and if these sets are topological spaces, then with the product topology, we have seen in Proposition 2.2.13 that to give a continuous map to the product of topological spaces is to give a collection of continuous maps to the individual spaces. However, to give a function from a product of sets to a set is not the same as given a collection of maps from the individual set to the set. One might ask if there's another set that does this, and the answer is yes, the disjoint union. In fact, when the individual sets are topological spaces, we can put a topology on the disjoint union so that we can make the same connection with continuous maps.

DEFINITION 2.4.1. The *disjoint union* of a collection $\{A_i \mid i \in I\}$ of sets is the set $\coprod_{i\in I} A_i$ that contains the A_i as mutually disjoint subsets and is equal to the union $\bigcup_{i\in I} A_i$.

REMARK 2.4.2. To give a function $f\colon \coprod_{i\in I} A_i \to B$ from a disjoint union of sets A_i to a set B is exactly to give a collection of maps $f_i\colon A_i \to B$. These satisfy $f_i(a_i) = f(a_i)$ for all $i \in I$, so f determines the f_i and conversely.

DEFINITION 2.4.3. Let $\{X_i \mid i \in I\}$ be a collection of topological spaces. The *disjoint union* $\coprod_{i\in I} X_i$ of the spaces X_i is the topological space with underlying set indicated disjoint union and with the topology under which a subset U is open if and only if $U \cap X_i$ is open for all $i \in I$.

REMARK 2.4.4. Under this definition, we have continuous inclusion maps $\iota_i \colon X_i \hookrightarrow \coprod_{i \in I} X_i$ given by $\iota_i(a_i) = a_i$ for $a_i \in X_i$.

PROPOSITION 2.4.5. *Let $f_i \colon X_i \to Y$ be a collection of functions from topological spaces X_i to a topological space Y. The map $f \colon \coprod_{i \in I} X_i \to Y$ that restricts to f_i on X_i for all i is continuous if and only if every f_i is continuous.*

PROOF. Let V be an open subset of Y. Then $f^{-1}(V) \cap X_i = f_i^{-1}(V)$, so $f^{-1}(V)$ is open if and only if $f_i^{-1}(V)$ is open for all i. □

LEMMA 2.4.6. *Let $X = \coprod_{i \in I} X_i$ be a disjoint union of topological spaces X_i. Then each X_i is open and closed in X.*

PROOF. We have $X_i \cap X_j = \varnothing$ if $i \neq j$ and $X_i \cap X_i = X_i$, so X_i is open in X. As for its complement, we have

$$X - X_i = \bigcup_{j \in I - \{j\}} X_j,$$

which is open as a union of open sets. □

EXAMPLE 2.4.7. Any union A of the two parallel lines in the plane $\mathbb{R}^2$ is homeomorphic to $\mathbb{R} \amalg \mathbb{R}$.

This example generalizes considerably.

LEMMA 2.4.8. *Let $\{A_i \mid i \in I\}$ be a collection of subspaces of a space X. Then the continuous map $\coprod_{i \in I} A_i \to X$ with restriction to A_i the embedding of A_i as a subspace of X is a homeomorphism if the A_i are mutually disjoint with union X and every A_i is open (and therefore closed) in X.*

PROOF. The continuous map f in the statement is onto if and only if $\bigcup_{i \in I} A_i = X$ and is one-to-one if and only if $A_i \cap A_j = \varnothing$ for all $i \neq j$. If f is open, then every X_i is open in X (and then closed as well, since A_i is the complement of the union of the A_j for $j \neq i$). Conversely, if every A_i is both open and closed in X and U is open in $\coprod_{i \in I} A_i$, then $f(U) = \bigcup_{i \in I}(U \cap A_i)$ is open as a union of open sets. □

Lemma 2.4.8 justifies the following terminology.

TERMINOLOGY 2.4.9. If X is a topological space and $\{A_i \mid i \in I\}$ is a collection of disjoint open subsets with union X, then we say that X is the disjoint union of the subspaces A_i and write $X = \amalg_{i \in I} A_i$.

EXAMPLES 2.4.10.

a. The subspace $X = \{(x,y) \in \mathbb{R}^2 \mid y \in \{0,1\}\}$ of $\mathbb{R}^2$ is the disjoint union of the lines $y = 0$ and $y = 1$ in the plane.

b. Any discrete spaces is the disjoint union of its singleton subsets.

EXAMPLE 2.4.11. Let $\{X_i \mid i \in I\}$ be a collection of topological spaces, and choose $x_i \in X_i$ for each $i \in I$. We have a quotient of $X = \coprod_{i \in I} X_i$ given by collapsing the subset $A = \{x_i \mid i \in I\}$ to a point. This will be a Hausdorff space if X is. This space is called a one-point union of the spaces X_i. In the case, for instance, that $X = S^1 \amalg S^1 \amalg \cdots \amalg S^1$, we get a finite collection of circles joined at a point.

CHAPTER 3

Connected and compact spaces

3.1. Connectedness and path connectedness

DEFINITION 3.1.1. A topological space X is *connected* if it is not a disjoint union of any two nonempty open subspaces. Otherwise, X is said to be *disconnected*.

LEMMA 3.1.2. *A topological space X is connected if its only subsets that are both open and closed are $\varnothing$ and X.*

PROOF. If A is a subset of X that is both open and closed, then so is A^c, and then $X = A \amalg A^c$. Conversely, if $X = U \amalg V$ with U and V nonempty and open in X, then U and V are also closed. $\square$

EXAMPLE 3.1.3. Topological spaces with one element are always connected, but discrete topological spaces with more than one element are disconnected.

EXAMPLE 3.1.4. The union $(-\infty,0) \cap (0,\infty)$ of intervals in $\mathbb{R}$ is disconnected as a subspace of $\mathbb{R}$, as both $(-\infty,0)$ and $(0,\infty)$ are open and closed in the subspace topology. However, $\mathbb{R}$ itself is connected.

REMARK 3.1.5. If X is the disjoint union of subspaces A and B, then A and B are their own closures, so A and B contain no limit points of each other. In fact, if A and B are any two disjoint subsets of X with union X such that A contains no limit points of B and vice-versa, then A and B are open and closed in X.

LEMMA 3.1.6. *If $X = U \amalg V$ for subspaces U and V, and A is a connected subset of X, then either $A \subseteq U$ or $A \subseteq V$.*

PROOF. We have that $A \cap U$ and $A \cap V$ are open and closed in A, so A is the disjoint union of these intersections. If A is connected, this forces one of the $A \cap U$ and $A \cap V$ to be empty, and then the other is A. $\square$

PROPOSITION 3.1.7. *The closure of a connected subset of a topological space is connected.*

PROOF. Let X be a topological space and A a connected subset. Suppose that $\overline{A} = U \amalg V$ for disjoint subspaces U and V in X. By Lemma 3.1.6, we have that A is contained in either U or V: without loss of generality, we suppose $A \subseteq U$. As U is closed, we have that $\overline{A} \subseteq U$ as well, and therefore $V = \varnothing$. $\square$

We also have the following statements.

PROPOSITION 3.1.8. *The image of a connected space under a continuous map is connected.*

PROOF. Suppose X is connected and $f\colon X \to Y$ is continuous. If Y is the disjoint union of (open) subspaces U and V, then X is the union of the disjoint open subsets $f^{-1}(U)$ and $f^{-1}(V)$, hence equal to their disjoint union as topological spaces. $\square$

PROPOSITION 3.1.9. *Let $X = \bigcup_{i\in I} A_i$ for a collection of connected subsets A_i of X for $i \in I$, and suppose that $\bigcap_{i\in I} A_i$ is nonempty. Then X is connected.*

PROOF. Let $a \in \bigcap_{i\in I} A_i$. If $X = U \amalg V$ for subspaces U and V, then without loss of generality we may suppose $a \in U$. Since A_i is connected and contains a, we must then have $A_i \subseteq U$ for all $i \in I$, forcing $V = \varnothing$. $\square$

The following is an immediate corollary of Proposition 3.1.9.

COROLLARY 3.1.10. *The union of all connected subsets of a topological space X that contain a given point $x \in X$ is connected.*

PROPOSITION 3.1.11. *The relation on a topological space X given by $x \sim y$ for $x, y \in X$ if y lies in the connected component of x is an equivalence relation on X.*

PROOF. Clearly $\sim$ is reflexive. For symmetry, it's enough show that for $x, y \in X$, the connected components A of x and B of y are either equal or disjoint. Suppose that $z \in A \cap B$. Then $A \cup B$ is connected by Proposition 3.1.9, but in that A (resp., B) is the largest connected subset of X containing x (resp., y), we must have $A = A \cap B = B$. Finally, if $x \sim y$ and $y \sim z$ for $x, y, z \in X$, then we've just seen that x and y have the same connected component, as do y and z, so x and z do as well. $\square$

By Corollary 3.1.10, we can always find a largest connected subset containing a given point.

DEFINITION 3.1.12. A *connected component* of a topological space X is a connected subset of X that is not properly contained in any larger connected subset of X. The *connected component of a point* $x \in X$ is the unique connected component of X containing x.

LEMMA 3.1.13. *A space X is the union of its distinct connected components, which are closed and disjoint. If every connected component of X is open, then X is the disjoint union of them. In particular, if X has only finitely many connected components, then it X is the disjoint union of its connected components.*

PROOF. We know that X is the union of its connected components, which are disjoint by Proposition 3.1.11. It follows from Proposition 3.1.7 that connected components are closed, being the largest connected subsets containing a given point. The second statement follows from the definition of a disjoint union of topological spaces. If X has finitely many connected components, then any union of all but one of them is closed, and then they are all open as complements of these unions. $\square$

EXAMPLE 3.1.14. Consider the subspace $A = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ of $\mathbb{R}$. Every $\{\frac{1}{n}\}$ is both open and closed in A, so is a connected component (in that it is connected). The set $\{0\}$ is also then a connected component, being that it is not contained in any larger connected subset. However, it is closed but not open, so A is not the disjoint union of its connected components.

EXAMPLE 3.1.15. Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. It is disconnected as $\mathbb{Q}$ is the union of its intersections with the intervals $(-\infty, \pi)$ and (π, ∞), for instance (as π is irrational). In fact, since there exists an irrational number between any two distinct rational numbers, the connected components of $\mathbb{Q}$ are just its singleton subsets.

REMARK 3.1.16. The property of being in the same connected component gives an equivalence relation on the points of a topological space, and the connected components are the equivalence classes.

DEFINITION 3.1.17. A topological space is said to be *totally disconnected* if its connected components are its singleton subsets.

EXAMPLE 3.1.18. Let $A = \{0,1\}$ with the discrete topology, and consider $X = \prod_{n=1}^{\infty} A$ with the product topology. Given any two distinct points $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ in X, there exists $n \geq 1$ such that $x_n \neq y_n$. Letting π_n denote the nth projection map, we have that $U = \pi_n^{-1}(x_n)$ and $V = \pi_n^{-1}(y_n)$ are disjoint basic open sets in X with union X, so $X = U \amalg V$, and x and y lie in distinct connected components. Thus, X is totally disconnected.

DEFINITION 3.1.19. A *path* γ from a point x to a point y in a topological space X is a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The points x and y are called the *endpoints* of γ: in particular, x is its *initial endpoint* and y is its *final endpoint*.

DEFINITION 3.1.20. Let X be a topological space.

a. We say that two points x and y in a topological space X can be *connected by a path* if there exists a path γ from x to y.

b. We say that a topological space X is *path connected* if every two points x and y can be connected by a path.

c. The *path component* of a point $x \in X$ is the set of all points $y \in X$ such that x and y are connected by a path.

REMARK 3.1.21. The relation of there exists a path from a point x to a point y on a topological space X is an equivalence relation on a topological space X. Thus, X is a disjoint union of its path components.

PROPOSITION 3.1.22. *Every path connected space is connected.*

PROOF. Let X be a path connected space. Suppose that X is a disjoint union of nonempty open subspaces U and V, and let $x \in X$ and $y \in Y$. Let γ be a path from x to y, and let $A = \gamma([0,1])$.

Then A is the disjoint union of the nonempty sets $A \cap U$ and $A \cap V$, so A is disconnected, but it is the image of a connected space under a continuous function. $\square$

EXAMPLE 3.1.23. Consider the subset $A = \{(x, \sin(\frac{1}{x})) \mid 0 < x \leq 1\}$ of $\mathbb{R}^2$. We have $\overline{A} = A \cup \{(0,y) \mid -1 \leq y \leq 1\}$. Note that $\overline{A}$ is connected as a subspace of $\mathbb{R}^2$ since A is. On the other hand, $\overline{A}$ is not path connected. Actually, this is easily reduced to proving that the topologist's sine curve $B = A \cup \{(0,0)\} \subset \overline{A}$ is connected, but not path connected. Let us explain this.

Suppose that γ is a path from $(0,0)$ to $(1, \sin(1))$ on B. We may assume without loss of generality that $\gamma(t) \neq (0,0)$ for $t > 0$. By definition, $\lim_{t \to 0^+} \gamma(t) = (0,0)$. On the other hand, for every $\varepsilon > 0$, there exists an $x < \varepsilon$ such that $\sin(\frac{1}{x}) = 1$. By the continuity of γ, for every $\delta > 0$, there then exists $t < \delta$ such that the second coordinate of $\gamma(t)$ equals 1. But this contradicts that $\lim_{t \to 0^+} \gamma(t) = (0,0)$.

3.2. Compactness

DEFINITION 3.2.1. Let $\mathscr{U}$ be a cover of a subset A of a topological space X. A *subcover* of $\mathscr{U}$ is a a subset of $\mathscr{U}$ that covers A.

DEFINITION 3.2.2. A topological space X is *compact* if every open cover of X has a finite subcover.

EXAMPLES 3.2.3.

a. Any finite topological space is compact.

b. Any topological space with the trivial topology is compact.

c. The real line $\mathbb{R}$ is not compact, since the collection of open intervals of length 1 is an open cover with no finite subcover.

d. The interval $(0,1]$ is not compact, since the connection of intervals $(\varepsilon, 1]$ with $\varepsilon > 0$ has no finite subcover.

PROPOSITION 3.2.4. *Any closed interval in $\mathbb{R}$ is compact.*

PROOF. As all closed intervals of finite length are homeomorphic, we can and will consider the interval $[0,1]$. Let $\mathscr{U}$ be an open cover of $[0,1]$, and let A be the subset of $[0,1]$ consisting of those x such that $[0,x]$ has a finite subcover by elements in $\mathscr{U}$. Let b be the supremum of of the elements of A. If $b < 1$, then let U be an element $\mathscr{U}$ containing b. Since U contains an interval $(b - \varepsilon, b + \varepsilon)$ for some $\varepsilon > 0$ with $b + \varepsilon \leq 1$, we can find a finite subcover $\mathscr{V}$ of $[0, b - \frac{\varepsilon}{2}]$ inside $\mathscr{U}$. Ten $\mathscr{V} \cup \{U\}$ is a finite subcover of $[0, b + \frac{\varepsilon}{2}]$ in $\mathscr{U}$. This means that $b + \frac{\varepsilon}{2} \in A$, contradicting the fact that b is the supremum of all elements of A. Thus $b = 1$, and therefore $\mathscr{U}$ has a finite subcover. $\square$

Let's establish a few basic statements regarding compact spaces.

LEMMA 3.2.5. *A subspace A of a topological space X is compact if and only if every open cover of A in X has a finite subcover.*

PROOF. Given an open cover $\mathscr{V}$ of A by open sets in A, we can find a set $\mathscr{U}$ of open sets in X covering A such that $\mathscr{V} = \{A \cap U \mid U \in \mathscr{U}\}$. Conversely, given a collection $\mathscr{U}$ of open sets in X covering A, we may define a cover $\mathscr{V}$ of A by open sets in A by taking intersections with A as in the latter formula. Any (finite) subset $\mathscr{C}$ of $\mathscr{U}$ covers A if and only if the (finite) subset $\{A \cap U \mid A \in \mathscr{C}\}$ of $\mathscr{V}$ covers A. □

PROPOSITION 3.2.6. *Every closed subset of a compact space is compact.*

PROOF. Let X be compact, and let $A \subseteq X$ be closed. Let $\mathscr{U}$ be an open cover of A in X. Then $\mathscr{U} \cup \{A^c\}$ is an open cover of X, so it has a finite subcover $\mathscr{V}$. If $A^c \in \mathscr{V}$, then $\mathscr{V} - \{A^c\}$ is an open cover of A in X, and otherwise, $\mathscr{V}$ is an open cover of A in X. □

LEMMA 3.2.7. *Let A be a compact subset of a Hausdorff space, and let $x \in A^c$. Then there exist disjoint open sets U and V with $A \subseteq U$ and $x \in V$.*

PROOF. For each $a \in A$, choose open disjoint neighborhoods U_a of a and V_a of x in X. The collection $\{U_a \mid a \in A\}$ is an open cover of A, and it has a finite subcover, say by $U_{a_1}, \dots, U_{a_n}$. Then $U = \bigcup_{i=1}^n U_{a_i}$ and $V = \bigcap_{i=1}^n V_{a_i}$ are the desired open subsets of X. □

PROPOSITION 3.2.8. *Every compact subset of a Hausdorff space is closed.*

PROOF. Let X be Hausdorff, and let $A \subseteq X$ be compact. By Lemma 3.2.7, for each $x \in A^c$, there exists an open neighborhood V_x of x in A^c. The union of the V_x is A^c, so A^c is open, and thus A is closed. □

PROPOSITION 3.2.9. *Let $f \colon X \to Y$ be a continuous map. If X is compact, then so is the image of f.*

PROOF. Let $\mathscr{V}$ be an open cover of $f(X)$ in Y. Then $\mathscr{U} = \{f^{-1}(V) \mid V \in \mathscr{V}\}$ is an open cover of X. Since X is compact, it has a finite subcover $\{f^{-1}(V_1), \dots, f^{-1}(V_n)\}$ with each $V_i \in \mathscr{V}$, and then $\{V_1, \dots, V_n\}$ is an open cover of $f(X)$. □

We have seen that continuous bijections need not be homeomorphisms. However, continuous bijections from compact spaces are.

THEOREM 3.2.10. *Let $f \colon X \to Y$ be a continuous surjection. If X is compact and Y is Hausdorff, then f is closed and a quotient map.*

PROOF. To see that f is a quotient map, we may show that $B \subseteq Y$ is closed if $f^{-1}(B)$ is closed. Since f is continuous and surjective, we may write $B = f(f^{-1}(B))$, so it suffices to show that f is a closed map.

Let A be a closed set in X, which is necessarily compact. As f is continuous, its image $f(A)$ is compact as well. As Y is Hausdorff, we then have that $f(A)$ is closed. □

COROLLARY 3.2.11. *Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

We next prove that a finite product of compact spaces is compact. First, we require the following lemma.

LEMMA 3.2.12. *Let X and Y be topological spaces, and suppose that X is compact. For any $y \in Y$ and open set W in $X \times Y$ containing $X \times \{y\}$, there exists an open neighborhood V of y in Y such that $X \times V \subseteq W$.*

PROOF. For each $x \in X$, let U_x be an open neighborhood of x in X and V_x be an open neighborhood of y in Y such that $U_x \times V_x \subseteq W$, which exist since $(x,y) \in W$ and W is open. The collection $\{U_x \mid x \in X\}$ covers X so has a finite subcover, say consisting of $U_{x_1}, \ldots, U_{x_n}$. If we set $V = \bigcap_{i=1}^{n} V_{x_i}$, then V is an open neighborhood of y in Y, and since each $U_{x_i} \times V$ is contained in W, so is their union $X \times V$. □

THEOREM 3.2.13. *Let $X_1, \ldots, X_n$ be compact spaces for some $n \geq 1$. Then $\prod_{i=1}^{n} X_i$ is compact.*

PROOF. It suffices by recursion to consider the case $n = 2$, so the product of compact spaces X and Y. Let $\mathscr{W}$ be an open covering of $X \times Y$. For any $y \in Y$, the set $X \times \{y\}$ is compact, being homeomorphic to X via the projection map, and so there exist $Z_1, \ldots, Z_m \in \mathscr{W}$ that together cover $X \times \{y\}$. Set $W_y = \bigcup_{i=1}^{m} Z_i$. We then have by Lemma 3.2.12 that there exists an open neighborhood of V_y of y such that $X \times V_y \subseteq W_y$. The collection $\{V_y \mid y \in Y\}$ then covers Y, and it has a finite subcover as Y is compact. But then $X \times Y$ is covered by the finitely many W_y, and each of these is in turn a union of finitely many sets in $\mathscr{W}$. Thus, $\mathscr{W}$ has a finite subcover. □

PROPOSITION 3.2.14. *A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

PROOF. Fix a subset A of $\mathbb{R}^n$. We may consider the cover of A by all open balls in $\mathbb{R}^n$ of radius 1. If A is unbounded, then it cannot have a finite subcover, since the union of any finite number of balls of radius 1 is bounded. Therefore, only bounded subsets can be compact.

Now suppose that A is bounded. It is then contained in a direct product of closed intervals, which is compact as a finite product of compact sets. If A is also closed, then it is compact by Proposition 3.2.6. On the other hand, if A is not closed, then it cannot be compact by Proposition 3.2.8. □

We give another criterion for a topological space to be compact.

DEFINITION 3.2.15. A collection $\mathscr{A}$ of subsets of X is said to have the *finite intersection property*, or *FIP*, if for every $n \geq 1$ and $A_1, \ldots, A_n \in \mathscr{A}$, we have that $\bigcap_{i=1}^{n} A_i$ is nonempty.

THEOREM 3.2.16. *A topological space X is compact if and only if every collection $\mathscr{A}$ of closed subsets of X having the finite intersection property satisfies $\bigcap_{A \in \mathscr{A}} A$ is nonempty.*

PROOF. Let $\mathscr{A}$ be a collection of closed subsets of X with the finite intersection property. Then $\mathscr{U} = \{A^c \mid A \in \mathscr{A}\}$ is a collection of open subsets of X with the property that no finite subset of $\mathscr{U}$ covers X. If X is compact, this forces $\mathscr{U}$ not to cover X, which is exactly to say that the intersection of elements of $\mathscr{A}$ is nonempty.

Conversely, if X is not compact, there exists an open cover $\mathscr{U}$ of X which has no finite subcover, which is to say that the collection $\mathscr{A} = \{U^c \mid U \in \mathscr{U}\}$ of closed sets has the finite intersection property but has empty intersection. □

Let us give an example of the use of this theorem.

THEOREM 3.2.17. *Let X be a nonempty compact Hausdorff space with no singleton open sets. Then X is uncountable.*

PROOF. If X is finite Hausdorff, then it is discrete, so has singleton open sets. So, suppose by way of contradiction that X is countably infinite. Label its points x_i for $i \geq 1$. Let $X = U_0$, which is an open neighborhood of x_1. By Lemma 1.5.13, we may recursively let U_i be an open neighborhood of x_{i+1} contained in U_{i-1} with $x_i \notin \overline{U_i}$ for each $i \geq 1$.

The intersection of any finite subcollection of $\{U_i \mid i \geq 1\}$ is some U_n for $n \geq 1$ since these U_i are nested. Since $x_{n+1} \in U_n$, this collection satisfies the FIP. In particular, the collection $\{\overline{U_i} \mid i \geq 1\}$ of closed sets does as well. Now, X being compact, we then have that $\bigcap_{i=1}^{\infty} \overline{U_i}$ is nonempty. At the same time, since $x_i \notin \overline{U_i}$, this intersection does not contain any of the points of $X = \{x_i \mid i \geq 1\}$, providing the desired contradiction. □

We can use this to give a proof of the uncountability of the real numbers.

COROLLARY 3.2.18. *The set of real numbers is uncountable.*

PROOF. By Theorem 3.2.17, the interval $[0,1]$ is uncountable, and therefore so is $\mathbb{R}$. □

3.3. Sequential and limit point compactness

We next introduce related notions to compactness.

DEFINITION 3.3.1. A space X is said to be *limit point compact* if every infinite subset of X has a limit point.

PROPOSITION 3.3.2. *Compact spaces are limit point compact.*

PROOF. Let X be a compact space, and let A be a subset without a limit point. Then A is necessarily closed. Moreover, for each $a \in A$, there exists an open set U_a containing a and no other point of A. Then X has an open cover by $\{U_a \mid a \in A\} \cup \{A^c\}$. Since X has a finite subcover, A is contained in a finite union of sets $U_{a_1}, \ldots, U_{a_n}$, which implies that $A = \{a_1, \ldots, a_n\}$, so A is finite. □

DEFINITION 3.3.3. A space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

The following is fairly immediate from the definitions.

PROPOSITION 3.3.4. *Sequentially compact spaces are limit point compact.*

PROOF. Let A be an infinite set in X, and let $(a_n)_{n\geq 1}$ be a sequence of distinct elements in A. It has a convergent subsequence, say with limit a. Then every neighborhood of a contains some point of the subsequence not equal to a. Thus, a is a limit point of A. $\square$

To be limit point compact is a weaker notion than being either compact or sequentially compact.

EXAMPLE 3.3.5. Let $Y = \{a,b\}$ the the two-point space with the trivial topology, and consider $X = \mathbb{Z} \times Y$, where $\mathbb{Z}$ has the discrete topology. Then any open neighborhood of a point (n,a) in X contains (n,b) and conversely, so every nonempty set in X has a limit point. In particular, X is limit point compact. However, X is not compact, and it is covered by the disjoint open sets $\{n\} \times Y$ with $n \in \mathbb{Z}$. It is also not sequentially compact, as the sequence of points (n,a) has no convergent subsequence.

In general, neither compactness nor sequential compactness implies the other. However, for metric spaces, all three of these notions of compactness are equivalent, as we shall show.

DEFINITION 3.3.6. Let A be a bounded subset of a metric space X. The *diameter* of A is the supremum of the distances between points in A.

DEFINITION 3.3.7. Let $\mathscr{U}$ be an open cover of a space X. Its *Lebesgue number* is the supremum of all $\varepsilon > 0$ such that every subset A of X of diameter less than ε is contained in an element of $\mathscr{U}$, if such an ε exists, and otherwise, we say $\mathscr{U}$ has *infinite Lebesgue number*.

LEMMA 3.3.8. *Let X be a sequentially compact metric space. Then every open cover of X has finite Lebesgue number.*

PROOF. Suppose that some open cover $\mathscr{U}$ of X has infinite Lebesgue number. For each $n \geq 1$, there exists a subset A_n of X of diameter less than $\frac{1}{n}$ not contained in any element of $\mathscr{U}$. For each such A_n, choose $x_n \in A_n$. The sequence $(x_n)_{n\geq 1}$ has a convergent subsequence $(x_{n_k})_{k\geq 1}$, say with limit x. Let $U \in \mathscr{U}$ be an open neighborhood of x, and let $B(x,\delta)$ be an open ball inside of it. Let k be sufficiently large such that $\frac{1}{n_k} < \frac{\delta}{2}$ and $d(x,x_{n_k}) < \frac{\delta}{2}$. By the triangle inequality, we then have $A_{n_k} \subseteq B(x,\delta) \subseteq U$, providing the desired contradiction. $\square$

DEFINITION 3.3.9. We say that a metric space X is *totally bounded* if for every $\varepsilon > 0$, there exists a finite cover of X by open balls of radius ε.

LEMMA 3.3.10. *Every sequentially compact metric space is totally bounded.*

PROOF. Let X be a metric space, and suppose that $\varepsilon > 0$ is such that X cannot be covered by finitely many balls of radius ε. Choose $x_1 \in X$ and then recursively choose x_n in the complement of

$\bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$. The sequence $(x_n)_{n\geq 1}$ satisfies $d(x_m, x_n) \geq \varepsilon$ for all $m > n$, and consequently it cannot have a convergent subsequence (since every convergent subsequence is necessarily Cauchy). Thus, X not sequentially compact. □

THEOREM 3.3.11. *Let X be a metrizable space. Then the following are equivalent:*

i. X is compact,

ii. X is limit point compact,

iii. X is sequentially compact.

PROOF. Let X be a limit point compact space. Let $(x_n)_{n\geq 1}$ be a sequence in X. If the set $A = \{x_n \mid n \geq 1\}$ of values of the sequence is finite, then $(x_n)_{n\geq 1}$ has a constant, hence convergent, subsequence. Otherwise, A is infinite, so has a limit point $x \in X$. Inductively, we have that every ball among the $B(x, \frac{1}{k})$ for $k \geq 1$ contains some x_{n_k} with $n_k \geq n_{k-1}$ if $k \geq 2$. The sequence $(x_{n_k})_{k\geq 1}$ then converges to x. Thus, X is sequentially compact.

Now suppose that X is sequentially compact, and let $\mathscr{U}$ be an open cover of X. Let $\varepsilon > 0$ be such that every subset of diameter less than ε is contained in an element of $\mathscr{U}$. Choose a finite open cover of X by open balls of radius $\frac{\varepsilon}{3}$, and note that they have diameter at most $\frac{2\varepsilon}{3}$, hence are each contained in some element of $\mathscr{U}$. Then $\mathscr{U}$ has a finite subcover by these elements, and therefore X is compact. □

3.4. Tychonoff's theorem

We briefly recall a few notions from set theory. In particular, recall that a relation on a set X is a subset of $X \times X$, and if R is such a relation, we often write aRb to denote $(a,b) \in R$. We have already used the notion of an equivalence relation earlier in the notes without comment. Another useful sort of relation is known as a partial ordering.

DEFINITION 3.4.1. A *partial ordering* on a set X is a relation $\leq$ on X that satisfies the following properties.

i. (*reflexivity*) For all $x \in X$, we have $x \leq x$.

ii. (*antisymmetry*) If $x, y \in X$ satisfy $x \leq y$ and $y \leq x$, then $x = y$.

iii. (*transitivity*) If $x, y, z \in X$ satisfy $x \leq y$ and $y \leq z$, then $x \leq z$.

A set X together with a partial ordering $\leq$ is referred to as a *partially ordered set*.

DEFINITION 3.4.2. A *total ordering* on a set X is a partial ordering $\leq$ such that for all $x, y \in X$, one has either $x \leq y$ or $y \leq x$. In this case, X together with $\leq$ is called a *totally ordered set*.

EXAMPLES 3.4.3.

a. The relation $\leq$ on $\mathbb{R}$ is a total ordering, as is $\geq$.

b. The relation $<$ on $\mathbb{R}$ is not a partial ordering, as it is not reflexive.

c. The relation $\subseteq$ on the set of subsets $\mathscr{P}_X$ of any set X, which is known as the *power set* of X, is a partial ordering. It is not a total ordering if X contains more than one element.

d. The relation $=$ is a partial ordering on any set.

Given a partial ordering $\leq$ on a set X, we can speak of minimal and maximal elements of X.

DEFINITION 3.4.4. Let X be a set with a partial ordering $\leq$.

a. A *minimal element* in X (under $\leq$) is an element $x \in X$ such that if $z \in X$ and $z \leq x$, then $z = x$.

b. A *maximal element* $y \in X$ is an element such that if $z \in X$ and $y \leq z$, then $z = y$.

Minimal and maximal elements need not exist, and when they exist, they need not be unique. Here are some examples.

EXAMPLES 3.4.5.

a. The set $\mathbb{R}$ has no minimal or maximal elements under $\leq$.

b. The interval $[0,1)$ in $\mathbb{R}$ has the minimal element 0 but no maximal element under $\leq$.

c. The power set $\mathscr{P}_X$ of X has the minimal element $\varnothing$ and maximal element X under $\subseteq$.

d. Under $=$ on X, every element is both minimal and maximal.

e. Consider the set $S \subset \mathscr{P}_X$ of nonempty subsets of a set X, with the partial ordering $\subseteq$. The minimal elements of S are exactly the singleton sets in X.

One can ask for a condition under which maximal (or minimal) elements exist. To phrase such a condition, we need two more notions.

DEFINITION 3.4.6. Let X be a set with a partial ordering $\leq$. A *chain* in X is a subset of X that is totally ordered under $\leq$.

DEFINITION 3.4.7. Let X be a set with a partial ordering $\leq$. Let A be a subset of X. An *upper bound* on A under $\leq$ is an element $x \in X$ such that $a \leq x$ for all $a \in A$.

EXAMPLES 3.4.8.

a. The subset $[0,1)$ of $\mathbb{R}$ has an upper bound $1 \in \mathbb{R}$ under $\leq$. In fact, any element $x \geq 1$ is an upper bound for $[0,1)$. The subset $[0,1]$ has the same upper bounds.

b. The subset $\mathbb{Q}$ of $\mathbb{R}$ has no upper bound under $\leq$.

We now come to Zorn's lemma, which is equivalent to the axiom of choice. We omit the proof of this fact.

THEOREM 3.4.9 (Zorn's lemma). *Let X be a nonempty set with a partial ordering $\leq$, and suppose that every chain in X has an upper bound. Then X contains a maximal element.*

We use Zorn's lemma to prove the following.

THEOREM 3.4.10 (Alexander subbase theorem). *A space X is compact if and only if there exists a subbase $\mathscr{S}$ for its topology such that every open cover of X by elements of $\mathscr{S}$ has a finite subcover.*

PROOF. Suppose that X is not compact, and let $\mathscr{S}$ be a subbase of X. We need to show that $\mathscr{S}$ contains an open cover $\mathscr{U}$ that does not have a finite subcover.

Let Q be the set of all open covers of X that have no finite subcover, and note that $Q \neq \varnothing$ by the noncompactness of X. Let $\mathscr{C}$ be a chain in Q, and let $\mathscr{W} = \bigcup_{\mathscr{V} \in \mathscr{C}} \mathscr{V}$ be the union of all covers in $\mathscr{C}$. Any finite collection of elements of $\mathscr{W}$, being each contained in some element of $\mathscr{C}$, are all contained in the largest such element $\mathscr{V}$ under inclusion. Being that $\mathscr{V}$ has no finite subcover, such a finite collection cannot cover X, so $\mathscr{W}$ has no finite subcover, which is to say that $\mathscr{W} \in Q$. Thus, every chain in Q has an upper bound, and so by Zorn's lemma, Q has a maximal element $\mathscr{M}$.

Now consider the subset $\mathscr{U} = \mathscr{S} \cap \mathscr{M}$ of $\mathscr{S}$. Being a subset of $\mathscr{M}$, no finite subset of $\mathscr{U}$ covers $\mathscr{M}$. We claim that $\mathscr{U}$ covers X, which will finish the proof. Let $x \in X$, let $U \in \mathscr{M}$ with $x \in U$. Let $V_1, \ldots, V_n \in \mathscr{S}$ be such that $x \in V_1 \cap \cdots \cap V_n \subseteq U$, which exist as $\mathscr{S}$ is a subbase. If $V_i \notin \mathscr{M}$ for all $1 \leq i \leq n$, we can find by the maximality of $\mathscr{M}$ finite subsets $\mathscr{N}_i$ of $\mathscr{M}$ such that the collections $\mathscr{N}_i \cap \{V_i\}$ cover X. Then $\mathscr{N} = \bigcup_{i=1}^{n} \mathscr{N}_i$ and $V_1 \cap \ldots \cap V_n$ together cover X, so $\mathscr{N} \cup \{U\} \subset \mathscr{M}$ covers X as well, contradicting the fact that $\mathscr{M}$ has no finite subcover. Thus, there exists i such that $V_i \in \mathscr{U}$, and V_i contains x by definition. Thus $\mathscr{U}$ is a cover of X. $\square$

THEOREM 3.4.11 (Tychonoff's theorem). *Any product of compact spaces is compact under the product topology.*

PROOF. Let $\{X_i \mid i \in I\}$ be a collection of compact spaces, and let $X = \prod_{i \in I} X_i$. For each $i \in I$, let $\mathscr{T}_i$ be the topology on X_i, and let

$$\mathscr{S} = \bigcup_{i \in I} \{\pi_i^{-1}(U_i) \mid U_i \in \mathscr{T}_i\},$$

where $\pi_i \colon X \to X_i$ is the ith projection map. Then $\mathscr{S}$ is a subbase for the topology on X. We claim that every open cover of X by elements in $\mathscr{S}$ has a finite subcover, which by the Alexander subbase theorem will finish the proof.

Let $\mathscr{U} \subseteq \mathscr{S}$ be an open cover of X. Then

$$\mathscr{U} = \bigcup_{i \in I} \{\pi_i^{-1}(U_i) \mid U_i \in \mathscr{U}_i\}$$

for some collections $\mathscr{U}_i$ of open sets in X_i for each $i \in I$. If no $\mathscr{U}_i$ covers X_i, then for each $i \in I$, we may by the axiom of choice find $x_i \in X_i$ such that $x_i \notin \bigcup_{U_i \in \mathscr{U}_i} U_i$. Set $x = (x_i)_{i \in I}$. Then $x \notin U$ for any $U \in \mathscr{U}$, for any such U has the form $U = \pi_i^{-1}(U_i)$ for some $U_i \in \mathscr{U}_i$ for some $i \in I$, and $x_i = \pi_i(x) \notin U_i$. Thus there exists $i \in I$ such that $\mathscr{U}_i$ covers X_i. It has a finite subcover $\mathscr{V}$, and the finite set $\{\pi_i^{-1}(V) \mid V \in \mathscr{V}\}$ then covers X. $\square$

3.5. Local connectedness and compactness

Local properties of a space are those which happen within a small enough neighborhood of a point. Here are the definitions of local connectedness and local compactness.

DEFINITION 3.5.1. A topological space X is called *locally connected at* $x \in X$ if every neighborhood of x contains a connected open neighborhood of x. A topological space is called *locally connected* if it is locally connected at each of its points.

PROPOSITION 3.5.2. *A topological space X is locally connected if and only if every connected component of every open set in X is open in X.*

PROOF. Suppose X is locally connected. Let U be open in X, and let A be a connected component of U. If $x \in A$, then there is a connected open neighborhood V of x that is contained in U, and V is contained in A by its connectedness. Since x was arbitrary, A is open.

Conversely, suppose that all connected components of open sets in X are open in X. Let $x \in X$, and let V be an open neighborhood of x. Let A be the connected component of x in V. Note that A is open by assumption, so it is a neighborhood of x. Thus, X is locally connected. $\square$

EXAMPLE 3.5.3. Let $X = \prod_{n=1}^{\infty}\{0,1\}$ with the product topology, as in Example 3.1.18. Its basic open sets have the form $\prod_{n=1}^{N} U_i \times \prod_{m=N+1}^{\infty}\{0,1\}$, which are in particular infinite. As its connected components of X are singletons, X is not locally connected.

We mention in passing that we have a similar notion of local path connectedness.

DEFINITION 3.5.4. A topological space X is called *locally path connected at* $x \in X$ if every neighborhood of x contains a path connected open neighborhood of x. A topological space is called *locally path connected* if it is locally path connected at each of its points.

Of course, locally path connected spaces are locally connected, but the converse does not hold in general.

DEFINITION 3.5.5. A topological space X is said to be *locally compact at* $x \in X$ if x has a compact neighborhood. A topological space is *locally compact* if it is locally compact at all of its points.

Compact spaces are of course locally compact, as are many other spaces.

EXAMPLE 3.5.6. The space $\mathbb{R}^n$ is locally compact, since every closed ball of positive radius about a point is compact. However, $\prod_{n=1}^{\infty}\mathbb{R}$ is not locally compact, since its basic open sets all have closure that is a product of finitely many closed intervals with infinitely many copies of $\mathbb{R}$, and these are not compact as $\mathbb{R}$ is not.

DEFINITION 3.5.7. *A compactification* of a topological space X is a pair (Y, ι) consisting of a compact topological space Y and an embedding ι of X in Y.

REMARK 3.5.8. We may view X as a subspace of a compactification Y via identifying it with its homeomorphic image. Two compactifications Y and Z of a space X are then said to be equivalent if there exists a homeomorphism $f\colon Y \to Z$ that restricts to the identity map on X. This gives an equivalence relation on the compactifications of X.

Topological spaces can be compactified by adding in a single point.

THEOREM 3.5.9. *Let X be a topological space. The set Y containing X and one additional element called ∞ has a topology consisting of the open sets in X and the complements $Y - A$ of closed, compact subsets A of X. Moreover, Y is a compact space under this topology, and any other compactification Z of X with $Z - X$ a singleton set is equivalent to Y.*

PROOF. Note that arbitrary intersections of closed, compact subsets A of X are closed, and then compact, and the union of an open subset U of X and the complement $Y - A$ of a closed, compact subset A of X is the complement in Y of a smaller closed, and then compact, subset of X. It follows that the collection $\mathscr{T}$ of open sets defined in the theorem is closed under arbitrary unions. Similarly, finite unions of closed, compact subsets of X are closed and compact, and the intersection of U and $Y - A$ as above is an open subset of X. Hence, $\mathscr{T}$ is closed under arbitrary intersections, so $\mathscr{T}$ is a topology.

Given an open cover $\mathscr{V}$ of Y, we may choose an element in it of the form $Y - A$ with A closed, compact in X, as such a set is needed to cover ∞. Then $\mathscr{V} - \{Y - A\}$ is an open cover of A, which has a finite subcover as A is compact, so $\mathscr{V}$ has a finite subcover as well. Thus, Y is compact.

Finally, for any other space Y' as in the theorem, with ∞' its additional point, is in canonical bijection with Y via the map $f\colon Y \to Y'$ such that $f(x) = x$ for $x \in X$ and $f(\infty) = \infty'$. If V is open in Y and $\infty' \notin Y'$, then $f(V) = V$ is open in X, hence in Y'. If V is an open neighborhood of ∞ in Y, then its complement A is closed, hence compact in Y, and of course also contained in X. Then $f(V) = Y' - f(A)$ is the complement of a compact, closed subset of X, so is open in Y' as well. Thus, f is continuous, and then f^{-1} is continuous too, as we did not distinguish Y and Y'. □

DEFINITION 3.5.10. For a space X, the compact space of Theorem 3.5.9 is called the *one-point compactification* of X.

PROPOSITION 3.5.11. *A topological space X is locally compact and Hausdorff if and only if its one-point compactification is Hausdorff.*

PROOF. Let Y be the one-point compactification of X. Suppose that X is locally compact and Hausdorff. To show that Y is Hausdorff, it suffices to consider some $x \in X$ and ∞. Since X is locally compact, we can find a compact neighborhood A of x, which then contains some open neighborhood U. Then U is disjoint from the open neighborhood $Y - A$ of ∞.

If X is not Hausdorff, then there exist points $u, v \in X$ not contained in disjoint open sets in X. No two complements of compact sets in X are disjoint, since they contain the added point ∞. Moreover, if $u \notin A$ and $v \in V$ for a compact subset A of X and an open set V in X, then the open complement $X-A$ contains u and is not disjoint from V, so $Y-A$ is not disjoint from v either. Thus, u and v are not contained in disjoint open subsets of Y, and Y is not Hausdorff.

If X is not locally compact, then it is not locally compact at ∞. Given an open neighborhood $Y-A$ of ∞, where A is closed in X and compact, and an open neighborhood U of x in X, we cannot have $U \subseteq A$, since if this were the case, then $\overline{U} \subseteq A$ would be a compact neighborhood of x. Thus, Y is again not Hausdorff. □

REMARK 3.5.12. The one-point compactification of a locally compact Hausdorff space X has the property of being a quotient space of each Hausdorff compactification of X via the unique surjection that is the identity on X.

EXAMPLE 3.5.13. The one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n. To see this, view S^n as the unit sphere centered at $\infty = (0,\dots,0,1) \in \mathbb{R}^{n+1}$. This is realized via the embedding $\mathbb{R}^n \to S^n$ sending $x = (x_1,\dots,x_n)$ to the unique point in $S^n - \{\infty\}$ on the line between $(x_1,\dots,x_n,0)$ and ∞.

Our definition of local compactness differs from our definition of local connectedness, as we only ask for a compact neighborhood, not one contained in an arbitrarily small neighborhood. For Hausdorff spaces, these notions are the same.

PROPOSITION 3.5.14. *A Hausdorff space X is locally compact if and only if every open neighborhood U of a point $x \in X$ contains a compact neighborhood A with $A \subseteq U$.*

PROOF. As the other direction is immediate, we may suppose that X is locally compact. Let U be open in X, and let $x \in U$. Let Y be the one-point compactification of X, which is compact Hausdorff since X is locally compact Hausdorff. Let $A = Y - U$, which is closed in Y, hence compact. By Lemma 3.2.7, we may find an open neighborhood V of x and an open set W containing A that are disjoint. The closure of V in Y is compact and disjoint from A, so is contained in U. □

COROLLARY 3.5.15. *Any open or closed subspace of a locally compact Hausdorff space is locally compact.*

PROOF. Let X be locally compact Hausdorff. If A is closed in X and $a \in A$, then there is a compact neighborhood C of a in X, and $C \cap A$ is then a compact neighborhood of a in A. If U is open in X and $x \in U$, then by Proposition 3.5.14, we can find a compact neighborhood B of x in U. □

COROLLARY 3.5.16. *A Hausdorff space X is locally compact if and only if it is homeomorphic to an open subspace of a compact Hausdorff space.*

PROOF. If X is locally compact but not compact, take its one-point compactification, in which X is open. If X is open in a compact Hausdorff space, then it is locally compact by Corollary 3.5.15. □

DEFINITION 3.5.17. A map $f\colon X \to Y$ of topological spaces is said to be a *local homeomorphism* if it is continuous, open, and for each $x \in X$, there exists an open neighborhood U of x such that the restriction of f to U is a homeomorphism onto its image.

EXAMPLE 3.5.18. The inclusion map of an open set in a topological space is a local homeomorphism.

EXAMPLE 3.5.19. The map $f\colon \mathbb{R} \to S^1$ given by $f(x) = (\cos x, \sin x)$ is a local homeomorphism, as is its restriction to any open interval. Note that there is, however, no local homeomorphism $g\colon S^1 \to \mathbb{R}$.

CHAPTER 4

Countability and separation axioms

4.1. Countability axioms

DEFINITION 4.1.1. A topological space X is said to be *first-countable* if every point of X has a countable basis of open neighborhoods.

EXAMPLE 4.1.2. If X is a metrizable space, then X is first-countable. For instance, if d is a metric on X, then $x \in X$ has the countable basis $\{B(x, \frac{1}{n}) \mid n \geq 1\}$ of open neighborhoods.

PROPOSITION 4.1.3. *Let A be a subset of a first-countable topological space X. Then $\overline{A}$ is the set of limits of convergent sequences in A.*

PROOF. If there exists a sequence $(a_n)_{n\geq 1}$ in A with limit $x \in X$, then every neighborhood of x contains all a_n for n sufficiently large, so $x \in \overline{A}$ by definition. Conversely, if $x \in \overline{A}$ and $\{U_n \mid n \geq 1\}$ is a basis of open neighborhoods of x, then we may pick $a_n \in A \cap U_n$ for each $n \geq 1$, and the sequence $(a_n)_{n\geq 1}$ has limit x. $\square$

PROPOSITION 4.1.4. *Let X be a first-countable space. Let $x \in X$. If $f: X \to Y$ is a function such that for every convergent sequence $(x_n)_{n\geq 1}$ in X with limit x, the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x)$, then f is continuous at x.*

PROOF. Let V be an open set in Y, let $x \in f^{-1}(V)$, and let $\{U_n \mid n \geq 1\}$ be a countable basis of open neighborhoods of x. If $f^{-1}(V)$ does not contain an open neighborhood of x, then it cannot contain any U_n. Assume this, and for each $n \geq 1$, choose $x_n \in U_n$ such that $x_n \notin f^{-1}(V)$. Then the sequence $(x_n)_{n\geq 1}$ converges to x, and by assumption, $(f(x_n))_{n\geq 1}$ converges to $f(x)$. Therefore, there exists $N \geq 1$ such that $f(x_n) \in V$ for all $n \geq N$, so $x_n \in f^{-1}(V)$ for such n, contradicting our assumption. $\square$

DEFINITION 4.1.5. A topological space X is *second-countable* if X has a countable base for its topology.

EXAMPLE 4.1.6. The space $\mathbb{R}^n$ is second-countable, since it has the base $\{B(x, \frac{1}{k}) \mid x \in \mathbb{Q}^n,\ k \geq 1\}$.

EXAMPLE 4.1.7. The space $X = \prod_{i=1}^{\infty} \mathbb{R}$ is second-countable in the product topology but not in the uniform topology. In the product topology, we can take the countable base consisting of products

$$\prod_{i=1}^{N}(a_i, b_i) \times \prod_{i=N+1}^{\infty} \mathbb{R}$$

with $a_i, b_i \in \mathbb{Q}$ and $a_i < b_i$ for $1 \le i \le N$ for some $N \ge 0$.

In the uniform topology, we have the uncountable discrete subspace

$$A = \{(a_n)_{n\ge1} \mid a_n \in \{0,1\}\}.$$

every two points being distance 1 apart in the uniform metric. For any base of X, the open balls of radius $\frac{1}{2}$ about the points of A would each contain an element of the base, which is therefore uncountable. In particular, metrizable spaces are not necessarily second-countable.

DEFINITION 4.1.8. A topological space X is *Lindelöf* if every open cover of X has a countable subcover.

PROPOSITION 4.1.9. *Second-countable spaces are Lindelöf.*

PROOF. Let $\mathscr{U}$ be an open cover of X, and let $\mathscr{B} = \{B_n \mid n \ge 1\}$ be a countable base of X. Let S be the set of positive integers n such that B_n is contained in an element of U. For each $n \in S$, let $U_n \in \mathscr{U}$ with $B_n \subseteq U_n$. We claim that $\{U_n \mid n \in S\}$ covers X.

If $x \in X$, then $x \in U$ for some $U \in \mathscr{U}$. Since $\mathbb{B}_n$ is a base for the topology on X, there exists $n \ge 1$ such that $x \in B_n \subseteq U$. By definition of S, we then have $n \in S$, and so $x \in U_n$. Thus, we have proven the claim. □

Recall that a subset A of a topological space is X is dense if $\overline{A} = X$.

DEFINITION 4.1.10. A topological space X is *separable* if it has a countable dense subset.

PROPOSITION 4.1.11. *Second-countable spaces are separable.*

PROOF. Let $\mathscr{B} = \{U_n \mid n \ge 1\}$ be a countable base of X. For each $n \ge 1$, choose $x_n \in U_n$, and set $A = \{x_n \mid n \ge 1\}$. Then for any $x \in X$ and open neighborhood V of x, there exists $n \ge 1$ such that $x \in U_n \subseteq V$, so in particular $x_n \in V$. Thus, $x \in \overline{A}$, so A is dense in X. □

For metric spaces, the converses to Propositions 4.1.9 and 4.1.11 hold.

THEOREM 4.1.12. *For a metric space X, the following are equivalent:*

i. X is second-countable,

ii. X is Lindelöf, and

iii. X is separable.

PROOF. By Propositions 4.1.9 and 4.1.11, it suffices to show that Lindeöf spaces are second-countable and separable spaces are second-countable. Let X be a metric space.

Suppose first that X is Lindelöf. For positive integers $n \ge 1$, let $\mathscr{B}_n$ be a countable subcover of the open cover of X by balls of radius $\frac{1}{n}$. Let $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$, which is also countable. We claim that $\mathscr{B}$ is a base for the metric topology on X. Let $x \in X$ and $\varepsilon > 0$, and consider the open ball $B(x,\varepsilon)$. Choose n with $\frac{2}{n} < \varepsilon$, and let $y \in X$ be such that $x \in B(y,\frac{1}{n}) \in \mathscr{B}_n$. Then $B(y,\frac{1}{n}) \subseteq B(x,\varepsilon)$ by the triangle inequality, hence the claim.

Next, suppose that X is separable, and let $S \subseteq X$ be a countable dense subset. We claim that the countable collection

$$\mathscr{B} = \{B(y, \tfrac{1}{n}) \mid y \in S, n \geq 1\}$$

of open balls is a base for the metric topology on X. Let $x \in X$ and $\varepsilon > 0$. As S is dense, $B(x, \frac{\varepsilon}{3})$ contains a point $y \in S$, and if we take $n \geq 1$ such that $\frac{\varepsilon}{3} \leq \frac{1}{n} \leq \frac{2\varepsilon}{3}$, then $x \in B(y, \frac{1}{n}) \subseteq B(x, \varepsilon)$ by the triangle inequality, hence the claim. $\square$

4.2. Separation axioms

In this section, we consider both weaker and stronger conditions in terms of the separation of points in a topological space than our much-used Hausdorff condition. We begin by defining the word "separated" in the sense of topology.

DEFINITION 4.2.1. We say that two subsets A and B of a topological space are *separated* if $\overline{A} \cap B = \varnothing$ and $A \cap \overline{B} = \varnothing$.

DEFINITION 4.2.2. A *separation* of X is a set of two disjoint open subspaces of X with union X.

REMARK 4.2.3. A space X has a separation if and only if it is disconnected, in which case it is the topological disjoint union of the two spaces A and B in the separation. Moreover, these A and B are clearly separated. Similarly, if X is the union of two separated subsets, then they are both closed, hence open, and therefore constitute a separation of X.

Here then are several weaker conditions that being Hausdorff.

DEFINITION 4.2.4.

a. A space X is said to be T_0 if every two distinct points of X have distinct sets of open neighborhoods.

b. A space X is said to be T_1 if points of X are closed.

c. A space X is said to be T_2 if it is Hausdorff.

EXAMPLE 4.2.5. Spaces that are T_0 but not T_1 are ubiquitous in modern algebraic geometry. For instance, what is known as the spectrum $\operatorname{Spec}\mathbb{Z}$ of the ring of integers of $\mathbb{Z}$ is a space consisting of one point (0) and another point (p) for each prime number p. It has a base consisting of (0) and the complements of finite sets $\{(p_1), \ldots, (p_n)\}$ with each p_i prime. The closure of $\{(0)\}$ is $\operatorname{Spec}\mathbb{Z}$. Unfortunately, the explanation for why algebraic geometers consider this space lies beyond the scope of this course.

DEFINITION 4.2.6. A space in which every two points with distinct sets of open neighborhoods are separated is called *symmetric*.

LEMMA 4.2.7. *A space is T_1 if and only if it is symmetric and T_0, i.e., every pair of distinct points is separated.*

PROOF. Suppose X is T_1. If $x, y \in X$ are distinct, then $X - \{y\}$ and $X - \{x\}$ are distinct open neighborhoods of x and y, respectively, so X is T_0. Moreover, $\overline{\{x\}} = \{x\}$ for all $x \in X$, so for $y \neq x$, the sets $\{x\}$ and $\{y\}$ are separated. That is, X is symmetric.

Conversely, if X is T_0 and $x \in X$, then for any $y \in X$, there exists an open neighborhood U containing x and not y. If X is also symmetric, then since x and y have distinct sets of neighborhoods, $\{x\}$ and $\{y\}$ are separated. In particular, $y \notin \overline{\{x\}}$, which is to say that $\{x\}$ is closed. □

DEFINITION 4.2.8.

a. A topological space X is *regular* if for every point $x \in X$ and closed subset A of X with $x \notin A$, there exist disjoint open sets U and V with $x \in U$ and $A \subseteq V$.

b. A topological space X is *normal* if for every two closed, disjoint subsets A and B of X, there exist disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

Since in a topological space, points need not be closed (i.e., the space need not be T_1), it is not clear that either regular or normal should implies Hausdorff. In general, they do not, and normal does not imply regular. So, we make the following definitions.

DEFINITION 4.2.9.

a. A space X is said to be T_3 if it is *regular* and T_0.

b. A space X is said to be T_4 if it is *normal* and T_1.

The following tells us that a T_3-space is Hausdorff and a T_4-space is regular.

LEMMA 4.2.10. *If a space is T_i for some $1 \leq i \leq 4$, then it is T_{i-1}.*

PROOF. By Lemma 4.2.7, a T_1-space is T_0, and by Lemma 1.5.9, a T_2-space is T_1.

Suppose X is T_3. Take $x, y \in X$ with $x \neq y$. By the T_0-axiom, there is without loss of generality an open neighborhood W of x than does not contain y. Then W^c and x are disjoint, hence by the regularity condition, are contained in disjoint open sets U and V, respectively. Then $x \in U$ and $y \in V$, so X is Hausdorff.

Similarly, suppose that X is T_4. Let $x \in X$, and let $A \subset X$ be closed with $x \notin A$. Since X is T_1, the set $\{x\}$ is closed. Therefore, by normality there exist open disjoint sets U containing $\{x\}$ and V containing A, as desired. □

The reader can check the following, which is most interesting for and stops at T_3-spaces.

LEMMA 4.2.11. *For $0 \leq i \leq 3$, every subspace of a T_i-space is T_i.*

EXAMPLE 4.2.12. The space $X = \mathbb{R}$ with the topology consisting of sets of the form $U - C$, where U is open in $\mathbb{R}$ in the Euclidean topology and C is a countable subset of U, is Hausdorff but not regular. It is Hausdorff since its topology is finer than the Euclidean topology on $\mathbb{R}$. It is not regular, as the set $\mathbb{Q}$ is closed in its topology, and the open sets containing $\mathbb{Q}$ in $\mathbb{R}$ are the complements U in $\mathbb{R}$ of countable sets of irrational numbers. An open neighborhood V of an irrational number contains all but countably many points in an open interval around it, and so does U, but such an interval is uncountable, so $U \cap V$ cannot be empty.

LEMMA 4.2.13. *A space X is regular (resp., normal) if and only if every open set of X containing a point x (resp., closed set A) contains the closure of an open set containing x (resp., A).*

PROOF. Let X be regular (resp., normal). Let A be a singleton (resp., closed) subset of X, and let W be an open subset containing A. Then $B = W^c$ is disjoint from A, so by regularity (resp., normality) of X, there exist disjoint open sets U containing A and V containing B. Then $A \subseteq U \subseteq V^c \subseteq W$. Since V^c is closed, W contains $\overline{U}$, so U is the desired open set.

Conversely, suppose that for any singleton (resp., closed) subset A of X, every open set of X containing A contains the closure of an open set containing A. Let B be a closed set disjoint from A, and let U be an open subset of B^c containing A and such that $\overline{U} \subseteq B^c$. Take $V = \overline{U}^c$ so that $B \subseteq V$, and note that U and V are disjoint. Thus, X is regular (resp., normal). □

We have seen that a direct product of Hausdorff spaces is Hausdorff. The analogous statement holds for regular Hausdorff spaces (i.e., T_3-spaces).

PROPOSITION 4.2.14. *A product of T_3-spaces is T_3 in the product topology.*

PROOF. Let $X = \prod_{i \in I} X_i$, where $\{X_i \mid i \in I\}$ is a collection of T_3-topological spaces. Then X is Hausdorff, and in particular, its points are closed. Let $x = (x_i)_{i \in I} \in X$, and let U be an open neighborhood of x. Then U contains a basic open neighborhood $\prod_{i \in I} U_i$, where each U_i is open in X_i and all but finitely many U_i equal X_i. For every i, we may choose an open neighborhood V_i of x_i with $\overline{V_i} \subseteq U_i$ by Lemma 4.2.13. If $U_i = X_i$, we take $V_i = X_i$ as well. Then $\prod_{i \in I} V_i$ contains x and has closure $\prod_{i \in I} \overline{V_i} \subseteq \prod_{i \in I} U_i$, as desired. □

The following example shows that even a finite direct product of T_4-spaces need not be T_4.

EXAMPLE 4.2.15. Consider $X = \mathbb{R}$ with the lower-limit topology generated by the base of open sets $[a,b)$ with $a < b$. Note that these sets are closed as well, since $(-\infty, a) \cup [b, \infty)$ has complement $[a,b)$. Then X is normal Hausdorff. To see normality, take disjoint closed sets A and B in X. For each $a \in A$, we pick $x_a > a$ with $[a, x_a)$ in the complement of B and let U be the union of these half-open intervals. Similarly, for each $b \in B$, we pick $y_b > b$ with $[b, y_b)$ in the complement of A and let V be the union of these intervals. Take $a \in A$ and $b \in B$, and suppose without loss of generality that $a < b$. Then $x_a < b$ since $b \notin [a, x_a)$, so the intersection $[a, x_a) \cap [b, y_b)$ is empty, and therefore U and V are disjoint.

On the other hand, the product $X^2 = X \times X$ is regular Hausdoff by Proposition 4.2.14, but it is not normal. The subspace $D = \{(x, -x) \mid x \in \mathbb{R}\}$ of $X \times X$ has the discrete topology and is closed in X^2, and if we take the subset $A = \{(x, -x) \mid x \in \mathbb{Q}\}$, then A and $D - A$ are closed subsets of X^2 that are not contained in disjoint open neighborhoods of X^2. We omit the nontrivial proofs of these facts.

THEOREM 4.2.16. *Regular, second-countable spaces are normal.*

PROOF. Let X be regular and second-countable. Let $\mathscr{B}$ be a countable base for the topology on X. Let A and B be disjoint closed subsets of X. By regularity, for each $x \in X$ we can find an open neighborhood U of x with $\overline{U}$ disjoint from B, and contained in U we can find some neighborhood in $\mathscr{B}$ of x. Together, these basis elements give a countable covering $\mathscr{U} = \{U_n \mid n \geq 1\}$ of A with $\overline{U_n} \cap B = \varnothing$ for each $n \geq 1$. Similarly, we can find a countable covering $\mathscr{V} = \{V_n \mid n \geq 1\}$ of B with $\overline{V_n} \cap A = \varnothing$. For each $n \geq 1$, consider the open sets

$$U_n' = U_n \cap \bigcap_{i=1}^{n} \overline{V_i}^c \quad \text{and} \quad V_n' = V_n \cap \bigcap_{i=1}^{n} \overline{U_i}^c.$$

If $a \in A$, then $a \in U_n$ for some $n \geq 1$, and $a \notin \overline{V_i}$ for all i, so $a \in U_n'$. The open sets $U = \bigcup_{n \geq 1} U_n'$ and $V = \bigcup_{n \geq 1} V_n'$ contain A and B, respectively, and they are disjoint, since if $u \in U_n'$ for some $n \geq 1$, then $u \notin V_i$ for $i \leq n$ by definition of U_n' and $u \notin V_i'$ for $i > n$ by definition of V_n'. □

THEOREM 4.2.17. *Metrizable spaces are normal.*

PROOF. Let A and B be disjoint, closed subsets of a metrizable space X, and let d be a metric on X. For each $a \in A$, there exists $\varepsilon_a > 0$ with $B(a, \varepsilon_a) \cap B = \varnothing$, and similarly, for each $b \in B$, there exists $\delta_b > 0$ with $B(b, \delta_b) \cap A = \varnothing$. Set

$$U = \bigcup_{a \in A} B\left(a, \frac{\varepsilon_a}{2}\right) \quad \text{and} \quad V = \bigcup_{b \in B} B\left(b, \frac{\delta_b}{2}\right).$$

Then U and V are open containing A and B, respectively. If $x \in U \cap V$, then there exist $a \in A$ and $b \in B$ such that $d(a,x) < \frac{\varepsilon_a}{2}$ and $d(x,b) < \frac{\delta_b}{2}$. By the triangle inequality, we then have $d(a,b) < \frac{\varepsilon_a}{2} + \frac{\delta_b}{2} \leq \max(\varepsilon_a, \delta_b)$. If $d(a,b) < \varepsilon_a$, then $b \in B(a, \varepsilon_a)$, a contradiction, and if $d(a,b) < \delta_b$, then $a \in B(b, \delta_b)$, contradiction. Thus U and V are disjoint. □

THEOREM 4.2.18. *Compact Hausdorff spaces are normal.*

PROOF. Let X be a compact Hausdorff space, and let A and B be disjoint closed subsets of X, which are necessarily compact. Lemma 3.2.7 implies that compact Hausdorff spaces are regular. That is, for each $b \in B$, we may choose disjoint open sets U_b containing A and V_b containing b. Then the sets V_b cover B, hence have a finite subcover, say by $V_{b_1}, \ldots, V_{b_n} \in B$. Then $U = \bigcap_{i=1}^n U_{b_i}$ and $V = \bigcup_{i=1}^n V_{b_i}$ are disjoint open sets containing A and B, respectively. □

Using the one-point compactification, we may use Theorem 4.2.13 to give a quick proof of the analogous result for locally compact Hausdorff spaces.

COROLLARY 4.2.19. *Locally compact Hausdorff spaces are regular.*

PROOF. Let X be locally compact and Hausdorff. Let Y be its one-point compactification, which is compact Hausdorff and therefore normal. As Y is T_4, it is also T_3, and X is T_3 as a subspace of Y. □

4.3. Urysohn's lemma

THEOREM 4.3.1 (Urysohn's lemma). *A topological space X is normal if and only if for every pair of disjoint nonempty closed sets A and B of X, there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

PROOF. Let X be a normal space. Set $U_1 = B^c$, and by normality of X, pick an open set U_0 containing A with $\overline{U_0} \subseteq U_1$. We will construct open sets U_q for $q \in \mathbb{Q} \cap (0,1)$ such that if $q, r \in \mathbb{Q} \cap [0,1]$ with $q < r$, then $\overline{U_q} \subseteq U_r$.

Fix any bijection $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{Q} \cap [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$. For each $n \geq 2$, let q_n be the largest value of $\phi(m)$ with $m < n$ that is less than $\phi(n)$, and let r_n be the smallest value of $\phi(m)$ with $m < n$ that is greater than $\phi(n)$. Suppose by induction that we have constructed $U_{\phi(m)}$ for all $m < n$. By Lemma 4.2.13, we may choose an open set $U_{\phi(n)}$ containing $\overline{U_{q_n}}$ and such that $\overline{U_{\phi(n)}}$ is contained in U_{r_n}. For negative $r \in \mathbb{Q}$, set $U_r = \varnothing$, and for $r \in \mathbb{Q}$ greater than 1, set $U_r = X$. In this way, we have constructed open sets U_q for all rational numbers q such that $\overline{U_q} \subseteq U_r$ if $q < r$ are rational numbers.

We now define f by setting

$$f(x) = \inf\{q \in \mathbb{Q} \mid x \in U_q\}$$

for $x \in X$. If $x \in A$, then $x \in U_0$, so $f(x) = 0$. If $x \in B$, then $x \notin U_1$ but $x \in U_r$ for all rationals $r > 1$, so $f(x) = 1$. If $x \in \overline{U_q}$, then $x \in U_r$ for all rationals $r > q$, so $f(x) \leq q$. Given a nonempty open interval (a,b) in $\mathbb{R}$ and x in its inverse image, we then have $q, r \in \mathbb{Q}$ with $a < q < f(x) < r < b$. Since $f(x) < r$, we have $x \in U_r$, and since $f(x) > q$, we have $x \notin \overline{U_q}$. That is, we have $x \in U_r - \overline{U_q}$, which is an open neighborhood with image under f contained in $[q,r] \subset (a,b)$. Thus, f is continuous.

As for the converse, note that if A and B are disjoint closed sets in X and $f : X \to [0,1]$ is a continuous function with $f(A) = \{0\}$ and $f(B) = \{1\}$, then $U = f^{-1}([0,1/2))$ and $V = f^{-1}((1/2,1])$ are disjoint open sets containing A and B, respectively. □

DEFINITION 4.3.2. A space X is *completely regular* if for every closed set A in X and $x \in A^c$, there exists a continuous function $f : X \to [0,1]$ with $f(A) = \{0\}$ and $f(x) = 1$. A space X is said to be *Tychonoff*, or $T_{3\frac{1}{2}}$, if it is completely regular and T_1.

REMARK 4.3.3. A normal Hausdorff space is Tychonoff, and a completely regular space is regular.

We leave it to the reader to check that the following hold, the condition on subspaces between the motivation for the word "completely".

PROPOSITION 4.3.4. *Subspaces and products of completely regular spaces are completely regular.*

Regularity, together with second-countability, implies complete regularity (since it implies normality). In fact, we have the following.

LEMMA 4.3.5. *Let X be a regular, second-countable space. Then there exists a countable collection $\{f_n \mid n \geq 1\}$ of continuous functions $f_n \colon X \to [0,1]$ such that for each pair (A,x) of a closed subset A of X and $x \in A^c$, there exists n such that $f_n(A) = \{0\}$ and $f_n(x) = 1$.*

PROOF. Since X is second-countable and regular, it is normal. Let $\{U_n \mid n \geq 1\}$ be a countable base of open sets in X. If $\overline{U_m} \subseteq U_n$, then by Urysohn's lemma, there exists a continuous function $f_{m,n} \colon X \to [0,1]$ with $f_{m,n}(\overline{U_m}) = \{1\}$ and $f(X - U_n) = \{0\}$.

The set A^c contains some neighborhood U_n of x, which contains the closure of some neighborhood U_m of x by the regularity of X. Then $f_{m,n}(x) = 1$ and $f_{m,n}(A) = \{0\}$. Since the collection of functions $f_{m,n}$ is countable, we have the lemma. □

THEOREM 4.3.6 (Urysohn metrization theorem). *Second-countable regular Hausdorff spaces are metrizable and are exactly the spaces that are homeomorphic to subspaces of the product space $[0,1]^J$, where J is countably infinite.*

PROOF. We may take $J = \{n \mid n \geq 1\}$. Recall that $\mathbb{R}^J$ is metrizable by Proposition 2.2.16, so to show metrizability, it suffices to embed our second-countable regular space X in $[0,1]^J$. If we can do this for a given X, then $[0,1]^J$ is regular Hausdorff and second-countable, and then so is X.

Let $(f_n)_{n \geq 1}$ be a sequence of functions $f_n \colon X \to [0,1]$ as in Lemma 4.3.5 and use them to define a function $f = (f_n)_{n \geq 1} \colon X \to [0,1]^J$, which is continuous as each f_n is continuous. Since X is Hausdorff, for any $x, y \in X$ with $x \neq y$, the set $\{x\}^c$ is an open neighborhood of y, so we can for each $x \neq y$ in X find an $n \geq 1$ such that $f_n(x) = 0$ and $f_n(y) = 1$. Thus, f is injective.

It remains to show that f is an open map to its image. Let U be an open set in X, let $a = (a_n)_{n \geq 1} \in f(U)$, and let $x \in U$ with $f(x) = a$. Choose an $m \geq 1$ such that $f_m(x) = 1$ and $f_m(y) = 0$ for all $y \notin U$. Let V be the open set $\pi_m^{-1}((0,1]) \cap f(X)$ in the image of f, where π_m is the mth projection map from $\pi_m \colon [0,1]^J \to [0,1]$. Then $\pi_m(a) = f_m(x) = 1$, so $a \in V$. For $b \in V$, we have $b = f(y)$ for some y with $\pi_m(b) = f_m(y) > 0$, so $y \in U$ and therefore $b \in f(U)$. That is, we have $a \in f(U) \subseteq V$. Thus, $f(U)$ is open. □

Another application of Urysohn's lemma is found in the following theorem.

THEOREM 4.3.7 (Tietze extension theorem). *Let A be a closed subspace of a normal topological space X. Any continuous map of A into an interval in $\mathbb{R}$ can be extended to a continuous map of X into the same interval in $\mathbb{R}$.*

PROOF. It is sufficient to prove this result for maps into $[-1,1]$, $[-1,1)$ and $(-1,1)$, since all intervals in $\mathbb{R}$ are homeomorphic to one of these. We next show how the result reduces to maps to $[-1,1]$. Given any $g \colon X \to [-1,1]$, let $B = g^{-1}(1)$ and $C = g^{-1}(-1)$. By Urysohn's lemma, there exist continuous functions $\phi \colon X \to [0,1]$ and $\phi' \colon X \to [0,1]$ with $\phi(B) \subseteq \{0\}$, $\phi(C) = \phi'(C) \subseteq \{0\}$, and $\phi(A) = \phi'(A) \subseteq \{1\}$. If $g(A) \subseteq [-1,1)$, then set $g' = \phi g$, and note that $g'|_A = g|_A$. Moreover, it satisfies $g'(x) = 0$ if $x \in C$ and $g'(x) < 1$ if $x \notin C$, so $g'(X) \subseteq [-1,1)$. Similarly, if $g(A) \subseteq (-1,1)$, then set $g' = \phi' g$. Then $g'|_A = g|_A$, and $g'(X) \subseteq (-1,1)$.

We next prove a weaker result. Suppose that $F\colon A \to [-1,1]$ is continuous. Let

$$B = F^{-1}([-1,-\tfrac{1}{3}]) \quad \text{and} \quad C = F^{-1}([\tfrac{1}{3},1]).$$

By Urysohn's lemma, there exists a continuous function

$$G\colon X \to [-\tfrac{1}{3},\tfrac{1}{3}]$$

with $G(B) = \{-\frac{1}{3}\}$ and $G(C) = \{\frac{1}{3}\}$. Then

$$|G(a) - F(a)| \le \frac{2}{3}$$

for all $a \in A$. That is, for $a \in B \cup C$, we have $|G(a) - F(a)| \le 1 - \frac{1}{3} = \frac{2}{3}$, and for $x \notin B \cup C$, we have $|G(a) - F(a)| \le \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$.

Now let $f\colon A \to [-1,1]$ be continuous. Suppose by induction that for some $n \ge 0$ we have found continuous functions

$$g_i\colon X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}, \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}\right]$$

for $1 \le i \le n$ with

$$\left| f(a) - \sum_{i=1}^{n} g_i(a) \right| \le \left(\frac{2}{3}\right)^n$$

for all $a \in A$. Then the argument of the previous paragraph applied to $F = (\frac{3}{2})^n(f - \sum_{i=1}^n g_i)$ yields the next function $g_{n+1} = (\frac{2}{3})^n G$ in the induction. Since $\sum_{n=1}^\infty \frac{1}{3}(\frac{2}{3})^{n-1} = 1$, the infinite series

$$g = \sum_{i=1}^{\infty} g_i\colon X \to [-1,1]$$

is well-defined, and by definition, it restricts to f on A. Moreover, g is continuous as the partial sums $\sum_{i=1}^n g_i$ converge uniformly to g on X: that is, $|\sum_{i=n+1}^\infty g_n(x)| \le (\frac{2}{3})^n$ for all $x \in X$. □

DEFINITION 4.3.8. The *support* of a continuous function f from a topological space X to $\mathbb{R}$ is the closure

$$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

DEFINITION 4.3.9. Let $\mathscr{U} = \{U_i \mid 1 \le i \le n\}$ be an open cover of X for some $n \ge 1$. A finite collection $\{\phi_i\colon X \to [0,1] \mid 1 \le i \le n\}$ of continuous functions is a *partition of unity* on X *subordinate to*, or *dominated by*, $\mathscr{U}$ if $\operatorname{supp}\phi_i \subseteq U_i$ for each i and $\sum_{i=1}^n \phi_i(x) = 1$ for all $x \in X$.

LEMMA 4.3.10. *Let $\mathscr{U} = \{U_i \mid 1 \le i \le n\}$ be a finite open cover of a normal space X for some $n \ge 1$. Then there exists an open cover $\mathscr{V} = \{V_i \mid 1 \le i \le n\}$ of X with $\overline{V_i} \subseteq U_i$ for all $1 \le i \le n$.*

PROOF. It suffices to show the existence of V_1 open with $\overline{V_1} \subset U_1$ and such that $\{U_2,\dots,U_n,V_1\}$ covers X, since then we can repeat with this new cover, replacing U_2 and so forth. Let $A = X - \bigcup_{i=2}^n U_i$. By normality of X, there exists an open set V_1 containing A with $A \subset V_1 \subset \overline{V_1} \subset U_1$. Then the collection $\{V_1,U_2,\dots,U_n\}$ covers X. □

PROPOSITION 4.3.11. *Let $\mathscr{U} = \{U_i \mid 1 \le i \le n\}$ be a finite open cover of a normal space X. Then there exists a partition of unity on X subordinate $\mathscr{U}$.*

PROOF. By Lemma 4.3.10, we can find an open cover $\mathscr{V} = \{V_i \mid 1 \le i \le n\}$ with $\overline{V_i} \subseteq U_i$ for $1 \le i \le n$ and an open cover $\mathscr{W} = \{W_i \mid 1 \le i \le n\}$ with $\overline{W_i} \subseteq V_i$ for each $1 \le i \le n$. By Urysohn's lemma, there exist functions $\psi_i \colon X \to [0,1]$ such that $\psi_i(\overline{W_i}) \subseteq \{1\}$ and $\psi_i(V_i^c) \subseteq \{0\}$. Since $\psi_i^{-1}(\mathbb{R} - \{0\}) \subseteq V_i$, we have $\operatorname{supp} \psi_i \subseteq \overline{V_i} \subseteq U_i$. For each $x \in X$, we have $x \in W_i$ for some i, and therefore $\sum_j \psi_j(x) \ge \psi_i(x) = 1 > 0$. We may then define

$$\phi_i(x) = \frac{\psi_i(x)}{\sum_{j=1}^n \psi_j(x)}$$

for $x \in X$. Then the ϕ_i form the desired partition of unity. □

DEFINITION 4.3.12. A second-countable Hausdorff space M is said to be a *manifold* if there exists $n \ge 0$ such that every point of M has an open neighborhood homeomorphic to $\mathbb{R}^n$. We then say that M is *n-dimensional*, or an *n-manifold*.

REMARK 4.3.13. To say that a point in M has an open neighborhood homeomorphic to $\mathbb{R}^n$ is to say that it has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ (in particular, as open balls in $\mathbb{R}^n$ are homeomorphic to $\mathbb{R}^n$). Equivalently, there is a local homeomorphism $\mathbb{R}^n \to M$ with image containing the point.

EXAMPLE 4.3.14. Open sets in $\mathbb{R}^n$ and the sphere S^n are n-manifolds. The torus is an example of a 2-manifold, or surface.

We have the following application of Urysohn's lemma and partitions of unity to manifolds.

THEOREM 4.3.15. *Any compact manifold can be embedded in $\mathbb{R}^N$ for some $N \ge 1$.*

PROOF. Since X is a manifold, it has an open cover $\mathscr{U}$ by open sets that may be embedded in $\mathbb{R}^n$ for some fixed n. Since X is compact, there exist $U_1, \ldots, U_m \in \mathscr{U}$ that together cover X. Let $f_i \colon U_i \to \mathbb{R}^n$ be an open embedding. By Theorem 4.2.18, the space X is normal, so there exists a partition of unity $\{\phi_i \mid 1 \le i \le m\}$ subordinate to $\{U_i \mid 1 \le i \le m\}$. For each i, set $A_i = \operatorname{supp} \phi_i$ and define $g_i \colon X \to \mathbb{R}^n$ by $g_i(x) = \phi_i(x) f_i(x)$ for $x \in U_i$ and $g_i(x) = 0$ for $x \in A_i^c$. Note that g_i is well-defined as $\phi_i(x) = 0$ for $x \in A^c$.

We set

$$F = ((\phi_1, g_1), \ldots, (\phi_m, g_m)) \colon X \to (\mathbb{R}^{1+n})^m$$

which is continuous as a product of continuous functions. We claim that F is an embedding, which will finish the proof with $N = m(1+n)$. Since X is compact, it is enough to show it is injective. If $x, y \in X$ and $F(x) = F(y)$, then $\phi_i(x) = \phi_i(y)$ and $g_i(x) = g_i(y)$ for all $1 \le i \le n$. Let i be such that $\phi_i(x) > 0$ so that $x, y \in U_i$. Then $g_i(x) = \phi_i(x) f_i(x)$ and $g_i(y) = \phi_i(x) f_i(y)$, so $f_i(x) = f_i(y)$. As $f_i \colon U_i \to \mathbb{R}^n$ is injective, we have $x = y$. □

CHAPTER 5

Homotopy theory

5.1. Path homotopies

Recall that a path on a topological space X is a continuous function $\gamma\colon [0,1] \to X$.

DEFINITION 5.1.1. Let X be a space and $a,b \in X$. Let γ and γ' be paths in X from a to b. A *path homotopy* from γ to γ' is a continuous function $F\colon [0,1]^2 \to X$ from γ to γ' such that

$$F(s,0) = \gamma(s) \quad \text{and} \quad F(s,1) = \gamma'(s)$$

for all $s \in [0,1]$ and

$$F(0,t) = a \quad \text{and} \quad F(1,t) = b$$

for all $t \in [0,1]$.

EXAMPLE 5.1.2. Consider the two paths γ, γ' from $(1,0)$ to $(-1,0)$ in $\mathbb{C}$ given by $\gamma(s) = e^{\pi i s}$ and $\gamma'(s) = e^{-\pi i s}$. We have a path homotopy between them given by $F(s,t) = \cos(\pi s) + i(1-2t)\sin(\pi s)$. However, no such path homotopy exists in $\mathbb{C} - \{0\}$, the idea being that for any path homotopy F, there must exist a t such that the path $\gamma_t(s) = F(s,t)$ for $s \in [0,1]$ passes through 0. This may be intuitively clear, but it takes some work to show.

DEFINITION 5.1.3. We say that two paths γ, γ' in X from a point a to a point b are *path homotopic* if there exists a path homotopy from γ to γ'.

NOTATION 5.1.4. We write $\gamma \sim \gamma'$ if two paths with the same endpoints are path homotopic.

PROPOSITION 5.1.5. *The relation of path homotopy on the set $\Pi(X,a,b)$ of paths with fixed endpoints $a,b \in X$ forms an equivalence relation.*

PROOF. If $\gamma \in \Pi(X,a,b)$, then the map $F\colon [0,1]^2 \to X$ given by $F(s,t) = \gamma(s)$ for all $s,t \in [0,1]$ is a path homotopy from γ to itself, so $\sim$ is reflective. If $\gamma' \in \Pi(X,a,b)$ with $\gamma \sim \gamma'$ and $F\colon [0,1]^2 \to X$ is a path homotopy from γ to γ', then $G(s,t) = F(s,1-t)$ is a path homotopy from γ' to γ, so $\gamma' \sim \gamma$. Thus, $\sim$ is symmetric. Finally, if $\gamma'' \in \Pi(X,a,b)$ as well and we have both $\gamma \sim \gamma'$ and $\gamma' \sim \gamma''$, with F a path homotopy from γ to γ' and G a path homotopy from γ' to γ'', then $H\colon [0,1]^2 \to X$ defined by

$$H(s,t) = \begin{cases} F(s,2t) & t \in [0,\frac{1}{2}], \\ G(s,2t-1) & t \in [\frac{1}{2},1] \end{cases}$$

is a path homotopy from γ to γ', so $\sim$ is transitive. To see this, one should note that $F(s,1) = \gamma'(s) = G(s,0)$ for all $s \in [0,1]$. □

NOTATION 5.1.6. Let $\pi_1(X,a,b)$ denote the set of path homotopy classes of paths on X from a to b. We write $[\gamma]$ for the path homotopy class of a path γ.

DEFINITION 5.1.7. If $\gamma \in \Pi(X,a,b)$ and $\mu \in \Pi(X,b,c)$, we define the composition of γ and μ to be the path $\gamma \star \mu \in \Pi(X,a,c)$ given by

$$(\gamma \star \mu)(s) = \begin{cases} \gamma(2s) & s \in [0,\frac{1}{2}], \\ \mu(2s-1) & s \in [\frac{1}{2},1]. \end{cases}$$

If one replaces γ, μ with path homotopic paths, the result of composition is path homotopic to $\gamma * \mu$.

PROPOSITION 5.1.8. *Let $\gamma, \gamma' \in \Pi(X,a,b)$ and $\mu, \mu' \in \Pi(X,b,c)$ with $\gamma \sim \gamma'$ and $\mu \sim \mu'$. Then $\gamma \star \mu \sim \gamma' \star \mu'$.*

PROOF. Let F be a path homotopy from γ to γ' and G be a path homotopy from μ to μ'. Define $H\colon [0,1]^2 \to X$ by

$$H(s,t) = \begin{cases} F(2s,t) & s \in [0,\frac{1}{2}], \\ G(2s-1,t) & s \in [\frac{1}{2},1] \end{cases}$$

for $s,t \in [0,1]$. Then H is a path homotopy from $\gamma \star \mu$ to $\gamma' \star \mu'$. For this, one should note that $F(1,t) = b = G(0,t)$ for all $t \in [0,1]$. □

REMARK 5.1.9. By Proposition 5.1.8, composition of paths induces product maps

$$\pi_1(X,a,b) \times \pi_1(X,b,c) \xrightarrow{\cdot} \pi_1(X,a,c),$$

and we have $[\gamma] \cdot [\mu] = [\gamma \star \mu]$ under these products.

5.2. The fundamental group

Let us briefly explore the properties of the products that we have constructed on paths and their path homotopy classes.

NOTATION 5.2.1.

a. For $a \in X$, let $e_a \in \Pi(X,a,a)$ denote the constant path $e_a(s) = a$ for $s \in [0,1]$.

b. For $a,b \in X$ and $\gamma \in \Pi(X,a,b)$, let $\widetilde{\gamma} \in \Pi(X,b,a)$ denote the reversed path $\widetilde{\gamma}(s) = \gamma(1-s)$ for $s \in [0,1]$.

REMARK 5.2.2. If $\gamma \in \Pi(X,a,b)$ and $\mu \in \Pi(X,b,c)$, then $\widetilde{\gamma \star \mu} = \widetilde{\mu} \star \widetilde{\gamma}$.

We check some useful path homotopy relations among compositions of paths.

LEMMA 5.2.3. *Let $\gamma \in \Pi(X,a,b)$ with $a,b \in X$.*

a. If $\gamma' \in \Pi(X,a,b)$ with $\gamma \sim \gamma'$, then $\widetilde{\gamma} \sim \widetilde{\gamma'}$.

b. We have $e_a \star \gamma \sim \gamma \sim \gamma \star e_b$,

c. We have $\gamma \star \widetilde{\gamma} \sim e_a$ and $\widetilde{\gamma} \star \gamma \sim e_b$.

d. If $\mu \in \Pi(X,b,c)$ and $\nu \in \Pi(X,c,d)$ for some $c,d \in X$, then $(\gamma \star \mu) \star \nu \sim \gamma \star (\mu \star \nu)$.

PROOF.

a. If F is a homotopy from γ to γ', then we define $G\colon [0,1]^2 \to X$ by

$$G(s,t) = F(1-s,t),$$

and this is a homotopy from $\widetilde{\gamma}$ to $\widetilde{\gamma'}$.

b. Define $F_a\colon [0,1]^2 \to X$ by

$$F_a(s,t) = \begin{cases} a & s \in [0, \frac{1-t}{2}], \\ \gamma(\frac{2s-1+t}{1+t}) & s \in [\frac{1-t}{2}, 1]. \end{cases}$$

Note that for $s = \frac{1-t}{2}$, the second case yields $\gamma(0) = a$, so we have continuity. We then check that $F_a(s,0) = a = e_a(2s)$ for $s \in [0, \frac{1}{2}]$ while $F_a(s,0) = \gamma(2s-1)$ for $s \in [\frac{1}{2}, 1]$, so $F_a(s,0) = (e_a \star \gamma)(s)$ for all $s \in [0,1]$. We also have $F_a(s,1) = \gamma(s)$ for all $s \in [0,1]$ and $F_a(0,t) = a$ and $F_a(1,t) = \gamma(1) = b$ for all $t \in [0,1]$. Thus F_a is a path homotopy from $e_a \star \gamma$ to γ.

If we replace γ by $\widetilde{\gamma}$, we get $e_b \star \widetilde{\gamma} \sim \widetilde{\gamma}$, and then

$$\gamma \star e_b = \widetilde{e_b \star \widetilde{\gamma}} \sim \widetilde{\widetilde{\gamma}} = \gamma.$$

c. Define $H\colon [0,1]^2 \to X$ by

$$H(s,t) = \begin{cases} \gamma(2s(1-t)) & s \in [0, \frac{1}{2}], \\ \gamma((2-2s)(1-t)) & s \in [\frac{1}{2}, 1]. \end{cases}$$

The H is a path homotopy from $\gamma \star \widetilde{\gamma}$ to e_a. We have $\widetilde{\gamma} \star \gamma \sim e_b$ by replacing γ by $\widetilde{\gamma}$.

d. Define $I\colon [0,1]^2 \to X$ by

$$I(s,t) = \begin{cases} \gamma(4(1+t)^{-1}s) & s \in [0, \frac{1}{4}(1+t)], \\ \mu(4s-1-t) & s \in [\frac{1}{4}(1+t), \frac{1}{4}(2+t)], \\ \nu((2-t)^{-1}(4s-2-t)) & s \in [\frac{1}{4}(2+t), 1]. \end{cases}$$

We leave it to the reader to check that I is a path homotopy from $(\gamma \star \mu) \star \nu$ to $\gamma \star (\mu \star \nu)$.

□

DEFINITION 5.2.4. A *loop* in a topological space X *based at a point* $a \in X$ is a path in X from a to a. The point a is called the *basepoint* of the loop.

NOTATION 5.2.5. For $a \in X$, we set $\Pi(X,a) = \Pi(X,a,a)$ and $\pi_1(X,a) = \pi_1(X,a,a)$.

If we restrict our product maps on set of path homotopies to loops based at a point $x_0 \in X$, we obtain an operation on the classes of paths

$$\pi_1(X,x_0) \times \pi_1(X,x_0) \xrightarrow{\star} \pi_1(X,x_0).$$

This operation makes $\pi_1(X,x_0)$ into what is known as a group.

DEFINITION 5.2.6. A *group* G is a set together with an operation $G \times G \xrightarrow{\cdot} G$ such that

i. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a,b,c \in G$,

ii. there exists an *identity element* $e \in G$ such that $e \cdot a = a = a \cdot e$ for all $a \in G$, and

iii. for every $a \in G$, there exists an *inverse element* $a^{-1} \in G$ such that $a \cdot a^{-1} = e$.

Here are just a few interesting groups.

EXAMPLES 5.2.7.

a. The integers $\mathbb{Z}$ with the operation $+$ forms a group. In this group, $e = 0$ and the inverse of a is $-a$.

b. The nonzero real numbers $\mathbb{R} - \{0\}$ together with the operation $\cdot$ forms a group. In it, $e = 1$ and the inverse of a is a^{-1}.

c. Given a set X, the set S_X of bijections $f \colon X \to X$ forms a group with respect to the operation of composition. In it, the identity element is id_X and the inverse of a bijection f is its inverse function f^{-1}.

PROPOSITION 5.2.8. *For any $x_0 \in X$, the set $\pi_1(X,x_0)$ is a group under the operation induced by composition of paths.*

PROOF. The operation in question is given on the classes of loops $\gamma, \mu \in \Pi(X,x_0)$ by $[\gamma] \cdot [\mu] = [\gamma \star \mu]$. That this makes G into a group follows from the various parts of Lemma 5.2.3: that is, the operation is associative by part d, the identity element is $[e_a]$ by part b, and the inverse of $[\gamma]$ is $[\tilde{\gamma}]$ by part c. $\square$

DEFINITION 5.2.9. The *fundamental group* of a space X *relative to a basepoint* $x_0 \in X$ is the group $\pi_1(X,x_0)$ together with the operation induced by composition of paths.

One might ask how the fundamental group depends upon the choice of basepoint. For this, we need a notion of equivalence among groups. Such an equivalence should be a bijection that respects the operation on its domain and codomain. A function between groups that respects these operations is called a homomorphism.

DEFINITION 5.2.10. A function $f \colon G \to G'$ of groups is a *homomorphism* from the group G to the group G' if $f(a \cdot b) = f(a) \cdot f(b)$ for all $a,b \in G$.

Note that in the latter definition, the operation on the left is the operation on G and the operation on the ring is the operation on G'.

EXAMPLES 5.2.11.

a. For any group G, the constant map taking value the identity of a group is a homomorphism called a trivial homomorphism.

b. For $n \in \mathbb{Z}$, the map $n\colon \mathbb{Z} \to \mathbb{Z}$ given by multiplication by n is a homomorphism. It is trivial if $n = 0$.

DEFINITION 5.2.12. A homomorphism from G to G' is an *isomorphism* if it is a bijection.

LEMMA 5.2.13. *If $f\colon G \to G'$ is an isomorphism of groups, then so is the inverse function $f^{-1}\colon G' \to G$.*

PROOF. Since f is a bijection, its inverse f^{-1} is as well. We must show that f^{-1} is a homomorphism. Let $a', b' \in G'$, and note that there exist unique $a, b \in G$ with $f(a) = a'$ and $f(b) = b'$. We then have

$$f^{-1}(a' \cdot b') = f^{-1}(f(a) \cdot f(b)) = f^{-1}(f(a \cdot b)) = a \cdot b.$$

□

EXAMPLE 5.2.14. The function $\exp\colon \mathbb{R} \to \mathbb{R}_{>0}$ given by $\exp(x) = e^x$ is an isomorphism from the real numbers with the operation of addition to the positive real numbers with the operation of multiplication. That is, it is clearly bijective, and we have $e^{a+b} = e^a e^b$ for $a, b \in \mathbb{R}$. Its inverse is the logarithm function $\log\colon \mathbb{R}_{>0} \to \mathbb{R}$.

DEFINITION 5.2.15. We say that two groups G and G' are *isomorphic* if there exists an isomorphism $f\colon G \to G'$, in which case we write $G \cong G'$.

There is no such thing as the set of all groups, as it is too large. However, the following still makes sense.

PROPOSITION 5.2.16. *The relation $\cong$ is an equivalence relation on any set of groups.*

PROOF. Let G, H and K be groups. Then $G \cong G$ via the identity map. If $G \cong H$, then $H \cong G$ by Lemma 5.2.13. If $G \cong H$ and $H \cong K$, then we have isomorphisms $f\colon G \to H$ and $f'\colon H \to K$. Then $f' \circ f$ is still a bijection, and $f'(f(a \cdot b)) = f'(f(a) \cdot f(b)) = f'(f(a)) \cdot f'(f(b))$ for $a, b \in G$, so $f' \circ f$ is a homomorphism as well, and therefore $G \cong K$. □

So, we can now answer our question regarding fundamental groups relative to different basepoints.

PROPOSITION 5.2.17. *The fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ of a space X relative to basepoints x_0 and x_1 are isomorphic if there exists a path λ in X from x_0 to x_1. Explicitly, the isomorphism $\psi_\lambda\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ determined by λ is*

$$\psi_\lambda([\gamma]) = [\tilde{\lambda}] \cdot [\gamma] \cdot [\lambda].$$

PROOF. Let $\lambda\colon [0,1] \to X$ be a path with $\lambda(0) = x_0$ and $\lambda(1) = x_1$. Define a map

$$\Pi(X,x_0) \to \Pi(X,x_1), \qquad \gamma \mapsto (\tilde{\lambda} \star \gamma) \star \lambda$$

for $\gamma \in \Pi(X,x_0)$. If $\gamma \sim \gamma'$, then $(\tilde{\lambda} \star \gamma) \star \lambda \sim (\tilde{\lambda} \star \gamma') \star \lambda$, so this map induces the function ψ_λ. This is a bijection since it has an inverse induced by $\mu \mapsto (\lambda \star \mu) \star \tilde{\lambda}$. It is then an isomorphism, since

$$\psi_\lambda([\gamma\gamma']) = [\tilde{\lambda}][\gamma\gamma'][\lambda] = [\tilde{\lambda}][\gamma][\gamma'][\lambda] = [\tilde{\lambda}][\gamma][\lambda][\tilde{\lambda}][\gamma'][\lambda] = \psi_\lambda([\gamma])\psi_\lambda([\gamma']).$$

□

TERMINOLOGY 5.2.18. We call ψ_λ as in Lemma 5.2.17 *conjugation by the path* λ.

REMARK 5.2.19. When X is path connected, we often refer to the fundamental group of X to mean the fundamental group relative to some basepoint, since all choices are isomorphic.

REMARK 5.2.20. The isomorphism we constructed in the proof of Proposition 5.2.17 depends on the choice of a path from one basepoint to another. It is not in general unique, nor is it even necessarily the identity if the two points are the same.

DEFINITION 5.2.21. A space X is *simply connected* if it is path connected and $\pi_1(X,x_0)$ is the trivial group for some (equivalently, all) $x_0 \in X$.

LEMMA 5.2.22. *If X is a simply connected space and $a,b \in X$, then any two paths in X from a to b are path homotopic.*

PROOF. Let $\gamma, \gamma' \in \Pi(X,a,b)$. Then $\gamma^{-1} \star \gamma' \in \Pi(X,a)$. Since X is simply connected, $\gamma^{-1} \star \gamma' \sim e_a$, so $\gamma \star (\gamma^{-1} \star \gamma') \sim e_a \star \gamma$, from which it follows that $\gamma' \sim \gamma$. □

Continuous maps between topological spaces give rise to maps between homotopy groups.

LEMMA 5.2.23. *Let $f\colon X \to Y$ be continuous, and let $x_0 \in X$.*

a. The path homotopy class of $f \circ \gamma$ in $\Pi(Y, f(x_0))$ depends only on the path homotopy class of $\gamma \in \Pi(X,x_0)$.

b. If $\mu \in \Pi(X,x_0)$, then $f \circ (\gamma \star \mu) = (f \circ \gamma) \star (f \circ \mu)$.

PROOF. If $\gamma \sim \gamma'$ for some $\gamma' \in \Pi(X,x_0)$, and $F\colon [0,1]^2 \to X$ is a path homotopy from γ to γ', then $f \circ F$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$. Part b is immediate from the definition of composition of paths. □

By Lemma 5.2.23, the following definition makes sense.

DEFINITION 5.2.24. For a continuous function $f\colon X \to Y$ and $x_0 \in X$, the map

$$f_*\colon \pi_1(X,x_0) \to \pi_1(Y, f(x_0))$$

given by $f_*([\gamma]) = [f \circ \gamma]$ is the *homomorphism induced by f on fundamental groups* based at x_0 and $f(x_0)$.

The following property is immediate from the definitions.

LEMMA 5.2.25. *If $f\colon X\to Y$ and $g\colon Y\to Z$ are continuous functions of topological spaces, then $(g\circ f)_* = g_*\circ f_*$ for any $x_0\in X$. Moreover, $(\mathrm{id}_X)_*$ is the identity homomorphism on $\pi_1(X,x_0)$ for any $x_0\in X$.*

In particular, if f is a homeomorphism, then f_* is an isomorphism.

COROLLARY 5.2.26. *Let $f\colon X\to Y$ be a homeomorphism of topological spaces. Then for any $x_0\in X$, the homomorphism $f_*\colon \pi_1(X,x_0)\to\pi_1(Y,f(x_0))$ is an isomorphism with inverse $(f^{-1})_*$.*

PROOF. Since $f^{-1}\circ f = \mathrm{id}_X$, we have $(f^{-1})_*\circ f_* = \mathrm{id}_{\pi_1(X,x_0)}$ by Lemma 5.2.25, and similarly for the other composition. □

DEFINITION 5.2.27. A *retraction* of X onto a subspace A is continuous function $r\colon X\to A$ such that $r(a)=a$ for all $a\in A$.

EXAMPLES 5.2.28.

a. Let X be a topological space, and for $x_0\in X$. Then the unique map $r\colon X\to\{x_0\}$ is a retraction of X onto x_0.

b. Consider the closed disk $D=\overline{B}(0,1)$ in $\mathbb{R}^2$. Then the map $r\colon\mathbb{R}^2\to D$ given by the identity on D and

$$r(x,y)=\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right)$$

for $(x,y)\notin D$ is a retraction.

LEMMA 5.2.29. *If $r\colon X\to A$ is a retraction of a space X onto a subspace A, then for any $a_0\in A$, the map $r_*\colon\pi_1(X,a_0)\to\pi_1(A,a_0)$ is surjective, and for the inclusion map $\iota\colon A\to X$, the map $\iota_*\colon\pi_1(A,a_0)\to\pi(X,a_0)$ is injective with $r_*\circ\iota_* = \mathrm{id}_{\pi_1(A,a_0)}$.*

PROOF. Let γ be a loop in A based at a_0. Then γ is also a loop in X based at a_0, and $r\circ\gamma=\gamma$, so r_* is surjective. Since $r\circ\iota=\mathrm{id}_A$, we have the last equality of the statement, which forces ι_* to be injective. □

5.3. Covering spaces

DEFINITION 5.3.1. Let $f\colon C\to X$ be a continuous map of topological spaces, and let U be an open set in X contained in $f(C)$. We say that U is *evenly covered* by f if $f^{-1}(U)$ is a disjoint union of open subspaces of C, each of which is mapped homeomorphically onto U by f.

REMARK 5.3.2. If U is an open subset of X evenly covered by $f\colon C\to X$ and $x\in U$, then

$$f^{-1}(U)=\coprod_{c\in f^{-1}(x)} V_c,$$

where V_c is an open neighborhood of c in C such that $f|_{V_c}\colon V_c\to U$ is a homeomorphism.

DEFINITION 5.3.3. We say that a continuous surjective function $p\colon C \to X$ between topological spaces is a *covering map* if for each $x \in X$, there exists an open neighborhood U of $x \in X$ such that U is evenly covered by p. The space C, together with its covering map, is then said to be a *covering space* of X.

EXAMPLE 5.3.4. View S^1 as the unit circle in $\mathbb{C}$. The function $p\colon \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi i x}$ is a covering map. Inside any open neighborhood of $1 \in S^1$, we have an open set $U = p(-\varepsilon, \varepsilon)$ for a sufficiently small $\varepsilon < \frac{1}{2}$. The set $p^{-1}(U)$ is the disjoint union of the open sets $(n-\varepsilon, n+\varepsilon)$ for $n \in \mathbb{Z}$. This U is a disjoint union of open neighborhoods of the points n forming the inverse image $p^{-1}(1)$.

EXAMPLE 5.3.5. The function $f\colon S^1 \to S^1$ defined by $f(z) = z^n$ is a covering map. Since the polynomial $x^n - a$ for $a \in S^1$ has exactly n roots in $\mathbb{C}$, all of which have complex absolute value 1, every point has n points in its inverse image. (If $a = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R}$, then these roots have the form $e^{2\pi i(\theta+j)/n}$, where $0 \le j \le n-1$.) The inverse image of any proper open arc centered at a is the disjoint union of open arcs centered at these points of $f^{-1}(a)$, where the latter arcs are of arc length $\frac{1}{n}$ times that of the original arc.

The following is easily verified.

LEMMA 5.3.6. *Covering maps are surjective local homeomorphisms. In particular, they are open maps.*

REMARK 5.3.7. The converse to Lemma 5.3.6 not hold. For instance, consider the restriction f of the map $p\colon \mathbb{R} \to S^1$ of Example 5.3.4 to $\mathbb{R}_{>0}$. It is a surjective local homeomorphism. Let $U = p(-\varepsilon, \varepsilon)$ be an arbitrarily small neighborhood of 1 as in said example. Then $p^{-1}(U) \cap (-\varepsilon, \varepsilon) = (0, \varepsilon)$, and it does not map homeomorphically onto U: in fact, its image does not even contain $1T$. Thus, p is not a covering map.

LEMMA 5.3.8. *Let $p\colon C \to X$ be a covering map, and let $B = p^{-1}(Y)$ be the inverse image in C of an open subspace Y of X. Then $p|_B\colon B \to Y$ is a covering map.*

PROOF. By definition, the restriction $p|_B$ is a continuous surjective map. Let $x \in Y$, and let U be an open neighborhood of x in Y (hence in X) contained in $p(B)$. By Remark 5.3.2, the open set $p^{-1}(U)$ is a disjoint union of open sets V_c in C for each $c \in p^{-1}(x)$ with $p|_{V_c}\colon V_c \to U$ a homeomorphism. For $b \in p^{-1}(x) \subseteq B$, since $p(V_b) = U \subseteq Y$, we have $V_b \subseteq B$. That is, $p|_B$ takes V_b homeomorphically onto its image U. $\square$

LEMMA 5.3.9. *Let $p\colon C \to X$ and $p'\colon C' \to X'$ be covering maps. Then the product map $P\colon C \times C' \to X \times X'$ with $P(c, c') = (p(c), p(c'))$ is a covering map as well.*

PROOF. For $(x, x') \in X \times X'$, let U and U' be open neighborhoods of x and x' in X and X' respectively such that $p^{-1}(U)$ and $(p')^{-1}(U')$ are disjoint unions of open neighborhoods of the points in the inverse images of x and x', respectively, such that the images of these open neighborhoods map homeomorphically under p and p' to U and U', again respectively. Then $P^{-1}(U \times U')$ is a disjoint union of all products of these sets, one for each point in $P^{-1}(x,x')$, and again, they each map homeomorphically to $U \times U'$ under P by construction. □

EXAMPLE 5.3.10. Consider the product map $\mathbb{R}^2 \to (S^1)^2$ of of the map p of Example 5.3.4 with itself. By Lemma 5.3.9, it is a covering map. That is, the plane is a covering space of the torus.

We next discuss the notion of lifting of paths to covering spaces.

DEFINITION 5.3.11. Let $f: X \to Y$ be a continuous function, and let $p: C \to Y$ be a surjective continuous function. A *(continuous) lifting* of f to C is a continuous function $\tilde{f}: X \to C$ such that $p \circ \tilde{f} = f$. We say that $\tilde{f}$ *lifts* f if $\tilde{f}$ is a lifting of f.

PROPOSITION 5.3.12. *Let $p: C \to X$ be a covering map, let $c_0 \in C$, and set $x_0 = p(c_0)$. If $\gamma: [0,1] \to X$ is a path in X with initial point x_0, then there exists a unique lifting $\tilde{\gamma}: [0,1] \to C$ of γ to a path in C with initial point c_0.*

PROOF. Since p is a covering map, there exists an open cover $\mathscr{U}$ of X by sets that are evenly covered by p. Then $\mathscr{V} = \{\gamma^{-1}(U) \mid U \in \mathscr{U}\}$ is an open cover of $[0,1]$. By Lemma 3.3.8, there exists $N \geq 1$ such that the intervals $A_i = [\frac{i}{N} : \frac{i+1}{N}]$ with $0 \leq i \leq N-1$ are each contained in some element of $\mathscr{V}$, which has the form $\gamma^{-1}(U_i)$ for some $U_i \in \mathscr{U}$.

We define $\tilde{\gamma}$ with $\tilde{\gamma}(0) = c_0$ on each A_i recursively. Suppose we have defined $\tilde{\gamma}$ on $[0, \frac{i}{N}]$ for some $i \geq 0$. Let $c_i = \tilde{\gamma}(\frac{i}{N})$. Since p evenly covers U_i, we have an open neighborhood V_i of c_i which maps homeomorphically to U_i under p. Let $f_i = (p|_{V_i})^{-1}: U_i \to V_i$ be the inverse homeomorphism, and define $\tilde{\gamma}(s) = f_i \circ \gamma(s)$ for $s \in A_i$. The map $\tilde{\gamma}$ on $[0, \frac{i+1}{N}]$ is then continuous by Lemma 2.1.12. Note that this is the only continuous extension of $\tilde{\gamma}(s)$ from $[0, \frac{i}{N}]$ to $[0, \frac{i+1}{N}]$. That is, $\tilde{\gamma}(\frac{i}{N}) \in V_i$ and $f^{-1}(U_i)$ is the disjoint union of V_i and its complement in the inverse image, so in that $\tilde{\gamma}|_{A_i}$ must have connected image, its entire image must lie in V_i. But then the map $p|_{V_i}: V_i \to U_i$ is bijective, so $f_i \circ \gamma|_{A_i}$ is the only continuous lift of $\gamma|_{A_i}$ to C in V_i. Thus, we have defined our unique path $\tilde{\gamma}: [0,1] \to C$ lifting γ with $\tilde{\gamma}(0) = c_0$. □

Similarly, we have the following, which we leave unproven. The proof is similar to that Proposition 5.3.12, replacing the even subdivision of $[0,1]$ into intervals with the subdivision of $[0,1]^2$ into N^2 squares with vertices $(\frac{i}{N}, \frac{j}{N})$ with $0 \leq i, j \leq N$, for some N.

LEMMA 5.3.13. *Let $p: C \to X$ be a covering map with $p(c_0) = x_0$ for some $c_0 \in C$ and $x_0 \in X$. If $F: [0,1]^2 \to X$ is a continuous map with $F(0,0) = c_0$, then there exists a unique lifting $\tilde{F}: [0,1]^2 \to C$ with $\tilde{F}(0,0) = c_0$.*

We use this to prove the following.

PROPOSITION 5.3.14. *Let $p\colon C \to X$ be a covering, let $\gamma, \gamma'\colon [0,1] \to X$ be path homotopic paths. Set $x_0 = \gamma(0) = \gamma'(0)$, and let $c_0 \in p^{-1}(x_0)$. Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be the unique lifts of γ and γ', respectively, to paths in C with initial point c_0. Then $\tilde{\gamma}$ and $\tilde{\gamma}'$ satisfy $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$ and are path homotopic.*

PROOF. Let F be a path homotopy between γ and γ', and use Lemma 5.3.13 to uniquely lift it to $\tilde{F}\colon [0,1]^2 \to C$ with $\tilde{F}(0,0) = c_0$. We repeatedly use the uniqueness of Proposition 5.3.12. Then $\tilde{F}(s,0)$ is a path lifting γ with initial point c_0, so must be $\tilde{\gamma}$. Similarly, $\tilde{F}(0,t)$ is a path lifting the constant path e_{x_0} with initial point c_0, so must be the constant path e_{c_0}. Then $\tilde{F}(0,1) = c_0$, and $\tilde{F}(s,1)$ is a path lifting γ' with initial point c_0, so must be $\tilde{\gamma}'$. Finally, $\tilde{F}(1,t)$ is a path lifting the constant path $e_{\gamma(1)}$ with $\tilde{F}(1,t) = \tilde{\gamma}(1)$, so must also be constant. Thus, $\tilde{\gamma}$ and $\tilde{\gamma}'$ have the same final point, and $\tilde{F}$ is a path homotopy from $\tilde{\gamma}$ to $\tilde{\gamma}'$. □

We are now ready to prove a theorem connecting covering spaces and the fundamental group.

THEOREM 5.3.15. *Let $p\colon C \to X$ be a covering, let $x_0 \in X$ and $c_0 \in C$ with $p(c_0) = x_0$. The function $\phi_{c_0}\colon \pi_1(X,x_0) \to p^{-1}(x_0)$ taking $[\gamma]$ for $\gamma \in \Pi(X,x_0)$ to the endpoint of the unique lift of γ to C with initial point c_0 is well-defined. If C is path connected, then ϕ_{c_0} is surjective, and if C is simply connected, then ϕ_{c_0} is bijective.*

PROOF. That ϕ_{c_0} is well-defined is an immediate consequence of Proposition 5.3.14. If C is path connected and $c_1 \in p^{-1}(x_0)$, then choose a path $\lambda \in \Pi(C,c_0,c_1)$. Then λ is a lift of $\gamma = p \circ \lambda \in \Pi(X,x_0)$, and $\phi_{c_0}([\gamma]) = c_1$.

Suppose that C is simply connected, and let $\gamma, \mu \in \Pi(X,x_0)$ with $\phi_{c_0}([\gamma]) = \phi_{c_0}([\mu])$. Let c_1 denote the latter point. Let $\tilde{\gamma}$ and $\tilde{\mu}$ be the unique lifts to C of γ and μ, respectively, with initial point c_0. Then have final point c_1 by assumption. Since C is simply connected, there then exists a homotopy $\tilde{F}$ from $\tilde{\gamma}$ to $\tilde{\mu}$, and then $p \circ \tilde{F}$ is a homotopy from γ to μ. In other words, we have $[\gamma] = [\mu]$, so ϕ_{c_0} is injective. □

We can now compute the fundamental group of S^1 relative to any basepoint.

THEOREM 5.3.16. *The fundamental group of S^1 is isomorphic to the integers $\mathbb{Z}$ with the operation of addition.*

PROOF. The map $p\colon \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi i x}$ is a covering. Since $\mathbb{R}$ is simply connected and the inverse image of 1 is $\mathbb{Z} \subset \mathbb{R}$, Theorem 5.3.15 tells us that the map $\phi_0\colon \pi_1(S^1,1) \to \mathbb{Z}$ is a bijection.

We need only show that ϕ_0 is a homomorphism. So, let $\gamma, \mu \in \Pi(S^1,1)$, and set $n = \phi_0([\gamma])$ and $m = \phi_0([\mu])$. Let $\tilde{\gamma}$ be a lift of γ with initial point 0; its final point is then n. Let $\tilde{\mu}$ be the unique lift of μ with initial point 0, and note that the function $\tilde{\mu}' = n + \tilde{\mu}$ is a lift of μ with initial point n and final point $n + \tilde{\mu}(1) = n + m$. Then $\tilde{\gamma} \star \tilde{\mu}$ lifts $\gamma \star \mu$ and has final point $n+m$, so $\phi_0([\gamma][\mu]) = n + m = \phi_0([\gamma]) + \phi_0([\mu])$. □

This has some fascinating applications. We give a few.

EXAMPLE 5.3.17. There is no retract from the closed unit disk (i.e., ball) D about the origin in $\mathbb{R}^2$ to S^1. That is, D is simply connected, and S^1 has nontrivial fundamental group. Any retract would induce a nonexistent surjection from the fundamental group of D based at a point of S^1, which is trivial, to the nontrivial fundamental group of S^1 based at that point.

DEFINITION 5.3.18. A *fixed point* of a function $f\colon S \to S$ from a set S to itself is $x \in S$ such that $f(x) = x$.

The following is the Brouwer fixed point theorem for the unit disk in $\mathbb{R}^2$.

THEOREM 5.3.19 (Brouwer fixed-point theorem). *Let D be the closed unit disk in $\mathbb{R}^2$. If $f\colon D \to D$ is continuous, then h has a fixed point.*

PROOF. Suppose $f\colon D \to D$ has no fixed point. Define $g\colon D \to S^1$ by letting $g(x)$ for $x \in D$ be the unique point of S^1 on the ray from $f(x)$ to x in $\mathbb{R}^2$. Then g is a retract from D onto S^1, as the reader will verify. But this contradicts Example 5.3.17. □

5.4. Homotopies

We now consider a more general notion of homotopy than path homotopy, which is less restrictive (and therefore somewhat simpler to define) even in the case of paths.

DEFINITION 5.4.1. Let X and Y be topological spaces, and let $f, f'\colon X \to Y$ be continuous functions. A *homotopy* from f to f' is a continuous function $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = f'(x)$ for all $x \in X$.

DEFINITION 5.4.2. We say that two continuous functions $f, f'\colon X \to Y$ are *homotopic* if there exists a homotopy from f to f'.

LEMMA 5.4.3. *The property of being homotopic is an equivalence relation on the set $C(X,Y)$ of continuous functions from a topological space X to a topological space Y.*

PROOF. Let $\sim$ denote the relation on $C(X,Y)$ given by $f \sim f'$ if there exists a homotopy from f to f'. Given $f \in C(X,Y)$, the function $F\colon X \times [0,1] \to Y$ given by $F(x,t) = f(x)$ is a homotopy from f to itself, so $f \sim f$. If $F\colon X \times [0,1] \to Y$ is a homotopy from f to f', then $\tilde{F}\colon X \times [0,1] \to Y$ defined by $\tilde{F}(x,t) = F(x,1-t)$ is a homotopy from f' to f, so $\sim$ is symmetric. If $f \sim f'$ and $f' \sim f''$ and F is a homotopy from f to f' and G is a homotopy from f' to f'', then $H\colon X \times [0,1] \to Y$ defined by

$$H(x,t) = \begin{cases} F(x,2t) & \text{if } t \in [0,\frac{1}{2}] \\ G(x,2t-1) & \text{if } t \in [\frac{1}{2},1] \end{cases}$$

is a homotopy from f to f''. □

EXAMPLES 5.4.4.

a. Any two continuous functions $f, f' \colon X \to \mathbb{R}^n$ are homotopic. That is, $F \colon X \times [0,1] \to \mathbb{R}^n$ given by $F(x,t) = tf(x) + (1-t)f'(x)$ provides a homotopy.

b. Let $X = \{x\}$ and Y be a discrete space. Then no two distinct functions $f, f' \colon X \to Y$ are homotopic. That is, any continuous function $F \colon X \times [0,1] \to Y$ has connected image, and is therefore constant.

c. Let γ and γ' be the paths in $\mathbb{C}$ given by $\gamma(s) = e^{\pi i s}$ and $\gamma'(s) = e^{-\pi i s}$, as in Example 5.1.2. Since $\tilde{\gamma}' \circ \gamma$ is a simple loop in the complex unit circle S^1, we see from Theorem 5.3.16 that γ and γ' are not path homotopic. On the other hand, they are homotopic via $F \colon S^1 \times [0,1] \to S^1$ given by $F(e^{2\pi i\theta}, t) = \gamma(\theta)^{1-2t}$. Thus, homotopic paths need not be path homotopic.

LEMMA 5.4.5. *Let X and Y be topological spaces, and let $x_0 \in X$. Let $F \colon X \times [0,1] \to Y$ be a homotopy from a continuous $f \colon X \to Y$ to a continuous function $f' \colon X \to Y$ such that $F(x_0,t)$ is a constant function of $t \in [0,1]$. Set $y_0 = F(x_0,t)$. Then the homomorphisms*

$$f_*, f'_* \colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$$

are equal.

PROOF. Let $\gamma \in \Pi(X,x_0)$. We claim that

$$H = F \circ (\gamma, \mathrm{id}_{[0,1]}) \colon [0,1]^2 \to Y$$

is a path homotopy from $f \circ \gamma$ to $f' \circ \gamma$. That is, $H(s,0) = F(\gamma(s),0) = f(\gamma(s))$ and $H(s,1) = F(\gamma(s),1) = f'(\gamma(s))$, while $H(0,t) = F(x_0,t) = y_0$ and $H(1,t) = F(x_0,t) = y_0$. We then have

$$f_*([\gamma]) = [f \circ \gamma] = [f' \circ \gamma] = f'_*([\gamma]).$$

□

In fact, we may improve Lemma 5.4.5 as follows.

PROPOSITION 5.4.6. *Let X and Y be topological spaces, and let $x_0 \in X$. Let $F \colon X \times [0,1] \to Y$ be a homotopy from a continuous $f \colon X \to Y$ to a continuous function $f' \colon X \to Y$. Then $\lambda \colon [0,1] \to Y$ defined by $\lambda(t) = F(x_0,t)$ for $t \in [0,1]$ is a path from $y_0 = f(x_0)$ to $y'_0 = f'(x_0)$ in Y, and we have*

$$f'_* = \psi_\lambda \circ f_* \colon \pi_1(X,x_0) \to \pi_1(X,y'_0),$$

where ψ_λ is conjugation by λ as in Lemma 5.2.17.

PROOF. Let $\gamma \in \Pi(X,x_0)$. We wish to show that $[f' \circ \gamma] = [\tilde{\lambda}][f \circ \gamma][\lambda]$, or equivalently, that there is a homotopy

$$\lambda \star (f' \circ \gamma) \sim (f \circ \gamma) \star \lambda$$

of paths from y_0 to y_0'. Note that $f(\gamma(s)) = F(\gamma(s),0)$ and $f'(\gamma(s)) = F(\gamma(s),1)$ are loops in Y based at y_0 and y_0', respectively. We define $G\colon [0,1]^2 \to Y$ by

$$G(s,t) = \begin{cases} \lambda(2s) & \text{if } s \in [0, \frac{1-t}{2}], \\ F(\gamma(2s-1+t), 1-t) & \text{if } s \in [\frac{1-t}{2}, 1-\frac{t}{2}], \\ \lambda(2s-1) & \text{if } s \in [1-\frac{t}{2}, 1]. \end{cases}$$

Then G is continuous as the values

$$\lambda(1-t) = F(\gamma(0), 1-t) \quad \text{and} \quad F(\gamma(1), 1-t) = \lambda(1-t)$$

of $G(s,t)$ agree at $s = \frac{1-t}{2}$ and $s = 1 - \frac{t}{2}$, respectively. At $s \in \{0,1\}$, we have

$$G(0,t) = \lambda(0) = y_0 \quad \text{and} \quad G(1,t) = \lambda(1) = y_1.$$

And at $t \in \{0,1\}$, we have $G(s,0) = \lambda(2s)$ and $G(s,1) = F(\gamma(2s),0) = f(\gamma(2s))$ if $s \in [0, \frac{1}{2}]$, and $G(s,0) = F(\gamma(2s-1),1) = f'(\gamma(2s-1))$ and $G(s,1) = \lambda(2s-1)$ if $s \in [\frac{1}{2}, 1]$. Thus, G is the desired path homotopy. □

COROLLARY 5.4.7. *If f and f' are homotopic continuous maps $X \to Y$, then for any basepoint in X, the map f_* is injective (resp., surjective) if and only if f'_* is.*

DEFINITION 5.4.8. A continuous map $f\colon X \to Y$ of topological spaces is *nullhomotopic* if f is homotopic to a constant map from X to Y.

COROLLARY 5.4.9. *If $f\colon X \to Y$ is nullhomotopic, then the homomorphism f_* is trivial.*

PROOF. Suppose that f is homotopic to the constant function c_y with value $y \in Y$. Fix $x_0 \in X$, set $y_0 = f(x_0)$, and consider $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$. By Proposition 5.4.6, we have $f_* = \psi_\lambda \circ (c_y)_*$ where λ is a path in y from y_0, and note that $\psi_\lambda \circ (c_y)_* = (c_{y_0})^*$, which is trivial. □

We also have the following rather profound consequence.

THEOREM 5.4.10 (Fundamental theorem of algebra). *Every nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$.*

PROOF. Suppose that p is a polynomial of degree n with leading coefficient 1 (by scaling, without loss of generality). Then $p(z) - z^n$ is a polynomial of degree less than n, which means that there exists $r > 0$ such that $r^n > |p(z) - z^n|$ for all $z \in \mathbb{C}$ with $|z| \geq r$: i.e., we can choose any r greater than the sum of the absolute values of the coefficients of $p(z)$.

We claim that the map $f\colon S^1 \to S^1$ given by

$$f(z) = \frac{p(rz)}{|p(rz)|},$$

which is well-defined as p has no roots, is homotopic to $g\colon S^1 \to S^1$ given by $g(z) = z^n$. Consider $H\colon S^1 \times [0,1] \to S^1$ given by

$$H(z,t) = \frac{(1-t)(p(rz)-(rz)^n)+(rz)^n}{|(1-t)(p(rz)-(rz)^n)+(rz)^n|}.$$

It satisfies $H(z,0) = f(z)$ and $H(z,1) = g(z)$ for all $z \in S^1$. Also, for $z \in S^1$ and $t \in [0,1]$, we have

$$|(1-t)(p(rz)-(rz)^n)| \le |p(rz)-(rz)^n| < r^n = |(rz)^n|,$$

and therefore the denominator in H is always nonzero. That is, the function H is a well-defined homotopy from f to g.

If p has no roots in $\mathbb{C}$, then f extends to a map $\tilde{f}\colon \mathbb{C} \to S^1$ on the simply connected space $\mathbb{C}$ by the same formula, so if $\iota\colon S^1 \to \mathbb{C}$ is the inclusion, then $f_* = \tilde{f}_* \circ \iota_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$ factors through the trivial group $\pi(\mathbb{C},1)$, so is trivial. On the other hand, g_* induces multiplication by n on $\pi_1(S^1,1) \cong \mathbb{Z}$. By Corollary 5.4.7, this forces $n = 0$. That is, p is constant. □

DEFINITION 5.4.11. A *deformation retraction* from a topological space X to a subspace A is a homotopy $R\colon X \times [0,1] \to X$ from the identity map on X to a retraction from X to A such that $R(a,t) = a$ for all $a \in A$ and $t \in [0,1]$. If a deformation retraction from X to A exists, we say that A is a *deformation retract* of X.

EXAMPLE 5.4.12. The unit circle S^1 in $\mathbb{C}$ is a deformation retract of $\mathbb{C} - \{0\}$ via the function $R\colon (\mathbb{C} - \{0\}) \times [0,1] \to \mathbb{C} - \{0\}$ given by

$$R(re^{i\theta},t) = ((1-t)r+t)e^{i\theta}$$

for $r > 0$ and $\theta \in [0,2\pi)$.

PROPOSITION 5.4.13. *Let A be a deformation retract of X, and let $a_0 \in A$. Then the inclusion map $\iota\colon A \to X$ gives rise to an isomorphism*

$$\iota_*\colon \pi_1(A,a_0) \to \pi_1(X,a_0).$$

PROOF. Let $R\colon X \times [0,1] \to X$ be a deformation retraction from X to A. It satisfies the conditions of Lemma 5.4.5 for the basepoint $a_0 \in A$, and therefore $r\colon X \to A$ defined by $r(x) = R(x,1)$ is a retraction satisfying

$$\iota_* \circ r_* = (\iota \circ r)_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,a_0)}.$$

Since $r \circ \iota = \mathrm{id}_A$, we also have

$$r_* \circ \iota_* = (r \circ \iota)_* = (\mathrm{id}_A)_* = \mathrm{id}_{\pi_1(A,a_0)}.$$

Thus, ι_* is an isomorphism with inverse r_*. □

As a special case, we have the notion of a contractible space.

DEFINITION 5.4.14. A space X is *contractible*, or *contractible to a point* $x \in X$ if the subspace $\{x\}$ is a deformation retract of X.

REMARK 5.4.15. If X is contractible to a point in X, then it is contractible to every point in X.

The following is immediate from Proposition 5.4.13.

COROLLARY 5.4.16. *If X is contractible, then its fundamental group is trivial.*

In Corollary 5.2.26, we may obtain an isomorphism on fundamental groups under a weaker condition on our continuous map. For this, we make the following definition.

DEFINITION 5.4.17. A continuous map $f\colon X \to Y$ of topological spaces is a *homotopy equivalence* if there exists a continuous function $g\colon Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. We say that g is a *homotopy inverse* to f.

EXAMPLES 5.4.18.

a. If $f\colon X \to Y$ is a homeomorphism, then it is a homotopy equivalence with homotopy inverse its inverse f^{-1}.

b. If A is deformation retract of X, then the inclusion map $\iota\colon A \to X$ is a homotopy equivalence with homotopy inverse a retraction $r\colon X \to A$.

c. The inclusion map of the open unit disk B about the origin in $\mathbb{R}^2$ into $\mathbb{R}^2$ is a homotopy equivalence. Note that any continuous map from $\mathbb{R}^2$ to itself that fixes B also fixes $\bar{B}$, so there does not exist a retraction from $\mathbb{R}^2$ to B.

PROPOSITION 5.4.19. *Let $f\colon X \to Y$ be a homotopy equivalence, let $x_0 \in X$ and $y_0 = f(x_0)$. Then*

$$f_*\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$$

is an isomorphism for any $x_0 \in X$.

PROOF. Let $g\colon Y \to X$ be a homotopy inverse to f. For a homotopy F from id_X to $g \circ f$, set $x_1 = g(f(x_0))$. Let $\lambda(t) = F(x_0,t)$ for $t \in [0,1]$, which is a path from x_0 to x_1 in X. By Propsition 5.4.6, we have

$$g_* \circ f_* = \psi_\lambda \circ \mathrm{id}_{\pi_1(X,x_0)} = \psi_\lambda,$$

but $\psi_\lambda\colon \pi_1(X,x_0) \to \pi_1(X,x_1)$ is an isomorphism with inverse $\psi_{\bar{\lambda}}$, so $g_* \circ f_*$ is an isomorphism. Switching the roles of f and g, we see that $f_* \circ g_*$ is also an isomorphism, so f_* and g_* are mutually inverse. $\square$