POINT-SET TOPOLOGY

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Introduction

It is rather difficult to say what topology is, in part because the very language needed to describe it requires certain basic definitions. It is also difficult to describe in terms of its initial aims, in part because there are many branches of topology. What we will mostly be studying is point-set topology, which is a major part of the foundation of modern mathematics. It is a subject rather broad in its definitions, which aims to study not just the Euclidean spaces we know, but far more general objects, from a general axiomatic perspective.

The key objects in the study of topology are topological spaces and continuous functions. Among the topological spaces considered are the spaces one considers in multivariable calculus: \( \mathbb{R}^n \), spheres, tori, and other real manifolds (which are spaces that look locally like \( \mathbb{R}^n \)). Some might say that topology aims to study geometric objects while essentially forgetting about geometry: that is, in topology, a circle and a square are in some sense the same thing, for there exists a bijection from the square to the circle that is both continuous and has a continuous inverse. To a topologist, any collection of points, all separated from each other, is a perfectly good space too.

The definition of a topological space is rather simple: it is a set together with a collection of subsets known as its open sets. The axioms that this collection of open sets must satisfy aim to make the definition of a continuous function between topological spaces reasonable. In topology, a function is said to be continuous if it the inverse image of every open set is open. This matches with the usual definition of continuity of maps on \( \mathbb{R}^n \), for instance.

Consider for instance the topological space \( \mathbb{R}^2 \), i.e., the plane. Around every point in the plane are centered discs without their boundaries: these are open sets. The open sets in \( \mathbb{R}^2 \) are arbitrary unions of these open disks. Suppose \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) sends \( x \) to \( y \). To say that \( f \) is continuous at \( x \) is to say that given an open disk \( B \) centered at \( y \), the inverse image \( f^{-1}(B) \) contains an open ball centered at \( x \). Then \( f \) is continuous if it is continuous at every point. But looking closely at the definition of a continuous function between topological spaces, one sees that this is in fact the same definition: \( f \) is continuous if and only if \( f^{-1}(U) \) is open for every open set \( U \) in \( \mathbb{R}^2 \). The key point here is that any open set \( U \) contains an open disk about any point that it contains.

There are several key notions that will arise in our study of topology, among them separability, closure, connectedness, and compactness. Spaces can have rather strange topologies. For instance, on any set \( X \), one could put the “trivial” topology that the only open sets are \( \emptyset \) and \( X \). The only continuous functions from this space \( X \) to any other space are constant. In topological terms, this space is highly inseparable, and so we often put requirements on our spaces that prevent this, e.g., that the space be what is known as Hausdorff. A space is Hausdorff if given any two distinct points, one can find open sets containing each that do not intersect each other. Any space with the trivial topology and at least two elements will of course not be Hausdorff.
As for closure, we should define what it means for a set to be closed, which is exactly to say that its complement will be open. In $\mathbb{R}^2$, a closed disk, i.e., a disk including its boundary, is closed. And in fact, that brings us quickly to the notion of closure of a set, which is to say the smallest closed set containing a given set. In $\mathbb{R}^2$, the closure of an open disk is the closed disk that is the open disk and its boundary. A space is is connected if it is not a disjoint union of two nonempty open sets, which is to say it has no nonzero subsets which are both open and closed. Then $\mathbb{R}^2$ is connected, but inside it, the union of open disks of radius 1 about 0 and 2 is not.

Some spaces are highly disconnected: for instance, for any set $X$, we may put on it the “discrete” topology, whereby every subset is open. With this topology, every open set is also closed, and so every point is what is known as a connected component. The whole space itself is then called totally disconnected. Finally, we have a notion of compactness. In $\mathbb{R}^2$, a set is compact if it is closed and bounded, the latter meaning that the whole set is contained within a large enough disk about the origin. In fact, the abstract definition is rather different: it says that a space is compact if whenever it is a union of some collection of open sets, only finitely many of these open sets are needed. To see why these might be the same, try thinking about a closed interval $[a, b]$ in $\mathbb{R}$. Cover it by a union of open intervals, and since each has some finite length, only finitely many of them are needed.

Among the most important of topological spaces are metric spaces, which are sets with a distance function satisfying the triangle inequality. For any metric space, we define a topology on it by again considering the open balls of varying radius about a point, and defining an arbitrary open set to be a union of these. When we consider this underlying topological space, we may then forget about the metric. So, one should consider a metric space as being more than just a topological space: i.e., it has a notion of distance that a topological space itself does not. This notion of distance is crucial for many applications in analysis, of course, but it is also useful to have the flexibility to work with spaces that have no natural metric. A topological space that has a metric that gives rise to the original topology is called metrizable.

Toward the end of these notes, we will turn from point-set topology to the very beginnings of algebraic topology, which motivated many of the latter definitions. In particular, we shall explain the notion of homotopy, which allows one to transform one space into another. From the perspective of homotopy theory, two spaces are homotopic if one can be continuously transformed into another. From the perspective of homotopy, a closed disk in $\mathbb{R}^2$ is just a point, in that its radius can be shrunk at a continuous rate until it becomes a point. This perspective is, for instance, quite useful for classifying compact surfaces by how many “holes” they have: a sphere and a torus are topologically different as they are not homotopic. The first object of abstract algebra that can be used to measure this difference is known as the fundamental group, and it consists of “homotopy classes” of loops on a space beginning and ending at a fixed point. For instance, the fundamental group of a circle is just the integers, with the integer determined by how many times a loop wraps around the circle and in what direction.
CHAPTER 1

Topological spaces

1.1. Topologies

We begin by defining topological spaces.

**Definition 1.1.1.** A topology on a set $X$ is a set $\mathcal{T}$ of subsets of $X$ such that
i. the empty set $\emptyset$ and the set $X$ are contained in $\mathcal{T}$,
ii. if $\mathcal{U}$ is a subset of $\mathcal{T}$, then $\bigcup_{U \in \mathcal{U}} U \in \mathcal{T}$, and
iii. if $U_1, \ldots, U_n \in \mathcal{T}$ for some $n \geq 1$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

In other words, a topology $\mathcal{T}$ is a collection of subsets of $X$ containing $\emptyset, X$ and which is closed under arbitrary unions and finite intersections.

**Definition 1.1.2.** A topological space is a pair $(X, \mathcal{T})$ consisting of a set $X$ and a topology $\mathcal{T}$ on $X$.

**Definition 1.1.3.** Let $X$ be a set.

a. The **discrete topology** on $X$ is the topology equal to the set of all subsets (i.e., the power set) of $X$. We say that a topological space $X$ is **discrete** if its topology is the discrete topology on $X$.

b. The **trivial topology** on $X$ is the topology $\{\emptyset, X\}$.

**Definition 1.1.4.** A subset $U$ of a topological space $(X, \mathcal{T})$ is called open if $U \in \mathcal{T}$.

**Remark 1.1.5.** We often omit the notation of a topology $\mathcal{T}$ on a topological space $X$ and simply refer to $X$ as a topological space when its topology $\mathcal{T}$ is understood. At times, we say that a topological space $X$ is endowed with (or has) a topology $\mathcal{T}$. We sometimes refer to a topological space more simply as a space.

**Definition 1.1.6.** An element of a topological space $X$ is called a **point** of $X$.

**Example 1.1.7.** A topological space $X$ has the discrete topology if and only if every subset of $X$ is open.

**Examples 1.1.8.**

a. Let $X = \{a, b\}$ be a two-point set. Then there are 4 distinct topologies on $X$, all equal to $\{\emptyset, X\} \cup S$, where $S$ is some subset of $\{\{a\}, \{b\}\}$.

b. Let $X = \{a, b, c\}$ be a three-point set. Then $\{\emptyset, \{a\}, \{b\}, X\}$ is not a topology on $X$, as it is not closed under unions, while $\{\emptyset, \{a, b\}, \{a, c\}, X\}$ is not a topology on $X$, as it is not closed under finite intersections.
Example 1.1.9. Let \( X \) be any set. Then \( X \) has a topology under which a nonempty subset \( U \) is open if and only if its complement \( X - U = \{ x \in X \mid x \notin U \} \) is finite. This topology is known as the finite complement topology on \( X \).

Example 1.1.10. The Euclidean topology on \( \mathbb{R} \) is the unique topology under which a set is open if it is a union of open intervals. This is a topology as both \( \emptyset \) and \( \mathbb{R} \) are open intervals, any union of unions of open intervals is a union of open intervals, and any finite intersection of open intervals is an open interval (which, as the reader will check, implies that any finite intersection of unions of open intervals is a union of open intervals).

Definition 1.1.11. If \( \mathcal{T} \) and \( \mathcal{T}' \) be topologies on a set \( X \) with \( \mathcal{T} \subseteq \mathcal{T}' \), we say that \( \mathcal{T}' \) is finer (or stronger) that \( \mathcal{T} \) and \( \mathcal{T} \) is coarser (or weaker) than \( \mathcal{T}' \). If, in addition, \( \mathcal{T} \neq \mathcal{T}' \), we say that \( \mathcal{T}' \) is strictly finer (or strictly stronger) than \( \mathcal{T} \), and \( \mathcal{T} \) is strictly coarser (or strictly weaker) than \( \mathcal{T}' \).

Remark 1.1.12. We think of a topology with more elements as being finer in that we think of open sets as separating points from each other, so a topology with more open-sets is more “fine-grained”, in a sense. The discrete topology is the finest topology on any set, while the trivial topology is the coarsest.

Remark 1.1.13. The terminology of a “finer” topology including one that may be the same is in some sense unfortunate, but it is the most standard usage.

Example 1.1.14. Consider the three-point set \( X = \{a, b, c\} \) with topologies \( \mathcal{T}_1 = \{\emptyset, X\} \), \( \mathcal{T}_2 = \{\emptyset, \{a\}, X\} \), \( \mathcal{T}_3 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \), and \( \mathcal{T}_4 \) the discrete topology. Then \( \mathcal{T}_{i+1} \) is strictly finer than \( \mathcal{T}_i \) for each \( 1 \leq i \leq 3 \). If we set \( \mathcal{T}'_3 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \), then \( \mathcal{T}'_3 \) is strictly finer than \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) and strictly coarser than \( \mathcal{T}_4 \) but has no such relation with \( \mathcal{T}_3 \).

Definition 1.1.15. An open neighborhood of a point \( x \) in a topological space \( X \) is an open set containing \( x \). We say that an open neighborhood \( V \) of \( x \) is in a subset \( A \) if \( V \subseteq A \).

At times, we may wish to speak of closed neighborhoods, in which case the following definition is useful.

Definition 1.1.16. A neighborhood of a point \( x \) in a topological space \( X \) is any subset of \( X \) containing an open neighborhood of \( x \).

Proposition 1.1.17. A set of subsets \( \mathcal{T} \) of \( X \) is a topology on \( X \) if and only if \( \emptyset, X \in \mathcal{T} \), the set \( \mathcal{T} \) is closed under arbitrary unions, and for all \( U_1, \ldots, U_n \in \mathcal{T} \) with \( n \geq 1 \), there exists an open neighborhood of \( x \) contained in \( \bigcap_{i=1}^{n} U_i \).

Proof. Suppose that \( U_1, \ldots, U_n \in \mathcal{T} \) for some \( n \geq 1 \), and set \( W = \bigcap_{i=1}^{n} U_i \). If \( \mathcal{T} \) is a topology, then \( W \) is open, so we may take \( W \) to be the open neighborhood of the proposition. If on the other hand we have that for each \( x \in W \), there exists an open neighborhood \( V_x \) of \( x \) in \( W \), then \( W = \bigcup_{x \in W} V_x \) is an element of \( \mathcal{T} \) if \( \mathcal{T} \) is closed under unions. Thus, \( \mathcal{T} \) is a topology under the conditions of the proposition. \( \square \)

Notation 1.1.18. Given a set \( X \) and a subset \( A \), we write \( A^c \) for the complement \( X - A \) of \( A \) when \( X \) is understood.
1.2. SUBSPACES

Definition 1.1.19. A subset $A$ of a topological space $X$ is \textit{closed}, or a \textit{closed subset} of $X$, if its complement $X - A$ is an open.

Examples 1.1.20.

a. Every subset of a discrete space is closed.

b. The only closed subsets of a space $X$ with the trivial topology are $\emptyset$ and $X$.

c. In $\mathbb{R}$ with its Euclidean topology, the closed sets are the intersections of closed intervals.

Proposition 1.1.21. Let $X$ be a topological space. Then $\emptyset$ and $X$ are closed subsets of $X$, all intersections of closed sets are closed, and all finite unions of closed sets are closed.

Proof. We have $\emptyset = X^c$ and $X = X^c$, so $\emptyset$ and $X$ are closed. If $\mathcal{A}$ is a set of closed subsets of $X$, then

$$\bigcap_{A \in \mathcal{A}} A = \left( \bigcup_{A \in \mathcal{A}} A^c \right)^c,$$

which is closed as the complement of an open set. If $A_1, \ldots, A_n$ are closed for some $n \geq 1$, then

$$\bigcup_{i=1}^{n} A_i = \left( \bigcap_{i=1}^{n} A_i^c \right)^c,$$

which is again closed. □

1.2. Subspaces

Definition 1.2.1. Let $(X, \mathcal{T})$ be a topological space and $A$ be a subset of $X$. The subspace topology on $A$ is the set $\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T} \}$ of subsets of $A$.

Definition 1.2.2. A topological space $A$ with underlying set a subset of a topological space $X$ is called a \textit{subspace} if its topology is the subspace topology from $X$.

We verify that the subspace topology on a topological space is in fact a topology.

Proposition 1.2.3. Let $X$ be a topological space and $A$ be a subset of $X$. Then the subspace topology on $A$ is a topology on $A$.

Proof. Let $\mathcal{T}$ denote the topology on $X$ and $\mathcal{T}_A$ the subspace topology on $A$. We have $\emptyset = \emptyset \cap A \in \mathcal{T}_A$ and $A = X \cap A \in \mathcal{T}_A$, so $\mathcal{T}_A$ satisfies property (i) of a topology.

If $\mathcal{V} \subset \mathcal{T}_A$, then for each $V \in \mathcal{V}$, there exists $U_V \in \mathcal{T}$ with $V = U_V \cap A$. We then have

$$\bigcup_{V \in \mathcal{V}} V = \bigcup_{V \in \mathcal{V}} (U_V \cap A) = \left( \bigcup_{V \in \mathcal{V}} U_V \right) \cap A \in \mathcal{T}_A$$

since $\bigcup_{V \in \mathcal{V}} U_V \in \mathcal{T}$ as $\mathcal{T}$ is a topology. Thus, $\mathcal{T}_A$ satisfies property (ii) of a topology.

If $V_1, \ldots, V_n \in \mathcal{T}_A$ for some $n \geq 1$, then $V_i = U_i \cap A$ for some $U_i \in \mathcal{T}$ for $1 \leq i \leq n$. We then have

$$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (U_i \cap A) = \left( \bigcap_{i=1}^{n} U_i \right) \cap A \in \mathcal{T}_A.$$
since $\bigcap_{i=1}^n U_i \in \mathcal{T}$ as $\mathcal{T}$ is a topology. Thus, $\mathcal{T}_A$ satisfies property (iii) of a topology.

The subspace topology on an open subset of a topological space is a subset of the topology on the space.

**Proposition 1.2.4.** If $X$ is a topological space and $U$ is an open subset of $X$, then the subspace topology on $A$ is the set of open subsets of $X$ that are contained in $A$.

**Proof.** If $V$ is an open subset of $A$, then $V = U \cap A$ for some open subset $U$ of $A$, so $V$ is open in $X$ as an intersection of two of its open subsets. Conversely, if $U$ is an open subset of $X$ contained in $A$, then $U = U \cap A$, so $U$ is open in $A$ as well.

**Examples 1.2.5.**

a. Every subspace of a discrete space $X$ is discrete.

b. The subspace topology on an open interval $(a,b) \in \mathbb{R}$ with $a < b$ consists of $\emptyset$ and all unions of open intervals $(a',b')$ with $a \leq a' \leq b' \leq b$.

c. Consider the four-point set $X = \{a,b,c,d\}$ with the topology $\{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. Then the subspace topology on $\{a,b\}$ is the discrete topology, while the subspace topology on $\{c,d\}$ is the trivial topology.

The reader may check the following lemma.

**Lemma 1.2.6.** Let $A$ be a closed subset of a topological space $X$. Then the closed sets of $A$ under the subspace topology are exactly the intersections of closed subsets of $X$ with $A$.

**Terminology 1.2.7.** An open (resp., closed) subset of a topological space $X$, when endowed with the subspace topology, is called an open (resp., closed) subspace of $X$.

### 1.3. Bases

**Definition 1.3.1.** A subset $\mathcal{B}$ of a topology $\mathcal{T}$ on a topological space $X$ is said to be a base, or basis, for the topology $\mathcal{T}$ on $X$ if every open set $U \in \mathcal{T}$ is a (possibly empty) union of elements of $\mathcal{B}$.

**Example 1.3.2.** The set of open intervals in $\mathbb{R}$ is a base for the Euclidean topology on $\mathbb{R}$.

**Example 1.3.3.** If $X$ is a set with the discrete topology, then the set $\{\{x\} \mid x \in X\}$ is a base of the topology on $X$.

**Definition 1.3.4.** We say that a collection $\mathcal{U}$ of subsets of $\mathcal{T}$ covers a subset $A$ of $X$ if $A \subseteq \bigcup_{U \in \mathcal{U}} U$, and $\mathcal{U}$ is a cover of $A$ if $\mathcal{U}$ covers $A$.

**Proposition 1.3.5.** Let $X$ be a set, and let $\mathcal{B}$ be a set of subsets of $X$ such that

i. $\mathcal{B}$ covers $X$ and

ii. for every $U,V \in \mathcal{B}$ and $x \in U \cap V$, there exists $W \in \mathcal{B}$ with $x \in W$ and $W \subseteq U \cap V$.

Then the collection

$$\mathcal{T} = \{\bigcup_{U \in \mathcal{U}} U \mid \mathcal{U} \subseteq \mathcal{B}\}$$

of arbitrary unions of elements of $\mathcal{B}$ is a topology on $X$, and $\mathcal{B}$ is a base for the topology $\mathcal{T}$. 

1.3. BASES

PROOF. If $\mathcal{T}$ is a topology, then $\mathcal{B}$ will by definition be a base, so we need only verify that $\mathcal{T}$ is a topology. Note that $\emptyset$ is the empty union of elements of $\mathcal{B}$ and $X \in \mathcal{T}$ by assumption. We also have that $\mathcal{T}$ is closed under arbitrary unions, as its elements are just the arbitrary unions of elements of $\mathcal{B}$. If $U_1, \ldots, U_n \in \mathcal{T}$ and $V = \bigcap_{i=1}^n U_i$ contains a point $x$, then by recursion on condition (ii) of the proposition, we have that there exists $W \in \mathcal{B}$ with $x \in W$ and $V \subseteq W$. By Proposition 1.1.17, the set $\mathcal{T}$ is a topology on $X$. □

DEFINITION 1.3.6. The topology $\mathcal{T}$ on a set $X$ given by arbitrary unions of elements of a collection of subsets $\mathcal{B}$ of $X$ satisfying the two conditions of Proposition 1.3.5 is called the topology generated by the base $\mathcal{B}$. We say that $\mathcal{B}$ generates the topology $\mathcal{T}$.

We may now easily define the Euclidean topology on $\mathbb{R}^n$.

EXAMPLE 1.3.7. The Euclidean topology on $\mathbb{R}^n$ is the topology generated by the base consisting of open balls in $\mathbb{R}^n$ of finite radius. To see this is a topology, one need only note that around any point inside any nonempty intersection of two open balls, there exists another open ball.

DEFINITION 1.3.8. Let $X$ be a topological space. A base of open neighborhoods of a point $x$ is a set $\mathcal{B}_x$ of open neighborhoods of $x$ such that for every open neighborhood $U$ of $x$ in $X$, there exists $V \in \mathcal{B}_x$ with $U \subseteq V$. An element of $\mathcal{B}_x$ is said to be a basic open neighborhood of $x$.

LEMMA 1.3.9. Let $X$ be a topological space, and for each $x \in X$, let $\mathcal{B}_x$ be a base of open neighborhoods of $x$. Then $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ is a base of open neighborhoods of $X$.

PROOF. Since each $\mathcal{B}_x$ is nonempty, we have $X = \bigcup_{U \in \mathcal{B}} U$. If $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ for some $x, y \in X$ and $z \in U \cap V$, then there exists $W \in \mathcal{B}_z$ with $W \subseteq U \cap V$. So, $\mathcal{B}$ is a base for the topology on $X$. □

EXAMPLE 1.3.10. In $\mathbb{R}^n$, the set of open balls centered at a point $x$ forms a base of open neighborhoods of $x$.

In fact, any set of subsets of $X$ with union $X$ gives rise to a topology on $X$.

DEFINITION 1.3.11. A collection $\mathcal{I}$ of open subsets of a topological space $X$ is called a subbase, or subbasis, for the topology on $X$ if its union is $X$ and every open set in $X$ is a union of finite intersections of elements of $\mathcal{I}$.

The following is an simple consequence of Proposition 1.3.5.

PROPOSITION 1.3.12. Let $\mathcal{I}$ be a set of subsets of $X$ with union $X$. Let $\mathcal{B}$ be the set of all finite intersections of elements of $\mathcal{I}$. Then $\mathcal{B}$ is a base for a topology on $X$ for which $\mathcal{I}$ is a subbase.

EXAMPLE 1.3.13. The set of all intervals $(a, \infty)$ and $(-\infty, b)$ with $a, b \in \mathbb{R}$ is a subbase for the Euclidean topology on $\mathbb{R}$. 
1.4. Closure

There is a smallest closed set containing a given subset of a topological space, known as its closure.

**Definition 1.4.1.** The closure $\overline{A}$ of a subset $A$ of a topological space $X$ is the intersection of all closed subsets of $X$ containing $A$.

**Lemma 1.4.2.** The closure $\overline{A}$ of a subset $A$ of $X$ is the smallest closed subset of $X$ containing $A$ in the sense that $A$ is closed and contains $A$ and, if $B$ is a closed subset of $X$ with $A \subseteq B$, then $\overline{A} \subseteq B$.

**Proof.** First, we remark that $\overline{A}$ is closed as an intersection of closed sets and contains $A$ and contains $A$ as all of these closed sets contain $A$. Moreover, if $B$ is closed and contains $A$, then $B$ contains the intersection $\overline{A}$ of all closed sets containing $A$, since $B$ is one of the sets over which the intersection is taken. □

**Example 1.4.3.** The closure of an open interval $(a, b)$ in $\mathbb{R}$ is the closed interval $[a, b]$, as $[a, b]$ is a closed set containing $(a, b)$, and none of $(a, b)$, $[a, b)$, and $(a, b]$ is closed.

We have the following alternative characterization of the closure.

**Proposition 1.4.4.** Let $A$ be a subset of a topological space $X$. Then $x \in \overline{A}$ if and only if every (open) neighborhood of $x$ in $X$ has nonempty intersection with $A$.

**Proof.** We have $x \in \overline{A}$ if and only if $x \in B$ for all closed sets $B$ containing $A$. The latter holds if and only if $x \notin U$ for all open sets $U \subseteq A^c$. And this holds if and only if every open set $U$ containing $x$ is not contained in $A^c$, hence has nonempty intersection with $A$. □

**Example 1.4.5.** If $x \in (a, b)$, then clearly every open neighborhood of $x$ intersects $(a, b)$, in particular in $x$. Any open interval of the form $(a-\epsilon, a+\epsilon)$ intersects $(a, b)$ in $\{a+\delta \mid 0 < \delta < \epsilon\}$. Similarly, any interval $(b-\epsilon, b+\epsilon)$ intersects $(a, b)$ in $\{b-\delta \mid 0 < \delta < \epsilon\}$. On the other hand, if $x > b$ or $x < a$, then there exists a sufficiently small interval centered at $x$ that does not intersect $(a, b)$. Since every open neighborhood of a point in $\mathbb{R}$ contains an open interval (centered at the point), Proposition 1.4.4 again tells us that the closure of $(a, b)$ is $[a, b]$.

We also have the notion of an interior of a set.

**Definition 1.4.6.** The interior $A^\circ$ of a subset $A$ of a topological space $A$ is the union of all open sets contained in $A$.

Note that $A^\circ = (\overline{A})^c$ by its definition. We then have the following as a consequence of Lemma 1.4.2.

**Lemma 1.4.7.** The interior of a subset $A$ of a topological space $X$ is the largest open subset contained in $A$.

We also have a notion of boundary.

**Definition 1.4.8.** The boundary $\partial A$ of a subset $A$ of $X$ is the complement of the interior $A^\circ$ of $A$ in the closure $\overline{A}$.
Example 1.4.9. In \( \mathbb{R} \), the closure of an open interval \((a, b)\) with \( a, b \in \mathbb{R} \) and \( a < b \) is the closed interval \([a, b]\), and the interior of \([a, b]\) is \((a, b)\). The boundary of \((a, b)\), or \([a, b]\), is \( \{a\} \cup \{b\} \).

Example 1.4.10. In \( \mathbb{R}^n \), the closure of the open ball of radius \( \varepsilon \) about a point \( x \) is the closed ball of radius \( \varepsilon \) about the point \( x \), while the boundary is the sphere of radius \( \varepsilon \) centered at \( x \).

Example 1.4.11. If \( X \) has the trivial topology, then the closure of any nonempty subset \( A \) is \( X \), while the interior of \( A \) is empty unless \( A = X \). So, any nonempty, proper subset \( A \) of \( X \) is its own boundary, whereas the boundary of \( X \) is empty.

Example 1.4.12. Consider the three-point set \( X = \{a, b, c\} \) with topology
\[
\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}.
\]
The closed sets of \( \mathcal{T} \) are \( \emptyset, \{b\}, \{c\}, \{b, c\}, \) and \( X \). Thus, the closure of \( \{a\} \) is \( X \), while \( \{b\} \) and \( \{c\} \) are closed.

Remark 1.4.13. The taking of subspaces can change interiors and closures. For instance, the interior of the closed interval \([a, b]\) with \( a < b \) in \( \mathbb{R} \) is \((a, b)\), but its interior in \([a, b]\) is \([a, b]\), since \([a, b]\) is open in \([a, b]\).

1.5. Limit points

Definition 1.5.1. A point \( x \) of a subset \( A \) of a topological space \( X \) is called a limit point of \( A \) if every (open) neighborhood of \( x \) intersects \( A \) in a point other than \( x \).

The following is a corollary of Proposition 1.4.4.

Corollary 1.5.2. The closure of a subset \( A \) of a topological space \( X \) is the union of \( A \) and the set of its limit points.

Proof. By Proposition 1.4.4, if \( x \in \overline{A} \), then every open neighborhood of \( x \) intersects \( A \) in a point other than \( x \). If \( x \notin A \), these intersections cannot contain \( x \), so \( x \) is a limit point. □

Since closed sets are their own closures, we have the following.

Corollary 1.5.3. Any closed subset \( A \) of a topological space contains all of its limit points.

Remark 1.5.4. In Proposition 1.4.4, we may replace the condition on every open neighborhood with the same condition restricted to open neighborhoods in any base.

Definition 1.5.5. A subset \( A \) of a topological space \( X \) is dense in \( X \) if \( \overline{A} = X \).

Example 1.5.6. The rational numbers \( \mathbb{Q} \) are dense in \( \mathbb{R} \) with its Euclidean topology. That is, every open interval containing a real number contains a rational number, being that the interval has finite nonzero length.

One might ask how the notion of a limit point compares to the notion of points to which sequences converge. For this, we need the following definition.
Definition 1.5.7. A sequence \((x_n)_{n \geq 1}\) of points of a topological space \(X\) converges to a point \(x \in X\) if for every open neighborhood \(U\) of \(x\), there exists \(N \geq 1\) such that \(x_n \in U\) for all \(n \geq N\). The point \(x\) is said to be a limit of the sequence \(x_n\).

Remark 1.5.8. By definition, a limit of a convergent sequence \((x_n)_{n \geq 1}\) that is not eventually constant is a limit point of the set \(\{x_n \mid n \geq 1\}\). However, the latter set may have more than one limit point even if \((x_n)_{n \geq 1}\) converges.

Example 1.5.9. If \(X\) has the trivial topology, then every sequence in \(X\) converges to every point of \(X\).

To avoid such pathologies as in the previous example, it is useful to put the following condition on a space.

Definition 1.5.10. A topological space \(X\) is called Hausdorff if for every two distinct points \(a, b \in X\), there exist open neighborhoods \(U\) of \(a\) and \(V\) of \(b\) such that \(U \cap V = \emptyset\).

More briefly, \(X\) is Hausdorff if every two distinct points of \(X\) have disjoint neighborhoods. In Hausdorff spaces, points are closed.

Lemma 1.5.11. In a Hausdorff space \(X\), every singleton set \(\{x\}\) for \(x \in X\) is closed.

Proof. If \(x \in X\) and \(a \in X\) with \(x \neq a\), then there exists an open neighborhood \(U_a\) of \(a\) not containing \(x\). As the union of all such open sets \(U_a\) is the complement of \(\{x\}\), the set \(\{x\}\) is closed. \(\square\)

The property of being Hausdorff is stronger than that of points being closed, however.

Example 1.5.12. In an infinite set \(X\) with the finite complement topology, points are closed as the complement of open sets. However, any two nonempty open sets in \(X\) intersect in all but finitely many elements of \(X\), so \(X\) is not Hausdorff. Moreover, every non-repeating sequence \((x_n)_{n \geq 1}\) in \(X\) converges to every point of \(X\), as the reader should check using the fact that open sets have finite complements.

Even better, in Hausdorff spaces, every convergent sequence has a unique limit.

Proposition 1.5.13. Every convergent sequence in a Hausdorff space has a unique limit.

Proof. If \(x \in X\) is a limit of a convergent sequence \((x_n)_{n \geq 1}\) in a Hausdorff space \(X\), then for any \(y \in X - \{x\}\), we have disjoint open neighborhoods \(U\) of \(x\) and \(V\) of \(y\). For sufficiently large \(n\), the point \(x_n\) are all in \(U\), hence not in \(V\), and therefore \(y\) is not a limit point of the sequence. \(\square\)

The following is immediate from the definitions.

Lemma 1.5.14. Every subspace of a Hausdorff space is Hausdorff.
1.6. Metric spaces

Metric spaces, and the open balls inside of them, provide fundamental examples of topological spaces. We review the definition here.

**Definition 1.6.1.** A metric on a set $X$ is a function $d : X \times X \to \mathbb{R}_\geq 0$ such that for all $a, b, c \in X$, one has

i. $d(a, b) = 0$ if and only if $x = y$,

ii. $d(a, b) = d(b, a)$, and

iii. $d(a, c) \leq d(a, b) + d(b, c)$.

**Terminology 1.6.2.** For a set $X$, the condition $d(a, c) \leq d(a, b) + d(b, c)$ on $d : X \times X \to \mathbb{R}_\geq 0$ for all $a, b, c \in X$ is called the **triangle inequality**.

**Definition 1.6.3.** A pair $(X, d)$ consisting of a set $X$ and a metric $d$ on $X$ is called a **metric space**.

**Notation 1.6.4.** When the metric $d$ on a metric space $(X, d)$ is understood, we often write $X$ for the metric space.

**Example 1.6.5.** The set $\mathbb{R}^n$ is a metric space for the distance function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_\geq 0$ defined by the Euclidean metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for $x = (x_i)_{i=1}^{n}$ and $y = (y_i)_{i=1}^{n}$. We remark that any three points $x, y,$ and $z$ form the vertices of a triangle, and the distances between them are the lengths of the sides, so the triangle inequality reduces to the usual triangle inequality of Euclidean geometry.

**Definition 1.6.6.** In a metric space $(X, d)$, the **open ball** of radius $\varepsilon > 0$ about a point (or with center) $x \in X$ is the set

$$B(x, \varepsilon) = \{ y \in X \mid d(x, y) < \varepsilon \}.$$  

The **closed ball** of radius $\varepsilon$ is

$$\overline{B}(x, \varepsilon) = \{ y \in X \mid d(x, y) \leq \varepsilon \}.$$

**Proposition 1.6.7.** The set of open balls in a metric space $X$ forms a base of a topology on $X$, and the set of open balls with center $x$ is a base of open neighborhoods of $x$ in this topology.

**Proof.** Clearly, the union of open balls in a metric space is $X$, so by Proposition 1.3.5, it suffices to show that for any two open balls $U = B(x, \varepsilon)$ and $V = B(y, \delta)$ in $X$ and any point $z \in U \cap V$, there exists an open ball containing $z$ and contained in $U \cap V$. For this, choose any positive real number

$$\rho < \min\{ \varepsilon - d(x, z), \delta - d(y, z) \}.$$  

Then $B(z, \rho) \subset U \cap V$ by the triangle inequality. That is, if $d(z, w) < \rho$, then

$$d(w, x) = d(w, z) + d(z, x) < \rho + d(x, z) < \varepsilon,$$
and similarly \( d(w, y) \leq d(w, z) + d(z, y) < \delta \), as desired. 

**Definition 1.6.8.** The metric topology on a set \( X \) induced by a metric \( d \) on \( X \) is the topology on \( X \) generated by the set of open balls under \( d \).

**Lemma 1.6.9.** Let \( X \) be a metric space. Then every closed ball \( \bar{B}(x, \varepsilon) \) is closed in the metric topology.

**Proof.** It suffices to show that the complement of \( \bar{B}(x, \varepsilon) \) is open. If \( y \notin \bar{B}(x, \varepsilon) \), then \( \delta = d(x, y) - \varepsilon > 0 \), and \( B(y, \delta) \) and \( B(x, \varepsilon) \) are disjoint by the triangle inequality. That is, if \( z \in B(y, \delta) \), then \( d(x, z) + \delta > d(x, z) + d(z, y) \geq d(x, y) \), so \( d(x, z) > \varepsilon \). Thus, \( x \notin B(x, \varepsilon) \).

Two different metrics on a set \( X \) can have the same metric topology. Take the following example.

**Example 1.6.10.** Consider the Euclidean metric \( d \) on \( \mathbb{R}^n \) and the box metric \( d' \) on \( \mathbb{R}^n \) defined by

\[
    d'(x, y) = \max \{ |x_i - y_i| \mid 1 \leq i \leq n \}
\]

for \( x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^n \in \mathbb{R}^n \). The reader should check that \( d'(x, y) \) is in fact a metric.

We have bases \( B(x, \varepsilon) \) and \( B'(x, \varepsilon) \) of open balls about a point \( x \) with respect to these respective metrics. For any \( y \in \mathbb{R}^n \), we have

\[
    \max \{ |x_i - y_i|^2 \mid 1 \leq i \leq n \} \leq \sum_{i=1}^n (x_i - y_i)^2 \leq n \max \{ |x_i - y_i|^2 \mid 1 \leq i \leq n \},
\]

so

\[
    B'(x, \varepsilon) \subseteq B(x, \varepsilon) \subseteq B'(x, \sqrt{n} \varepsilon)
\]

for all \( \varepsilon > 0 \). Thus, the two metric topologies coincide.

We examine the topological properties of metric spaces.

**Proposition 1.6.11.** Metric spaces are Hausdorff spaces.

**Proof.** Let \( X \) be a metric space, and suppose that \( x, y \in X \) are distinct points. Let \( \varepsilon = \frac{1}{2} d(x, y) \). Then \( B(x, \varepsilon) \) and \( B(y, \varepsilon) \) are disjoint by the triangle inequality. 

**Definition 1.6.12.** A topological space \((X, \mathcal{T})\) is metrizable if there exists a metric \( d \) on \( X \) such that the metric topology induced by \( d \) is the topology \( \mathcal{T} \) on \( X \).

The following is a direct corollary of Proposition 1.6.11.

**Corollary 1.6.13.** If \( X \) is a metrizable topological space, then \( X \) is Hausdorff.

Discrete spaces are metric spaces as well.

**Definition 1.6.14.** Let \( X \) be a set. The discrete metric \( d \) on \( X \) is defined by

\[
    d(x, y) = \begin{cases} 
    1 & \text{if } x \neq y \\
    0 & \text{if } x = y.
    \end{cases}
\]

We have the following.
Lemma 1.6.15. The discrete metric on a set $X$ is a metric, and the topology induced by this metric is the discrete topology.

Proof. That $d$ is a metric is straightforward. Since $B(x, \frac{1}{2}) = \{x\}$ for all $x \in X$, singleton sets are open in this topology, so $X$ is discrete.

The following example shows that even metric spaces can defy our intuition from Euclidean geometry.

Example 1.6.16. While in any metric space, the closure of the open ball $B(x, \epsilon)$ is contained in the closed ball $\overline{B}(x, \epsilon)$, the closure can in fact be smaller. For instance, if $(X, d)$ is a metric space with the discrete metric $d'$, then the set $B(x, 1) = \{x\}$ is both closed and open, while $B(x, \epsilon) = X$ for all $\epsilon > 1$.

Definition 1.6.17. A subset $A$ of a metric space $X$ is bounded if there exists $N > 0$ such that $d(x, y) \leq N$ for all $x, y \in A$.

This notion of boundedness is not a topological one.

Lemma 1.6.18. If $(X, d)$ is a metric space, then the function $d': X \times X \to \mathbb{R}_{\geq 0}$ given by

$$d'(x, y) = \min\{d(x, y), 1\}$$

is a metric on $X$, and the metric topologies on $X$ from $d$ and $d'$ are the same.

Proof. Let $x, y, z \in X$. If $d(x, y) \leq 1$, then

$$d'(x, y) = d(x, y) \leq \min\{d(x, z) + d(z, y), 1 + d(z, y), d(x, z) + 1 + 1\} = d'(x, z) + d'(z, y),$$

by the triangle inequality for the first of the terms in the set and the fact that $d(x, y) \leq 1$ for the others. If $d(x, y) \geq 1$, then

$$d'(x, y) = 1 \leq \min\{d(x, z) + d(z, y), 1 + d(z, y), d(x, z) + 1 + 1\} = d'(x, z) + d'(z, y)$$

by the triangle inequality for the first term in the set (in that $1 \leq d(x, y) \leq d(x, z) + d(z, y)$) and the fact that the other terms are clearly at least one. Thus, $d'$ satisfies the triangle inequality, and it clearly satisfies the other two conditions for being a metric.

The open balls of radius less than 1 form a base of any metric topology, as any open ball contains one of these. Since these sets of open balls coincide for $d$ and $d'$, these metrics induce the same topology on $X$.

Example 1.6.19. Under the Euclidean metric, $\mathbb{R}$ is not bounded, but it is with respect to the metric $d'(x, y) = \min\{|x - y|, 1\}$ on $\mathbb{R}$. Nevertheless, both of these metrics induce the Euclidean topology.

Sequences in metric spaces behave as one might expect.

Proposition 1.6.20. Let $(X, d)$ be a metric space. A sequence $(x_n)_{n \geq 1}$ in $X$ converges to $a \in X$ if and only if $\lim_{n \to \infty} d(a, x_n) = 0$. 

PROOF. If $U$ is an open neighborhood of $a$, then $U$ contains some open ball $B(a, \epsilon)$. If $\lim_{n \to \infty} d(a, x_n) = 0$, then there exists $N \geq 1$ such that $d(a, x_n) < \epsilon$ for all $n \geq N$, so $x_n \in U$. Conversely, if $(x_n)_{n \geq 1}$ converges to $a$, then for any $\epsilon$, there exists $N \geq 1$ such that $x_n \in B(a, \epsilon)$ for $n \geq N$, which is to say $d(a, x_n) < \epsilon$. Thus, the limit $\lim_{n \to \infty} d(a, x_n)$ is 0. \hfill \Box

PROPOSITION 1.6.21. Let $X$ be a metrizable space, let $A \subseteq X$, and let $x \in X$. Then $x \in \overline{A}$ if and only if it is the limit of a convergent sequence of elements of $A$.

PROOF. We need only see that any $x \in \overline{A}$ is such a limit. Fix a metric $d$ on $X$ so that we may consider open balls in $X$. If $x \in \overline{A}$, then for any $n \geq 1$, there exists $x_n \in B(x, \frac{1}{n}) \cap A$ by definition of $\overline{A}$. Then the $x_n$ converge to $x$ by Proposition 1.6.20. \hfill \Box
CHAPTER 2

Continuous functions

2.1. Continuous functions

In this section, \( X \) and \( Y \) will denote topological spaces.

**Definition 2.1.1.** A function \( f : X \to Y \) is said to be continuous if \( f^{-1}(V) \) is open for every open subset \( V \) of \( Y \).

**Definition 2.1.2.** We say that \( f : X \to Y \) between topological spaces is continuous at a point \( x \in X \) (or continuous at \( x \)) if for every open neighborhood \( V \) of \( f(x) \), there exists an open neighborhood \( U \) of \( x \) such that \( f(U) \subseteq V \).

**Remark 2.1.3.** As every open subset is a union of open neighborhoods of its points, a function is continuous if and only if it is continuous at every point.

We can check continuity on basis elements of \( Y \).

**Proposition 2.1.4.** Let \( \mathcal{B}_X \) and \( \mathcal{B}_Y \) be fixed bases of the topologies on \( X \) and \( Y \), respectively. Let \( f : X \to Y \) be a function.

- a. A function \( f : X \to Y \) is continuous if and only if \( f^{-1}(V) \) is open in \( X \) for all \( V \in \mathcal{B}_Y \).

- b. A function \( f : X \to Y \) is continuous at \( x \in X \) if and only if for every basic open neighborhood \( V \in \mathcal{B}_Y \) of \( f(x) \), there exists a basic open neighborhood \( U \in \mathcal{B}_X \) of \( x \) with \( f(U) \subseteq V \).

**Proof.** If \( f \) is continuous, then \( f^{-1}(V) \) is open for all \( V \in \mathcal{B}_Y \) by definition. Conversely, suppose \( f^{-1}(V) \) is open for all \( V \in \mathcal{B}_Y \). As \( \mathcal{B}_Y \) is a base, for any open subset \( W \) of \( Y \), there exists \( C \subseteq \mathcal{B}_Y \) such that \( W = \bigcup_{V \in C} V \). Then \( f^{-1}(W) = \bigcup_{V \in C} f^{-1}(V) \), so \( f^{-1}(W) \) is open as a union of open sets. Thus, we have proven part (a).

If \( f \) is continuous at \( x \in X \) and \( V \) is a basic open neighborhood of \( x \), then \( f^{-1}(V) \) contains an open neighborhood of \( x \), and such an open neighborhood contains a basic open neighborhood of \( x \). Conversely, if for every \( V \in \mathcal{B}_Y \) with \( f(x) \in V \), the set \( f^{-1}(V) \) contains a (basic) open neighborhood of \( x \), then for any open neighborhood \( W \) of \( f(x) \), we have that \( W \) contains some such \( V \). Thus, \( f \) is continuous at \( x \). \( \square \)

Here is the fundamental example, which states that a map between metric spaces is continuous with respect to their metric topologies if and only if it is continuous in the usual sense.

**Proposition 2.1.5.** Let \( X \) and \( Y \) be metric spaces, which we endow with their metric topologies. A function \( f : X \to Y \) is continuous if and only if for every \( x \in X \) and \( \varepsilon \in \mathbb{R}_{>0} \), there exists \( \delta \in \mathbb{R}_{>0} \) such that if \( f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \).
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PROOF. The sets $B_X$ and $B_Y$ of open balls of positive radius in $X$ and $Y$, respectively, are bases for the metric topologies on $X$ and $Y$. The result is therefore a direct consequence of the equivalence of (i) and (iii) in Proposition 2.1.4. 

We give a few more examples.

EXAMPLES 2.1.6.

a. If $X$ is discrete, then any map $f : X \to Y$ is continuous.

b. If $Y$ has the trivial topology, then any map $f : X \to Y$ is continuous.

c. If the set $X$ has two topologies $T_X$ and $T'_X$ with $T'_X$ finer than $T_X$ and $f$ is continuous for $T_X$, then $f$ is continuous for $T'_X$.

d. If $Y$ has two topologies $T_Y$ and $T'_Y$ with $T'_Y$ finer than $T_Y$ and $f$ is continuous for $T_Y$, then it is continuous for $T'_Y$.

e. As a special case of the two latter examples, the identity map $id_X : X \to X$ given by $id_X(x) = x$ is continuous with potentially different topologies on the domain and codomain if and only if the topology of the domain is finer than the topology on the codomain.

We can also express continuity in terms of closed sets and closures.

LEMMA 2.1.7. A function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is closed for every closed subset $B$ of $Y$.

PROOF. We have $f^{-1}(B) = X - f^{-1}(B^c)$, so $f^{-1}(B)$ is closed if and only if $f^{-1}(B^c)$ is open. The result follows since $B^c$ runs over the open sets of $Y$ as $B$ runs over the closed sets. 

PROPOSITION 2.1.8. A function $f : X \to Y$ is continuous if and only if for every $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$.

PROOF. If $f$ is continuous and $A' = f^{-1}(f(\overline{A}))$, then $A'$ is closed and $A \subseteq f^{-1}(f(A)) \subseteq A'$, so $\overline{A} \subseteq A'$. Thus $f(\overline{A}) \subseteq f(A') = f(\overline{A})$.

Conversely, suppose that $f(\overline{A}) \subseteq f(A)$ for all $A$. Take $B$ to be a closed subset of $Y$, and set $A = f^{-1}(B)$. Then $f(A) \subseteq B$, and if $x \in \overline{A}$, then $f(x) \in f(\overline{A}) \subseteq B$. In other words, $\overline{A} \subseteq f^{-1}(B) = A$, so $f^{-1}(B)$ is closed. Therefore, $f$ is continuous.

EXAMPLES 2.1.9.

a. Constant functions are continuous.

b. The inclusion map $t_A : A \to X$ of a subspace $A$ of a space $X$ is continuous. That is, if $V$ is open in $X$, then $t_A^{-1}(V) = A \cup V$ is open in $A$ in the subspace topology.

c. A composite of continuous maps is continuous.

d. If $f : X \to Y$ is a continuous function, then its restriction $f|_{A}$ to any subset $A$ of $X$ given by $f|_{A}(a) = f(a)$ for all $a \in A$ is continuous for the subspace topology on $A$, since $f|_{A} = f \circ t_A$. (We say that $f$ restricts to $g = f|_{A}$ on $A$, that $f$ extends $g$ to $X$, and that $f$ is an extension of $g$ from $A$ to $X$.)
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If \( f : X \to Y \) is a continuous function, then the map \( g : X \to f(X) \) is continuous for \( g(x) = f(x) \) for all \( x \in X \), where \( f(X) \) is endowed with the subspace topology from \( Y \). If \( V \subseteq f(X) \) is open, then \( V = U \cap f(X) \) for some open subset \( U \) of \( Y \), so \( g^{-1}(V) = f^{-1}(U) \) is open.

**Definition 2.1.10.** An open cover of a subset \( A \) of a topological space \( X \) is a collection \( \mathcal{U} \) of open subsets of \( X \) that covers \( A \).

**Lemma 2.1.11.** A function \( f : X \to Y \) is continuous if and only if there is an open cover \( \mathcal{U} \) of \( X \) such that \( f|_U \) is continuous for all \( U \in \mathcal{U} \).

**Proof.** We have seen that each \( f|_U \) is continuous if \( f \) is. On the other hand, suppose \( f|_U \) is continuous for all \( U \in \mathcal{U} \). If \( V \subseteq Y \) is an open subset, then \( f^{-1}(V) \cap U = f|_U^{-1}(V) \) is open. Thus
\[
\bigcup_{U \in \mathcal{U}} (f^{-1}(V) \cap U)
\]
is open, so \( f \) is continuous. \( \square \)

**Lemma 2.1.12.** If \( f : X \to Y \) is a function, \( A \) and \( B \) are closed subspaces of \( X \) such that \( A \cup B = X \), and \( f|_A \) and \( f|_B \) are continuous, then \( f \) is continuous.

**Proof.** Let \( C \subseteq Y \) be closed. Then \( f^{-1}(C) = f|_A^{-1}(C) \cup f|_B^{-1}(C) \), and \((f|_A)^{-1}(C)\) is closed in \( A \) and \((f|_B)^{-1}(C)\) is closed in \( B \). Since \( A \) and \( B \) are closed in \( X \), the latter two inverse images are closed in \( X \), so \( f^{-1}(C) \) is closed as a union of two closed sets. \( \square \)

**Definition 2.1.13.** A function \( f : X \to Y \) is a homeomorphism if it is a continuous bijection and its inverse is continuous as well.

**Definition 2.1.14.** Two spaces \( X \) and \( Y \) are homeomorphic if there exists a homeomorphism \( f : X \to Y \).

We leave the following for the reader to verify.

**Lemma 2.1.15.** The relation \( \simeq \) on any set of topological spaces given by \( X \simeq Y \) if \( X \) is homeomorphic to \( Y \) is an equivalence relation.

**Examples 2.1.16.**

a. The function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^3 \) is a homeomorphism, as it is continuous with continuous inverse \( f^{-1}(x) = x^{1/3} \).

b. The function \( f : (0, 1) \to \mathbb{R}_{>0} \) given by \( f(x) = \frac{x}{1-x} \) is a homeomorphism (where \( (0, 1) \) has the subspace topology from \( \mathbb{R} \)).

**Definition 2.1.17.** A function \( f : X \to Y \) is an embedding if it is an injective function that is a homeomorphism onto its image \( f(X) \) with the subspace topology from \( Y \).

**Definition 2.1.18.** A function \( f : X \to Y \) is an open map, or open, if \( f(U) \) is open for every open subset \( U \) of \( X \). A function \( f : X \to Y \) is a closed map, or closed, if \( f(U) \) is closed for every closed subset \( U \) of \( X \).

**Remark 2.1.19.** If \( f : X \to Y \) is a bijection, then \( f \) is open if and only if \( f \) is closed.
By definition, homeomorphisms are open maps; that is, they are the continuous, open bijections. However, in general, continuous bijections may not be open.

**Examples 2.1.20.**

a. Let \( f : S^1 \to \mathbb{R}^2 \) be the natural embedding, where
\[
S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.
\]
Then \( S^1 \) is open itself, but \( f(S^1) = S^1 \) is not open in \( \mathbb{R}^2 \).

b. Let \( X \) be a set with two topologies \( \mathcal{T} \) and \( \mathcal{T}' \) with \( \mathcal{T}' \) strictly finer than \( \mathcal{T} \). Then the identity map \( \text{id}_X : X \to X \) with the domain having the topology \( \mathcal{T}' \) and the codomain having the topology \( \mathcal{T} \). Then \( f \) is continuous, but its inverse \( f^{-1} \) is not, so \( f \) is not an open map. On the other hand, \( f^{-1} \) is an open map.

We next consider the behavior of continuous functions and sequences.

**Proposition 2.1.21.** If \( f : X \to Y \) be a function. If \( f \) is continuous, then for every sequence \( (x_n)_{n \geq 1} \) in \( X \) that converges to a point \( x \in X \), the sequence \( (f(x_n))_{n \geq 1} \) converges to \( f(x) \). If \( X \) is a metrizable space such that for every sequence \( (x_n)_{n \geq 1} \) in \( X \) that converges to some \( x \in X \), then \( (f(x_n))_{n \geq 1} \) converges to \( f(x) \), then \( f \) is continuous.

**Proof.** Suppose that \( f \) is continuous. Let \( (x_n)_{n \geq 1} \) be a sequence in \( X \) converging to \( x \). Let \( V \) be an open neighborhood of \( f(x) \) in \( Y \). Then \( f^{-1}(V) \) is an open neighborhood of \( x \in X \), so contains all \( x_n \) for \( n \geq N \) for some \( N \geq 1 \). Thus \( f(x_n) \in V \) for all such \( n \geq N \), and therefore the sequence of \( f(x_n) \) converges to \( f(x) \).

Now let be \( X \) metrizable, and suppose that whenever \( (x_n)_{n \geq 1} \) converges to \( x \in X \), the sequence \( (f(x_n))_{n \geq 1} \) converges to \( f(x) \). Let \( A \) be a subset of \( X \) and \( x \in \overline{A} \). It suffices by Proposition 2.1.8 to show that \( f(x) \in \overline{f(A)} \). For this, we need only exhibit a sequence \( (x_n)_{n \geq 1} \) in \( A \) such that \( f(x) \) is the limit of the \( f(x_n) \). But such a sequence exists by the metrizability of \( X \) and Proposition 1.6.20.

**Definition 2.1.22.** Let \( (f_n)_{n \geq 1} \) be a sequence of functions from a set \( X \) to a metric space \( Y \) with metric \( d_Y \). The sequence \( (f_n) \) converges uniformly to \( f : X \to Y \) if for every \( \varepsilon > 0 \) there exists an integer \( N \) such that \( d_Y(f(x), f_n(x)) < \varepsilon \) for all \( n \geq N \) and \( x \in X \).

**Proposition 2.1.23.** Let \( X \) be a topological space and \( Y \) be a metric space with metric \( d_Y \). Let \( (f_n)_{n \geq 1} \) be a sequence of functions \( f_n : X \to Y \) that converges uniformly to a function \( f : X \to Y \). Then \( f \) is continuous.

**Proof.** Let \( \varepsilon > 0 \) and \( a \in X \). Choose \( N \geq 1 \) such that \( d_Y(f_n(x), f(x)) < \frac{\varepsilon}{3} \) for all \( n \geq N \) and \( x \in X \). By the continuity of \( f_N \), we can find an open neighborhood \( U \) of \( a \) such that if \( x \in U \), then \( d_Y(f_N(a), f_N(x)) < \frac{\varepsilon}{3} \). For such \( x \in X \), we then have
\[
d_Y(f(a), f(x)) < d_Y(f(a), f_N(a)) + d_Y(f_N(a), f_N(x)) + d_Y(f_N(x), f(x)) < \varepsilon.
\]
2.2. Product spaces

Proposition 2.2.1. Let $I$ be an indexing set, and for each $i \in I$, let $X_i$ be a topological space. Then the set $\mathcal{B}$ of product sets $\prod_{i=1}^{n} U_i$ with each $U_i$ open in $X_i$ and $U_i = X_i$ for all but finitely many $i \in I$ forms a base of a topology on $X = \prod_{i \in I} X_i$.

Proof. Let $U = \prod_{i \in I} U_i$ and $V = \prod_{i \in I} V_i$ be two sets in $\mathcal{B}$. Then

$$\left(\prod_{i \in I} U_i\right) \cap \left(\prod_{i \in I} V_i\right) = \prod_{i \in I} (U_i \cap V_i),$$

each $U_i \cap V_i$ is open in $X_i$, and all but finitely many $U_i \cap V_i$ equal $X_i$. Thus $U \cap V \in \mathcal{B}$. Since $X \in \mathcal{B}$ as well, it follows that $\mathcal{B}$ is a base. \hfill \Box

Definition 2.2.2. Let $I$ be an indexing set, and for each $i \in I$, let $X_i$ be a topological space. The product topology on $X$ is the topology generated by the base of sets $\prod_{i \in I} U_i$ with each $U_i$ open in $X_i$ and all but finitely many $U_i = X_i$.

Remark 2.2.3. The product topology on a finite product $\prod_{i=1}^{n} X_i$ of topological spaces $X_1, \ldots, X_n$ with $n \geq 1$ has a base $\prod_{i=1}^{n} U_i$ of open sets with $U_i$ open in $X_i$ for $1 \leq i \leq n$. I.e., the condition that all but finitely many $U_i$ be $X_i$ is vacuous, since $\{1, \ldots, n\}$ is a finite set.

Definition 2.2.4. Let $I$ be an indexing set, and for each $i \in I$, let $X_i$ be a topological space. The $j$th projection map $\pi_j: \prod_{i \in I} X_i \to X_j$ for $j \in I$ is the function defined by $\pi_j((x_i)_{i \in I}) = x_j$ for $(x_i)_{i \in I} \in \prod_{i \in I} X_i$.

The following proposition explains the seemingly strange choice of the product topology on $X$.

Proposition 2.2.5. Let $I$ be an indexing set, and for each $i \in I$, let $X_i$ be a topological space. The product topology on $X = \prod_{i \in I} X_i$ is the coarsest topology on $X$ such that each projection map $\pi_j: X \to X_j$ with $j \in I$ is continuous.

Proof. Let $i_1, \ldots, i_n \in I$ be elements for some $n \geq 1$, and for $1 \leq k \leq n$, let $U_{i_k} \subset X_{i_k}$ be an open subset. Then

$$\bigcap_{k=1}^{n} \pi_k^{-1}(U_{i_k}) = \prod_{i \in I} V_i$$

with $V_i = X_i$ unless $i \in \{i_1, \ldots, i_n\}$ and $V_i$ open in $X_i$ if so. The collection of sets $\pi_j^{-1}(U_j)$ with $j \in I$ and $U_j \subset X_j$ therefore forms a subbase for the product topology on $X$. In other words, the product topology is the coarsest topology such that these sets are open in $X$, which is to say such that every $\pi_j$ is continuous. \hfill \Box

Proposition 2.2.6. Let $X$ be a topological space, let $Y_i$ be topological spaces for each $i$ in an indexing set $I$, and endow $Y = \prod_{i \in I} Y_i$ with the product topology. A function $f = (f_i)_{i \in I}: X \to Y$ is continuous if and only if each $f_i: X \to Y_i$ is continuous.

Proof. If $f$ is continuous, then each $f_i = \pi_i \circ f$ is continuous, where $\pi_i: Y \to Y_i$ is the projection map in the $i$-coordinate. Conversely, suppose that each $f_i$ is continuous. For $J$ be a
finite subset of $I$, and for each $j \in J$, let $V_j$ be an open subset of $Y_j$. Let $V = \prod_{i \in I} V_i$ where we set $V_i = X_i$ for $i \in I - J$. Then

$$f^{-1}(V) = \bigcap_{i \in I} f_i^{-1}(V_i) = \bigcap_{j \in J} f_j^{-1}(V_j),$$

which is open as a finite intersection of open sets. Therefore $f$ is continuous. \hfill \Box

PROPOSITION 2.2.7. Let $I$ be an indexing set, and let $X_i$ be a Hausdorff topological space for each $i \in I$. Then $X = \prod_{i \in I} X_i$ is Hausdorff in the product topology.

PROOF. Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be distinct elements of $X$, and let $j \in I$ be such that $x_j \neq y_j$. Let $U_j$ and $V_j$ be disjoint open neighborhoods of $x_j$ and $y_j$. Then $U = \pi_j^{-1}(U_j)$ and $V = \pi_j^{-1}(V_j)$ are open in $X$ in the product topology, and $U \cap V = \emptyset$. \hfill \Box

REMARK 2.2.8. From our definition of the product topology on a direct product $X = \prod_{i \in I} X_i$ of topological spaces, it is not hard to see that if $J$ is any finite subset of $I$ and $U$ is any open subset of $\prod_{j \in J} X_j$ in the product topology, then $U \times \prod_{i \in I - J} X_i$ is open in $X$. However, not every open subset need have this form. For instance, consider $\prod_{n=0}^\infty \mathbb{R}$, in which we consider the open subsets $U_i = \{(x_n)_{n \geq 0} \mid x_0, x_i \in (0, 1)\}$ for $i \geq 1$. Then the union $U = \bigcup_{i=1}^\infty U_i$ is open in $X$ but is not a product of the stated form.

REMARK 2.2.9. Just to speak of an element of an arbitrary product $A = \prod_{i \in I} A_i$ for an uncountable set $I$ and nonempty sets $A_i$, we run into the issue of needing to make uncountably many choices. If we cannot, we may not be able to say that that $A$ contains a single element! That one can in fact do this is equivalent to an axiom of set theory, called the axiom of choice. It states that, given a collection of disjoint sets $\{A_i \mid i \in I\}$ for some indexing set $I$, there exists a function $f : I \to \prod_{i \in I} A_i$ to the disjoint union of the sets $A_i$ such that $f(i) \in A_i$ for each $i \in I$. In other words, we can pick one element from each set $A_i$.

There is a more simply described topology on a product that agrees with the product topology for finite products but is rather finer for infinite products. It might constitute a first guess at a natural topology on the product.

DEFINITION 2.2.10. Let $X_i$ be a topological space for each $i$ in an indexing set $I$, and set $X = \prod_{i \in I} X_i$. The box topology on $X$ is the topology generated by the base consisting of products $\prod_{i \in I} U_i$ of open sets $U_i \subseteq X_i$ for each $i \in I$.

REMARK 2.2.11. Suppose that $X$ is an infinite product of spaces that do not have the trivial topology. Then the box topology on $X$ is strictly finer than the product topology.

PROPOSITION 2.2.12. Let $I$ be an indexing set. Let $X_i$ be a topological space and let $A_i$ be a subset of $X_i$ for each $i \in I$. The closure of $\prod_{i \in I} A_i$ in the box or product topology on $\prod_{i \in I} X_i$ is $\prod_{i \in I} \overline{A_i}$.

PROOF. If $a = (a_i)_{i \in I} \in \prod_{i \in I} X_i$ and $U = \prod_{i \in I} U_i$ is an open neighborhood of $a$ in the product (i.e., all but finitely many $U_i = X_i$) or box topologies, then $U \cap A = \prod_{i \in I} (U_i \cap A_i)$ is nonempty if and only if each $U_i \cap A_i$ is nonempty. So, if $a \in \overline{A_i}$ for all $i \in I$, then $a$ lies in the closure of
\[ \prod_{i \in I} A_i. \] Conversely, if \( a \) lies in the latter closure for all possible choices of \( U \), then \( a \in \overline{A_i} \) for each \( i \in I \).

The following tells us when a product of maps is continuous.

**Proposition 2.2.13.** For each \( i \) in an indexing set \( I \), let \( f_i : X_i \to Y_i \) be a function between topological spaces. Set \( X = \prod_{i \in I} X_i \) and \( Y = \prod_{i \in I} Y_i \) and \( f = (f_i)_{i \in I} : X \to Y \). Then \( f \) is continuous with respect to the product (resp., box) topology on \( X \) and \( Y \) if and only if each \( f_i \) is continuous.

**Proof.** Let \( J \) be a finite subset of \( I \), and let \( V_j \) be open in \( Y_j \) for each \( j \in J \). Let \( V = \prod_{i \in I} V_i \) where \( V_i = X_i \) for \( i \in I - J \). Then \( f^{-1}(V) = \prod_{i \in I} f_i^{-1}(V_i) \) is open if and only if \( f_j^{-1}(V_j) \) is open for every \( j \in J \). Thus in the product topology, \( f \) is continuous if and only if each \( f_j \) is continuous. In the box topology, we may simply replace \( J \) by \( I \) in the above argument.

**Example 2.2.14.** Consider \( \mathbb{R}^I = \prod_{i \in I} \mathbb{R} \). The metric topology for the uniform metric given by

\[
d(x, y) = \sup\{\min\{|x_i - y_i|, 1\} \mid i \in I\},
\]

is called the uniform topology. The reason for the name is as follows: the space of functions \( f : I \to \mathbb{R} \) from a set \( I \) to \( \mathbb{R} \) is in bijection with \( \mathbb{R}^I \) via the map which takes a function \( f \) to \( (f(i))_{i \in I} \).

A sequence of functions \( (f_n)_{n \geq 1} \) with \( f_n : I \to \mathbb{R} \) converges uniformly to some \( f : I \to \mathbb{R} \) if and only if the corresponding sequence in \( \mathbb{R}^I \) converges to \( (f(i))_{i \in I} \) in the uniform topology.

**Example 2.2.15.** If \( I \) is finite, the product, uniform, and box topologies on \( \mathbb{R}^I \) are all simply the Euclidean topology. If \( I \) is infinite, then the box topology is strictly finer than the uniform topology, which is strictly finer than the product topology, as we next explain.

The product topology on \( \mathbb{R}^I \) has a basis of open neighborhoods of 0 consisting of a product \( P_\varepsilon \) of open intervals \((-\varepsilon, \varepsilon)\) in finitely many coordinates and \( \mathbb{R} \) in the others. The uniform topology has a basis of open balls \( B_\varepsilon = B(0, \varepsilon) \) of radius \( \varepsilon \) < 1. We have \( B_\varepsilon \subset P_\varepsilon \), but \( P_\varepsilon \) contains no \( B_\varepsilon \). The box topology has a basis of open neighborhoods consisting of products of open intervals centered at 0 of lengths depending on the coordinates. Inside \( B_\varepsilon \), we have the product of open intervals \((-\varepsilon, \varepsilon)\) in every coordinate, which is open in the box topology. On the other hand, a product of open intervals centered at 0 the infimum of the lengths of which is 0 contains no \( B_\varepsilon \).

**Proposition 2.2.16.** Let \( J \) be an indexing set. If \( J \) is infinite, then \( \mathbb{R}^J \) with the box topology is not metrizable. The space \( \mathbb{R}^J \) with the product topology is metrizable if and only if \( J \) is countable.

**Proof.** For the first statement, it suffices to consider the case that \( J \) is countable, which we can then take to be the set of positive integers. Consider the subset \( U = \mathbb{R}_{>0}^J \) of \( \mathbb{R}^J \) in the box topology. The 0 \( \in \overline{U} \), as the reader can check. However, if \( (a_n)_{n \geq 1} \) is a sequence in \( U \) and \( a_{m, n} \) is its \( m \)th coordinate, then the product \( \prod_{n=1}^\infty (-a_{n, n}, a_{n, n}) \) is an open neighborhood of 0 that does not contain any \( a_n \). By Proposition 1.6.21, the space \( \mathbb{R}^J \) with the box topology is not metrizable.

Now, take the product topology on \( \mathbb{R}^J \) with \( J \) uncountable, and consider the set

\[
A = \{(x_j)_{j \in J} \mid x_j = 1 \text{ for all but finitely many } j \in J\}.
\]
Then $0 \in \overline{A}$, since any product of open intervals $(-\varepsilon, \varepsilon)$ in finitely many coordinates and $\mathbb{R}$ in the others clearly intersects $A$. On other other hand, if $(a_n)_{n \geq 1}$ is a sequence in $A$, let $I$ be the subset of $J$ consisting of those $j \in J$ such that the $j$-coordinate of some $a_n$ is not 1. This is a countable set, so $J \neq I$. But then there exists $j \in J$ such that the $j$-coordinate of $a_n$ is 1 for all $n \geq 1$, which means in particular the neighborhood of 0 that is $\mathbb{R}$ in all coordinates but the $j$-coordinate and $(-\frac{1}{2}, \frac{1}{2})$ in the $j$-coordinate does not contain any $a_n$.

For the product topology on $\mathbb{R}^J$ with $J = \mathbb{Z}_{\geq 1}$, the reader may check that we have a metric $d$ defined by

$$d(x,y) = \sup \left\{ \frac{1}{n} \min \{|x_n - y_n|, 1\} \mid n \geq 1 \right\}.$$  

\[ \square \]

### 2.3. Quotient spaces

Recall that we say that a function $f : X \to Y$ is surjective (or a surjection) if $f(X) = Y$.

**Definition 2.3.1.** A quotient map $\pi : X \to Y$ of topological spaces $X$ and $Y$ is a surjection such that a subset $V$ of $Y$ is open if and only if $\pi^{-1}(V)$ is open in $X$.

**Remark 2.3.2.** A surjective map $\pi : X \to Y$ is a quotient map if and only if a subset $B$ of $Y$ is closed if and only if $\pi^{-1}(B)$ is closed in $X$.

**Examples 2.3.3.** Let $X$ be a topological space.

a. The identity map $\text{id}_X : X \to X$ is a quotient map.

b. Any constant map from $X$ is a quotient map onto its singleton image.

c. Let $X = \prod_{i \in I} X_i$ be a product of topological spaces. Then the projection maps $\pi_i : X \to X_i$ are quotient maps, as the reader can check.

**Example 2.3.4.** Let $S^1$ denote the unit open circle in $\mathbb{C}$. Consider the map $f : \mathbb{R} \to S^1$ given by $f(x) = e^{2\pi i x}$. Then $f$ is a quotient map.

The product map $F : \mathbb{R}^2 \to S^1 \times S^1$ given by $F(x,y) = (f(x), f(y))$ is a quotient map as well, realizing a torus (which has the shape of the surface of a donut) as a quotient of the plane. In fact, if we restrict this function to the square $X = [0,1]^2$, the resulting map $F : X \to S^1 \times S^1$ is also a quotient map. It satisfies $F(0,y) = F(1,y)$ and $F(x,0) = F(x,1)$ for all $x, y \in [0,1]$. In particular,

$$F(0,0) = F(1,0) = F(0,1) = F(1,1) = (1,1).$$

One may think of the quotient map as identifying the left and right sides of the square with each other and the top and bottom sides of the square with each other, which in the process identifies the four corners with each other.

The following example illustrates that quotient maps need not be open maps.

**Example 2.3.5.** Let $X$ be the subspace of $\mathbb{R}^2$ consisting of points with $y$-coordinate 0 or 1. A subset of $X$ is open if and only if its intersection with each of the two lines in $X$ is open in the Euclidean topology.
Define \( f: X \to \mathbb{R} \) by
\[
f(x,y) = \begin{cases} 
x & \text{if } y = 0 \\
|x| & \text{if } y = 1.
\end{cases}
\]
Then \( f \) is a quotient map. Let \( U \) be the open subset of \( X \) consisting of those \((x,y)\) with \( y = 1 \), or with \( y = 0 \) and \( x = 0 \). Then \( f(U) = [0,\infty) \), so \( f \) is not open.

Now consider the restriction \( f|_U: U \to \mathbb{R} \). It is a continuous surjection. However, \( f|_U^{-1}([0,\infty)) = \{(x,1) \mid x \in \mathbb{R}\} \), but \([0,\infty)\) is not open, so \( f|_U \) is not a quotient map.

**Lemma 2.3.6.** Let \( X \) be a topological space, and let \( \pi: X \to Y \) be a surjective function onto a set \( Y \). There exists a unique topology on \( Y \) such that \( \pi \) is a quotient map.

**Proof.** The set of subsets of \( Y \) such that \( \pi^{-1}(V) \) is open is easily seen to be a topology on \( Y \), since \( \pi^{-1} \) commutes with the taking of intersections and unions. Moreover, this is the only topology such that \( \pi \) is a quotient map, since \( \pi \) being a quotient map means exactly that the \( V \subseteq Y \) such that \( \pi^{-1}(V) \) is open are the open sets.

Given Lemma 2.3.6, we may make the following definition.

**Definition 2.3.7.** Let \( X \) be a topological space and \( \pi: X \to Y \) a surjective function to a set \( Y \). The *quotient topology* on \( Y \) is the unique topology on \( Y \) such that \( \pi \) is a quotient map.

**Definition 2.3.8.** If \( \pi: X \to Y \) is a quotient map between topological spaces, then \( Y \) is said to be a *quotient space* of \( X \).

**Example 2.3.9.** Let \( Y = \{a,b\} \) be the two-point set. Consider the function \( f: \mathbb{R} \to Y \) given by \( f(x) = a \) if \( x < 0 \) and \( f(x) = b \) if \( x \geq 0 \). The quotient topology on \( Y \) contains the sets \( \emptyset, \{a\} \) and \( \{a,b\} \) but not \( \{b\} \), as \( f^{-1}(b) \) is not open. Then \( Y \) is not Hausdorff, though \( \mathbb{R} \) is.

**Proposition 2.3.10.** Let \( \pi: X \to Y \) be a quotient map. Let \( B \) be a subset of \( Y \) and \( A = \pi^{-1}(B) \). Consider the map \( p: A \to \pi(A) \) given by \( p(a) = \pi(a) \) for \( a \in A \).

- **a.** If \( A \) is open or closed in \( X \), then \( p \) is a quotient map.
- **b.** If \( \pi \) is open or closed, then \( p \) is a quotient map.

**Proof.** By definition, \( p \) is continuous and surjective. If \( V \) is a subset of \( \pi(A) \), then \( p^{-1}(V) = \pi^{-1}(V) \) is open (resp., closed) in \( X \) if and only if \( V \) is open (resp., closed) in \( X \). If \( A \) is open (resp., closed) in \( X \), then \( p^{-1}(V) \) is open (resp., closed) in \( X \) if and only if \( p^{-1}(V) \) is closed in \( A \), and therefore, \( p \) is a quotient map. If \( \pi \) is open (resp., closed) and \( p^{-1}(V) \) is open (resp., closed) in \( A \), then \( V = \pi(p^{-1}(V)) \) is open (resp., closed) in \( \pi(A) \), so \( p \) is a quotient map.

The quotient map has a certain universality property, as expressed in the following theorem.

**Theorem 2.3.11.** Let \( \pi: X \to Y \) be a quotient map, and let \( f: X \to Z \) be any continuous map such that \( f \) is constant on \( \pi^{-1}\{y\} \) for all \( y \in Y \). Then there exists a unique function \( g: Y \to Z \) such that \( f = g \circ \pi \), and \( g \) is continuous.

**Proof.** The function \( g \) is determined by \( g(y) = f(x) \) where \( \pi(x) = y \), which is well-defined by the constancy of \( f \) on the nonempty set \( \pi^{-1}\{y\} \). Let \( W \) be open in \( Z \), and note that
\( \pi^{-1}(g^{-1}(W)) = f^{-1}(W) \). The set \( f^{-1}(W) \) is open in \( X \), so as \( \pi \) is a quotient map, \( g^{-1}(W) \) is open in \( Y \). Thus, \( g \) is continuous. \( \square \)

**Example 2.3.12.** Let \( X \) be a topological space, and let \( A \) be a subset. Let \( X/A = (X - A) \sqcup \{\ast\} \) as sets, and define \( \pi: X \to X/A \) by

\[
\pi(x) = \begin{cases} 
  x & \text{if } x \in X - A \\
  \ast & \text{if } a \in A.
\end{cases}
\]

We give \( Y \) the quotient topology. This is the space given by collapsing \( A \) to a point. It may not be Hausdorff even if \( X \) is: for instance, if we take \( X = \mathbb{R} \) and \( A = (0, \infty) \), then 0 is in every open neighborhood of * in \( Y \). However, \( Y \) will be Hausdorff if for every \( x \notin A \), there exist disjoint open sets \( U \) containing \( x \) and \( V \) containing \( A \).

**Example 2.3.13.** Let \( X \) be a topological space. The cone \( C(X) \) on \( X \) is the quotient space of (the cylinder) \( X \times [0, 1] \) in which we collapse \( X \times \{0\} \) to a point. If \( X = S^1 \), this is homeomorphic to a cone in \( \mathbb{R}^2 \), e.g.,

\[
D = \{(rx, ry, r) \mid r \in [0, 1], (x, y) \in S^1 \},
\]

where \( S^1 \) is the unit circle in \( \mathbb{R}^2 \). Here, the homeomorphism \( D \to C(S^1) \) takes \((rx, ry, r)\) to the image of \(((x, y), r) \in X \times [0, 1] \) in \( C(S^1) \).

### 2.4. Disjoint unions

To give maps to a direct product of sets is to give a collection of maps to the individual sets, and if these sets are topological spaces, then with the product topology, we have seen in Proposition 2.2.13 that to give a continuous map to the product of topological spaces is to give a collection of continuous maps to the individual spaces. However, to give a function from a product of sets to a set is not the same as given a collection of maps from the individual set to the set. One might ask if there’s another set that does this, and the answer is yes, the disjoint union. In fact, when the individual sets are topological spaces, we can put a topology on the disjoint union so that we can make the same connection with continuous maps.

**Definition 2.4.1.** The *disjoint union* of a collection \( \{A_i \mid i \in I\} \) of sets is the set \( \bigsqcup_{i \in I} A_i \) that contains the \( A_i \) as mutually disjoint subsets and is equal to the union \( \bigcup_{i \in I} A_i \).

**Remark 2.4.2.** To give a function \( f: \bigsqcup_{i \in I} A_i \to B \) from a disjoint union of sets \( A_i \) to a set \( B \) is exactly to give a collection of maps \( f_i: A_i \to B \). These satisfy \( f_i(a_i) = f(a_i) \) for all \( i \in I \), so \( f \) determines the \( f_i \) and conversely.

**Definition 2.4.3.** Let \( \{X_i \mid i \in I\} \) be a collection of topological spaces. The *disjoint union* \( \bigsqcup_{i \in I} X_i \) of the spaces \( X_i \) is the topological space with underlying set indicated disjoint union and with the topology under which a subset \( U \) is open if and only if \( U \cap X_i \) is open for all \( i \in I \).

**Remark 2.4.4.** Under this definition, we have continuous inclusion maps \( \iota_i: X_i \hookrightarrow \bigsqcup_{i \in I} X_i \) given by \( \iota_i(a_i) = a_i \) for \( a_i \in X_i \).
PROPPOSITION 2.4.5. Let \( f_i : X_i \to Y \) be a collection of functions from topological spaces \( X_i \) to a topological space \( Y \). The map \( f : \bigsqcup_{i \in I} X_i \to Y \) that restricts to \( f_i \) on \( X_i \) for all \( i \) is continuous if and only if every \( f_i \) is continuous.

PROOF. Let \( V \) be an open subset of \( Y \). Then \( f^{-1}(V) \cap X_i = f_i^{-1}(V) \), so \( f^{-1}(V) \) is open if and only if \( f_i^{-1}(V) \) is open for all \( i \). \( \square \)

LEMMA 2.4.6. Let \( X = \bigsqcup_{i \in I} X_i \) be a disjoint union of topological spaces \( X_i \). Then each \( X_i \) is open and closed in \( X \).

PROOF. We have \( X_i \cap X_j = \emptyset \) if \( i \neq j \) and \( X_i \cap X_i = X_i \), so \( X_i \) is open in \( X \). As for its complement, we have
\[
X - X_i = \bigcup_{j \in I - \{i\}} X_j,
\]
which is open as a union of open sets. \( \square \)

EXAMPLE 2.4.7. Any union \( A \) of the two parallel lines in the plane \( \mathbb{R}^2 \) is homeomorphic to \( \mathbb{R} \sqcup \mathbb{R} \).

This example generalizes considerably.

LEMMA 2.4.8. Let \( \{A_i \mid i \in I\} \) be a collection of subspaces of a space \( X \). Then the continuous map \( \bigsqcup_{i \in I} A_i \to X \) with restriction to \( A_i \) the embedding of \( A_i \) as a subspace of \( X \) is a homeomorphism if the \( A_i \) are mutually disjoint with union \( X \) and every \( A_i \) is open (and therefore closed) in \( X \).

PROOF. The continuous map \( f \) in the statement is onto if and only if \( \bigcup_{i \in I} A_i = X \) and is one-to-one if and only if \( A_i \cap A_j = \emptyset \) for all \( i \neq j \). If \( f \) is open, then every \( X_i \) is open in \( X \) (and then closed as well, since \( A_i \) is the complement of the union of the \( A_j \) for \( j \neq i \)). Conversely, if every \( A_i \) is both open and closed in \( X \) and \( U \) is open in \( \bigsqcup_{i \in I} A_i \), then \( f(U) = \bigcup_{i \in I}(U \cap A_i) \) is open as a union of open sets. \( \square \)

Lemma 2.4.8 justifies the following terminology.

TERMINOLOGY 2.4.9. If \( X \) is a topological space and \( \{A_i \mid i \in I\} \) is a collection of disjoint open subsets with union \( X \), then we say that \( X \) is the disjoint union of the subspaces \( A_i \) and write \( X = \bigsqcup_{i \in I} A_i \).

EXAMPLES 2.4.10.

a. The subspace \( X = \{(x,y) \in \mathbb{R}^2 \mid y \in \{0,1\}\} \) of \( \mathbb{R}^2 \) is the disjoint union of the lines \( y = 0 \) and \( y = 1 \) in the plane.

b. Any discrete spaces is the disjoint union of its singleton subsets.

EXAMPLE 2.4.11. Let \( \{X_i \mid i \in I\} \) be a collection of topological spaces, and choose \( x_i \in X_i \) for each \( i \in I \). We have a quotient of \( X = \bigsqcup_{i \in I} X_i \) given by collapsing the subset \( A = \{x_i \mid i \in I\} \) to a point. This will be a Hausdorff space if \( X \) is. This space is called a one-point union of the spaces \( X_i \). In the case, for instance, that \( X = S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1 \), we get a finite collection of circles joined at a point.
Chapter 3

Topological properties

3.1. Connectedness

Definition 3.1.1. A topological space $X$ is connected if it cannot be written as a disjoint union of two nonempty open sets. Otherwise, $X$ is said to be disconnected.

Lemma 3.1.2. A topological space $X$ is connected if its only subsets that are both open and closed are $\emptyset$ and $X$. 

Proof. If $A$ is a subset of $X$ that is both open and closed, then so is $A^c$, and then $X = A \sqcup A^c$. Conversely, if $X = U \sqcup V$ with $U$ and $V$ nonempty and open in $X$, then $U$ and $V$ are also closed. \hfill $\square$

Example 3.1.3. Topological spaces with one element are always connected, but discrete topological spaces with more than one element are disconnected.

Example 3.1.4. The union $(-\infty, 0) \cap (0, \infty)$ of intervals in $\mathbb{R}$ is disconnected as a subspace of $\mathbb{R}$, as both $(-\infty, 0)$ and $(0, \infty)$ are open and closed in the subspace topology. However, $\mathbb{R}$ itself is connected.

Remark 3.1.5. If $X$ is the disjoint union of subspaces $A$ and $B$, then $A$ and $B$ are their own closures, so $A$ and $B$ contain no limit points of each other. In fact, if $A$ and $B$ are any two disjoint subsets of $X$ with union $X$ such that $A$ contains no limit points of $B$ and vice-versa, then $A$ and $B$ are open and closed in $X$.

Lemma 3.1.6. If $X = U \sqcup V$ for subspaces $U$ and $V$, and $A$ is a connected subset of $X$, then either $A \subseteq U$ or $A \subseteq V$.

Proof. We have that $A \cap U$ and $A \cap V$ are open and closed in $A$, so $A$ is the disjoint union of these intersections. If $A$ is connected, this forces one of the $A \cap U$ and $A \cap V$ to be empty, and then the other is $A$. \hfill $\square$

Proposition 3.1.7. The closure of a connected subset of a topological space is connected.

Proof. Let $X$ be a topological space and $A$ a connected subset. Suppose that $\overline{A} = U \sqcup V$ for disjoint subspaces $U$ and $V$ in $X$. By Lemma 3.1.6, we have that $A$ is contained in either $U$ or $V$: without loss of generality, we suppose $A \subseteq U$. As $U$ is closed, we have that $\overline{A} \subseteq U$ as well, and therefore $V = \emptyset$. \hfill $\square$

We also have the following statements.

Proposition 3.1.8. The image of a connected space under a continuous map is connected.
PROOF. Suppose $X$ is connected and $f : X \to Y$ is continuous. If $Y$ is the disjoint union of (open) subspaces $U$ and $V$, then $X$ is the union of the disjoint open subsets $f^{-1}(U)$ and $f^{-1}(V)$, hence equal to their disjoint union as topological spaces.

**Proposition 3.1.9.** Let $X = \bigcup_{i \in I} A_i$ for a collection of connected subsets $A_i$ of $X$ for $i \in I$, and suppose that $\bigcap_{i \in I} A_i$ is nonempty. Then $X$ is connected.

**Proof.** Let $a \in \bigcap_{i \in I} A_i$. If $X = U \amalg V$ for subspaces $U$ and $V$, then without loss of generality we may suppose $a \in U$. Since $A_i$ is connected and contains $a$, we must then have $A_i \subseteq U$ for all $i \in I$, forcing $V = \emptyset$. □

The following is an immediate corollary of Proposition 3.1.9.

**Corollary 3.1.10.** The union of all connected subsets of a topological space $X$ that contain a given point $x \in X$ is connected.

By Corollary 3.1.10, we can always find a largest connected subset containing a given point.

**Definition 3.1.11.** A connected component of a topological space $X$ is a connected subset of $X$ that is not properly contained in any larger connected subset of $X$. The connected component of a point $x \in X$ is the unique connected component of $X$ containing $x$.

**Lemma 3.1.12.** A space $X$ is the union of its distinct connected components, which are closed and disjoint. If every connected component of $X$ is open, then $X$ is the disjoint union of them. In particular, if $X$ has only finitely many connected components, then it $X$ is the disjoint union of its connected components.

**Proof.** We know that $X$ is the union of its connected components, which are disjoint. It follows from Proposition 3.1.7 that connected components are closed, being the largest connected subsets containing a given point. The second statement follows from the definition of a disjoint union of topological spaces. If $X$ has finitely many connected components, then any union of all but one of them is closed as the complement of an open set, and then they are all open as finite intersections of these unions. □

**Example 3.1.13.** Consider the subspace $A = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ of $\mathbb{R}$. Every $\{\frac{1}{n}\}$ is both open and closed in $A$, so is a connected component (in that it is connected). The set $\{0\}$ is also then a connected component, being that it is not contained in any larger connected subset. However, it is closed but not open, so $A$ is not the disjoint union of its connected components.

**Example 3.1.14.** Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. It is disconnected as $\mathbb{Q}$ is the union of its intersections with the intervals $(-\infty, \pi)$ and $(\pi, \infty)$, for instance (as $\pi$ is irrational). In fact, since there exists an irrational number between any two distinct rational numbers, the connected components of $\mathbb{Q}$ are just its singleton subsets.

**Remark 3.1.15.** The property of being in the same connected component gives an equivalence relation on the points of a topological space, and the connected components are the equivalence classes.

**Definition 3.1.16.** A topological space is said to be totally disconnected if its connected components are its singleton subsets.
EXAMPLE 3.1.17. Let $A = \{0, 1\}$ with the discrete topology, and consider $X = \prod_{n=1}^{\infty} A$ with the product topology. Given any two distinct points $x = (x_n)_{n \geq 1}$ and $y = (y_n)_{n \geq 1}$ in $X$, there exists $n \geq 1$ such that $x_n \neq y_n$. Letting $\pi_n$ denote the $n$th projection map, we have that $U = \pi_n^{-1}(x_n)$ and $V = \pi_n^{-1}(x_n)$ are disjoint basic open sets in $X$ with union $X$, so $X = U \amalg V$, and $x$ and $y$ lie in distinct connected components. Thus, $X$ is totally disconnected.

DEFINITION 3.1.18. A path $\gamma$ from a point $x$ to a point $y$ in a topological space $X$ is a continuous function $\gamma: [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The points $x$ and $y$ are called the endpoints of $\gamma$; in particular, $x$ is its initial endpoint and $y$ is its final endpoint.

DEFINITION 3.1.19. Let $X$ be a topological space.

a. We say that two points $x$ and $y$ in a topological space $X$ can be connected by a path if there exists a path $\gamma$ from $x$ to $y$.

b. We say that a topological space $X$ is path connected if every two points $x$ and $y$ can be connected by a path.

PROPOSITION 3.1.20. Every path connected space is connected.

PROOF. Let $X$ be a path connected space. Suppose that $X$ is a disjoint union of nonempty open subspaces $U$ and $V$, and let $x \in X$ and $y \in Y$. Let $\gamma$ be a path from $x$ to $y$, and let $A = \gamma([0, 1])$. Then $A$ is the disjoint union of the nonempty sets $A \cap U$ and $A \cap V$, so $A$ is disconnected, but it is the image of a connected space under a continuous function.

EXAMPLE 3.1.21. Consider the subset $A = \{(x, \sin(\frac{1}{x})) \mid 0 < x \leq 1\}$ of $\mathbb{R}^2$. We have $\overline{A} = A \cup \{(0, y) \mid 0 \leq y \leq 1\}$. Note that $\overline{A}$ is connected as a subspace of $\mathbb{R}^2$ since $A$ is. On the other hand, $\overline{A}$ is not path connected. Suppose $\gamma$ were a path from $a = (0, 0)$ to $b = (1, \sin(1))$. Then $\lim_{t \to 0^+} \gamma(t) = (0, 0)$. On the other hand, for every $\epsilon > 0$, there exists an $x < \epsilon$ such that $\sin(\frac{1}{x}) = 1$. By the continuity of $\gamma$, for every $\delta > 0$, there then exists $t < \delta$ such that the second coordinate of $\gamma(t)$ equals 1. But this contradicts that $\lim_{t \to 0^+} \gamma(t) = (0, 0)$.

3.2. Compactness

DEFINITION 3.2.1. Let $\mathcal{U}$ be a cover of a subset $A$ of a topological space $X$. A subcover of $\mathcal{U}$ is a a subset of $\mathcal{U}$ that covers $A$.

DEFINITION 3.2.2. A topological space $X$ is compact if every open cover of $X$ has a finite subcover.

EXAMPLES 3.2.3.

a. Any finite topological space is compact.

b. Any topological space with the trivial topology is compact.

c. The real line $\mathbb{R}$ is not compact, since the collection of open intervals of length 1 is an open cover with no finite subcover.

d. The interval $(0, 1]$ is not compact, since the connection of intervals $(\epsilon, 1]$ with $\epsilon > 0$ has no finite subcover.
PROPOSITION 3.2.4. Any closed interval in \( \mathbb{R} \) is compact.

PROOF. As all closed intervals of finite length are homeomorphic, we can and will consider the interval \([0, 1]\). Let \( \mathcal{U} \) be an open cover of \([0, 1]\), and let \( A \) be the subset of \([0, 1]\) consisting of those \( x \) such that \([0, x]\) has a finite subcover by elements in \( \mathcal{U} \). Let \( b \) be the supremum of \( A \). If \( b < 1 \), then let \( U \) be an element \( \mathcal{U} \) containing \( b \). Since \( U \) contains an interval \((b - \varepsilon, b + \varepsilon)\) for some \( \varepsilon > 0 \) with \( b + \varepsilon \leq 1 \), we can find a finite subcover \( \mathcal{V} \) of \([0, b - \varepsilon]\) in \( \mathcal{U} \), and then \( \mathcal{V} \cup \{U\} \) is a finite subcover of \([0, b + \varepsilon]\) in \( \mathcal{U} \). But then \( b + \frac{\varepsilon}{2} \in A \), contradicting the definition of \( b \). Thus \( b = 1 \), and therefore \( \mathcal{U} \) has a finite subcover. \( \square \)

Let’s establish a few basic statements regarding compact spaces.

LEMMA 3.2.5. A subspace \( A \) of a topological space \( X \) is compact if and only if every open cover of \( A \) in \( X \) has a finite subcover.

PROOF. Given an open cover \( \mathcal{V} \) of \( A \) by open sets in \( X \), we can find a set \( \mathcal{U} \) of open sets in \( X \) covering \( A \) such that \( \mathcal{V} = \{A \cap U \mid U \in \mathcal{U}\} \). Conversely, given a collection \( \mathcal{U} \) of open sets in \( X \) covering \( A \), we may define a cover \( \mathcal{V} \) of \( A \) by open sets in \( X \) by taking intersections with \( A \) as in the latter formula. Any (finite) subset \( \mathcal{C} \) of \( \mathcal{U} \) covers \( A \) if and only if the (finite) subset \( \{A \cap U \mid A \in \mathcal{C}\} \) of \( \mathcal{V} \) covers \( A \). \( \square \)

PROPOSITION 3.2.6. Every closed subset of a compact space is compact.

PROOF. Let \( X \) be compact, and let \( A \subseteq X \) be closed. Let \( \mathcal{U} \) be an open cover of \( A \) in \( X \). Then \( \mathcal{U} \cup \{A^c\} \) is an open cover of \( X \), so it has a finite subcover \( \mathcal{V} \). If \( A^c \in \mathcal{V} \), then \( \mathcal{V} - \{A^c\} \) is an open cover of \( A \) in \( X \), and otherwise, \( \mathcal{V} \) is an open cover of \( A \) in \( X \). \( \square \)

LEMMA 3.2.7. Let \( A \) be a compact subset of a Hausdorff space, and let \( x \in A^c \). Then there exist disjoint open sets \( U \) and \( V \) with \( A \subseteq U \) and \( x \in V \).

PROOF. For each \( a \in A \), choose open disjoint neighborhoods \( U_a \) of \( a \) and \( V_a \) of \( x \) in \( X \). The collection \( \{U_a \mid a \in A\} \) is an open cover of \( A \), and it has a finite subcover, say by \( U_{a_1}, \ldots, U_{a_n} \). Then \( U = \bigcup_{i=1}^n U_{a_i} \) and \( V = \bigcap_{i=1}^n V_{a_i} \) are the desired open subsets of \( X \). \( \square \)

PROPOSITION 3.2.8. Every compact subset of a Hausdorff space is closed.

PROOF. Let \( X \) be Hausdorff, and let \( A \subseteq X \) be compact. By Lemma 3.2.7, for each \( x \in X \), there exists an open neighborhood \( V_x \) of \( x \) in \( A^c \). The union of the \( V_x \) is \( A^c \), so \( A^c \) is open, and thus \( A \) is closed. \( \square \)

PROPOSITION 3.2.9. Let \( f : X \to Y \) be a continuous map. If \( X \) is compact, then so is the image of \( f \).

PROOF. Let \( \mathcal{V} \) be an open cover of \( f(X) \) in \( Y \). Then \( \mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\} \) is an open cover of \( X \). Since \( X \) is compact, it has a finite subcover \( \{f^{-1}(V_1), \ldots, f^{-1}(V_n)\} \) with each \( V_i \in \mathcal{V} \), and then \( \{V_1, \ldots, V_n\} \) is an open cover of \( f(X) \). \( \square \)

We have seen that continuous bijections need not be homeomorphisms. However, continuous bijections from compact spaces are.
**Theorem 3.2.10.** Let \( f : X \to Y \) be a continuous surjection. If \( X \) is compact and \( Y \) is Hausdorff, then \( f \) is a quotient map.

**Proof.** It suffices to show that \( f \) is a closed map. Let \( A \) be a closed set in \( X \), which is necessarily compact. As \( f \) is continuous, its image \( f(A) \) is compact as well. As \( Y \) is Hausdorff, we then have that \( f(A) \) is closed. \( \square \)

**Corollary 3.2.11.** Every continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

We next prove that a finite product of compact spaces is compact. First, we require the following lemma.

**Lemma 3.2.12.** Let \( X \) and \( Y \) be topological spaces, and suppose that \( X \) is compact. For any \( y \in Y \) and open set \( W \) in \( X \times Y \) containing \( X \times \{ y \} \), there exists an open neighborhood \( V \) of \( y \) in \( Y \) such that \( X \times V \subseteq W \).

**Proof.** For each \( x \in X \), let \( U_x \) be an open neighborhood of \( x \) in \( X \) and \( V_x \) be an open neighborhood of \( y \) in \( Y \) such that \( U_x \times V_x \subseteq W \), which exist since \( (x, y) \in W \) and \( W \) is open. The collection \( \{ U_x \mid x \in X \} \) covers \( X \) so has a finite subcover, say consisting of \( U_{x_1}, \ldots, U_{x_n} \). If we set \( V = \bigcap_{i=1}^n V_{x_i} \), then \( V \) is an open neighborhood of \( y \) in \( Y \), and since each \( U_{x_i} \times V \) is contained in \( W \), so is their union \( X \times V \). \( \square \)

**Theorem 3.2.13.** Let \( X_1, \ldots, X_n \) be compact spaces for some \( n \geq 1 \). Then \( \prod_{i=1}^n X_i \) is compact.

**Proof.** It suffices by recursion to consider the case \( n = 2 \), so the product of compact spaces \( X \) and \( Y \). Let \( \mathcal{W} \) be an open covering of \( X \times Y \). For any \( y \in Y \), the set \( X \times \{ y \} \) is compact, being homeomorphic to \( X \) via the projection map, and so there exist \( Z_1, \ldots, Z_n \in \mathcal{W} \) that together cover \( X \times \{ y \} \). Set \( W_y = \bigcup_{i=1}^n Z_i \). We then have by Lemma 3.2.12 that there exists an open neighborhood \( V_y \) of \( y \) such that \( X \times V_y \subseteq W_y \). The collection \( \{ V_y \mid y \in Y \} \) then covers \( Y \), and it has a finite subcover as \( Y \) is compact. But then \( X \times Y \) is covered by the finitely many \( W_y \), and each of these is in turn a union of finitely many sets in \( \mathcal{W} \). Thus, \( \mathcal{W} \) has a finite subcover. \( \square \)

**Proposition 3.2.14.** A subspace of \( \mathbb{R}^n \) is compact if and only if it is closed and bounded.

**Proof.** Fix a subset \( A \) of \( \mathbb{R}^n \). We may consider the cover of \( A \) by all open balls in \( \mathbb{R}^n \) of radius 1. If \( A \) is unbounded, then it cannot have a finite subcover, since the union of any finite number of balls of radius 1 is bounded. Therefore, only bounded subsets can be compact.

Now suppose that \( A \) is bounded. It is then contained in a direct product of closed intervals, which is compact as a finite product of compact sets. If \( A \) is also closed, then it is compact by Proposition 3.2.6. On the other hand, if \( A \) is not closed, then it cannot be compact by Proposition 3.2.8. \( \square \)

We give another criterion for a topological space to be compact.

**Definition 3.2.15.** A collection \( \mathcal{A} \) of subsets of \( X \) is said to have the finite intersection property if for every \( n \geq 1 \) and \( A_1, \ldots, A_n \in \mathcal{A} \), we have that \( \bigcap_{i=1}^n A_i \) is nonempty.
3. TOPOLOGICAL PROPERTIES

**Theorem 3.2.16.** A topological space $X$ is compact if and only if every collection $\mathcal{A}$ of closed subsets of $X$ having the finite intersection property satisfies $\bigcap_{A \in \mathcal{A}} A$ is nonempty.

**Proof.** Let $\mathcal{A}$ be a collection of closed subsets of $X$ with the finite intersection property. Then $\mathcal{U} = \{A^c \mid A \in \mathcal{A}\}$ is a collection of open subsets of $X$ with the property that no finite subset of $\mathcal{U}$ covers $X$. If $X$ is compact, this forces $\mathcal{U}$ not to cover $X$, which is exactly to say that the intersection of elements of $\mathcal{A}$ is nonempty.

Conversely, if $X$ is not compact, there exists an open cover $\mathcal{U}$ of $X$ which has no finite subcover, which is to say that the collection $\mathcal{A} = \{U^c \mid U \in \mathcal{U}\}$ of closed sets has the finite intersection property but has empty intersection. \hfill $\Box$

We next introduce related notions to compactness.

**Definition 3.2.17.** A space $X$ is said to be limit point compact if every infinite subset of $X$ has a limit point.

**Proposition 3.2.18.** Compact spaces are limit point compact.

**Proof.** Let $X$ be a compact space, and let $A$ be a subset without a limit point. Then $A$ is necessarily closed. Moreover, for each $a \in A$, there exists an open set $U_a$ containing $a$ and no other point of $A$. Then $X$ has an open cover by $\{U_a \mid a \in A\} \cup \{A^c\}$. Since $X$ has a finite subcover, $A$ is contained in a finite union of sets $U_{a_1}, \ldots, U_{a_n}$, which implies that $A = \{a_1, \ldots, a_n\}$, so $A$ is finite. \hfill $\Box$

**Definition 3.2.19.** A space $X$ is said to be sequentially compact if every sequence in $X$ has a convergent subsequence.

The following is fairly immediate from the definitions.

**Proposition 3.2.20.** Sequentially compact spaces are limit point compact.

**Proof.** Let $A$ be an infinite set in $X$, and let $(a_n)_{n \geq 1}$ be a sequence of distinct elements in $A$. It has a convergent subsequence, say with limit $a$. Then every neighborhood of $a$ contains some point of the subsequence not equal to $a$. Thus, $a$ is a limit point of $A$. \hfill $\Box$

To be limit point compact is a stronger notion than being either compact or sequentially compact.

**Example 3.2.21.** Let $Y = \{a, b\}$ the the two-point space with the trivial topology, and consider $X = \mathbb{Z} \times Y$, where $\mathbb{Z}$ has the discrete topology. Then any open neighborhood of a point $(n, a)$ in $X$ contains $(n, b)$ and conversely, so every nonempty set in $X$ has a limit point. In particular, $X$ is limit point compact. However, $X$ is not compact, and it is covered by the disjoint open sets $\{n\} \times Y$ with $n \in \mathbb{Z}$. It is also not sequentially compact, as the sequence of points $(n, a)$ has no convergent subsequence.

In general, neither compactness nor sequential compactness implies the other. However, for metric spaces, all three of these notions of compactness are equivalent, as we shall show.

**Definition 3.2.22.** Let $A$ be a bounded subset of a metric space $X$. The diameter of $A$ is the supremum of the distances between points in $A$. 

3.3. Local properties

Definition 3.2.23. Let \( \mathcal{U} \) be an open cover of a space \( X \). Its Lebesgue number is the smallest \( \varepsilon > 0 \) such that every subset \( A \) of \( X \) of diameter less than \( \varepsilon \) is contained in an element of \( \mathcal{U} \), if such an \( \varepsilon \) exists, and otherwise, we say \( \mathcal{U} \) has infinite Lebesgue number.

Lemma 3.2.24. Let \( X \) be a sequentially compact metric space. Then every open cover of \( X \) has finite Lebesgue number.

Proof. Suppose that some open cover \( \mathcal{U} \) of \( X \) has infinite Lebesgue number. For each \( n \geq 1 \), there exists a subset \( A_n \) of \( X \) of diameter less than \( \frac{1}{n} \) not contained in any element of \( \mathcal{U} \). For each such \( A_n \), choose \( x_n \in A_n \). The sequence \( (x_n)_{n \geq 1} \) has a convergent subsequence \( (x_{n_k})_{k \geq 1} \), say with limit \( x \). Let \( U \in \mathcal{U} \) be an open neighborhood of \( x \), and let \( B(x, \delta) \) be an open ball inside of it. Let \( k \) be sufficiently large such that \( \frac{1}{n_k} < \frac{\delta}{2} \) and \( d(x, x_{n_k}) < \frac{\delta}{2} \). By the triangle inequality, we then have \( A_{n_k} \subseteq B(x, \delta) \subseteq U \), providing the desired contradiction.

Definition 3.2.25. We say that a metric space \( X \) is totally bounded if for every \( \varepsilon > 0 \), there exists a finite cover of \( X \) by open balls of radius \( \varepsilon \).

Lemma 3.2.26. Let \( X \) be a sequentially compact metric space. Then for every \( \varepsilon > 0 \), there exists a finite cover of \( X \) by open balls of radius \( \varepsilon \).

Proof. Let \( X \) be a metric space, and suppose that \( \varepsilon > 0 \) is such that \( X \) cannot be covered by finitely many balls of radius \( \varepsilon \). Choose \( x_1 \in X \) and then recursively choose \( x_n \) in the complement of \( \bigcup_{i=1}^{n-1} B(x_i, \varepsilon) \). The sequence \( (x_n)_{n \geq 1} \) satisfies \( d(x_m, x_n) \geq \varepsilon \) for all \( m \geq n \), and consequently it cannot have a convergent subsequence (since every convergent subsequence is necessarily Cauchy). Thus, \( X \) not sequentially compact.

Theorem 3.2.27. Let \( X \) be a metrizable space. Then the following are equivalent:

i. \( X \) is compact,

ii. \( X \) is limit point compact,

iii. \( X \) is sequentially compact.

Proof. Let \( X \) be a limit point compact space. Let \( (x_n)_{n \geq 1} \) be a sequence in \( X \). If the set \( A = \{x_n \mid n \geq 1\} \) of values of the sequence is finite, then \( (x_n)_{n \geq 1} \) has a constant, hence convergent, subsequence. Otherwise, \( A \) is infinite, so has a limit point \( x \in X \). Inductively, we have that every ball among the \( B(x, \frac{1}{k}) \) for \( k \geq 1 \) contains some \( x_n \) with \( n \geq n_{k-1} \) if \( k \geq 2 \). The sequence \( (x_{n_k})_{k \geq 1} \) then converges to \( x \). Thus, \( X \) is sequentially compact.

Now suppose that \( X \) is sequentially compact, and let \( \mathcal{U} \) be an open cover of \( X \). Let \( \varepsilon > 0 \) be such that every subset of diameter less than \( \varepsilon \) is contained in an element of \( \mathcal{U} \). Choose a finite open cover of \( X \) by open balls of radius \( \frac{\varepsilon}{4} \), and note that they have diameter at most \( \frac{\varepsilon}{2} \), hence are each contained in some element of \( \mathcal{U} \). Then \( \mathcal{U} \) has a finite subcover by these elements, and therefore \( X \) is compact.

3.3. Local properties

Local properties of a space are those which happen within a small enough neighborhood of a point. Here are the definitions of local connectedness and local compactness.
**Definition 3.3.1.** A topological space $X$ is called *locally connected at* $x \in X$ if every neighborhood of $x$ contains a connected open neighborhood of $x$. A topological space is called *locally connected* if it is locally connected at each of its points.

**Proposition 3.3.2.** A topological space $X$ is locally connected if and only if every connected component of every open set in $X$ is open in $X$.

**Proof.** Suppose $X$ is locally connected. Let $U$ be open in $X$, and let $A$ be a connected component of $U$. If $x \in A$, then there is a connected open neighborhood $V$ of $x$ that is contained in $U$, and $V$ is contained in $A$ by its connectedness. Since $x$ was arbitrary, $A$ is open.

Conversely, suppose that all connected components of open sets in $X$ are open in $X$. Let $x \in X$, and let $V$ be an open neighborhood of $x$. Let $A$ be the connected component of $x$ in $V$. Note that $A$ is open by assumption, so it is a neighborhood of $x$. Thus, $X$ is locally connected. □

**Example 3.3.3.** Let $X = \prod_{n=1}^{\infty} \{0, 1\}$ with the product topology, as in Example 3.1.17. Its basic open sets have the form $\prod_{n=1}^{N} U_i \times \prod_{m=N+1}^{\infty} A$, which are in particular infinite. As its connected components of $X$ are singletons, $X$ is not locally connected.

**Definition 3.3.4.** A topological space $X$ is said to be *locally compact at* $x \in X$ if $x$ has a compact neighborhood. A topological space is *locally compact* if it is locally compact at all of its points.

Compact spaces are of course locally compact, as are many other spaces.

**Example 3.3.5.** The space $\mathbb{R}^n$ is locally compact, since every closed ball of positive radius about a point is compact. However, $\prod_{n=1}^{\infty} \mathbb{R}$ is not locally compact, since its basic open sets all have closure that is a product of finitely many closed intervals with infinitely many copies of $\mathbb{R}$, and these are not compact as $\mathbb{R}$ is not.

**Definition 3.3.6.** A *compactification* of a topological space $X$ is a pair $(Y, \iota)$ consisting of a compact topological space $Y$ and an embedding $\iota$ of $X$ in $Y$.

**Remark 3.3.7.** We may view $X$ as a subspace of a compactification $Y$ via identifying it with its homeomorphic image. Two compactifications $Y$ and $Z$ of a space $X$ are then said to be equivalent if there exists a homeomorphism $f: Y \to Z$ that restricts to the identity map on $X$. This gives an equivalence relation on the compactifications of $X$.

Topological spaces can be compactified by adding in a single point.

**Theorem 3.3.8.** Let $X$ be a topological space. The set $Y$ containing $X$ and one additional element called $\infty$ has a topology consisting of the open sets in $X$ and the complements $Y - A$ of closed, compact subsets $A$ of $X$. Moreover, $Y$ is a compact space under this topology, and any other compactification $Z$ of $X$ with $Z - X$ a singleton set is equivalent to $Y$.

**Proof.** Note that arbitrary intersections of closed, compact subsets $A$ of $X$ are closed, and then compact, and the union of an open subset $U$ of $X$ and the complement $Y - A$ of a closed, compact subset $A$ of $X$ is the complement in $Y$ of a smaller closed, and then compact, subset of $X$. It follows that the collection $\mathcal{T}$ of open sets defined in the theorem is closed under arbitrary unions. Similarly, finite unions of closed, compact subsets of $X$ are compact, and the intersection
of \( U \) and \( Y - A \) as above is an open subset of \( X \). Hence, \( \mathcal{T} \) is closed under arbitrary intersections, so \( \mathcal{T} \) is a topology.

Given an open cover \( \mathcal{V} \) of \( Y \), we may choose an element in it of the form \( Y - A \) with \( A \) closed, compact in \( X \), as such a set is needed to cover \( \infty \). Then \( \mathcal{V} - \{ Y - A \} \) is an open cover of \( A \), which has a finite subcover as \( A \) is compact, so \( \mathcal{V} \) has a finite subcover as well. Thus, \( Y \) is compact.

Finally, for any other space \( Y' \) as in the theorem, with \( \infty' \) its additional point, is in canonical bijection with \( Y \) via the map \( f: Y \to Y' \) such that \( f(x) = x \) for \( x \in X \) and \( f(\infty) = \infty' \). If \( V \) is open in \( Y \) and \( \infty' \notin Y' \), then \( f(V) = V \) is open in \( X \), hence in \( Y' \). If \( V \) is an open neighborhood of \( \infty \) in \( Y \), then its complement \( A \) is closed, hence compact in \( Y \), and of course also contained in \( X \). Then \( f(V) = Y' - f(A) \) is the complement of a compact, closed subset of \( X \), so is open in \( Y' \) as well. Thus, \( f \) is continuous, and then \( f^{-1} \) is continuous too, as we did not distinguish \( Y \) and \( Y' \). \( \square \)

**Definition 3.3.9.** For a space \( X \), the compact space of Theorem 3.3.8 is called the one-point compactification of \( X \).

**Proposition 3.3.10.** A topological space \( X \) is locally compact and Hausdorff if and only if its one-point compactification is Hausdorff.

**Proof.** Let \( Y \) be the one-point compactification of \( X \). Suppose that \( X \) is locally compact and Hausdorff. To show that \( Y \) is Hausdorff, it suffices to consider some \( x \in X \) and \( \infty \). Since \( X \) is locally compact, we can find a compact neighborhood \( A \) of \( x \), which then contains some open neighborhood \( U \). Then \( U \) is disjoint from the open neighborhood \( Y - A \) of \( \infty \).

If \( X \) is not Hausdorff, then there exist points \( u, v \in X \) not contained in disjoint open sets in \( X \). No two complements of compact sets in \( X \) are disjoint, since they contain the added point \( \infty \). Moreover, if \( u \notin A \) and \( v \in V \) for a compact subset \( A \) of \( X \) and an open set \( V \) in \( X \), then the open complement \( X - A \) contains \( u \) and is not disjoint from \( V \), so \( Y - A \) is not disjoint from \( v \) either. Thus, \( u \) and \( v \) are not contained in disjoint open subsets of \( Y \), and \( Y \) is not Hausdorff.

Suppose \( X \) is not locally compact. Then there exists of a point \( x \in X \) that does not have a compact open neighborhood in \( X \). Given an open neighborhood \( Y - A \) of \( \infty \), where \( A \) is closed in \( X \) and compact, and an open neighborhood \( U \) of \( x \) in \( X \), we cannot have \( U \subseteq A \), since if this were the case, then \( \overline{U} \subseteq A \) would be a compact neighborhood of \( x \). Thus, \( Y \) is again not Hausdorff. \( \square \)

**Remark 3.3.11.** The one-point compactification of a locally compact Hausdorff space \( X \) has the property of being a quotient space of each Hausdorff compactification of \( X \) via the unique surjection that is the identity on \( X \).

**Example 3.3.12.** The one-point compactification of \( \mathbb{R}^n \) is homeomorphic to \( S^n \). To see this, view \( S^n \) as the unit sphere centered at \( z = (0, \ldots, 0, 1) \). This is realized via the embedding \( \mathbb{R}^n \to S^n \) sending \( x \) to the unique point in \( S^n \) on the line between \( x \) and \( z \).

Our definition of local compactness differs from our definition of local connectedness, as we only ask for a compact neighborhood, not one contained in an arbitrarily small neighborhood. For Hausdorff spaces, these notions are the same.

**Proposition 3.3.13.** A Hausdorff space \( X \) is locally compact if and only if for every open neighborhood \( U \) of a point \( x \in X \) contains a compact neighborhood \( A \) with \( A \subseteq U \).
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PROOF. As the other direction is immediate, we may suppose that \( X \) is locally compact. Let \( U \) be open in \( X \), and let \( x \in U \). Let \( Y \) be the one-point compactification of \( X \). Let \( A = Y - U \), which is closed in \( Y \), hence compact. By Lemma 3.2.7, we may find an open neighborhood \( V \) of \( x \) and an open set \( W \) containing \( A \) that are disjoint. The closure of \( V \) in \( Y \) is compact and disjoint from \( A \), so is contained in \( U \).

COROLLARY 3.3.14. Any open or closed subspace of a locally compact Hausdorff space is locally compact.

PROOF. Let \( X \) be locally compact Hausdorff. If \( A \) is closed in \( X \) and \( a \in A \), then there is a compact neighborhood \( C \) of \( x \) in \( X \), and \( C \cap A \) is then a compact neighborhood of \( a \) in \( A \). If \( U \) is open in \( X \) and \( x \in U \), then by Proposition 3.3.13, we can find a compact neighborhood \( B \) of \( x \) in \( U \).

COROLLARY 3.3.15. A Hausdorff space \( X \) is locally compact if and only if it is homeomorphic to an open subspace of a compact Hausdorff space.

PROOF. If \( X \) is locally compact but not compact, take its one-point compactification, in which \( X \) is open, thus locally compact by Corollary 3.3.14.

DEFINITION 3.3.16. A map \( f : X \to Y \) of topological spaces is said to be a local homeomorphism if it is continuous, open, and for each \( x \in X \), there exists an open neighborhood \( U \) of \( x \) such that the restriction of \( f \) to \( U \) is a homeomorphism onto its image.

EXAMPLE 3.3.17. The inclusion map of an open set in a topological space is a local homeomorphism.

EXAMPLE 3.3.18. The map \( f : \mathbb{R} \to S^1 \) given by \( f(x) = (\cos x, \sin x) \) is a local homeomorphism, as is its restriction to any open interval. Note that there is, however, no local homeomorphism \( g : S^1 \to \mathbb{R} \).

3.4. Countability axioms

DEFINITION 3.4.1. A topological space \( X \) is said to be first-countable if every point of \( X \) has a countable basis of open neighborhoods.

EXAMPLE 3.4.2. If \( X \) is a metrizable space, then \( X \) is first-countable. For instance, if \( d \) is a metric on \( X \), then \( x \in X \) has the countable basis \( \{B(x; \frac{1}{n}) \mid n \geq 1\} \) of open neighborhoods.

PROPOSITION 3.4.3. For a subset \( A \) of a first-countable topological space \( X \), every limit of a convergent sequence \((a_n)_{n \geq 1}\) in \( A \) lies in \( \overline{A} \).

PROOF. If there exists \((a_n)_{n \geq 1}\) with limit \( x \in X \), then every neighborhood of \( x \) contains all \( a_n \) for \( n \) sufficiently large, so \( x \in \overline{A} \) by definition. Conversely, if \( x \in \overline{A} \) and \( \{U_n \mid n \geq 1\} \) is a basis of open neighborhoods of \( x \), then we may pick \( a_n \in A \cap U_n \) for each \( n \geq 1 \), and the sequence \((a_n)_{n \geq 1}\) has limit \( x \).

PROPOSITION 3.4.4. Let \( X \) be a first-countable space. If \( f : X \to Y \) is a function such that for every convergent sequence \((x_n)_{n \geq 1}\) in \( X \) with limit \( x \), the sequence \((f(x_n))_{n \geq 1}\) converges to \( f(x) \), then \( f \) is continuous.
3.4. COUNTABILITY AXIOMS

PROOF. Let $V$ be an open set in $Y$, let $x \in f^{-1}(V)$, and let $\{U_n \mid n \geq 1\}$ be a countable basis of open neighborhoods of $x$. If $f^{-1}(V)$ does not contain an open neighborhood of $x$, then it cannot contain any $U_n$. Assume this, and for each $n \geq 1$, choose $x_n \in U_n$ such that $x_n \notin f^{-1}(V)$. Then the sequence $(x_n)_{n \geq 1}$ converges to $x$, and by assumption, $(f(x_n))_{n \geq 1}$ converges to $f(x)$. Therefore, there exists $N \geq 1$ such that $f(x_n) \in V$ for all $n \geq N$, so $x_n \in f^{-1}(V)$ for such $n$, contradicting our assumption. □

DEFINITION 3.4.5. A topological space $X$ is second-countable if $X$ has a countable base for its topology.

EXAMPLE 3.4.6. The space $\mathbb{R}^n$ is second-countable, since it has the base $\{B(x, \frac{1}{k}) \mid x \in \mathbb{Q}^n, k \geq 1\}$.

EXAMPLE 3.4.7. The space $X = \prod_{i=1}^{\infty} \mathbb{R}$ is second-countable in the product topology but not in the uniform topology. In the product topology, we can take the countable base consisting of products

$$\prod_{n=1}^{N} (a_i, b_i) \times \prod_{n=N+1}^{\infty} \mathbb{R}$$

with $a_i, b_i \in \mathbb{Q}$ and $a_i < b_i$ for $1 \leq i \leq N$ for some $N \geq 0$.

In the uniform topology, we have the uncountable discrete subspace

$$A = \{(a_n)_{n \geq 1} \mid a_n \in \{0, 1\}\}.$$

every two points being distance 1 apart in the uniform metric. For any base of $X$, the open balls of radius $\frac{1}{2}$ about the points of $A$ would each contain an element of the base, which is therefore uncountable. In particular, metrizable spaces are not necessarily second-countable.

DEFINITION 3.4.8. A topological space $X$ is Lindelöf if every open cover of $X$ has a countable subcover.

PROPOSITION 3.4.9. Second-countable spaces are Lindelöf.

PROOF. Let $\mathcal{U}$ be an open cover of $X$, and let $\mathcal{B}$ be a countable basis of $X$. For each element of $\mathcal{B}$, choose an element of $\mathcal{U}$ containing it. Then the subset of $\mathcal{U}$ consisting of these open sets is countable and covers $X$. □

Recall that a subset $A$ of a topological space is $X$ is dense if $\overline{A} = X$.

DEFINITION 3.4.10. A topological space $X$ is separable if it has a countable dense subset.

PROPOSITION 3.4.11. Second-countable spaces are separable.

PROOF. Let $\mathcal{B} = \{U_n \mid n \geq 1\}$ be a countable base of $X$. For each $n \geq 1$, choose $x_n \in U_n$, and set $A = \{x_n \mid n \geq 1\}$. Then for any $x \in X$ and open neighborhood $V$ of $x$, there exists $n \geq 1$ such that $x \in U_n \subseteq V$, so in particular $x_n \in V$. Thus, $x \in \overline{A}$, so $A$ is dense in $X$. □

For metric spaces, the converses to Propositions 3.4.9 and 3.4.11 hold.

THEOREM 3.4.12. For a metric space $X$, the following are equivalent:

i. $X$ is second-countable,
ii. \( X \) is Lindelöf, and

iii. \( X \) is separable.

**Proof.** By Propositions 3.4.9 and 3.4.11, it suffices to show that Lindelöf spaces are second-countable and separable spaces are second-countable. Let \( X \) be a metric space.

Suppose first that \( X \) is Lindelöf. For positive integers \( n \geq 1 \), let \( \mathcal{B}_n \) be a countable subcover of the open cover of \( X \) by balls of radius \( \frac{1}{n} \). Let \( \mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n \), which is also countable. We claim that \( \mathcal{B} \) is a base for the metric topology on \( X \). Let \( x \in X \) and \( \varepsilon > 0 \), and consider the open ball \( B(x, \varepsilon) \). Choose \( n \) with \( 2^n < \varepsilon \), and let \( y \in X \) be such that \( x \in B(y, \frac{1}{n}) \in \mathcal{B}_n \). Then \( B(x, \varepsilon) \subseteq B(y, \frac{1}{n}) \) by the triangle inequality, hence the claim.

Next, suppose that \( X \) is separable, and let \( S \subseteq X \) be a countable dense subset. We claim that the countable collection 
\[
\mathcal{B} = \{ B(y, \frac{1}{n}) \mid y \in S, n \geq 1 \}
\]
of open balls is a base for the metric topology on \( X \). Let \( x \in X \) and \( \varepsilon > 0 \). As \( S \) is dense, \( B(x, \frac{\varepsilon}{3}) \) contains a point \( y \in S \), and if we take \( n \geq 1 \) such that \( \frac{2^n}{n} \leq \frac{2\varepsilon}{3} \), then \( x \in B(y, \frac{1}{n}) \subseteq B(x, \varepsilon) \) by the triangle inequality, hence the claim. \( \square \)

### 3.5. Separation axioms

In this section, we consider both weaker and stronger conditions in terms of the separation of points in a topological space than our much-used Hausdorff condition.

**Definition 3.5.1.**

a. A topological space \( X \) is **regular** if for every point \( x \in X \) and closed subset \( A \) of \( X \) with \( x \notin A \), there exist disjoint open sets \( U \) and \( V \) with \( x \in U \) and \( A \subseteq V \).

b. A topological space \( X \) is **normal** if for every two closed, disjoint subsets \( A \) and \( B \) of \( X \), there exist disjoint open subsets \( U \) and \( V \) of \( X \) such that \( A \subseteq U \) and \( B \subseteq V \).

Here are several more of many separability conditions that are given names or symbols.

**Definition 3.5.2.**

a. A space \( X \) is said to be **\( T_0 \)** if every two distinct points of \( X \) have distinct sets of open neighborhoods.

b. A space \( X \) is said to be **\( T_1 \)** if points of \( X \) are closed.

c. A space \( X \) is said to be **\( T_2 \)** if it is Hausdorff.

d. A space \( X \) is said to be **\( T_3 \)** if it is **regular** and \( T_0 \).

e. A space \( X \) is said to be **\( T_4 \)** if it is **normal** and \( T_1 \).

**Remark 3.5.3.** Each condition \( T_i \) for \( 1 \leq i \leq 4 \) is stronger than \( T_{i-1} \). In particular, to say that a space is \( T_3 \) (resp., \( T_4 \)) is to say that it is regular Hausdorff (resp., normal Hausdorff). For instance, we check that a \( T_3 \)-space \( X \) is Hausdorff: take \( x, y \in X \) with \( x \neq y \). By the \( T_0 \)-axiom, there is an open neighborhood \( W \) of a point \( x \) that does not contain \( y \). Then \( W^c \) and \( x \) are disjoint, hence by the regularity condition, are contained in disjoint open sets \( U \) and \( V \), respectively. Then \( x \in U \) and \( y \in V \), so \( X \) is Hausdorff.
EXAMPLE 3.5.4. Spaces that are $T_0$ but not $T_1$ are ubiquitous in modern algebraic geometry. For instance, what is known as the spectrum $\text{Spec} \mathbb{Z}$ of the ring of integers of $\mathbb{Z}$ is a space consisting of one point $(0)$ and another point $(p)$ for each prime number $p$. It has a base consisting of $(0)$ and the complements of finite sets $\{(p_1), \ldots, (p_n)\}$ with each $p_i$ prime. The closure of $\{(0)\}$ is $\text{Spec} \mathbb{Z}$. Unfortunately, the explanation for why algebraic geometers consider this space lies beyond the scope of this course.

The reader can check the following, which we note stops at $T_3$-spaces.

LEMMA 3.5.5. For $0 \leq i \leq 3$, every subspace of a $T_i$-space is $T_i$.

EXAMPLE 3.5.6. The space $X = \mathbb{R}$ with the topology consisting of sets of the form $U - C$, where $U$ is open in $\mathbb{R}$ in the Euclidean topology and $C$ is a countable subset of $U$, is Hausdorff but not regular. It is Hausdorff since its topology is finer than the Euclidean topology on $\mathbb{R}$. It is not regular, as the set $\mathbb{Q}$ is closed in its topology, and the open sets containing $\mathbb{Q}$ in $\mathbb{R}$ are the complements $U$ in $\mathbb{R}$ of countable sets of irrational numbers. An open neighborhood $V$ of an irrational number $X$ contains all but countably many points in an open interval around it, and so does $U$, but such an interval is uncountable, so $U \cap V$ cannot be empty.

One might ask for the meaning of the word “separated” in the sense of topology.

DEFINITION 3.5.7. We say that two subsets $A$ and $B$ of a topological space are separated if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

DEFINITION 3.5.8. A separation of $X$ is a set of two disjoint open and closed subspaces of $X$ with union $X$.

REMARK 3.5.9. A space $X$ has a separation if and only if it is disconnected, in which case it is the topological disjoint union of the two spaces $A$ and $B$ in the separation. Moreover, these $A$ and $B$ are clearly separated. Similarly, if $X$ is the union of two separated subsets, then they are both open and closed, hence constitute a separation of $X$.

DEFINITION 3.5.10. A space in which every two points with distinct open neighborhoods are separated is called symmetric.

REMARK 3.5.11. A space is $T_1$ if and only if it is symmetric and $T_0$, i.e., every pair of distinct points is separated.

LEMMA 3.5.12. A space $X$ is regular (resp., normal) if and only if every open set of $X$ containing a point $x$ (resp., closed set $A$) contains the closure of an open set containing $x$ (resp., $A$).

PROOF. Let $X$ be regular (resp., normal). Let $A$ be a singleton (resp., closed) subset of $X$, and let $W$ be an open subset containing $A$. Then $B = W^c$ is disjoint from $A$, so by regularity (resp., normality) of $X$, there exist disjoint open sets $U$ containing $A$ and $V$ containing $B$. Then $A \subseteq U \subseteq V^c \subseteq W$. Since $V^c$ is closed, $W$ contains $\overline{U}$, so $U$ is the desired open set.

Conversely, suppose that for any singleton (resp., closed) subset $A$ of $X$, every open set of $X$ containing $A$ contains the closure of an open set containing $A$. Let $B$ be a closed set disjoint from $A$, and let $U$ be an open subset of $B^c$ containing $A$ and such that $\overline{U} \subseteq B^c$. Take $V = \overline{U}^c$ so that $B \subseteq V$, and note that $U$ and $V$ are disjoint. Thus, $X$ is regular (resp., normal).
We have seen that a direct product of Hausdorff spaces is Hausdorff. The analogous statement holds for regular Hausdorff spaces (i.e., $T_3$-spaces).

**Proposition 3.5.13.** A product of regular Hausdorff spaces is regular Hausdorff in the product topology.

**Proof.** Let $X = \prod_{i \in I} X_i$, where $\{X_i \mid i \in I\}$ is a collection of regular Hausdorff topological spaces. Then $X$ is Hausdorff, and in particular, its points are closed. Let $x = (x_i)_{i \in I} \in X$, and let $U$ be an open neighborhood of $x$. Then $U$ contains a basic open neighborhood $\prod_{i \in I} U_i$, where each $U_i$ is open in $X_i$ and all but finitely many $U_i$ equal $X_i$. For every $i$, we may choose an open neighborhood $V_i$ of $x_i$ with $\overline{V}_i \subseteq U_i$ by Lemma 3.5.12. If $U_i = X_i$, we take $V_i = X_i$ as well. Then $\prod_{i \in I} V_i$ contains $x$ and has closure $\prod_{i \in I} \overline{V}_i \subseteq \prod_{i \in I} U_i$, as desired. $\square$

The following example shows that even a finite direct product of $T_4$-spaces need not be $T_4$.

**Example 3.5.14.** Consider $X = \mathbb{R}$ with the lower-limit topology generated by the base of open sets $[a,b)$ with $a < b$. Note that these sets are closed as well, since $(-\infty,a) \cup [b,\infty)$ has complement $[a,b)$. Then $X$ is normal Hausdorff. To see normality, take disjoint closed sets $A$ and $B$ in $X$. For each $a \in A$, we pick $x_a > a$ with $[a,x_a)$ in the complement of $B$ and let $U$ be the union of these half-open intervals. Similarly, for each $b \in B$, we pick $y_b > b$ with $[b,y_b)$ in the complement of $A$ and let $V$ be the union of these intervals. Take $a \in A$ and $b \in B$, and suppose without loss of generality that $a < b$. Then $x_a < b$ since $b \notin [a,x_a)$, so the intersection $[a,x_a) \cap [b,y_b)$ is empty, and therefore $U$ and $V$ are disjoint.

On the other hand, the product $X^2 = X \times X$ is regular Hausdorff by Proposition 3.5.13, but it is not normal. The subspace $D = \{(x,-x) \mid x \in \mathbb{R}\}$ of $X \times X$ has the discrete topology and is closed in $X^2$, and if we take the subset $A = \{(x,-x) \mid x \in \mathbb{Q}\}$, then $A$ and $D - A$ are closed subsets of $X^2$ that are not contained in disjoint open neighborhoods of $X^2$. We omit the nontrivial proofs of these facts.

**Theorem 3.5.15.** Regular, second-countable spaces are normal.

**Proof.** Let $X$ be regular and second-countable. Let $\mathcal{B}$ be a countable base for the topology on $X$. Let $A$ and $B$ be disjoint closed subsets of $X$. By regularity, for each $x \in X$ we can find an open neighborhood $U$ of $x$ with $\overline{U}$ disjoint from $B$, and contained in $U$ we can find some neighborhood in $\mathcal{B}$ of $x$. Together, these basis elements give a countable covering $\mathcal{U} = \{U_n \mid n \geq 1\}$ of $A$ with $\overline{U}_n \cap B = \emptyset$ for each $n \geq 1$. Similarly, we can find a countable covering $\mathcal{V} = \{V_n \mid n \geq 1\}$ of $B$ with $\overline{V}_n \cap A = \emptyset$. For each $n \geq 1$, consider the open sets.

$$U'_n = U_n \cap \bigcap_{i=1}^n \overline{V}_i \quad \text{and} \quad V'_n = V_n \cap \bigcap_{i=1}^n \overline{U}_i.$$

If $a \in A$, then $a \in U_n$ for some $n \geq 1$, and $a \notin \overline{V}_i$ for all $i$, so $a \in U'_n$. The open sets $U = \bigcup_{n \geq 1} U'_n$ and $V = \bigcup_{n \geq 1} V'_n$ contain $A$ and $B$, respectively, and they are disjoint, since if $u \in U'_n$ for some $n \geq 1$, then $u \notin V_i$ for $i \leq n$ by definition of $U'_n$ and $u \notin V'_i$ for $i > n$ by definition of $V'_n$. $\square$

**Theorem 3.5.16.** Metrizable spaces are normal.
Let $A$ and $B$ be disjoint, closed subsets of a metrizable space $X$, and let $d$ be a metric on $X$. For each $a \in A$, there exists $\varepsilon_a > 0$ with $B(a, \varepsilon_a) \cap B = \emptyset$, and similarly, for each $b \in B$, there exists $\delta_b > 0$ with $B(b, \delta_b) \cap A = \emptyset$. Set

$$U = \bigcup_{a \in A} B\left(a, \frac{\varepsilon_a}{2}\right) \quad \text{and} \quad V = \bigcup_{b \in B} B\left(b, \frac{\delta_b}{2}\right).$$

Then $U$ and $V$ are open containing $A$ and $B$, respectively, and they are disjoint by the triangle inequality.

**Theorem 3.5.17.** Compact Hausdorff spaces are normal.

**Proof.** Let $X$ be a compact Hausdorff space, and let $A$ and $B$ be disjoint closed subsets of $X$, which are necessarily compact. Lemma 3.2.7 implies that compact Hausdorff spaces are regular. That is, for each $b \in B$, we may choose disjoint open sets $U_b$ containing $A$ and $V_b$ containing $b$. Then the sets $V_b$ cover $B$, hence have a finite subcover, say by $V_{b_1}, \ldots, V_{b_n} \in B$. Then $U = \bigcap_{i=1}^n U_{b_i}$ and $V = \bigcup_{i=1}^n V_{b_i}$ are disjoint open sets containing $A$ and $B$, respectively.

Using the one-point compactification, we may use Theorem 3.5.12 to give a quick proof of the analogous result for locally compact Hausdorff spaces.

**Corollary 3.5.18.** Locally compact Hausdorff spaces are regular.

**Proof.** Let $X$ be locally compact and Hausdorff. Let $Y$ be its one-point compactification, which is compact Hausdorff and therefore normal. As $Y$ is $T_4$, it is also $T_3$, and $X$ is $T_3$ as a subspace of $Y$. 

\[ \square \]
CHAPTER 4

Theorems

4.1. Urysohn’s lemma

PROOF. By the normality of $X$, there exist disjoint open sets $W$ and $Z$ containing the closed sets $\overline{U}$ and $V^c$, respectively. As $Z^c$ is closed, we have $W \subseteq Z^c \subseteq V$. □

THEOREM 4.1.1 (Urysohn’s lemma). A topological space $X$ is normal if and only if for every pair of disjoint closed sets $A$ and $B$ of $X$, there exists a continuous function $f: X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

PROOF. Let $X$ be a normal space. Set $U_1 = B^c$, and by normality of $X$, pick an open set $U_0$ containing $A$ with $\overline{U_0} \subseteq U_1$. Fix any bijection $\phi: \mathbb{Z}_{\geq 0} \to \mathbb{Q} \cap [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$. For each $n \geq 2$, let $q_n$ be the largest value of $\phi(m)$ with $m < n$ that is less than $\phi(n)$, and let $r_n$ be the smallest value of $\phi(m)$ with $m < n$ that is greater than $\phi(n)$. Suppose by induction that we have constructed $U_{\phi(m)}$ for all $m < n$. By Lemma 3.5.12, we may choose an open set $U_{\phi(n)}$ containing $\overline{U_{q_n}}$ and such that $\overline{U_{\phi(n)}}$ is contained in $U_{r_n}$. For negative $r \in \mathbb{Q}$, set $U_r = \emptyset$, and for $r \in \mathbb{Q}$ greater than 1, set $U_r = X$. In this way, we have constructed open sets $U_q$ for all rational numbers $q$ such that $\overline{U_q} \subseteq U_r$ if $q < r$ are rational numbers.

We now define $f$ by setting

$$f(x) = \inf\{q \in \mathbb{Q} \mid x \in U_q\}$$

for $x \in X$. If $x \in A$, then $x \in U_0$, so $f(x) = 0$. If $x \in B$, then $x \notin U_1$ but $x \in U_r$ for all rationals $r > 1$, so $f(x) = 1$. If $x \in U_q$, then $x \in U_r$ for all rationals $r > q$, so $f(x) \leq q$. Given a nonempty open interval $(a, b)$ in $\mathbb{R}$ and $x$ in its inverse image, we then have $q, r \in \mathbb{Q}$ with $a < q < f(x) < r < b$. Since $f(x) < r$, we have $x \in U_r$, and since $f(x) > q$, we have $x \notin \overline{U_q}$. That is, we have $x \in U_r - \overline{U_q}$, which is an open neighborhood with image under $f$ contained in $[q, r] \subset (a, b)$. Thus, $f$ is continuous.

As for the converse, note that if $A$ and $B$ are disjoint closed sets in $X$ and $f: X \to [0,1]$ is a continuous function with $f(A) = \{0\}$ and $f(B) = \{1\}$, then $U = f^{-1}([0,1/2))$ and $V = f^{-1}((1/2,0])$ are disjoint open sets containing $A$ and $B$, respectively. □

DEFINITION 4.1.2. A space $X$ is completely regular if for every closed set $A$ in $X$ and $x \in A^c$, there exists a continuous function $f: [0,1] \to X$ with $f(A) = \{0\}$ and $f(x) = 1$. A space $X$ is said to be Tychonoff, or $T_{3 \frac{1}{2}}$, if it is completely regular and $T_1$.

REMARK 4.1.3. A normal Hausdorff space is Tychonoff, and a completely regular space is regular.
We leave it to the reader to check that the following hold, the condition on subspaces between the motivation for the word “completely”.

**Proposition 4.1.4.** Subspaces and products of completely regular spaces are completely regular.

Regularity, together with second-countability, implies complete regularity (since it implies normality). In fact, we have the following.

**Lemma 4.1.5.** Let $X$ be a regular, second-countable space. Then there exists a countable collection \( \{f_n \mid n \geq 1 \} \) of continuous functions $f_n : X \to [0, 1]$ for $n \geq 1$ such that for each pair $(A, x)$ of a closed subset $A$ of $X$ and $x \in A^c$, there exists $n$ such that $f_n(A) = \{0\}$ and $f_n(x) = 1$.

**Proof.** Since $X$ is second-countable and regular, it is normal. Let \( \{U_n \mid n \geq 1 \} \) be a countable base of neighborhoods of $X$. If $\overline{U_m} \subseteq U_n$, then by Urysohn’s lemma, there exists a continuous function $f_{m,n} : X \to [0, 1]$ with $f_{m,n}(\overline{U_m}) = \{1\}$ and $f(X - U_n) = \{0\}$.

The set $A^c$ contains some neighborhood $U_n$ of $x$, which contains the closure of some neighborhood $U_m$ of $x$ by the regularity of $X$. Then $f_{m,n}(x) = 1$ and $f_{m,n}(A) = \{0\}$. Since the collection of functions $f_{m,n}$ is countable, we have the lemma.

**Theorem 4.1.6 (Urysohn metrization theorem).** Second-countable regular Hausdorff spaces are metrizable and are exactly the spaces that are homeomorphic to subspaces of the product space $[0, 1]^J$, where $J$ is countably infinite.

**Proof.** We may take $J = \{n \mid n \geq 1\}$. Recall that $\mathbb{R}^J$ is metrizable by Proposition 2.2.16, so to show metrizability, it suffices to embed our second-countable regular space $X$ in $[0, 1]^J$. If we can do this for a given $X$, then $[0, 1]^J$ is regular Hausdorff and second-countable, and then so is $X$.

Let $(f_n)_{n \geq 1}$ be a sequence of functions $f_n : X \to [0, 1]$ as in Lemma 4.1.5 and use them to define a function $f = (f_n)_{n \geq 1} : X \to [0, 1]^J$, which is continuous as each $f_n$ is continuous. Since $X$ is Hausdorff, for any $x, y \in X$ with $x \neq y$, the set $\{x\}^c$ is an open neighborhood of $y$, so we can for each $x \neq y$ in $X$ find an $n \geq 1$ such that $f_n(x) = 0$ and $f_n(y) = 1$. Thus, $f$ is injective.

It remains to show that $f$ is an open map to its image. Let $U$ be an open set in $X$, let \( a = (a_n)_{n \geq 1} \in f(U) \), and let $x \in U$ with $f(x) = a$. Choose an $m \geq 1$ such that $f_m(a) = 1$ and $f_m(a) = 0$ for all $x \not\in U$. Let $V$ be the open set $\pi_m^{-1}((0, 1]) \cap f(X)$ in the image of $f$, where $\pi_n$ is the $n$th projection map from $\prod_{n=1}^\infty [0, 1]$. Then $a \in V$ and $V \subseteq f(U)$, since $f^{-1}(0) \subseteq U^c$. Thus, $f(U)$ is open.

Another application of Urysohn’s lemma is found in the following theorem, for which we omit a proof.

**Theorem 4.1.7 (Tietze extension theorem).** Let $A$ be a closed subspace of a normal topological space $X$. Any continuous map of $A$ into a closed interval (possibly of infinite length) in $\mathbb{R}$ can be extended to a continuous map of $X$ into the same closed interval in $\mathbb{R}$.

**Definition 4.1.8.** A second-countable Hausdorff space $M$ is said to be a manifold if there exists $n \geq 0$ such that every point of $M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$. We then say that $M$ is $n$-dimensional, or an $n$-manifold.
4.2. Tychonoff’s theorem

Remark 4.1.9. To say that a point in $M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$ is to say that it has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ (in particular, as open balls in $\mathbb{R}^n$ are homeomorphic to $\mathbb{R}^n$). Equivalently, there is a local homeomorphism $\mathbb{R}^n \to M$ with image containing the point.

Example 4.1.10. Open sets in $\mathbb{R}^n$ and the sphere $S^n$ are $n$-manifolds. The torus is an example of a 2-manifold.

We have the following application of Urysohn’s lemma to manifolds.

Theorem 4.1.11. Any compact manifold can be embedded in $\mathbb{R}^N$ for some $N \geq 1$.

4.2. Tychonoff’s theorem

We briefly recall a few notions from set theory. In particular, recall that a relation on a set $X$ is a subset of $X \times X$, and if $R$ is such a relation, we often write $aRb$ to denote $(a, b) \in R$. We have already used the notion of an equivalence relation earlier in the notes without comment. Another useful sort of relation is known as a partial ordering.

Definition 4.2.1. A partial ordering on a set $X$ is a relation $\leq$ on $X$ that satisfies the following properties.

i. (reflexivity) For all $x \in X$, we have $x \leq x$.

ii. (antisymmetry) If $x, y \in X$ satisfy $x \leq y$ and $y \leq x$, then $x = y$.

iii. (transitivity) If $x, y, z \in X$ satisfy $x \leq y$ and $y \leq z$, then $x \leq z$.

A set $X$ together with a partial ordering $\leq$ is referred to as a partially ordered set.

Definition 4.2.2. A total ordering on a set $X$ is a partial ordering $\leq$ such that for all $x, y \in X$, one has either $x \leq y$ or $y \leq x$. In this case, $X$ together with $\leq$ is called a totally ordered set.

Examples 4.2.3.

a. The relation $\leq$ on $\mathbb{R}$ is a total ordering, as is $\geq$.

b. The relation $<$ on $\mathbb{R}$ is not a partial ordering, as it is not reflexive.

c. The relation $\subseteq$ on the set of subsets $\mathcal{P}_X$ of any set $X$, which is known as the power set of $X$, is a partial ordering. It is not a total ordering if $X$ contains more than one element.

d. The relation $=$ is a partial ordering on any set.

Given a partial ordering $\leq$ on a set $X$, we can speak of minimal and maximal elements of $X$.

Definition 4.2.4. Let $X$ be a set with a partial ordering $\leq$.

a. A minimal element in $X$ (under $\leq$) is an element $x \in X$ such that if $z \in X$ and $z \leq x$, then $z = x$.

b. A maximal element $y \in X$ is an element such that if $z \in X$ and $y \leq z$, then $z = y$.

Minimal and maximal elements need not exist, and when they exist, they need not be unique. Here are some examples.
EXAM PLES 4.2.5.

a. The set \( \mathbb{R} \) has no minimal or maximal elements under \( \leq \).

b. The interval \([0, 1)\) in \( \mathbb{R} \) has the minimal element 0 but no maximal element under \( \leq \).

c. The power set \( \mathcal{P}_X \) of \( X \) has the minimal element \( \emptyset \) and maximal element \( X \) under \( \subseteq \).

d. Under \( = \) on \( X \), every element is both minimal and maximal.

e. Consider the set \( S \subset \mathcal{P}_X \) of nonempty subsets of a set \( X \), with the partial ordering \( \subseteq \). The minimal elements of \( S \) are exactly the singleton sets in \( X \).

One can ask for a condition under which maximal (or minimal) elements exist. To phrase such a condition, we need two more notions.

**Definition 4.2.6.** Let \( X \) be a set with a partial ordering \( \leq \). A chain in \( X \) is a subset of \( X \) that is totally ordered under \( \leq \).

**Definition 4.2.7.** Let \( X \) be a set with a partial ordering \( \leq \). Let \( A \) be a subset of \( X \). An upper bound on \( A \) under \( \leq \) is an element \( x \in \mathcal{P}_X \) such that \( a \leq x \) for all \( a \in A \).

**Examples 4.2.8.**

a. The subset \([0, 1)\) of \( \mathbb{R} \) has an upper bound \( 1 \in \mathbb{R} \) under \( \leq \). In fact, any element \( x \geq 1 \) is an upper bound for \([0, 1)\). The subset \([0, 1]\) has the same upper bounds.

b. The subset \( \mathbb{Q} \) of \( \mathbb{R} \) has no upper bound under \( \leq \).

We now come to Zorn’s lemma, which is equivalent to the axiom of choice. We omit the proof of this fact.

**Theorem 4.2.9 (Zorn’s lemma).** Let \( X \) be a nonempty set with a partial ordering \( \leq \), and suppose that every chain in \( X \) has an upper bound. Then \( X \) contains a maximal element.

We use Zorn’s lemma to prove the following.

**Theorem 4.2.10 (Alexander subbase theorem).** A space \( X \) is compact if and only if there exists a subbase \( \mathcal{S} \) for its topology such that every open cover of \( X \) by elements of \( \mathcal{S} \) has a finite subcover.

**Proof.** Suppose that \( X \) is not compact, and let \( \mathcal{S} \) be a subbase of \( X \). We need to show that \( \mathcal{S} \) contains an open cover \( \mathcal{W} \) that does not have a finite subcover.

Let \( \mathcal{Q} \) be the set of all open covers of \( X \) that have no finite subcover, and note that \( \mathcal{Q} \neq \emptyset \) by the noncompactness of \( X \). Let \( \mathcal{C} \) be a chain in \( \mathcal{Q} \), and let \( \mathcal{W} = \bigcup_{\mathcal{V} \in \mathcal{C}} \mathcal{V} \) be the union of all covers in \( \mathcal{C} \). Any finite collection of elements of \( \mathcal{W} \), being each contained in some element of \( \mathcal{C} \), are all contained in the largest such element \( \mathcal{V} \) under inclusion. Being that \( \mathcal{V} \) has no finite subcover, such a finite collection cannot cover \( X \), so \( \mathcal{W} \) has no finite subcover, which is to say that \( \mathcal{W} \in \mathcal{Q} \). Thus, every chain in \( \mathcal{Q} \) has an upper bound, and so by Zorn’s lemma, \( \mathcal{Q} \) has a maximal element \( \mathcal{M} \).

Now consider the subset \( \mathcal{W} = \mathcal{S} \cap \mathcal{M} \) of \( \mathcal{S} \). Being a subset of \( \mathcal{M} \), no finite subset of \( \mathcal{W} \) covers \( \mathcal{M} \). We claim that \( \mathcal{W} \) covers \( X \), which will finish the proof. Let \( x \in X \), let \( U \in \mathcal{M} \) with
4.2. Tychonoff’s Theorem

Given $x \in U$, and let $V_1, \ldots, V_n \in \mathcal{I}$ such that $x \in V_1 \cap \cdots \cap V_n \subseteq U$, which exist as $\mathcal{I}$ is a subbase. If $V_i \notin \mathcal{M}$ for all $1 \leq i \leq n$, we can find by the maximality of $\mathcal{M}$ finite subsets $\mathcal{N}_i$ of $\mathcal{M}$ such that the collections $\mathcal{N}_i \cap \{V_i\}$ cover $X$. Then $\mathcal{N} = \bigcup_{i=1}^n \mathcal{N}_i$ and $V_1 \cap \cdots \cap V_n$ together cover $X$, so $\mathcal{N} \cup \{U\} \subset \mathcal{M}$ covers $\mathcal{M}$ as well, contradicting the fact that $\mathcal{M}$ has no finite subcover. Thus, there exists $i$ such that $V_i \in \mathcal{U}$, and $V_i$ contains $x$ by definition. Thus $\mathcal{U}$ is a cover of $X$. □

Theorem 4.2.11 (Tychonoff’s theorem). Any product of compact spaces is compact under the product topology.

Proof. Let $\{X_i \mid i \in I\}$ be a collection of compact spaces, and let $X = \prod_{i \in I} X_i$. For each $i \in I$, let $\mathcal{T}_i$ be the topology on $X_i$, and let

$$\mathcal{I} = \bigcup_{i \in I} \{\pi_i^{-1}(U_i) \mid U_i \in \mathcal{T}_i\},$$

where $\pi_i: X \to X_i$ is the $i$th projection map. Then $\mathcal{I}$ is a subbase for the topology on $X$. We claim that every open cover of $X$ by elements in $\mathcal{I}$ has a finite subcover, which by the Alexander subbase theorem will finish the proof.

Let $\mathcal{U} \subseteq \mathcal{I}$ be an open cover of $X$. Then

$$\mathcal{U} = \bigcup_{i \in I} \{\pi_i^{-1}(U_i) \mid U_i \in \mathcal{U}_i\}$$

for some collections $\mathcal{U}_i$ of open sets in $X_i$ for each $i \in I$. If no $\mathcal{U}_i$ covers $X$, then for each $i \in I$, we may by the axiom of choice find $x_i \in X_i$ such that $x_i \notin \bigcup_{U \in \mathcal{U}_i} U_i$. Set $x = (x_i)_{i \in I}$. Then $x \notin U$ for any $U \in \mathcal{U}$, for any such $U$ has the form $U = \pi_i^{-1}(U_i)$ for some $U_i \in \mathcal{U}_i$ for some $i \in I$, and $x_i = \pi_i(x) \notin U_i$. Thus there exists $i \in I$ such that $\mathcal{U}_i$ covers $X_i$. It has a finite subcover $\mathcal{V}$, and the finite set $\{\pi_i^{-1}(V) \mid V \in \mathcal{V}\}$ then covers $X$. □
CHAPTER 5

Homotopy theory

5.1. Homotopies and path homotopies

**Definition 5.1.1.** Let $X$ and $Y$ be topological spaces, and let $f, f' : X \to Y$ be continuous functions. A *homotopy* from $f$ to $f'$ is a continuous function $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in X$.

**Examples 5.1.2.**

a. Any two continuous functions $f, f' : X \to \mathbb{R}^n$ are homotopic. That is, $F : X \times [0, 1] \to \mathbb{R}^n$ given by $F(x, t) = tf(x) + (1-t)f'(x)$ provides a homotopy.

b. Let $X = \{x\}$ and $Y$ be a discrete space. Then no two distinct functions $f, f' : X \to Y$ are homotopic. That is, any continuous function $F : X \times [0, 1] \to Y$ has connected image, and is therefore constant.

**Definition 5.1.3.** We say that two continuous functions $f, f' : X \to Y$ are *homotopic* if there exists a homotopy from $f$ to $f'$.

**Lemma 5.1.4.** The property of being homotopic is an equivalence relation on the set of continuous functions $f : X \to Y$ between topological spaces $X$ and $Y$.

We leave this as an exercise, as we will be more focused on the case of path homotopies. Recall that a path on $X$ is a continuous function $\gamma : [0, 1] \to X$.

**Definition 5.1.5.** Let $X$ be a space and $a, b \in X$. Let $\gamma$ and $\gamma'$ be paths in $X$ from $a$ to $b$. A *path homotopy* from $\gamma$ to $\gamma'$ is a homotopy $F : [0, 1]^2 \to X$ from $\gamma$ to $\gamma'$ such that $F(0, t) = a$ and $F(1, t) = b$ for all $t \in [0, 1]$.

**Example 5.1.6.** Consider the two paths $\gamma, \gamma'$ from $(1, 0)$ to $(-1, 0)$ in $\mathbb{C}$ given by $\gamma(s) = e^{\pi is}$ and $\gamma'(s) = e^{-\pi is}$. We have a path homotopy between them given by $F(s, t) = \cos(\pi s) + i(1 - 2t)\sin(\pi s)$. However, no such path homotopy exists in $\mathbb{C} - \{0\}$, the idea being that for any path homotopy $F$, there must exist a $t$ such that the path $\gamma(s) = F(s, t)$ for $s \in [0, 1]$ passes through 0. This may be intuitively clear, but it takes some work to show. Note that $\gamma$ and $\gamma'$ are still homotopic in $\mathbb{C} - \{0\}$ (in fact, any two paths in $\mathbb{C} - \{0\}$ are homotopic), so being path homotopic really is stronger than being homotopic.

**Definition 5.1.7.** We say that two paths $\gamma, \gamma'$ in $X$ from a point $a$ to a point $b$ are *path homotopic* if there exists a path homotopy from $\gamma$ to $\gamma'$.

**Notation 5.1.8.** We write $\gamma \sim \gamma'$ if two paths with the same endpoints are path homotopic.
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**Proposition 5.1.9.** The relation of path homotopy on the set $\Pi(X,a,b)$ of paths with fixed endpoints $a,b\in X$ forms an equivalence relation.

**Proof.** If $\gamma \in \Pi(X,a,b)$, then the map $F: [0,1]^2 \to X$ given by $F(s,t) = \gamma(t)$ for all $s,t \in [0,1]$ is a path homotopy from $\gamma$ to itself, so $\sim$ is reflective. If $\gamma' \in \Pi(X,a,b)$ with $\gamma \sim \gamma'$ and $F': [0,1]^2 \to X$ is a path homotopy from $\gamma$ to $\gamma'$, then $G(s,t) = F(s,1-t)$ is a path homotopy from $\gamma'$ to $\gamma$, so $\gamma' \sim \gamma$. Thus, $\sim$ is symmetric. Finally, if $\gamma'' \in \Pi(X,a,b)$ as well and we have both $\gamma \sim \gamma'$ and $\gamma' \sim \gamma''$, with $F$ a path homotopy from $\gamma$ to $\gamma'$ and $G$ a path homotopy from $\gamma'$ to $\gamma''$, then $H: [0,1]^2 \to X$ defined by

$$H(s,t) = \begin{cases} F(s,2t) & t \in [0,\frac{1}{2}], \\ G(s,2t-1) & t \in [\frac{1}{2},1] \end{cases}$$

is a path homotopy from $\gamma$ to $\gamma'$, so $\sim$ is transitive. To see this, one should note that $F(s,1) = \gamma'(s) = G(s,0)$ for all $s \in [0,1]$. \hfill $\square$

**Notation 5.1.10.** Let $\pi_1(X,a,b)$ denote the set of path homotopy classes of paths on $X$ from $a$ to $b$. We write $[\gamma]$ for the path homotopy class of a path $\gamma$.

**Definition 5.1.11.** If $\gamma \in \Pi(X,a,b)$ and $\mu \in \Pi(X,b,c)$, we define the composition of $\gamma$ and $\gamma'$ to be the path $\gamma \ast \mu \in \Pi(X,a,c)$ given by

$$(\gamma \ast \mu)(s) = \begin{cases} \gamma(2s) & s \in [0,\frac{1}{2}], \\ \mu(2s-1) & s \in [\frac{1}{2},1]. \end{cases}$$

If one replaces $\gamma, \mu$ with path homotopic paths, the result of composition is path homotopic to $\gamma \ast \mu$.

**Proposition 5.1.12.** Let $\gamma, \gamma' \in \Pi(X,a,b)$ and $\mu, \mu' \in \Pi(X,b,c)$ with $\gamma \sim \gamma'$ and $\mu \sim \mu'$. Then $\gamma \ast \mu \sim \gamma' \ast \mu'$.

**Proof.** Let $F$ be a path homotopy from $\gamma$ to $\gamma'$ and $G$ be a path homotopy from $\mu$ to $\mu'$. Define $H: [0,1]^2 \to X$ by

$$H(s,t) = \begin{cases} F(2s,t) & s \in [0,\frac{1}{2}], \\ G(2s-1,t) & s \in [\frac{1}{2},1] \end{cases}$$

for $s,t \in [0,1]$. Then $H$ is a path homotopy from $\gamma \ast \mu$ to $\gamma' \ast \mu'$. For this, one should note that $F(1,t) = b = G(0,t)$ for all $t \in [0,1]$.

**Remark 5.1.13.** By Proposition 5.1.12, composition of paths induces product maps

$$\pi_1(X,a,b) \times \pi_1(X,b,c) \to \pi_1(X,a,c),$$

and we have $[\gamma] \cdot [\mu] = [\gamma \ast \mu]$ under these products.
5.2. The fundamental group

Let's briefly explore the properties of the products we've constructed on paths and their path homotopy classes.

**Notation 5.2.1.**

a. For \( a \in X \), let \( e_a \in \Pi(X, a, a) \) denote the constant path \( e_a(s) = a \) for \( s \in [0, 1] \).

b. For \( a, b \in X \) and \( \gamma \in \Pi(X, a, b) \), let \( \tilde{\gamma} \in \Pi(X, b, a) \) denote the reversed path \( \tilde{\gamma}(s) = \gamma(1-s) \) for \( s \in [0, 1] \).

**Remark 5.2.2.** If \( \gamma \in \Pi(X, a, b) \) and \( \mu \in \Pi(X, b, c) \), then \( \gamma \ast \mu = \mu \ast \tilde{\gamma} \).

We check some useful path homotopy relations among compositions of paths.

**Lemma 5.2.3.** Let \( \gamma \in \Pi(X, a, b) \) with \( a, b \in X \).

a. If \( \gamma' \in \Pi(X, a, b) \) with \( \gamma \sim \gamma' \), then \( \tilde{\gamma} \sim \tilde{\gamma}' \).

b. We have \( e_a \ast \gamma \sim \gamma \ast e_b \).

c. We have \( \gamma \ast \tilde{\gamma} \sim e_a \) and \( \tilde{\gamma} \ast \gamma \sim e_b \).

d. If \( \mu \in \Pi(X, b, c) \) and \( \nu \in \Pi(X, c, d) \) for some \( c, d \in X \), then \( (\gamma \ast \mu) \ast \nu \sim \gamma \ast (\mu \ast \nu) \).

**Proof.**

a. If \( F \) is a homotopy from \( \gamma \) to \( \gamma' \), then we define \( G : [0, 1]^2 \to X \) by
\[
G(s, t) = F(1-s, t),
\]
and this is a homotopy from \( \tilde{\gamma} \) to \( \tilde{\gamma}' \).

b. Define \( F_a : [0, 1]^2 \to X \) by
\[
F_a(s, t) = \begin{cases} 
  a & s \in [0, 1-t], \\
  \gamma(t^{-1}(s-1+t)) & s \in (1-t, 1].
\end{cases}
\]

Then \( F_a \) is a path homotopy from \( e_a \ast \gamma \) to \( \gamma \). If we replace \( \gamma \) by \( \tilde{\gamma} \), we get \( e_b \ast \tilde{\gamma} \sim \tilde{\gamma} \), and then
\[
\gamma \ast e_b = \tilde{\gamma} \ast \tilde{\gamma} \sim \tilde{\gamma} = \gamma.
\]

c. Define \( H : [0, 1]^2 \to X \) by
\[
H(s, t) = \begin{cases} 
  \gamma(2s \ast (1-t)) & s \in [0, \frac{1}{2}], \\
  \gamma((2-2s) \ast (1-t)) & s \in [\frac{1}{2}, 1].
\end{cases}
\]

The \( H \) is a path homotopy from \( \gamma \ast \tilde{\gamma} \) to \( e_a \). We have \( \tilde{\gamma} \ast \gamma \sim e_b \) by replacing \( \gamma \) by \( \tilde{\gamma} \).

d. Define \( I : [0, 1]^2 \to X \) by
\[
I(s, t) = \begin{cases} 
  \gamma(4(1-t)^{-1}s) & s \in [0, \frac{1}{4}(1+t)], \\
  \mu(4s - 1 - t) & s \in [\frac{1}{4}(1+t), \frac{1}{4}(2+t)], \\
  \nu((2-t)^{-1}(4s - 2 - t)) & s \in [\frac{1}{4}(2+t), 1].
\end{cases}
\]

We leave it to the reader to check that \( I \) is a path homotopy from \( (\gamma \ast \mu) \ast \nu \) to \( \gamma \ast (\mu \ast \nu) \).
Definition 5.2.4. A loop in a topological space $X$ based at a point $a \in X$ is a path in $X$ from $a$ to $a$. The point $a$ is called the basepoint of the loop.

Notation 5.2.5. For $a \in X$, we set $\Pi(X, a) = \Pi(X, a, a)$ and $\pi_1(X, a) = \pi_1(X, a, a)$.

If we restrict our product maps on set of path homotopies to loops based at a point $x_0 \in X$, we obtain an operation on the set of paths

$$\Pi(X, x_0) \times \Pi(X, x_0) \to \Pi(X, a).$$

This operation makes $\Pi(X, x_0)$ into what is known as a group.

Definition 5.2.6. A group $G$ is a set together with an operation $G \times G \to G$ such that

i. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$,

ii. there exists an identity element $e \in G$ such that $e \cdot a = a = a \cdot e$ for all $a \in G$, and

iii. for every $a \in G$, there exists an inverse element $a^{-1} \in G$ such that $a \cdot a^{-1} = e$.

Here are just a few interesting groups.

Examples 5.2.7.

a. The integers $\mathbb{Z}$ with the operation $+$ forms a group. In this group, $e = 0$ and the inverse of $a$ is $-a$.

b. The nonzero real numbers $\mathbb{R} - \{0\}$ together with the operation $\cdot$ forms a group. In it, $e = 1$ and the inverse of $a$ is $a^{-1}$.

c. Given a set $X$, the set $S_X$ of bijections $f : X \to X$ forms a group with respect to the operation of composition. In it, the identity element is $\text{id}_X$ and the inverse of a bijection $f$ is its inverse function $f^{-1}$.

Proposition 5.2.8. For any $x_0 \in X$, the set $\pi_1(X, x_0)$ is a group under the operation induced by composition of paths.

Proof. The operation in question is given on the classes of loops $\gamma, \mu \in \Pi(X, x_0)$ by $[\gamma] \cdot [\mu] = [\gamma \ast \mu]$. That this makes $G$ into a group follows from the various parts of Lemma 5.2.3: that is, the operation is associative by part d, the identity element is $[e_a]$ by part b, and the inverse of $[\gamma]$ is $[\tilde{\gamma}]$ by part c. 

Definition 5.2.9. The fundamental group of a space $X$ relative to a basepoint $x_0 \in X$ is the group $\pi_1(X, x_0)$ together with the operation induced by composition of paths.

One might ask how the fundamental group depends upon the choice of basepoint. For this, we need a notion of equivalence among groups. Such an equivalence should be a bijection that respects the operation on its domain and codomain. A function between groups that respects these operations is called a homomorphism.

Definition 5.2.10. A function $f : G \to G'$ of groups is a homomorphism from the group $G$ to the group $G'$ if $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in G$. 

\[\square\]
Note that in the latter definition, the operation on the left is the operation on $G$ and the operation on the ring is the operation on $G'$.

**Definition 5.2.11.** A homomorphism from $G$ to $G'$ is an isomorphism if it is a bijection.

**Lemma 5.2.12.** If $f : G \to G'$ is an isomorphism of groups, then so is the inverse function $f^{-1} : G' \to G$.

**Proof.** Since $f$ is a bijection, its inverse $f^{-1}$ is as well. We must show that $f^{-1}$ is a homomorphism. Let $a', b' \in G'$, and note that there exist unique $a, b \in G$ with $f(a) = a'$ and $f(b) = b'$. We then have
\[
f^{-1}(a' \cdot b') = f^{-1}(f(a) \cdot f(b)) = f^{-1}(f(a \cdot b)) = a \cdot b.
\]
\[
\square
\]

**Example 5.2.13.** The function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ given by $\exp(x) = e^x$ is an isomorphism from the real numbers with the operation of addition to the positive real numbers with the operation of multiplication. That is, it is clearly bijective, and we have $e^{a+b} = e^a e^b$ for $a, b \in \mathbb{R}$. Its inverse is the logarithm function $\log : \mathbb{R}_{>0} \to \mathbb{R}$.

**Definition 5.2.14.** We say that two groups $G$ and $G'$ are *isomorphic* if there exists an isomorphism $f : G \to G'$, in which case we write $G \cong G'$.

There is no such thing as the set of all groups, as it is too large. However, the following still makes sense.

**Proposition 5.2.15.** The relation $\cong$ is an equivalence relation on any set of groups.

**Proof.** Let $G$, $H$ and $K$ be groups. Then $G \cong G$ via the identity map. If $G \cong H$, then $H \cong G$ by Lemma 5.2.12. If $G \cong H$ and $H \cong K$, then we have isomorphisms $f : G \to H$ and $f' : H \to K$. Then $f' \circ f$ is still a bijection, and $f'(f(a \cdot b)) = f'(f(a) \cdot f(b)) = f'(f(a)) \cdot f'(f(b))$ for $a, b \in G$, so $f' \circ f$ is a homomorphism as well, and therefore $G \cong K$.

So, we can now answer our question regarding fundamental groups relative to different basepoints.

**Proposition 5.2.16.** The fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ of a space $X$ relative to basepoints $x_0$ and $x_1$ are isomorphic if there exists a path in $X$ from $x_0$ to $x_1$.

**Proof.** Let $\lambda : [0, 1] \to X$ be a path with $\lambda(0) = x_0$ and $\lambda(1) = x_1$. Define a map
\[
\Pi(X, x_0) \to \Pi(X, x_1), \quad \gamma \mapsto \lambda \star \gamma \star \tilde{\lambda}
\]
for $\gamma \in \Pi(X, x_0)$. If $\gamma \sim \gamma'$, then $\lambda \star \gamma \star \tilde{\lambda} \sim \lambda \star \gamma' \star \tilde{\lambda}$, so this map induces a function
\[
f : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad f([\gamma]) = [\lambda][\gamma][\lambda]^{-1}
\]
This is a bijection since it has an inverse induced by $\mu \mapsto \lambda \star \mu \star \lambda$. It is then an isomorphism, since
\[
f([\gamma']) = [\lambda][\gamma'][\lambda]^{-1} = [\lambda][\gamma'][\lambda]^{-1} = [\lambda][\gamma][\lambda]^{-1}[\lambda][\gamma'][\lambda]^{-1} = f([\gamma])f([\gamma']).
\]
\[
\square\]
Remark 5.2.17. When $X$ is path connected, we often refer to the fundamental group of $X$ to mean the fundamental group relative to some basepoint, since all choices are isomorphic.

Remark 5.2.18. The isomorphism we constructed in the proof of Proposition 5.2.16 depends on the choice of a path from one basepoint to another. It is not in general unique, nor is it even necessarily the identity if the two points are the same.

Definition 5.2.19. A space $X$ is simply connected if it is path connected and $\pi_1(X,x_0)$ is the trivial group for some (equivalently, all) $x_0 \in X$.

Lemma 5.2.20. If $X$ is a simply connected space and $a,b \in X$, then any two paths in $X$ from $a$ to $b$ are path homotopic.

Proof. Let $\gamma, \gamma' \in \Pi(X,a,b)$. Then $\gamma^{-1} * \gamma' \in \Pi(X,a)$. Since $X$ is simply connected, $\gamma^{-1} * \gamma' \sim e_a$, so $\gamma' (\gamma^{-1} * \gamma') \sim e_a * \gamma$, from which it follows that $\gamma' \sim \gamma$.

Continuous maps between topological spaces give rise to maps between homotopy groups.

Lemma 5.2.21. Let $f: X \to Y$ be a map of topological spaces, and let $x_0 \in X$.

a. The path homotopy class of $f \circ \gamma$ in $\Pi(Y,f(x_0))$ depends only on the path homotopy class of $\gamma \in \Pi(X,x_0)$.

b. If $\mu \in \Pi(X,x_0)$, then $f \circ (\gamma * \mu) = (f \circ \gamma) * (f \circ \mu)$.

Proof. If $\gamma \sim \gamma'$ for some $\gamma' \in \Pi(X,x_0)$, and $F: [0,1]^2 \to X$ is a path homotopy from $\gamma$ to $\gamma'$, then $f \circ F$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$. Part b is immediate from the definition of composition of paths.

By Lemma 5.2.21, the following definition makes sense.

Definition 5.2.22. For a continuous function $f: X \to Y$ of topological spaces and $x_0 \in X$, the map

$$f_*: \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$$

given by $f_*([\gamma]) = [f \circ \gamma]$ is the homomorphism induced by $f$ on fundamental groups based at $x_0$.

The following property is easily verified from the definitions.

Lemma 5.2.23. If $f: X \to Y$ and $g: Y \to Z$ are continuous functions of topological spaces, then $(g \circ f)_* = g_* \circ f_*$ for any $x_0 \in X$. Moreover, $(\text{id}_X)_*$ is the identity homomorphism on $\pi_1(X,x_0)$ for any $x_0 \in X$.

In particular, if $f$ is a homeomorphism, then since $f \circ f^{-1} = \text{id}_X$, we have that $f_*$ is an isomorphism with inverse $(f^{-1})_*$. 

Definition 5.2.24. A retraction of $X$ onto a subspace $A$ is a continuous function $r: X \to A$ such that $r(a) = a$ for all $a \in A$.

Lemma 5.2.25. If $r: X \to A$ is a retraction of a space $X$ onto a subspace $A$, then for any $a_0 \in A$, the map $r_*: \pi_1(X,x_0) \to \pi_1(A,a_0)$ is surjective.

Proof. Let $\gamma$ be a loop in $A$ based at $a_0$. Then $\gamma$ is also a loop in $X$ based at $a_0$, and $r \circ \gamma = \gamma$, so $r_*$ is surjective.
5.3. Covering spaces

**Definition 5.3.1.** Let \( f : C \rightarrow X \) be a continuous map of topological spaces, and let \( U \) be an open set in \( X \) contained in \( f(C) \). We say that \( U \) is **evenly covered** by \( f \) if \( f^{-1}(U) \) is a disjoint union of open subspaces of \( C \), each of which is mapped homeomorphically onto \( U \) by \( f \).

**Remark 5.3.2.** If \( U \) is an open subset of \( X \) evenly covered by \( f : C \rightarrow X \) and \( x \in U \), then

\[
f^{-1}(U) = \bigcup_{c \in f^{-1}(x)} V_c,
\]

where \( V_c \) is an open neighborhood of \( c \) in \( C \) such that \( f|_{V_c} : V_c \rightarrow U \) is a homeomorphism.

**Definition 5.3.3.** We say that a continuous surjective function \( p : C \rightarrow X \) between topological spaces is a **covering map** if for each \( x \in X \), there exists an open neighborhood \( U \) of \( x \) such that \( U \) is evenly covered by \( p \). The space \( C \), together with its covering map, is then said to be a **covering space** of \( X \).

**Example 5.3.4.** View \( S^1 \) as the unit circle in \( \mathbb{C} \). The function \( p : \mathbb{R} \rightarrow S^1 \) given by \( p(x) = e^{2\pi i x} \) is a covering map. Inside any open neighborhood of \( 1 \in S^1 \), we have an open set \( U = p(\varepsilon, \varepsilon) \) for a sufficiently small \( \varepsilon < \frac{1}{2} \). The set \( p^{-1}(U) \) is the disjoint union of the open sets \( (n - \varepsilon, n + \varepsilon) \) for \( n \in \mathbb{Z} \). This \( U \) is a disjoint union of open neighborhoods of the points \( n \) forming the inverse image \( p^{-1}(1) \).

**Example 5.3.5.** The function \( f : S^1 \rightarrow S^1 \) defined by \( f(z) = z^n \) is a covering map. Since the polynomial \( z^n - a \) for \( a \in S^1 \) has exactly \( n \) roots in \( \mathbb{C} \), all of which have complex absolute value 1, every point has \( n \) points in its inverse image. If \( a = e^{2\pi i \theta} \) for some \( \theta \in \mathbb{R} \), then these roots have the form \( e^{2\pi i (\theta + j)/n} \), where \( 0 \leq j \leq n - 1 \).

The following is easily verified.

**Lemma 5.3.6.** Covering maps are surjective local homeomorphisms. In particular, they are open maps.

**Remark 5.3.7.** The converse to Lemma 5.3.6 not hold. For instance, consider the restriction \( f \) of the map \( p : \mathbb{R} \rightarrow S^1 \) of Example 5.3.4 to \( \mathbb{R}_{>0} \). It is a surjective local homeomorphism. Let \( U = p(-\varepsilon, \varepsilon) \) be an arbitrarily small neighborhood of \( 1 \) as in said example. Then \( p^{-1}(U) \cap (0, \varepsilon) \), and it does not map homeomorphically onto \( U \): in fact, its image does not even contain \( e \). Thus, \( p \) is not a covering map.

**Lemma 5.3.8.** Let \( p : C \rightarrow X \) be a covering map, and let \( B = p^{-1}(Y) \) be the inverse image in \( C \) of an open subspace \( Y \) of \( X \). Then \( p|_B : B \rightarrow Y \) is a covering map.

**Proof.** By definition, the restriction \( p|_B \) is a continuous surjective map. Let \( x \in Y \), and let \( U \) be an open neighborhood of \( x \) in \( X \) contained in \( p(B) \). By Remark 5.3.2, the open set \( p^{-1}(U) \) is a disjoint union of open sets \( V_c \) in \( C \) for each \( c \in p^{-1}(x) \) with \( p|_{V_c} : V_c \rightarrow U \) a homeomorphism. Let \( u \in U \) and \( b \in p^{-1}(u) \subseteq B \). Since \( p(V_b) = U \subseteq Y \), we have \( V_b \subseteq B \). That is, \( p|_B \) takes \( V_b \) homeomorphically onto its image \( U \). □
Lemma 5.3.9. Let \( p : C \to X \) and \( p' : C' \to X' \) be covering maps. Then the product map \( P : C \times C' \to X \times X' \) with \( P(c, c') = (p(c), p(c')) \) is a covering map as well.

Proof. For \((x, x') \in X \times X'\), let \( U \) and \( U' \) be open neighborhoods of \( x \) and \( x' \) in \( X \) and \( X' \) respectively such that \( p^{-1}(U) \) and \( (p')^{-1}(U') \) are disjoint unions of open neighborhoods of the points in the inverse images of \( x \) and \( x' \), respectively, such that the images of these open neighborhoods map homeomorphically under \( p \) and \( p' \) to \( U \) and \( U' \), again respectively. Then \( P^{-1}(U \times U') \) is a disjoint union of all products of these sets, one for each point in \( P^{-1}(x, x') \), and again, they each map homeomorphically to \( U \times U' \) under \( P \) by construction.

Example 5.3.10. Consider the product map \( \mathbb{R}^2 \to (S^1)^2 \) of the map \( p \) of Example 5.3.4 with itself. By Lemma 5.3.9, it is a covering map. That is, the plane is a covering space of the torus.

We next discuss the notion of lifting of paths to covering spaces.

Definition 5.3.11. Let \( f : X \to Y \) be a continuous function, and let \( p : C \to Y \) be a surjective continuous function. A (continuous) lifting of \( f \) to \( C \) is a continuous function \( \tilde{f} : X \to C \) such that \( p \circ \tilde{f} = f \). We say that \( \tilde{f} \) lifts \( f \) if \( \tilde{f} \) is a lifting of \( f \).

Proposition 5.3.12. Let \( p : C \to X \) be a covering map, let \( c_0 \in C \), and set \( x_0 = p(c_0) \). If \( \gamma : [0, 1] \to X \) is a path in \( X \) with initial point \( x_0 \), then there exists a unique lifting \( \tilde{\gamma} : [0, 1] \to C \) of \( \gamma \) to a path in \( C \) with initial point \( c_0 \).

Proof. Since \( p \) is a covering map, there exists an open cover \( \mathcal{U} \) of \( X \) by sets that are evenly covered by \( p \). Then \( \mathcal{U} = \{ p^{-1}(U) \mid U \in \mathcal{U} \} \) is an open cover of \( [0, 1] \). By Lemma 3.2.24, there exists \( N \geq 0 \) such that the intervals \( A_i = \left[ \frac{i}{N}, \frac{i+1}{N} \right) \) with \( 0 \leq i \leq N - 1 \) are each contained in some element of \( \mathcal{U} \), which has the form \( p^{-1}(U) \) for some \( U \in \mathcal{U} \).

We define \( \tilde{\gamma} \) with \( \tilde{\gamma}(0) = c_0 \) on each \( A_i \) recursively. Suppose we have defined \( \tilde{\gamma} \) on \( [0, \frac{i}{N}] \) for some \( i \geq 0 \). Let \( c_i = \tilde{\gamma}(\frac{i}{N}) \). Since \( p \) evenly covers \( U_i \), we have an open neighborhood \( V_i \) of \( c_i \) which maps homeomorphically to \( U_i \) under \( p \). Let \( f_i = (p|_{V_i})^{-1} : U_i \to V_i \) be the inverse homeomorphism, and define \( \tilde{\gamma}(s) = f_i \circ \gamma(s) \) for \( s \in A_i \). The map \( \tilde{\gamma} \) on \( [0, \frac{i+1}{N}] \) is then continuous by Lemma 2.1.12. Note that this is the only continuous extension of \( \tilde{\gamma}(s) \) from \( [0, \frac{i}{N}] \) to \( [0, \frac{i+1}{N}] \). That is, \( \tilde{\gamma}(\frac{i}{N}) \in V_i \) and \( f_i^{-1}(U_i) \) is the disjoint union of \( V_i \) and its complement in the inverse image, so in that \( \tilde{\gamma}|_{A_i} \) must have connected image, its entire image must lie in \( V_i \). But then the map \( p|_{V_i} : V_i \to U_i \) is bijective, so \( f_i \circ \gamma|_{A_i} \) is the only continuous lift of \( \gamma|_{A_i} \) to \( C \) in \( V_i \). Thus, we have defined our unique path \( \tilde{\gamma} : [0, 1] \to C \) lifting \( \gamma \) with \( \tilde{\gamma}(0) = c_0 \).

Similarly, we have the following, which we leave unproven. The proof is similar to that Proposition 5.3.12, replacing the even subdivision of \( [0, 1] \) into intervals with the subdivision of \( [0, 1]^2 \) into \( N^2 \) squares with vertices \( \left( \frac{i}{N}, \frac{j}{N} \right) \) with \( 0 \leq i, j \leq N \), for some \( N \).

Lemma 5.3.13. Let \( p : C \to X \) be a covering map with \( p(c_0) = x_0 \) for some \( c_0 \in C \) and \( x_0 \in X \). If \( F : [0, 1]^2 \to X \) is a continuous map with \( F(0, 0) = c_0 \), then there exists a unique lifting \( \tilde{F} : [0, 1]^2 \to C \) with \( \tilde{F}(0, 0) = c_0 \).

We use this to prove the following.
Let \( p : C \to X \) be a covering, let \( \gamma, \gamma' : [0, 1] \to X \) be path homotopic paths. Set \( x_0 = \gamma(0) = \gamma'(0) \), and let \( c_0 \in p^{-1}(x_0) \). Let \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) be the unique lifts of \( \gamma \) and \( \gamma' \), respectively, to paths in \( C \) with initial point \( c_0 \). Then \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) satisfy \( \tilde{\gamma}(1) = \tilde{\gamma}'(1) \) and are path homotopic.

PROOF. Let \( F \) be a path homotopy between \( \gamma \) and \( \gamma' \), and use Lemma 5.3.13 to uniquely lift it to \( \tilde{F} : [0, 1]^2 \to C \) with \( \tilde{F}(0, 0) = c_0 \). We repeatedly use the uniqueness of Proposition 5.3.12. Then \( \tilde{F}(s, 0) \) is a path lifting \( \gamma \) with initial point \( c_0 \), so must be \( \tilde{\gamma} \). Similarly, \( \tilde{F}(0, t) \) is a path lifting the constant path \( e_{x_0} \) with initial point \( c_0 \), so must be the constant path \( e_{c_0} \). Then \( \tilde{F}(0, 1) = c_0 \), and \( \tilde{F}(s, 1) \) is a path lifting \( \gamma' \) with initial point \( c_0 \), so must be \( \tilde{\gamma}' \). Finally, \( \tilde{F}(1, t) \) is a path lifting the constant path \( e_{x_1} \) with \( \tilde{F}(1, t) = \tilde{\gamma}'(1) \), so must also be constant. Thus, \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) have the same final point, and \( \tilde{F} \) is a path homotopy from \( \tilde{\gamma} \) to \( \tilde{\gamma}' \).

We are now ready to prove a theorem connecting covering spaces and the fundamental group.

THEOREM 5.3.15. Let \( p : C \to X \) be a covering, let \( x_0 \in X \) and \( c_0 \in C \) with \( p(x_0) = c_0 \). The function \( \phi_{c_0} : \pi_1(X, x_0) \to p^{-1}(x_0) \) taking \( [\gamma] \) for \( \gamma \in \Pi(X, x_0) \) to the endpoint of the unique lift of \( \gamma \) to C with initial point \( c_0 \) is well-defined. If \( C \) is path connected, then \( \phi_{c_0} \) is surjective, and if \( C \) is simply connected, then \( \phi_{c_0} \) is bijective.

PROOF. That \( \phi_{c_0} \) is well-defined is an immediate consequence of Proposition 5.3.14. If \( C \) is path connected and \( c_1 \in p^{-1}(x_0) \), then choose a path \( \lambda \in \Pi(X, c_0, c_1) \). Then \( \lambda \) is a lift of \( \gamma = p \circ \lambda \in \Pi(X, x_0) \), and \( \phi_{c_0}([\gamma]) = c_1 \).

Suppose that \( C \) is simply connected, and let \( \gamma, \mu \in \Pi(X, x_0) \) with \( \phi_{c_0}([\gamma]) = \phi_{c_0}([\mu]) \). Let \( c_1 \) denote the latter point. Let \( \tilde{\gamma} \) and \( \tilde{\mu} \) be the unique lifts to \( C \) of \( \gamma \) and \( \mu \), respectively, with initial point \( c_0 \). Then have final point \( c_1 \) by assumption. Since \( C \) is simply connected, there then exists a homotopy \( \tilde{F} \) from \( \tilde{\gamma} \) to \( \tilde{\mu} \), and then \( p \circ \tilde{F} \) is a homotopy from \( \gamma \) to \( \mu \). In other words, we have \( [\gamma] = [\mu] \), so \( \phi_{c_0} \) is injective. We can now compute the fundamental group of \( S^1 \) relative to any basepoint.

THEOREM 5.3.16. The fundamental group of \( S^1 \) is isomorphic to the integers \( \mathbb{Z} \) with the operation of addition.

PROOF. The map \( p : \mathbb{R} \to S^1 \) given by \( p(x) = e^{2\pi i x} \) is a covering. Since \( \mathbb{R} \) is simply connected and the inverse image of 1 is \( \mathbb{Z} \subset \mathbb{R} \), Theorem 5.3.15 tells us that the map \( \phi_0 : \pi_1(S^1, 1) \to \mathbb{Z} \) is a bijection.

We need only show that \( \phi_0 \) is a homomorphism. So, let \( \gamma, \mu \in \Pi(S^1, 1) \), and set \( n = \phi_0([\gamma]) \) and \( m = \phi_0([\mu]) \). Let \( \tilde{\gamma} \) be a lift of \( \gamma \) with initial point 0; its final point is then \( n \). Let \( \tilde{\mu} \) be the unique lift of \( \mu \) with initial point 0, and note that the function \( \tilde{\mu}' = n + \tilde{\mu} \) is a lift of \( \mu \) with initial point \( n \) and final point \( n + \tilde{\mu}(1) = n + m \). Then \( \tilde{\gamma} \ast \tilde{\mu} \) lifts \( \gamma \ast \mu \) and has final point \( n + m \), so \( \phi_0([\gamma][\mu]) = n + m = \phi_0([\gamma]) + \phi_0([\mu]) \).