# Reciprocity maps with restricted ramification

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**Abstract**

We compare two maps that arise in study of the cohomology of global fields with ramification restricted to a finite set $S$ of primes. One of these maps, which we call an $S$-reciprocity map, interpolates the values of cup products in $S$-ramified cohomology. In the case that the field is the $p$th cyclotomic field for an odd prime $p$ and $S$ consists of the unique prime over $p$, we describe an application to the values of this cup product on cyclotomic $p$-units, as first studied in [McS], confirming a relationship observed in [Sh2] between pairing values in distinct eigenspaces for the action of Galois. For more general cyclotomic fields, we relate the cokernel of an $S$-reciprocity map to the dual Selmer groups of residual representations attached to newforms that satisfy congruences with Eisenstein series, motivated by a conjecture of the author's in [Sh3].

## 1 Introduction

The primary goal of this work of multiple aims is the comparison of two maps that arise in the Galois cohomology of global fields with restricted ramification. In this introduction, we introduce the two maps for a fixed global field. In the body of the paper, we shall study the maps that they induce up towers of such fields.

Fix a prime $p$, a global field $F$ of characteristic not equal to $p$, and a finite set of primes $S$ of $F$ including all those above $p$ and any real infinite places. We suppose that either $p$ is odd or $F$ has no real places. We will use the terms *S-ramified* and *S-split* to mean unramified outside of the primes in $S$ and completely split at the primes in $S$, respectively. Let $G_{F,S}$ denote the Galois group of the maximal $S$-ramified extension $\Omega_S$ of $F$.

We will be concerned with three objects related to the arithmetic of $F$:

- the pro-$p$ completion $\mathscr{U}_F$ of the $S$-units of $F$,
- the Galois group $\mathfrak{X}_F$ of the maximal $S$-ramified abelian pro-$p$ extension of $F$, and

- the Galois group $Y_F$ of the maximal unramified, $S$-split abelian $p$-extension $H_F$ of $F$.

Each of these three groups has a fairly simple cohomological description. That is, $\mathscr{U}_F$ is canonically isomorphic to the continuous cohomology group $H^1(G_{F,S}, \mathbf{Z}_p(1))$ via Kummer theory, and $\mathfrak{X}_F$ is canonically isomorphic to its Pontryagin double dual $H^1(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p)^\vee$. As a quotient of $\mathfrak{X}_F$, the group $Y_F$ is identified with the image of the Poitou-Tate map

$$H^1(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p)^\vee \to H^2(G_{F,S}, \mathbf{Z}_p(1)),$$

where $A^\vee$ denotes the Pontryagin dual of a topological abelian group $A$. Alternatively, we may use class field theory and Kummer theory to identify $Y_F$ with a subgroup of $H^2(G_{F,S}, \mathbf{Z}_p(1))$. (That these provide the same identification is non-obvious but known; see the note [Sh4] for a proof.)

The $S$-reciprocity map for $F$ is a homomorphism

$$\Psi_F \colon \mathscr{U}_F \to H^2(G_{F,S}, \mathfrak{X}_F(1))$$

that interpolates values of the cup products

$$H^1(G_{F,S}, \mathbf{Z}/p^n\mathbf{Z}) \times H^1(G_{F,S}, \mathbf{Z}_p(1)) \xrightarrow{\cup} H^2(G_{F,S}, \mu_{p^n})$$

in the sense that if $\rho \in \mathrm{Hom}(\mathfrak{X}_F, \mathbf{Z}/p^n\mathbf{Z}) \subseteq \mathfrak{X}_F^\vee$ and $\rho^*$ denotes the induced map

$$\rho^* \colon H^2(G_{F,S}, \mathfrak{X}_F(1)) \to H^2(G_{F,S}, \mu_{p^n}),$$

then for any $a \in \mathscr{U}_F$, we have

$$\rho \cup a = \rho^*(\Psi_F(a)).$$

In fact, $\Psi_F$ may itself be viewed as left cup product with the element of $H^1(G_{F,S}, \mathfrak{X}_F)$ that is the quotient map $G_{F,S} \to \mathfrak{X}_F$. In the case that $F$ is abelian, the $S$-reciprocity map turns out to be of considerable arithmetic interest, its values relating to $p$-adic $L$-values of cuspidal eigenforms that satisfy congruences with Eisenstein series (see [Sh3] and [FK2]).

We also have a second homomorphism

$$\Theta_F \colon \mathscr{U}_F \to H^1(G_{F,S}, Y_F^\vee)^\vee$$

that can be defined as follows. Let $\mathscr{Y}$ denote the quotient of the group ring $\mathbf{Z}_p[Y_F]$ by the square of its augmentation ideal, so that we have an exact sequence

$$0 \to Y_F \to \mathscr{Y} \to \mathbf{Z}_p \to 0$$

of $\mathbf{Z}_p[G_{F,S}]$-modules, the $G_{F,S}$-cocycle determining the class of the extension being the quotient map $G_{F,S} \to Y_F$. For each $v \in S$, choose a prime over $v$ in $\Omega_S$ and thereby a homomorphism $j_v \colon G_{F_v} \to G_{F,S}$ from the absolute Galois group of $F_v$. The exact sequence is split as a sequence of $\mathbf{Z}_p[G_{F_v}]$-modules for the action induced by $j_v$.

The target of the cup product pairing

$$H^i(G_{F,S}, \mathbf{Z}_p(1)) \times H^{2-i}(G_{F,S}, Y_F^\vee) \to H^2(G_{F,S}.Y_F^\vee(1))$$

is the quotient of $H^2(G_{F,S}, \mathscr{Y}^\vee(1))$ by the image of $H^2(G_{F,S}, \mu_{p^\infty})$. The dual of the local splitting of $\mathscr{Y} \to \mathbf{Z}_p$ and the sum of invariant maps of local class field theory provide maps

$$H^2(G_{F,S}, \mathscr{Y}^\vee(1)) \to \bigoplus_{v \in S} H^2(G_{F_v}, \mu_{p^\infty}) \to \mathbf{Q}_p/\mathbf{Z}_p,$$

and their composite is trivial on $H^2(G_{F,S}, \mu_{p^\infty})$. Thus, we have a well-defined pairing

$$H^i(G_{F,S}, \mathbf{Z}_p(1)) \times H^{2-i}(G_{F,S}, Y_F^\vee) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

For $i = 1$, this pairing yields $\Theta_F$ via Pontryagin duality. For $i = 2$, it induces a map

$$q_F \colon H^2(G_{F,S}, \mathbf{Z}_p(1)) \to Y_F.$$

An alternate, but equivalent, definition of these maps is given in Section 3.1 (see also Section 2.3). Note that $\Theta_F$ and $q_F$ need not be canonical, as they can depend on the choices of primes of $\Omega_S$ over each $v \in S$ (or rather, the primes of $H_F$ below them).

As $G_{F,S}$ acts trivially on the module $\mathfrak{X}_F$ (by conjugation upon restriction) and has $p$-cohomological dimension 2, we have

$$H^2(G_{F,S}, \mathfrak{X}_F(1)) \cong H^2(G_{F,S}, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathfrak{X}_F.$$

Moreover, there are canonical isomorphisms

$$H^1(G_{F,S}, Y_F^\vee)^\vee \cong \mathrm{Hom}(\mathfrak{X}_F, Y_F^\vee)^\vee \cong \mathrm{Hom}(Y_F \otimes_{\mathbf{Z}_p} \mathfrak{X}_F, \mathbf{Q}_p/\mathbf{Z}_p)^\vee \cong Y_F \otimes_{\mathbf{Z}_p} \mathfrak{X}_F,$$

through which we identify the leftmost and rightmost groups in the equation.

We may now state our key result, the Iwasawa-theoretic generalization of which is found in Theorem 3.1.4.

**Theorem.** *The map $q_F$ is a splitting of the injection of $Y_F$ in $H^2(G_{F,S}, \mathbf{Z}_p(1))$ given by Poitou-Tate duality, and the diagram*

$$\begin{tikzcd} \mathscr{U}_F \arrow[r, "\Psi_F"] \arrow[dr, "-\Theta_F"'] & H^2(G_{F,S}, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathfrak{X}_F \arrow[d, "q_F \otimes \mathrm{id}"] \\ & Y_F \otimes_{\mathbf{Z}_p} \mathfrak{X}_F \end{tikzcd}$$

*commutes.*

The case that $F = \mathbf{Q}(\mu_p)$ for an odd prime $p$ and $S$ contains only the prime over $p$ provides an entertaining application. Consider the cup product pairing

$$( \ , \ ) \colon \mathscr{U}_F \times \mathscr{U}_F \to Y_F \otimes_{\mathbf{Z}_p} \mu_p.$$

Let $\Delta = \mathrm{Gal}(F/\mathbf{Q})$, let $\omega \colon \Delta \to \mathbf{Z}_p^\times$ be the Teichmüller character, and for odd $i \in \mathbf{Z}$, let $\eta_i \in \mathscr{U}_F$ denote the projection of $(1-\zeta_p)^{p-1}$ to the $\omega^{1-i}$-eigenspace $\mathscr{U}_F^{(1-i)}$ of $\mathscr{U}_F$. Suppose that Vandiver's conjecture holds at $p$ and that two distinct eigenspaces $Y_F^{(1-k)}$ and $Y_F^{(1-k')}$ of $Y_F$ are nonzero for even $k, k' \in \mathbf{Z}$. Then a simple argument using Theorem 1 tells us that $(\eta_{p-k}, \eta_{k+k'-1})$ is nonzero if and only if $(\eta_{p-k'}, \eta_{k+k'-1})$ is nonzero (see Theorem 3.2.1). We find this rather intriguing, as the two values lie in distinct eigenspaces of $Y_F \otimes_{\mathbf{Z}_p} \mu_p$ that would at first glance appear to bear little relation to one another.

The second goal of this paper is the application of our abstract results to the study of Selmer groups of modular representations. In particular, we consider a main conjecture of sorts for a dual Selmer group of the family of residual representations attached to cuspidal eigenforms that satisfy mod $p$ congruences with ordinary Eisenstein series, proving it under certain assumptions (see Theorem 4.3.4). Our study in Section 4 has its roots in the work of Greenberg, who first considered residual Selmer groups at $p = 691$ for the weight 12 eigenform $\Delta$ for $\mathrm{SL}_2(\mathbf{Z})$ in [Gr1, Section 9] (see also [Gr2, Section 4]).

The author has previously conjectured a precise relationship between the reduction of the two-variable $p$-adic $L$-function of Mazur-Kitagawa modulo an Eisenstein ideal and the value of a reciprocity map on a norm-compatible system of $p$-units in a $p$-cyclotomic tower [Sh3, Conjecture 6.3]. Under hypotheses that imply the conjecture by work of Fukaya and Kato [FK2] and Wake and Wang Erickson [WWE], we elucidate the role that the map $\Theta_F$ plays in the structure of Selmer groups of certain twists of reducible residual modular representations. Here, we consider characters with parity opposite to that imposed in earlier well-known work of Greenberg and Vatsal [GV], which complicates the structure of the Selmer groups considerably. The theorem above allows us to relate the value of the reciprocity map, and hence the two-variable $p$-adic $L$-function, to the Selmer group (see Theorem 4.3.4). This may be seen as extending the results of [Sh1, Section 6].

In Section 4.4, we lift our results from the residual setting. Assuming residual irreducibility, Ochiai formulated a two-variable main conjecture [Oc] and proved a divisibility using Kato's Euler system [Ka] for Selmer groups of families of modular representations. A three-variable extension of the other divisibility was proven under various hypotheses in the groundbreaking work of Skinner and Urban [SU]. In our residually reducible setting, a main conjecture is necessarily dependent on a choice of lattice, so we first formulate the conjecture of interest to us in Conjecture 4.4.3. Fukaya and Kato have proven a result towards this main conjecture in a particular eigenspace in the case of tame level one [FK2, Section 11]. We set out to glean what information we can about modular Selmer groups from the Selmer groups of their residual representations. For this, we suppose that our version of the main conjecture holds and prove that Greenberg's conjecture on the bounded growth of plus parts of $p$-parts of class class groups and other mild hypotheses then imply that the dual Selmer groups of our modular representations are pseudo-cyclic (see Theorem 4.4.5).

The appendix of this paper contains generalities on topological modules and continuous cohomology preliminary to our results, but it may be safely skipped without sacrificing significant understanding of the rest of the paper. Section 2 concerns the abstract theory of the rather simple sorts of Selmer complexes that we will consider, including duality for such complexes. The maps $\Psi_F$ and $\Theta_F$ appear as connecting homomorphisms in the cohomology of such Selmer complexes.

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# 2 Cohomology

The following section provides background on the Iwasawa-theoretic generalizations of Tate and Poitou-Tate duality that we require. For the case of a complete local noetherian ring with finite residue field, these dualities are detailed in the work of Nekovar [Ne]. The case of a general profinite ring can be found in the work of Lim [Li]. We use the standard conventions for signs of differentials on complexes, as found in [Ne, Chapter 1].

## 2.1 Connecting maps and cup products

Let $G$ be a profinite group, and let $\Lambda$ be a profinite ring. We suppose that $\Lambda$ is a topological $R$-algebra for a commutative profinite ring $R$ in its center. For later use, we fix a set $\mathscr{I}$ of open ideals of $\Lambda$ that forms a basis of neighborhoods of 0 in $\Lambda$.

Let $\mathscr{T}_{\Lambda,G}$ denote the category of topological $\Lambda$-modules endowed with a continuous $\Lambda$-linear action of $G$, which is to say topological $\Lambda[\![G]\!]$-modules, with morphisms the continuous $\Lambda[G]$-module homomorphisms. Let $\mathscr{C}_{\Lambda,G}$ (resp., $\mathscr{D}_{\Lambda,G}$) denote the full subcategories of compact (resp., discrete) $\Lambda$-modules. Let $\mathscr{C}_{\Lambda}$ (resp., $\mathscr{D}_{\Lambda}$) denote the full subcategories of modules with trivial $G$-actions.

Let

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0 \tag{2.1}$$

be a short exact sequence in $\mathscr{C}_{\Lambda,G}$ or $\mathscr{D}_{\Lambda,G}$. Such a sequence gives rise to a short exact sequence

$$0 \to C(G,A) \to C(G,B) \to C(G,C) \to 0$$

of inhomogeneous continuous $G$-cochain complexes (see Appendix A.2), and the maps between the terms of the complexes are continuous. In the case that the modules of (2.1) are in $\mathscr{C}_{\Lambda,G}$, we suppose in what follows that $G$ has the property that the cohomology groups $H^i(G,M)$ of $C(G,M)$ are finite for every finite $\mathbf{Z}[G]$-module $M$. The following is standard.

**Lemma 2.1.1.** *Let $s\colon C \to B$ be a continuous function that splits $\pi$ of* (2.1)*. Define*

$$\partial^i\colon H^i(G,C) \to H^{i+1}(G,A)$$

*on the class of a cocycle $f$ to be the class of $\iota^{-1} \circ d_B^i(s \circ f)$, where $d_B^i$ denotes the ith differential on $C(G,B)$. The resulting sequence*

$$\cdots \to H^i(G,A) \to H^i(G,B) \to H^i(G,C) \xrightarrow{\partial^i} H^{i+1}(G,A) \to \cdots$$

*is exact in $\mathscr{C}_{\Lambda}$ (resp., $\mathscr{D}_{\Lambda}$).*

*Proof.* The existence of $s$ in the case of $\mathscr{C}_{\Lambda,G}$ is just [RZ, Proposition 2.2.2]. That $\partial^i$ is a homomorphism of $\Lambda$-modules and that the sequence is exact are standard. That the cohomology groups lie in the stated categories are Lemma A.2.1 and Proposition A.2.2. That the homomorphisms are continuous in the case of $\mathscr{C}_{\Lambda,G}$ follows from Proposition A.2.2. That is, the continuity of the maps induced by $\iota$ and $\pi$ on cochains implies their continuity on cocycles and hence on the quotient cohomology groups. As for $\partial^i$, its continuity then follows from the continuity of the maps induced by $s$ and the corresponding splitting of $\iota$ on cochains and the continuity of the $i$th differential $d_B^i$ on $C^i(G,B)$. □

Suppose now that $\pi$ has a continuous splitting $s\colon C \to B$ of $\Lambda$-modules. Then we obtain a 1-cocycle

$$\chi\colon G \to \mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A)$$

given by

$$\chi(\sigma)(c) = \sigma s(\sigma^{-1}c) - s(c) \tag{2.2}$$

for $c \in C$, the class of which is independent of $s$. Conversely, the Galois action on $B$ is prescribed by the cocycle using (2.2). As usual, $s$ gives rise to a $\Lambda$-module splitting $t\colon B \to A$ of $\iota$ such that $\iota \circ t + s \circ \pi$ is the identity.

The following is well-known (see, e.g., [Fl, Proposition 3]).

**Lemma 2.1.2.** *Endow* $\mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A)$ *with the compact-open topology.*

*a. The group* $\mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A)$ *is an object of* $\mathscr{T}_{R,G}$.

*b. The canonical evaluation map* $C \times \mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A) \to A$ *is continuous.*

*c. The* 1*-cocycle* $\chi\colon G \to \mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A)$ *is continuous.*

We wish to compare our connecting homomorphisms with cup products. We have continuous cup products

$$C^i(G,\mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A)) \times C^j(G,C) \xrightarrow{\cup} C^{i+j}(G,A)$$

that satisfy $\lambda(\rho \cup f) = \rho \cup \lambda f$ for all $\lambda \in \Lambda$, $\rho \in C^i(G,\mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A))$, and $f \in C^j(G,C)$. For $\rho = \chi$ (with $i = 1$), the cup product $\chi \cup f \in C^{j+1}(G,A)$ may be described explicitly by the $(j+1)$-cochain

$$(\chi \cup f)(\sigma,\tau) = \chi(\sigma)\sigma f(\tau)$$

for $\sigma \in G$ and $\tau$ a $j$-tuple in $G$. We set $\tilde{\chi}(f) = \chi \cup f$.

*Remark* 2.1.3. For a cochain complex $X$ and $j \in \mathbf{Z}$, the complex $X[j]$ is that with $X[j]^i = X^{i+j}$ and differential $d_{X[j]}$ on $X[j]$ that is $(-1)^j$ times the differential $d_X$ on $X$.

Since $d(\chi \cup f) = (-1)^j \chi \cup df$, we have maps of sequences of compact or discrete $\Lambda$-modules

$$\tilde{\chi}\colon C(G,C) \to C(G,A)[1]$$

The map $\tilde{\chi}$ commutes with the differentials of these complexes up to the sign $(-1)^{j+1}$ in degree $j$. The map $\tilde{\chi}$ may of course be made into a map of of complexes by adjusting the signs, but we have no cause to do this here.

**Lemma 2.1.4.** *For all $i \geq 0$, the cup product map $\tilde{\chi}^i\colon H^i(G,C) \to H^{i+1}(G,A)$ agrees with the connecting homomorphism $\partial^i$.*

*Proof.* Let $f$ be an $i$-cocycle on $G$ with values in $C$, and set $g = s \circ f$. The coboundary of $g$ has values in $A$, and one sees immediately that

$$dg(\sigma,\tau) = \sigma s(f(\tau)) - s(\sigma f(\tau)) = \chi(\sigma)(\sigma f(\tau)) = (\chi \cup f)(\sigma,\tau)$$

for $\sigma \in G$ and $\tau$ an $i$-tuple of elements of $G$. $\square$

If $X \in \mathscr{T}_{\Lambda,G}$ is locally compact, then its Pontryagin dual $X^\vee = \mathrm{Hom}_{\mathrm{cts}}(X,\mathbf{R}/\mathbf{Z})$ with $G$-action given by $(g\phi)(x) = \phi(g^{-1}x)$ for $g \in G$, $x \in X$, and $\phi \in X^\vee$ is a locally compact object of $\mathscr{T}_{\Lambda^\circ,G}$, where $\Lambda^\circ$ denotes the opposite ring to $\Lambda$. Pontryagin duality induces an exact equivalence of categories between $\mathscr{C}_{\Lambda,G}$ and $\mathscr{D}_{\Lambda^\circ,G}$.

Now suppose that the exact sequence (2.1) is of modules in $\mathscr{C}_{\Lambda,G}$, and suppose $A$ and $C$ are endowed with the $\mathscr{I}$-adic topology. The dual of (2.1) fits in an exact sequence

$$0 \to C^\vee \to B^\vee \to A^\vee \to 0 \tag{2.3}$$

in $\mathscr{D}_{\Lambda^\circ,G}$. The canonical isomorphism $\mathrm{Hom}_\Lambda(C,A) \cong \mathrm{Hom}_{\Lambda^\circ}(A^\vee,C^\vee)$ (see Proposition A.1.9) then produces a continuous 1-cocycle

$$\chi^*\colon G \to \mathrm{Hom}_{\Lambda^\circ}(A^\vee,C^\vee),$$

from $\chi$. A direct computation shows that $\chi^*$ satisfies

$$\chi^*(\sigma)(\phi) = -\sigma t^*(\sigma^{-1}\phi) + t^*(\phi)$$

for $\phi \in A^\vee$ and $t^*\colon A^\vee \to B^\vee$ the map associated to $t$. That is, the cocycle $-\chi^*$ defines the class of the dual extension (2.3) in $H^1(G,\mathrm{Hom}_{\Lambda^\circ}(A^\vee,C^\vee))$. By Lemma 2.1.4, we therefore have the following.

**Lemma 2.1.5.** *The cup product map $\widetilde{\chi^{*i}}$ is the negative of the connecting homomorphism $H^i(G,A^\vee) \to H^{i+1}(G,C^\vee)$.*

## 2.2 Global fields

Let $p$ be a prime number, and now suppose that $\Lambda$ is a pro-$p$ ring. Let $F$ be a global field of characteristic not equal to $p$, and let $S$ denote a finite set of primes of $F$ including those above $p$ and any real places of $F$. Let $S^f$ (resp., $S^\infty$) denote the set of finite (resp., real) places in $S$.

We use $G_{F,S}$ to denote the Galois group of the maximal unramified outside $S$ extension of $F$. Fix, once and for all, a local embedding of the algebraic closure of $F$ at a prime above each $v \in S$ and therefore a homomorphism $j_v \colon G_{F_v} \to G_{F,S}$, where $G_{F_v}$ denotes the absolute Galois group of the completion $F_v$ of $F$ at the prime over $v$.

For $M \in \mathscr{T}_{\Lambda,G_{F,S}}$, we consider the direct sum of local cochain complexes

$$C_l(G_{F,S},M) = \bigoplus_{v \in S^f} C(G_{F_v},M) \oplus \bigoplus_{v \in S^\infty} \widehat{C}(G_{F_v},M), \tag{2.4}$$

where $\widehat{C}(G_{F_v},M)$ denotes the total complex of the Tate complex of continuous cochains for $v \in S^\infty$. We then define (modified) continuous compactly supported cochain complex of $M$ by

$$C_c(G_{F,S},M) = \mathrm{Cone}(C(G_{F,S},M) \xrightarrow{\ell_S} C_l(G_{F,S},M))[-1], \tag{2.5}$$

where $\ell_S = (\ell_v)_{v \in S}$ is the sum of the localization maps $\ell_v$ defined by $\ell_v(f) = f \circ j_v$.

*Remark* 2.2.1. Let $X$ and $Y$ be cochain complexes. If $C = \mathrm{Cone}(f \colon X \to Y)[-1]$, then $C = X \oplus Y[-1]$ with differential

$$d_C(x,y) = (d_X(x), -f(x) - d_Y(y)).$$

The $i$th cohomology group of the complex $C_c(G_{F,S},M)$ (resp., $C_l(G_{F,S},M)$) is denoted $H^i_c(G_{F,S},M)$ (resp., $H^i_l(G_{F,S},M)$). Note that if $M$ is finite, then $H^i_l(G_{F,S},M)$ and $H^i(G_{F,S},M)$ are finite for all $i \in \mathbf{Z}$, and as a result, so are the $H^i_c(G_{F,S},M)$.

For $T \in \mathscr{C}_{\Lambda,G_{F,S}}$, we may then view each $H^i_*(G_{F,S},T)$ as an object of $\mathscr{C}_\Lambda$, where $*$ denotes either no symbol, $l$, or $c$. We then have a long exact sequence

$$\cdots \to H^i_c(G_{F,S},T) \to H^i(G_{F,S},T) \to H^i_l(G_{F,S},T) \to H^{i+1}_c(G_{F,S},T) \to \cdots$$

in $\mathscr{C}_\Lambda$. Similarly, we may view each $H^i_*(G_{F,S},A)$ for $A \in \mathscr{D}_{\Lambda,G_{F,S}}$ as an object of $\mathscr{D}_\Lambda$, and we obtain a corresponding long exact sequence for such an $A$.

Suppose that

$$\phi \colon M \times N \to L$$

for $M \in \mathscr{T}_{\Lambda,G_{F,S}}$, $N \in \mathscr{T}_{\Lambda^\circ,G_{F,S}}$, and $L \in \mathscr{T}_{R,G_{F,S}}$ is a continuous, $\Lambda$-balanced, $G_{F,S}$-equivariant homomorphism. Similarly to Lemma A.2.3, we have compatible continuous, $\Lambda$-balanced cup products

$$\begin{aligned}
C^i_l(G_{F,S},M) \times C^j_l(G_{F,S},N) &\xrightarrow{\cup_l} C^{i+j}_l(G_{F,S},L) \\
C^i(G_{F,S},M) \times C^j_c(G_{F,S},N) &\xrightarrow{\cup_c} C^{i+j}_c(G_{F,S},L) \\
C^i_c(G_{F,S},M) \times C^j(G_{F,S},N) &\xrightarrow{{}_c\cup} C^{i+j}_c(G_{F,S},L)
\end{aligned}$$

of $R$-modules for all $i, j \in \mathbf{Z}$ (see [Ne, Section 5.7]). As in Lemma A.2.3, if we introduce bimodule structures on $M$, $N$, and $L$ that commute with the $G_{F,S}$-action, and if the pairing $\phi$ is also linear for the new left and right actions on $M$ and $N$, respectively, then the cup products are likewise linear for them.

Class field theory yields that $H^i_c(G_{F,S}, T)$ (and $H^i_c(G_{F,S}, A)$) are trivial for $i > 3$ and

$$H^3_c(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p.$$

Taking the $\Lambda$-balanced pairing $T^\vee \times T(1) \to \mathbf{Q}_p/\mathbf{Z}_p(1)$, cup products then give rise to the compatible isomorphisms in $\mathscr{C}_\Lambda$ of Tate and Poitou-Tate duality

$$\begin{aligned} \beta^i_{l,T} &\colon H^i_l(G_{F,S}, T(1)) \to H^{2-i}_l(G_{F,S}, T^\vee)^\vee \\ \beta^i_T &\colon H^i(G_{F,S}, T(1)) \to H^{3-i}_c(G_{F,S}, T^\vee)^\vee \\ \beta^i_{c,T} &\colon H^i_c(G_{F,S}, T(1)) \to H^{3-i}(G_{F,S}, T^\vee)^\vee. \end{aligned}$$

(see [Li, Theorem 4.2.6]).

## 2.3 Connecting maps and cup products

Suppose that we are given an exact sequence of compact or discrete modules as in (2.1) (with $G = G_{F,S}$) and a continuous splitting $s\colon C \to B$ of $\Lambda$-modules. Let $\chi\colon G_{F,S} \to \mathrm{Hom}_{\Lambda,\mathrm{cts}}(C, A)$ be the induced continuous 1-cocycle.

We wish to define a certain Selmer complex for $B$ under the assumption that the class of $\chi$ is locally trivial in the sense that $\ell_v(\chi)$ is a $G_{F_v}$-coboundary for each $v \in S$. For this, we need the following lemma, the proof of which is straightforward and left to the reader.

**Lemma 2.3.1.** *Fix a continuous homomorphism $\varphi_v\colon C \to A$ of $\Lambda$-modules. Give $A$, $B$, and $C$ the $G_{F_v}$-actions induced by $j_v$ and their $G_{F,S}$-actions. The following are equivalent:*

*(i)* $\ell_v(\chi) = d\varphi_v$,

*(ii)* $s - \iota \circ \varphi_v$ *is* $G_{F_v}$*-equivariant,*

*(iii)* $t + \varphi_v \circ \pi$ *is* $G_{F_v}$*-equivariant.*

For each $v \in S$, assume that $\ell_v(\chi)$ is a coboundary, and choose $\varphi_v$ as in Lemma 2.3.1. Note that the following construction can be affected by these choices, even on cohomology. If $\mathrm{Hom}_{\Lambda,\mathrm{cts}}(C, A)$ has trivial $G_{F,S}$-action, then it is always possible to choose $\varphi_v = 0$. We then define

$$C_f(G_{F,S}, B) = \mathrm{Cone}(C(G_{F,S}, B) \xrightarrow{t_S} C_l(G_{F,S}, A))[-1],$$

where $t_S$ is defined as the composition of $\ell_S$ with the map of complexes determined by $t_v = t + \varphi_v \circ \pi$ in its $v$-coordinate. In that $\iota$ is locally split by $t_v$, the map induced by $t_S$ on cohomology is independent of the choices of $\varphi_v$.

It should be noted that the Selmer complex $C_f(G_{F,S}, B)$ depends in general upon the choices of local embeddings. In fact, we have the following lemma.

**Lemma 2.3.2.** *For $\sigma \in G_{F,S}$, consider the homomorphisms $j'_v$ for $v \in S$ that are defined on $\tau \in G_{F_v}$ by $j'_v(\tau) = \sigma j_v(\tau)\sigma^{-1}$, and form the corresponding localization maps $\ell'_v$. Define $\varphi'_v \colon C \to A$ on $c \in C$ by*

$$\varphi'_v(c) = \sigma\varphi_v(\sigma^{-1}c) - \chi(\sigma)(c),$$

*and let $t'_S$ be the map induced in the $v$-coordinate by the composition $t'_v \circ \ell'_v$, where $t'_v = t + \varphi'_v \circ \pi$. Then the diagram*

$$\begin{tikzcd} C(G_{F,S}, B) \arrow[r, "t_S"] \arrow[d, "\sigma^*"] & C_l(G_{F,S}, A) \arrow[d, "\sigma"] \\ C(G_{F,S}, B) \arrow[r, "t'_S"] & C_l(G_{F,S}, A) \end{tikzcd}$$

*of maps of complexes commutes. Here, $\sigma^*$ is the isomorphism induced by the standard action of $\sigma$ on cochains, and the right-hand vertical map is induced by the action of $\sigma$ on coefficients, where the action of $G_{F_v}$ on $A$ is understood to be attained through $j_v$ on the upper row and $j'_v$ on the lower.*

*Proof.* Let $v \in S$ and $\tau \in G_{F_v}$. We claim first that the above choice of $\varphi'_v$ makes $t'_v$ into a $G_{F_v}$-equivariant map for the $G_{F_v}$-actions induced by $j'_v$. By Lemma 2.3.1, it suffices to check the Galois-equivariance of $s'_v = s - \iota\varphi'_v$, given the Galois-equivariance of $s_v = s - \iota\varphi_v$. Set $\tau'_v = j'_v(\tau)$ and $\tau_v = j_v(\tau)$. As maps on $C$, we have

$$s'_v\tau'_v = s\tau'_v - \iota(\sigma\varphi_v\sigma^{-1}\tau'_v - \chi(\sigma)\tau'_v) = \sigma s_v\sigma^{-1}\tau'_v = \sigma s_v\tau_v\sigma^{-1} = \tau'_v\sigma s_v\sigma^{-1},$$

and we then check

$$\sigma s_v\sigma^{-1} = \sigma s\sigma^{-1} - \iota\sigma\varphi_v\sigma^{-1} = s + \iota\chi - \iota\sigma\varphi_v\sigma^{-1} = s'_v.$$

We verify commutativity of the diagram. For $f \in C^i(G_{F,S}, B)$, and now taking $\tau$ to be an $i$-tuple in $G_{F_v}$, we have

$$(t'_v \circ \ell'_v \circ \sigma^*(f))(\tau) = (t'_v \circ \sigma^*(f))(\sigma\tau_v\sigma^{-1}) = (t\sigma + \sigma\varphi_v\pi - \chi(\sigma)\sigma\pi)(f(\tau_v)),$$

and $\iota$ applied to the latter value is the following map applied to $f(\tau_v)$:

$$\iota t\sigma + \sigma\iota\varphi_v\pi + s\pi\sigma - \sigma s\pi = \sigma \circ (1 - s\pi + \iota\varphi_v\pi) = \sigma \circ (\iota t + \iota\varphi_v\pi) = \iota \circ \sigma \circ t_v.$$

In other words, we have

$$(t'_v \circ \ell'_v \circ \sigma^*(f))(\tau) = (\sigma \circ t_v)(f(\tau_v)) = (\sigma \circ t_v \circ \ell_v(f))(\tau).$$

$\square$

In particular, we see that, even when the Galois action on $\mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A)$ is trivial, the diagram in Lemma 2.3.2 (and the resulting diagram on cohomology) would not have to commute if instead we took our canonical choices $\varphi_v = \varphi'_v = 0$, since we could have $\chi(\sigma) \neq 0$. In other words, the Selmer complex depends upon our choice of local embeddings.

For later purposes, it is notationally convenient for us to work with the Tate twist of the sequence (2.1). We have the following exact sequences, with the third requiring the assumption that each $\ell_v(\chi)$ is a coboundary:

$$0 \to C(G_{F,S},A(1)) \to C(G_{F,S},B(1)) \to C(G_{F,S},C(1)) \to 0 \tag{2.6}$$

$$0 \to C_c(G_{F,S},A(1)) \to C_c(G_{F,S},B(1)) \to C_c(G_{F,S},C(1)) \to 0 \tag{2.7}$$

$$0 \to C_c(G_{F,S},A(1)) \to C_f(G_{F,S},B(1)) \to C(G_{F,S},C(1)) \to 0. \tag{2.8}$$

These yield connecting maps

$$\begin{aligned} \kappa_B^i &\colon H^i(G_{F,S},C(1)) \to H^{i+1}(G_{F,S},A(1)) \\ \kappa_{c,B}^i &\colon H_c^i(G_{F,S},C(1)) \to H_c^{i+1}(G_{F,S},A(1)) \\ \kappa_{f,B}^i &\colon H^i(G_{F,S},C(1)) \to H_c^{i+1}(G_{F,S},A(1)) \end{aligned}$$

of $\Lambda$-modules.

We have continuous cup products for any $i, j \in \mathbf{Z}$, given by

$$C^i(G_{F,S},\mathrm{Hom}_{\Lambda,\mathrm{cts}}(C,A)) \times C^j(G_{F,S},C(1)) \xrightarrow{\cup} C^{i+j}(G_{F,S},A(1)).$$

As before, setting $\tilde{\chi}(f) = \chi \cup f$, we obtain a map of sequences of $\Lambda$-modules

$$\tilde{\chi} \colon C(G_{F,S},C(1)) \to C(G_{F,S},A(1))[1]$$

that is a map of complexes up to the signs of the differentials. In a similar manner, we have maps of sequences of $\Lambda$-modules

$$\begin{aligned} \tilde{\chi}_c &\colon C_c(G_{F,S},C(1)) \to C_c(G_{F,S},A(1))[1] \\ \tilde{\chi}_f &\colon C(G_{F,S},C(1)) \to C_c(G_{F,S},A(1))[1] \end{aligned}$$

with the same property, given explicitly as follows. First, for $f = (f_1, f_2) \in C_c^i(G_{F,S}, C(1))$, where $f_1 \in C^i(G_{F,S}, C(1))$ and $f_2 \in C_l^{i-1}(G_{F_v}, C(1))$, we set

$$\tilde{\chi}_c(f) = \chi \cup_c f = (\chi \cup f_1, -\ell_S(\chi) \cup f_2),$$

where the second cup product is the sum of the local cup products at $v \in S$. In the case that the class of $\chi$ is locally a coboundary (of $\varphi_v$ at $v \in S$), we set $\varphi_S = (\varphi_v)_{v \in S}$ and let

$$\chi' = (\chi, -\varphi_S) \in C_c^1(G_{F,S}, \mathrm{Hom}_{\Lambda,\mathrm{cts}}(C, A))$$

be the canonical cocycle lifting $\chi$. For $f \in C^i(G_{F,S}, C(1))$, we then define

$$\tilde{\chi}_f(f) = \chi'{}_c\cup f = (\chi \cup f, -\varphi_S \cup \ell_S(f)) \in C_c^{i+1}(G_{F,S}, A(1)). \tag{2.9}$$

**Proposition 2.3.3.** *One has $\kappa_B^i = \tilde{\chi}^i$, $\kappa_{c,B}^i = \tilde{\chi}_c^i$, and $\kappa_{f,B}^i = \tilde{\chi}_f^i$ for all $i \in \mathbf{Z}$.*

*Proof.* We treat the cases one-by-one, showing that the maps agree on cocycles. The first equality is Lemma 2.1.4. For the next, let $f = (f_1, f_2) \in C_c^i(G_{F,S}, C(1))$ be a cocycle, which is to say that $df_1 = 0$ and $df_2 = -\ell_S(f_1)$. Set $g = (g_1, g_2) = (s \circ f_1, s \circ f_2)$. For $i \geq 1$, one then has that

$$dg_1(\sigma, \tau) = \chi(\sigma)(\sigma f_1(\tau))$$

for $\sigma \in G_{F,S}$ and $\tau$ an $i$-tuple in $G_{F,S}$ and

$$dg_2(\sigma, \tau) = -\ell_S(g_1)(\sigma, \tau) + \ell_S(\chi)(\sigma)(\sigma f_2(\tau)),$$

where $\sigma \in G_{F_v}$ and $\tau$ is an $(i-1)$-tuple in $G_{F_v}$. It follows that

$$dg = (d(g_1), -\ell_S(g_1) - d(g_2)) = (\chi \cup f_1, -\ell_S(\chi) \cup f_2),$$

for $i \geq 1$.

For $i = 0$, we remark that $Z^0(G_{F,S}, C(1)) \to Z^0(G_{F_v}, C(1))$ is injective for any prime $v$, so the only cocycles in $C_c^0(G_{F,S}, C(1))$ are sums of cocycles in $\hat{C}^{-1}(G_{F_v}, B(1))$ for real places $v$. For any $i \leq 0$ and any real $v$, we may then use repeated right cup product with a 2-cocycle with class generating $\hat{H}^2(G_{F_v}, \mathbf{Z}_p)$ to reduce the question to $i \geq 1$, as right cup product commutes not only with $\kappa_{c,B}$ by definition of the connecting homomorphism and the compatibility of left cup product and coboundary on the level of cocycles, but also with $\tilde{\chi}_c$ by associativity of the cup product on the level of complexes.

Finally, assume that the class of $\chi$ is locally trivial, and let $f$ be a cocycle in $C^i(G_{F,S}, C(1))$. We lift it to $(s \circ f, 0)$ in

$$C_f^i(G_{F,S}, B(1)) \cong C^i(G_{F,S}, B(1)) \oplus C_l^{i-1}(G_{F_v}, A(1)).$$

Using the first case, its coboundary is then

$$(\chi \cup f, -t_S(s \circ f)) = (\chi \cup f, -\varphi_S \circ \ell_S(f)) = (\chi \cup f, -\varphi_S \cup \ell_S(f)).$$

$\square$

## 2.4 Duality

Suppose now that our exact sequence (2.1) is in $\mathscr{C}_{\Lambda, G_{F,S}}$. We will assume that $A$ and $C$ are endowed with the $\mathscr{I}$-adic topology and that $\pi$ has a (continuous) splitting $s \colon C \to B$ of $\Lambda$-modules. The map of Proposition A.1.9 provides a cocycle

$$\chi^* \colon G_{F,S} \to \mathrm{Hom}_{\Lambda^\circ}(A^\vee, C^\vee)$$

attached to $\chi$. The cocycle $-\chi^*$ defines $B^\vee$ as an extension of $C^\vee$ by $A^\vee$. Let

$$\nu_B^i \colon H^{i+1}(G_{F,S}, A^\vee)^\vee \to H^i(G_{F,S}, C^\vee)^\vee$$

denote the Pontryagin dual map of $\kappa^i_{B(1)^\vee}$, and define $\nu^i_{c,B} = (\kappa^i_{c,B(1)^\vee})^\vee$ and $\nu^i_{f,B} = (\kappa^i_{f,B(1)^\vee})^\vee$ similarly. By Poitou-Tate duality and a careful check of signs, we have the following rather standard result.

**Lemma 2.4.1.** *The following diagram commutes:*

$$\begin{tikzcd}
H^i(G_{F,S}, C(1)) \arrow[r, "\kappa_B^i"] \arrow[d, "\beta_C^i"] & H^{i+1}(G_{F,S}, A(1)) \arrow[d, "\beta_A^{i+1}"] \\
H_c^{3-i}(G_{F,S}, C^\vee)^\vee \arrow[r, "(-1)^{i+1}\nu_{c,B}^{2-i}"] & H_c^{2-i}(G_{F,S}, A^\vee)^\vee,
\end{tikzcd}$$

*Similarly, we have*

$$\beta_{c,A}^{i+1} \circ \kappa_{c,B}^i = (-1)^{i+1} \nu_B^{2-i} \circ \beta_{c,C}^i$$

*Proof.* For $f \in H^i(G_{F,S}, C(1))$ and $g \in H_c^{2-i}(G_{F,S}, A^\vee)$, we have

$$((\beta_A^{i+1} \circ \kappa_B^i)(f))(g) = (\beta_A^{i+1}(\chi \cup f))(g) = g\,{}_c\!\cup (\chi \cup f) = (-1)^i (\chi^* \cup_c g)\,{}_c\!\cup f,$$

the latter equality by [Ne, 5.3.3.2-4], while

$$((\nu_{c,B}^{2-i} \circ \beta_C^i)(f))(g) = \beta_C^i(f)(\kappa_{c,B(1)^\vee}^{2-i}(g)) = -(\chi^* \cup_c g)\,{}_c\!\cup f.$$

$\square$

We also require the following general lemma.

**Lemma 2.4.2.** *Suppose that we have an exact triangle of cochain complexes of $\Lambda$-modules*

$$M \xrightarrow{(\iota,\iota')} N \oplus N' \xrightarrow{\pi+\pi'} O \to M[1].$$

*Then the map of complexes*

$$\mathrm{Cone}\,(M \xrightarrow{\iota} N) \xrightarrow{(-\pi,\iota')} \mathrm{Cone}\,(N' \xrightarrow{\pi'} O)$$

*is a quasi-isomorphism.*

*Proof.* On cohomology, we have a commutative diagram of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(M) \arrow[r, "\iota^i"] \arrow[d, "(\iota')^i"] & H^i(N) \arrow[r] \arrow[d, "-\pi^i"] & H^i(\mathrm{Cone}(\iota)) \arrow[r] \arrow[d, "(-\pi,\iota')^i"] & H^{i+1}(M) \arrow[r, "\iota^{i+1}"] \arrow[d, "(\iota')^{i+1}"] & H^{i+1}(N) \arrow[r] \arrow[d, "-\pi^{i+1}"] & \cdots \\
\cdots \arrow[r] & H^i(N') \arrow[r, "(\pi')^i"] & H^i(O) \arrow[r] & H^i(\mathrm{Cone}(\pi')) \arrow[r] & H^{i+1}(N') \arrow[r, "(\pi')^{i+1}"] & H^{i+1}(O) \arrow[r] & \cdots .
\end{tikzcd}$$

We have canonical commutative diagrams

$$\begin{tikzcd}
\mathrm{coker}((\iota')^i) \arrow[r] \arrow[d, hook] & \mathrm{coker}(\pi^i) \arrow[d, hook] \\
H^i(\mathrm{coker}\,\iota') \arrow[r] & H^i(\mathrm{coker}\,\pi)
\end{tikzcd} \qquad \begin{tikzcd}
H^i(\ker\iota') \arrow[r] \arrow[d, two heads] & H^i(\ker\pi) \arrow[d, two heads] \\
\ker((\iota')^i) \arrow[r] & \ker(\pi^i)
\end{tikzcd}$$

where the maps are the natural ones. Since the maps $\mathrm{coker}\,\iota' \to \mathrm{coker}\,\pi$ and $\ker\iota' \to \ker\pi$ are quasi-isomorphisms of complexes, the maps $(-\pi,\iota')^i$ are isomorphisms of $\Lambda$-modules for all $i$. □

Applying Lemma 2.4.2 to the exact triangle

$$C(G_{F,S},B(1))[-1] \to C_l(G_{F,S},B(1))[-1] \to C_c(G_{F,S},B(1)) \to C(G_{F,S},B(1))$$

(noting that $B = A \oplus C$ locally), we see that $C_f(G_{F,S},B(1))$ is quasi-isomorphic to

$$\mathrm{Cone}(C_l(G_{F_v},C(1))[-1] \to C_c(G_{F,S},B(1))). \tag{2.10}$$

Assuming that $\chi$ is locally trivial, we have that $\chi^*$ is locally trivial and

$$C_f(G_{F,S},B^\vee) = \mathrm{Cone}(C(G_{F,S},B^\vee) \xrightarrow{(s^\vee)_S} C_l(G_{F_v},C^\vee))[-1].$$

The following is now a consequence of Tate and Poitou-Tate duality.

**Proposition 2.4.3.** *There are natural isomorphisms of topological* $\Lambda$*-modules*

$$\beta^i_{f,B}\colon H^i_f(G_{F,S},B(1)) \to H^{3-i}_f(G_{F,S},B^\vee)^\vee$$

*that fit into a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_c(G_{F,S},A(1)) \arrow[r] \arrow[d, "\beta^i_{c,A}"] & H^i_f(G_{F,S},B(1)) \arrow[r] \arrow[d, "\beta^i_{f,B}"] & H^i(G_{F,S},C(1)) \arrow[r, "\kappa^i_{f,B}"] \arrow[d, "\beta^i_C"] & H^{i+1}_c(G_{F,S},A) \arrow[r] \arrow[d, "\beta^{i+1}_{c,A}"] & \cdots \\
\cdots \arrow[r] & H^{3-i}(G_{F,S},A^\vee)^\vee \arrow[r] & H^{3-i}_f(G_{F,S},B^\vee)^\vee \arrow[r] & H^{3-i}_c(G_{F,S},C^\vee)^\vee \arrow[r] & H^{2-i}(G_{F,S},A^\vee)^\vee \arrow[r] & \cdots,
\end{tikzcd}$$

*where the lower connecting map is* $(-1)^{i+1}\nu^{2-i}_{f,B}$.

## 2.5 Iwasawa cohomology

Let $K$ be a Galois extension of $F$ that is $S$-ramified over a finite extension of $F$ in $K$, and set $\Gamma = \mathrm{Gal}(K/F)$. Let $\Omega$ denote the maximal $S$-ramified extension of $K$. Let $R$ be a commutative pro-$p$ ring, and let $\mathfrak{A}$ be a topological basis of open ideals of $R$. We set $\Lambda = R[\![\Gamma]\!]$, which is itself a pro-$p$ ring. Typically, we will simply use $S$ to denote the set of primes $S_E$ in any extension $E$ of $F$ lying above those in $S$, and $G_{E,S}$ will denote the Galois group of the maximal $S$-ramified extension of $E$. If $E$ is contained in $K$, we set $\mathscr{G}_{E,S} = \mathrm{Gal}(\Omega/E)$. We let $\Lambda^\iota$ denote $\Lambda$ viewed as a $\Lambda[\mathscr{G}_{F,S}]$-module with $\Lambda$ acting by left multiplication and $\sigma \in \mathscr{G}_{F,S}$ acting by right multiplication by the image of $\sigma^{-1}$ in $\Gamma$.

Let us review a few results that may be found in [Li], extending work of Nekovar [Ne]. Suppose that $T \in \mathscr{C}_{\Lambda,\mathscr{G}_{F,S}}$. We define the Iwasawa cochain complex as

$$C_{*,S}(K,T) = C_*(\mathscr{G}_{F,S}, \Lambda^\iota \mathbin{\hat{\otimes}}_R T)$$

where $*$ again denotes no symbol, $c$, or $l$. Via a version of Shapiro's lemma for cochains, we have

$$C_{*,S}(K,T) \cong \varprojlim_{E\subset K} C_*(\mathscr{G}_{E,S},T),$$

with the inverse limit taken over number fields $E$ containing $F$ with respect to corestriction maps. We may replace $\mathscr{G}_{E,S}$ with $G_{E,S}$ in the above and obtain a quasi-isomorphic complex, but the above isomorphism makes the $\Lambda$-module structure more transparent, since $T$ is only assumed to be a $G_{E,S}$-module for sufficiently large $E$. We may view the Iwasawa cohomology group $H^i_{*,S}(K,T)$ as an object of $\mathscr{C}_\Lambda$ by endowing it with the initial topology. Since $T$ is compact, we have a canonical isomorphism

$$H^i_{*,S}(K,T) \xrightarrow{\sim} \varprojlim_{E\subset K} H^i_*(\mathscr{G}_{E,S},T)$$

of $\Lambda$-modules.

Similarly, for $A \in \mathscr{D}_{\Lambda,G_{F,S}}$, we can and will make the identification

$$C_*(G_{K,S},A) \cong C_*(\mathscr{G}_{F,S},\mathrm{Hom}_{R,\mathrm{cts}}(\Lambda,A)),$$

where we view $\Lambda$ here as a right $\Lambda$-module via right multiplication and a $\mathscr{G}_{F,S}$-module via left multiplication and put the discrete topology on the $\Lambda[\mathscr{G}_{F,S}]$-module $\mathrm{Hom}_{R,\mathrm{cts}}(\Lambda,A)$. Direct limits with respect to restriction maps provide an isomorphism

$$\varinjlim_{E \subset K} C_*(\mathscr{G}_{E,S},A) \to C_*(G_{K,S},A)$$

of complexes of discrete $\Lambda$-modules. Again, we may replace $\mathscr{G}_{E,S}$ with $G_{E,S}$ in this isomorphism. Direct limits being exact, we have resulting $\Lambda$-module isomorphisms

$$\varinjlim_{E \subset K} H^i_*(\mathscr{G}_{E,S},A) \xrightarrow{\sim} H^i_*(G_{K,S},A).$$

Taking $T \in \mathscr{C}_{\Lambda,G_{F,S}}$, the pairing $T^\vee \times T(1) \to \mu_{p^\infty}$ induced by composition gives rise to a continuous, perfect pairing

$$\mathrm{Hom}_{R,\mathrm{cts}}(\Lambda,T^\vee) \times (\Lambda^\iota \hat{\otimes}_R T(1)) \to \mu_{p^\infty}, \qquad (\phi,\lambda \otimes t) \mapsto \phi(\lambda)(t).$$

Cup product then induces the following isomorphisms in $\mathscr{C}_\Lambda$, much as before:

$$\begin{aligned} \beta^i_{l,T} &\colon H^i_{l,S}(K,T(1)) \to H^{2-i}_l(G_{K,S},T^\vee)^\vee \\ \beta^i_T &\colon H^i_S(K,T(1)) \to H^{3-i}_c(G_{K,S},T^\vee)^\vee \\ \beta^i_{c,T} &\colon H^i_{c,S}(K,T(1)) \to H^{3-i}(G_{K,S},T^\vee)^\vee \end{aligned}$$

(see [Li, Theorem 4.2.6]). These agree with the inverse limits of the duality maps over number fields in $K$.

For any complex $T$ in $\mathscr{C}_{R,G_{F,S}}$ on which the $G_{F,S}$-actions factor through $\Gamma$, let $T^\dagger$ be the complex in $\mathscr{C}_{\Lambda,G_{F,S}}$ that is $T$ as a complex of topological $R$-modules, has $\Gamma$-actions those induced by the $G_{F,S}$-action on $T$, and has trivial $G_{F,S}$-actions.

**Lemma 2.5.1.** *Let $T \in \mathscr{C}_{R,G_{F,S}}$ be such that the $G_{F,S}$-action factors through $\Gamma$. We have the following natural isomorphisms in $\mathscr{C}_\Lambda$:*

*a.* $H^3_{c,S}(K,T(1)) \cong T^\dagger$,

*b.* $H^2_{c,S}(K,T(1)) \cong H^2_{c,S}(K,R(1)) \hat{\otimes}_R T^\dagger$, *c.* $H^2_S(K,T(1)) \cong H^2_S(K,R(1)) \hat{\otimes}_R T^\dagger$ *if $p$ is odd or $K$ has no real places.*

*Proof.* By the above remark, we may replace $T$ by $T^\dagger$ in the cohomology groups in question, thereby assuming that $T \in \mathscr{C}_{\Lambda,G_{F,S}}$ has a trivial $G_{F,S}$-action. Write $T = \varprojlim T_\alpha$, where the finite $T_\alpha$ run over the $\Lambda$-quotients of $T$ by open submodules. We then have

$$H^i_{c,S}(K,T(1)) \cong \varprojlim_\alpha H^i_{c,S}(K,T_\alpha(1)).$$

Note that there is an exact sequence

$$0 \to \mathbf{Z}_p^r \xrightarrow{\theta} \mathbf{Z}_p^s \to T_\alpha \to 0$$

of compact $\mathbf{Z}_p$-modules for some $r,s \geq 0$. Since $H^3_{c,S}(K,\mathbf{Z}_p(1)) \cong \mathbf{Z}_p$ and $H^4_{c,S}(K,\mathbf{Z}_p(1))$ is trivial, the long exact sequence in cohomology yields that the composite map

$$T_\alpha \xrightarrow{\sim} H^3_{c,S}(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} T_\alpha \to H^3_{c,S}(K,T_\alpha(1))$$

of $\Lambda$-modules is an isomorphism. Part a then follows by taking inverse limits of these compatible isomorphisms.

Next, since the map $H^3_{c,S}(K,\mathbf{Z}_p^r(1)) \xrightarrow{\theta} H^3_{c,S}(K,\mathbf{Z}_p^s(1))$ is injective, $H^2_{c,S}(K,T_\alpha(1))$ is isomorphic to the cokernel of the map

$$H^2_{c,S}(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p^r \xrightarrow{\mathrm{id}\otimes\theta} H^2_{c,S}(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p^s.$$

Thus, the canonical map

$$H^2_{c,S}(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} T_\alpha \to H^2_{c,S}(K,T_\alpha(1))$$

is an isomorphism, and therefore so is

$$H^2_{c,S}(K,\mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} T \to H^2_{c,S}(K,T(1)).$$

This also holds for $T = R$, so the composite map

$$H^2_{c,S}(K,\mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} T \xrightarrow{\sim} H^2_{c,S}(K,\mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} R \hat{\otimes}_R T \to H^2_{c,S}(K,R(1)) \hat{\otimes}_R T$$

is an isomorphism as well, from which part b follows.

Finally, if $p$ is odd or $K$ has no real places, then $H^3_S(K,\mathbf{Z}_p(1)) = 0$, and part c follows by the analogous argument. $\square$

Suppose now that we have an exact sequence

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$$

in $\mathscr{C}_{R,\mathscr{G}_{F,S}}$ and that $\pi$ has a (continuous) splitting $s\colon C \to B$ of $R$-modules. Again, we have a continuous 1-cocycle

$$\chi\colon \mathscr{G}_{F,S} \to \mathrm{Hom}_{R,\mathrm{cts}}(C,A).$$

On cohomology, we again have connecting maps

$$\kappa_B^i\colon H_S^i(K,C(1)) \to H_S^{i+1}(K,A(1))$$
$$\kappa_{c,B}^i\colon H_{c,S}^i(K,C(1)) \to H_{c,S}^{i+1}(K,A(1))$$

of compact $\Lambda$-modules. If we require that each $\ell_v(\chi)$ be a local coboundary upon restriction to $G_{K,S}$, then we also have

$$\kappa_{f,B}^i\colon H_S^i(K,C(1)) \to H_{c,S}^{i+1}(K,A(1)),$$

but in general it is just a map of compact $R$-modules. Again, these agree with left cup product by $\chi$, which is to say, the inverse limits of the maps $\tilde{\chi}$, $\tilde{\chi}_c$ , and $\tilde{\chi}_f$ at the finite level, respectively.

Consider the dual exact sequence and the resulting cocycle

$$\chi^*\colon \mathscr{G}_{F,S} \to \mathrm{Hom}_R(A^\vee, C^\vee).$$

We again denote the Pontryagin duals of the resulting $i$th connecting maps by $\nu_B^i$, $\nu_{c,B}^i$ and, if $\chi$ is locally cohomologically trivial, $\nu_{f,B}^i$ on $G_{K,S}$-cohomology.

Suppose that $A$ and $C$ are endowed with the $\mathfrak{A}$-adic topology. As in Lemma 2.4.1, we have the following.

**Lemma 2.5.2.** *The maps $\kappa_B^i$ and $\kappa_{c,B}^i$ coincide with the maps $(-1)^{i+1}\nu_{c,B}^{2-i}$ and $(-1)^{i+1}\nu_B^{2-i}$, respectively, under Poitou-Tate duality.*

We define

$$C_{f,S}(K,B) = C_f(\mathscr{G}_{F,S}, \Lambda^\iota \hat{\otimes}_R B).$$

Similarly, define

$$C_f(G_{K,S}, B^\vee) = \varinjlim_{E \subset K} C_f(\mathscr{G}_{E,S}, B^\vee).$$

We have the following analogue of Proposition 2.4.3.

**Proposition 2.5.3.** *Suppose that $\chi$ is locally cohomologically trivial upon restriction to $G_{K,S}$. Then there are natural isomorphisms*

$$\beta^i_{f,B} \colon H^i_{f,S}(K,B(1)) \to H^{3-i}_f(G_{K,S},B^\vee)^\vee$$

*of topological R-modules that are compatible with $\beta^i_{c,A}$ and $\beta^i_C$ in the sense of Proposition 2.4.3, and we have $\beta^{i+1}_{c,A} \circ \kappa^i_{f,B} = (-1)^{i+1} \nu^{2-i}_{f,B} \circ \beta^i_C$.*

# 3 Reciprocity maps

## 3.1 The fundamental exact sequences

Let us now take $K$ to be an $S$-ramified Galois extension of $F$, and we set $\Gamma = \mathrm{Gal}(K/F)$. From now on, we assume that either $p$ is odd or that $K$ has no real places. We set $R = \mathbf{Z}_p$ and $\Lambda = \mathbf{Z}_p[\![\Gamma]\!]$. Note that $\Lambda$ is a pro-$p$ ring.

We briefly run through the Iwasawa modules (i.e., $\Lambda$-modules) of interest. First, we use $\mathfrak{X}_K$ denote the Galois group of the maximal $S$-ramified abelian pro-$p$ extension of $K$. Let $Y_K$ denote the Galois group of the maximal unramified abelian pro-$p$ extension of $K$ in which all primes above those in $S$ split completely. Let $\mathscr{U}_K$ be the inverse limit under norm maps of the $p$-completions of the $S$-units in number fields in $K$.

The map $\beta_{c,\mathbf{Z}_p}$ of complexes induced by cup product yields a canonical isomorphism

$$H^2_{c,S}(K,\mathbf{Z}_p(1)) \xrightarrow{\sim} H^1(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p)^\vee,$$

allowing us to identify the group on the left with $\mathfrak{X}_K$. Moreover, the kernel of the natural map

$$H^2_{c,S}(K,\mathbf{Z}_p(1)) \to H^2_S(K,\mathbf{Z}_p(1))$$

is exactly the kernel of the restriction map $\mathfrak{X}_K \to Y_K$. Therefore, we obtain a canonical injection

$$\iota_K \colon Y_K \to H^2_S(K,\mathbf{Z}_p(1)).$$

Finally, Kummer theory allows us to canonically identify $\mathscr{U}_K$ with $H^1_S(K,\mathbf{Z}_p(1))$.

Let $\mathscr{X}$ denote the quotient of $\mathbf{Z}_p[\![\mathfrak{X}_K]\!]$ by the square of its augmentation ideal $I_{\mathfrak{X}_K}$. With the standard isomorphism $\mathfrak{X}_K \xrightarrow{\sim} I_{\mathfrak{X}_K}/I^2_{\mathfrak{X}_K}$ that takes an element $\sigma$ to $(\sigma-1) \bmod I^2_{\mathfrak{X}_K}$, this gives rise to the exact sequence

$$0 \to \mathfrak{X}_K \to \mathscr{X} \to \mathbf{Z}_p \to 0 \tag{3.1}$$

of compact $\mathbf{Z}_p[\![G_{K,S}]\!]$-modules, where $\mathfrak{X}_K$ acts by left multiplication on $\mathscr{X}$ and trivially on $\mathfrak{X}_K$ and $\mathbf{Z}_p$. The character that this sequence defines on $G_{K,S}$ is simply the restriction map $\mu\colon G_{K,S} \to \mathfrak{X}_K$.

By part c of Lemma 2.5.1, our connecting map $\kappa^1_{\mathscr{X}}$ is a homomorphism

$$\Psi_K\colon \mathscr{U}_K \to H^2_S(K,\mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} \mathfrak{X}_K$$

that we call the $S$-reciprocity map for $K$. Since $\mu \in H^1(G_{K,S},\mathfrak{X}_K)^{G_{F,S}}$, the equality $\kappa^1_{\mathscr{X}} = \tilde{\mu}$ and Galois equivariance of cup products imply that the $S$-reciprocity map $\Psi_K$ is a homomorphism of $\Lambda$-modules.

*Remark* 3.1.1. With an additional hypothesis, we may give $\mathscr{X}$ a continuous, $\mathbf{Z}_p$-linear $G_{F,S}$-action that turns (3.1) into an exact sequence of $\Lambda$-modules. Let $M_K$ be the maximal $S$-ramified abelian pro-$p$ extension of $K$. Suppose that there exists a continuous homomorphism $\alpha\colon \Gamma \to \mathrm{Gal}(M_K/F)$ splitting the restriction map (e.g., that $\Gamma$ is free pro-$p$). As topological spaces, one then has

$$\mathrm{Gal}(M_K/F) \cong \mathfrak{X}_K \times \Gamma$$

via the inverse of the map that takes $(x,\gamma)$ to $x\alpha(\gamma)$. Using this identification, we define a map $p\colon \mathrm{Gal}(M_K/F) \to \mathfrak{X}_K$ by $p(x,\gamma) = x$. It is a cocycle as $\alpha$ is a homomorphism. The cocycle on $G_{F,S}$ resulting from inflation gives rise to a continuous $G_{F,S}$-action on $\mathscr{X}$, extending the natural $G_{K,S}$-action (and the conjugation action of $G_{F,S}$ on $\mathfrak{X}_K$). The class of $p$ is independent of the splitting, which means that the class of the extension (3.1) is as well.

The following lemma is easily proven by pushing out the exact sequence (3.1).

**Lemma 3.1.2.** *Let $M \in \mathscr{C}_{\mathbf{Z}_p,G_{F,S}}$ with trivial $G_{K,S}$-action, and write $M \cong \varprojlim_\alpha M_\alpha$ with $M_\alpha$ a finite quotient of $M$. For*

$$h = (h_\alpha)_\alpha \in \varprojlim_\alpha H^1(G_{K,S},M_\alpha),$$

*let*

$$\tilde{h}\colon \mathscr{U}_K \cong H^1_S(K,\mathbf{Z}_p(1)) \to H^2_S(K,M(1)) \cong H^2_S(K,\mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} M$$

*be the composite map induced by the inverse limit under corestriction of left cup products with cohomology classes that restrict to $h_\alpha$. Then*

$$(\mathrm{id} \otimes h) \circ \Psi_K = \tilde{h}.$$

As a quotient of $\mathscr{X}$, we obtain the extension

$$0 \to Y_K \to \mathscr{Y} \to \mathbf{Z}_p \to 0.$$

In this case, the cocycle $\chi\colon G_{K,S} \to Y_K$ defining the extension is unramified everywhere and $S$-split. Applying parts a and b of Lemma 2.5.1, the first connecting homomorphism is then a homomorphism

$$\Theta_K = \kappa^1_{f,\mathscr{Y}}\colon \mathscr{U}_K \to \mathfrak{X}_K \hat{\otimes}_{\mathbf{Z}_p} Y_K,$$

and the second is

$$q_K = \kappa^2_{f,\mathscr{Y}}\colon H^2_S(K,\mathbf{Z}_p(1)) \to Y_K.$$

**Proposition 3.1.3.** *The composition $q_K \circ \iota_K$ is the identity of $Y_K$.*

*Proof.* By Proposition 2.3.3, the map $q_K$ agrees with $\tilde{\chi}^2_f$. Precomposing it with the map

$$H^2_{c,S}(K,\mathbf{Z}_p(1)) \to H^2_S(K,\mathbf{Z}_p(1)),$$

the resulting map

$$H^2_{c,S}(K,\mathbf{Z}_p(1)) \to H^3_{c,S}(K,Y_K(1))$$

is exactly $\tilde{\chi}^2_c = \kappa^2_{c,\mathscr{Y}}$, again by Proposition 2.3.3. By Lemma 2.5.2, it is Pontryagin dual (under Poitou-Tate duality) to the map

$$H^0(G_{K,S},Y_K^\vee) \to H^1(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p)$$

given by $-\kappa^0_{\mathscr{Y}(1)^\vee} = \widetilde{\chi^{*0}}$, or in other words, the map $Y_K^\vee \to \mathfrak{X}_K^\vee$ that takes an element to its precomposition with the projection $\mathfrak{X}_K \to Y_K$. That being said, $q_K \circ \iota_K$ is as stated. $\square$

Let

$$\mathrm{sw}\colon Y_K \hat{\otimes}_{\mathbf{Z}_p} \mathfrak{X}_K \xrightarrow{\sim} \mathfrak{X}_K \hat{\otimes}_{\mathbf{Z}_p} Y_K$$

denote the isomorphism induced by commutativity of the tensor product.

**Theorem 3.1.4.** *The natural diagram*

$$\begin{tikzcd} \mathscr{U}_K \arrow[r, "\Psi_K"] \arrow[dr, "-\Theta_K"'] & H^2_S(K,\mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} \mathfrak{X}_K \arrow[d, "\mathrm{sw}\circ(q_K\otimes\mathrm{id})"] \\ & \mathfrak{X}_K \hat{\otimes}_{\mathbf{Z}_p} Y_K \end{tikzcd}$$

*commutes.*

*Proof.* The skew-symmetry of the cup product yields the commutativity of the outer square in the following diagram:

$$\begin{tikzcd}
 & H^2_S(K,\mathfrak{X}_K(1)) \arrow[d] \arrow[dr, "\tilde{\chi}^2_f"] & \\
H^1_S(K,\mathbf{Z}_p(1)) \arrow[ur, "\tilde{\mu}^1"] \arrow[dr, "-\tilde{\chi}^1_f"'] & \mathfrak{X}_K \hat{\otimes}_{\mathbf{Z}_p} Y_K & H^3_{c,S}(K,\mathfrak{X}_K \hat{\otimes}_{\mathbf{Z}_p} Y_K(1)) \arrow[l, "\sim"'] \\
 & H^2_{c,S}(K,Y_K(1)). \arrow[u, "\wr"] \arrow[ur, "\tilde{\mu}^2_c"'] &
\end{tikzcd}$$

Here, the downward arrow is the composition of the natural isomorphism

$$H^2_S(K,\mathfrak{X}_K(1)) \cong H^2_S(K,\mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} \mathfrak{X}_K$$

with $\mathrm{sw} \circ (q_K \otimes \mathrm{id})$. Recall that $q_K$ is $\tilde{\chi}^2_f$ on $H^2_S(K,\mathbf{Z}_p(1))$ followed by the invariant map. Since the leftward arrow is the invariant map, the upper-right triangle commutes.

The upward arrow is the composition of the isomorphism

$$H^2_{c,S}(K,Y_K(1)) \cong H^2_{c,S}(K,\mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} Y_K$$

with the natural isomorphism $H^2_{c,S}(K,\mathbf{Z}_p(1)) \cong \mathfrak{X}_K$ of Poitou-Tate duality (see Lemma 2.5.1b). In other words, noting that $(\mathfrak{X}_K \hat{\otimes}_{\mathbf{Z}_p} Y_K)^\vee$ may be canonically identified with $\mathrm{Hom}_{\mathrm{cts}}(\mathfrak{X}_K, Y_K^\vee)$, the Pontryagin dual of the upward arrow is the natural identification

$$\mathrm{Hom}_{\mathrm{cts}}(\mathfrak{X}_K, Y_K^\vee) \to H^1(G_{K,S}, Y_K^\vee).$$

Let $\mathscr{Z} = \mathscr{X} \hat{\otimes}_{\mathbf{Z}_p} Y_K$, and note that

$$0 \to \mathfrak{X}_K \hat{\otimes}_{\mathbf{Z}_p} Y_K \to \mathscr{Z} \to Y_K \to 0$$

is still exact. By Lemma 2.5.2, the Pontryagin dual of the map $\tilde{\mu}^2_c = \kappa^2_{c,\mathscr{Z}}$ in the diagram is the map

$$\tilde{\mu}^0 = -\kappa^0_{\mathscr{Z}^\vee(1)} \colon H^0(G_{K,S}, \mathrm{Hom}_{\mathrm{cts}}(\mathfrak{X}_K, Y_K^\vee)) \to H^1(G_{K,S}, Y_K^\vee),$$

that is cup product with the projection map $\mu \in H^1(G_{K,S}, \mathfrak{X}_K)$. The dual of the invariant map is just the identification of $H^0(G_{K,S}, \mathrm{Hom}_{\mathrm{cts}}(\mathfrak{X}_K, Y_K^\vee))$ with $\mathrm{Hom}_{\mathrm{cts}}(\mathfrak{X}_K, Y_K^\vee)$. The composite map is composition with the projection, so it is again the natural identification. In other words, the lower-right triangle commutes, so the left half of the diagram commutes, and that is what was claimed. $\square$

In general, suppose that $L$ is an $S$-ramified abelian pro-$p$ extension of $K$, and set $G = \mathrm{Gal}(L/K)$. The reciprocity map induces

$$\Psi_{L/K} \colon \mathscr{U}_K \to H^2_S(K, \mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} G,$$

equal to $(\mathrm{id} \otimes \pi_G) \circ \Psi_K$, where $\pi_G \colon \mathfrak{X}_K \to G$ is the restriction map on Galois groups. We again refer to $\Psi_{L/K}$ as an $S$-reciprocity map, for the extension $L/K$. Note that $\Psi_{L/K} = \widetilde{\pi_G}$ in the notation of Lemma 3.1.2.

Consider the localization map

$$\ell_S \colon H^2_S(K, \mathbf{Z}_p(1)) \hat{\otimes}_{\mathbf{Z}_p} G \to \prod_{v \in S_K} G$$

obtained from the isomorphisms

$$H^2_{\mathrm{cts}}(G_{E_v}, \mathbf{Z}_p(1)) \cong \mathbf{Z}_p$$

for $v \in S_E$ and $E$ a number field containing $F$. By Proposition 2.3.3 and a standard description of the local reciprocity map in local class field theory, the $v$-coordinate of the composition $\rho_{L/K} = \ell_S \circ \Psi_{L/K}$ is the composition of the natural maps

$$\mathscr{U}_K \to \varprojlim_{E_v \subset K_v} \varprojlim_n E_v^\times / E_v^{\times p^n}$$

with the local reciprocity map for the (fixed) local extension defined by $L/K$ at $v \in S_K$ and the canonical injection of its Galois group into $G$. We will let $\mathscr{U}_{L/K}$ denote the kernel of $\rho_{L/K}$.

Theorem 3.1.4 now has following corollary.

**Corollary 3.1.5.** *For any $S$-ramified abelian pro-$p$ extension $L$ of $K$ with Galois group $G$, one has a commutative diagram*

$$\begin{tikzcd} \mathscr{U}_{L/K} \arrow[r, "\Psi_{L/K}"] \arrow[dr, "(\pi_G \otimes \mathrm{id}) \circ \Theta_K"'] & Y_K \hat{\otimes}_{\mathbf{Z}_p} G \arrow[d, "-\mathrm{sw}_G"] \\ & G \hat{\otimes}_{\mathbf{Z}_p} Y_K, \end{tikzcd}$$

*where $\mathrm{sw}_G$ is the standard isomorphism induced by commutativity of the tensor product.*

This, in turn, has the following corollary for the extension defined by $Y_K$.

**Corollary 3.1.6.** *Let $H_K$ be the maximal unramified, $S$-split abelian pro-$p$ extension of $K$. The map*

$$\Psi_{H_K/K} \colon \mathscr{U}_K \to Y_K \hat{\otimes}_{\mathbf{Z}_p} Y_K$$

*has antisymmetric image.*

*Proof.* We apply Corollary 3.1.5. The map $\rho_{H_K/K}$ is a sum of local reciprocity maps $\rho_v\colon \mathscr{U}_K \to Y_K$ that are all trivial since the decomposition group at $v$ in $Y_K$ is trivial by definition. Hence, we have $\mathscr{U}_{H_K/K} = \mathscr{U}_K$. Furthermore, the map $(\pi_{Y_K} \otimes \mathrm{id}) \circ \Theta_K$ equals $\kappa^1_{\mathscr{Y}}$ by construction (see (2.9) and Proposition 2.3.3). The result follows immediately. $\square$

*Remark* 3.1.7. Let us check that $\Theta_F$ agrees with the likewise-denoted map of the introduction. If we compose $\Theta_F$ for $i=1$ (or $q_F$ for $i=2$) with the map $H^i_c(G_{F,S}, Y_F(1)) \to H^{3-i}(G_{F,S}, Y_F^\vee)^\vee$ of Poitou-Tate duality, we obtain a map $f \mapsto (g \mapsto \chi^*{}_c\cup f \cup g)$. On the other hand, $\widetilde{\chi^{*2}_f}$ induces a map

$$H^2(G_{F,S}, Y_F^\vee(1)) \to H^3_c(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p/\mathbf{Z}_p$$

which we may compose with the pairing

$$H^{2-i}(G_{F,S}, \mathbf{Z}_p(1)) \times H^i(G_{F,S}, Y_F^\vee) \to H^2(G_{F,S}, Y_F^\vee(1)) \tag{3.2}$$

to obtain the same map $f \mapsto (g \mapsto \chi^*{}_c\cup f \cup g)$ in the dual.

It thus suffices to see that the map $\rho\colon H^2(G_{F,S}, Y_F^\vee(1)) \to \mathbf{Q}_p/\mathbf{Z}_p$ described in the introduction agrees with $\widetilde{\chi^{*2}_f}$, which by Proposition 2.3.3 and the discussion of Section 2.4 is the negative of the connecting homomorphism $\kappa^2_{f,\mathscr{Y}(1)^\vee}$. The map $\rho$ is given by lifting from $H^2(G_{F,S}, Y_F^\vee(1))$ to $H^2(G_{F,S}, \mathscr{Y}^\vee(1))$, applying restriction and local splittings (i.e., $(s^\vee)_S$) to land in $H^2_l(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1))$, and then applying the sum of invariant maps. The map $H^2(G_{F,S}, \mathscr{Y}^\vee(1)) \to H^2_l(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1))$ that is the connecting homomorphism arising from the Selmer complex is $-(s^\vee)_S$ by definition. The map $\kappa^2_{f,\mathscr{Y}(1)^\vee}$ is given by lifting the cocycle in $Z^2(G_{F,S}, \mathscr{Y}^\vee(1))$ to $C^2_f(G_{F,S}, \mathscr{Y}^\vee(1))$, taking its coboundary, and lifting the resulting element of $B^3_f(G_{F,S}, \mathscr{Y}^\vee(1))$ to $Z^3_c(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1))$. For this, we can first lift to $Z^2_l(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1))$ and then map to $Z^3_c(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1))$. That is, $\kappa^2_{f,\mathscr{Y}(1)^\vee} = -\widetilde{\chi^{*2}_f}$ results from

$$H^2(G_{F,S}, Y_F^\vee(1)) \twoheadleftarrow H^2(G_{F,S}, \mathscr{Y}^\vee(1)) \xrightarrow{-(s^\vee)_S} H^2_l(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1)) \to H^3_c(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p(1)),$$

which by its description is $-\rho$. Thus, we have the desired equality.

## 3.2 A special case

Let $F = \mathbf{Q}(\mu_p)$, and let $\Delta = \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$. Let $S = S_{\mathbf{Q}} = \{p, \infty\}$, and let $\zeta_p$ denote a primitive $p$th root of unity. Let $\omega\colon \Delta \hookrightarrow \mathbf{Z}_p^\times$ denote the Teichmüller character. For a $\mathbf{Z}_p[\Delta]$-module $M$ and $j \in \mathbf{Z}$, we let $M^{(j)}$ denote its $\omega^j$-eigenspace. Finally, set

$$\eta_i = \prod_{\delta \in \Delta} (1 - \zeta_p^\delta)^{\omega^{i-1}(\delta)} \in \mathscr{U}_F^{(1-i)}$$

for odd $i \in \mathbf{Z}$.

We will consider the cup product pairing

$$( \ , \ )_{p,F,S} \colon H^1(G_{F,S}, \mu_p) \times H^1(G_{F,S}, \mu_p) \to Y_F \otimes \mu_p.$$

of [McS], which we can view as a pairing on the group of elements of $F^\times$ which have associated fractional ideals that are $p$th powers times a power of the prime over $p$.

Let us say that an integer $k$ is irregular for $p$ if $k$ is even, $2 \le k \le p-3$, and $p$ divides the $k$th Bernoulli number $B_k$. For primes of index of irregularity at least 2 that, for instance, satisfy Vandiver's conjecture, we have the following consequence of the antisymmetry expressed in Corollary 3.1.6.

**Theorem 3.2.1.** *Suppose that $k$ and $k'$ are irregular for $p$ with $k < k'$ and that $Y_F^{(k)} = Y_F^{(k')} = 0$. Then $(\eta_{p-k}, \eta_{k+k'-1})_{p,F,S} = 0$ if and only if $(\eta_{p-k'}, \eta_{k+k'-1})_{p,F,S} = 0$.*

*Proof.* For any $r$ that is irregular for $p$, we let $\chi_{p-r} \colon Y_F \to \mu_p$ denote the Kummer character attached to $\eta_{p-r}$. Supposing that $Y_F^{(r)} = 0$, fix an $\mathbf{F}_p$-basis of $\mathrm{Hom}(Y_F, \mu_p)$ which contains the nontrivial $\chi_{p-r}$ and a dual basis of $Y_F/p$ containing elements $\sigma_r \in Y_F^{(1-r)}$ for each such character with $\chi_{p-r}(\sigma_r) = \zeta_p$ and $\chi_{p-r}$ trivial on all other elements of the basis.

Since

$$\chi_{p-k}(\kappa^1_{\mathscr{Y}}(\eta_{k+k'-1})) = (\eta_{p-k}, \eta_{k+k'-1})_{p,F,S} \in Y_F^{(1-k')} \otimes \mu_p,$$

the coefficient of $\sigma_{k'} \otimes \sigma_k$ in the expansion of $\kappa^1_{\mathscr{Y}}(\eta_{k+k'-1})$ mod $p$ in terms of the standard basis of the tensor product is $[\eta_{p-k}, \eta_{k+k'-1}]_{k'}$, where

$$(\eta_{p-k}, \eta_{k+k'-1})_{p,F,S} = \sigma_{k'} \otimes \zeta_p^{[\eta_{p-k}, \eta_{k+k'-1}]_{k'}}.$$

Similarly, the coefficient of $\sigma_k \otimes \sigma_{k'}$ is the analogously defined $[\eta_{p-k'}, \eta_{k+k'-1}]_k$. The antisymmetry of Corollary 3.1.6 forces

$$[\eta_{p-k}, \eta_{k+k'-1}]_{k'} = -[\eta_{p-k'}, \eta_{k+k'-1}]_k,$$

and, in particular, the result. □

This phenomenon can be seen in the tables of the pairing values for $p < 25{,}000$ produced by the author and McCallum (see [McS]). What is remarkable about Theorem 3.2.1 is that it relates pairing values in distinct eigenspaces of $Y_F$.

*Remark* 3.2.2. Note that the condition for the vanishing of these pairing values appears in the statement of [Sh2, Theorem 5.2]. We remark that there is a mistake in said statement that is rendered inconsequential by Theorem 3.2.1. That is, it was accidentally only assumed there that $(\eta_{p-k}, \eta_{k+k'-1})_{p,F,S} = 0$ for all $k' > k$, while this vanishing for all $k' \neq k$ is used in the proof (although at one point the condition that $k' > k$ is again written therein).

# 4 Modular representations

## 4.1 Background

In this section, we briefly explore the application of the above results to the setting of Selmer groups of the residual representations of cusp forms satisfying mod $p$ congruences with Eisenstein series. We take $F = \mathbf{Q}(\mu_{Np})$, where $p \geq 5$ is prime and $N \geq 1$ is such that $p \nmid N\varphi(N)$. We will use $S$ to denote the set of primes of any number field consisting of the primes dividing $p$ and any infinite places and $S'$ to denote the union of $S$ with the primes over $N$. Let $K$ be the cyclotomic $\mathbf{Z}_p$-extension of $F$. Let

$$\mathbf{Z}_{p,N} = \varprojlim_r \mathbf{Z}/Np^r\mathbf{Z},$$

and identify $\mathrm{Gal}(K/\mathbf{Q})$ with $\mathbf{Z}_{p,N}^\times$ via the Galois action on roots of unity.

Let $\Lambda_N = \mathbf{Z}_p[\![\mathbf{Z}_{p,N}^\times/\langle -1 \rangle]\!]$. Let $R$ denote the ring generated over $\mathbf{Z}_p$ by the values of all $p$-adic characters of $(\mathbf{Z}/Np\mathbf{Z})^\times$. Pick a primitive, even character $\theta\colon (\mathbf{Z}/Np\mathbf{Z})^\times \to R^\times$ such that $\theta \neq \omega^2$ if $N = 1$ and either $\theta\omega^{-1}|_{(\mathbf{Z}/p\mathbf{Z})^\times} \neq 1$ or $\theta\omega^{-1}|_{(\mathbf{Z}/N\mathbf{Z})^\times}(p) \neq 1$. (Note that the primitivity forces $N$ to be odd or divisible by 4.) For a $\Lambda_N$-module $M$, we set $M^\theta = M \otimes_{\mathbf{Z}_p[(\mathbf{Z}/Np\mathbf{Z})^\times]} R$ for the map $\mathbf{Z}_p[(\mathbf{Z}/Np\mathbf{Z})^\times] \to R$ induced by $\theta$. We set $\Lambda_\theta = \Lambda_N^\theta$, and we note that if we forget the $(\mathbf{Z}/Np\mathbf{Z})^\times$-action, then each $\Lambda_\theta$ is simply $\Lambda = R[\![1+p\mathbf{Z}_p]\!]$.

Let $\mathfrak{h}$ denote Hida's cuspidal Hecke algebra of endomorphisms of the space $\mathscr{S}$ of ordinary $\Lambda_N$-adic cusp forms. Here, we make the convenient choice that the group element for $j \in \mathbf{Z}_{p,N}^\times$ acts on $\mathfrak{h}$ and therefore on $\mathscr{S}$ as the inverse diamond operator $\langle j \rangle^{-1}$, where $\langle -1 \rangle = 1$. By work of Hida and Ohta [Oh, Theorem 1.4.3], $\mathscr{S}$ is a projective $\Lambda_N$-module, and we have a perfect pairing $\mathscr{S} \times \mathfrak{h} \to \Lambda_N$ of $\Lambda_N$-modules. In fact, $\mathfrak{h}$ is Cohen-Macaulay and $\mathscr{S}$ is a dualizing module for $\mathfrak{h}$.

Let $I$ be the Eisenstein ideal of $\mathfrak{h}$ generated by $T_\ell - 1 - \ell\langle \ell \rangle$ for primes $\ell \nmid Np$ and $U_\ell - 1$ for primes $\ell \mid Np$, where $T_\ell$ and $U_\ell$ are the usual Hecke operators. We have $(\mathfrak{h}/I)_\theta \cong \Lambda_\theta/(\xi_\theta)$, where $\xi_\theta \in \Lambda_\theta$ is a twist of the $p$-adic $L$-function for $\omega^2\theta^{-1}$. Specifically, identifying $[u]-1 \in \Lambda_\theta$ for $u$ a topological generator of $1+p\mathbf{Z}_p \subset \mathbf{Z}_{p,N}^\times$ (and $[u]$ the corresponding group element) with a variable in a $\mathbf{Z}_p$-power series ring, we have

$$\xi_\theta(u^s - 1) = L_p(\omega^2\theta^{-1}, s-1).$$

We let $\mathfrak{h}$ act via the adjoint action on

$$\mathscr{T} = \varprojlim_r H^1_{\text{ét}}(X_1(Np^r)_{/\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))^{\text{ord}}.$$

Here, we use the model of the modular curve $X_1(Np^r)$ over $\mathbf{Z}[\frac{1}{Np}]$ in which the open subscheme $Y_1(Np^r)$ over a $\mathbf{Z}[\frac{1}{Np}]$-scheme $Z$ is the fine moduli scheme of elliptic curves $E$ over $Z$ together with an injective morphism $\mathbf{Z}/Np^r\mathbf{Z} \to E$ of $Z$-group schemes (see [FK2, Section 1.4] for the effect on the Galois action). Ohta [Oh, Definition 4.1.17] constructed a perfect pairing

$$\mathscr{T} \times \mathscr{T} \to \mathscr{S}(1) \tag{4.1}$$

of $\mathfrak{h}$-modules that is equivariant for the action of $\sigma \in G_{\mathbf{Q},S'}$ on $\mathscr{S}$ by $\langle \alpha(\sigma) \rangle$, where $\alpha \colon G_{\mathbf{Q},S'} \to \mathbf{Z}_{p,N}^{\times}$ is given by the action of $G_{\mathbf{Q},S'}$ on roots of unity. (Actually, Ohta's pairing takes values in $\Lambda$, but this pairing may be quickly derived from it: see [NP, 1.6.10] or [FK2, Proposition 1.6.6].)

We view $\overline{\mathbf{Q}}$ as sitting inside $\mathbf{C}$, so that we may speak of complex conjugation on any $\mathbf{Z}_p[G_{\mathbf{Q}}]$-module $M$, as well as the resulting fixed part $M^+$ under this involution. Moreover, this gives us a choice $\zeta_n = e^{2\pi i/n}$ of primitive $n$th root of unity in $K$ for each $n \geq 1$, as well as an isomorphism between $\mu_n$ and $\mathbf{Z}/n\mathbf{Z}$ taking $\zeta_n$ to 1.

For a compact $\Lambda_N$-module $A$, let $A^{\iota}$ denote the compact $\Lambda_N[\![\mathrm{Gal}(K/\mathbf{Q})]\!]$-module that is $A$ as a compact $\Lambda_N$-module and has continuous $\mathrm{Gal}(K/\mathbf{Q})$-action $\cdot^{\iota}$ given by $\gamma \cdot^{\iota} a = \gamma^{-1} a$ for $\gamma \in \mathbf{Z}_{p,N}^{\times}$ and $a \in A$. We let $G_{\mathbf{Q}}$ act trivially on $\mathscr{S}$. For a finitely generated $\mathfrak{h}$-module $A$, we will write $A_\theta$ for the localization of $A \hat{\otimes}_{\mathbf{Z}_p} R$ at the maximal ideal of $\mathfrak{h} \hat{\otimes}_{\mathbf{Z}_p} R$ containing $I$ and $\langle a \rangle^{-1} - \theta(a)$ for $a \in (\mathbf{Z}/Np\mathbf{Z})^{\times}$. Set

$$T = \mathscr{T}_\theta / I\mathscr{T}_\theta, \quad P = \mathscr{S}_\theta / I\mathscr{S}_\theta, \quad \text{and } Q = (\Lambda_\theta^{\iota}/(\xi_\theta))(1).$$

We have a canonical exact sequence

$$0 \to P \to T \to Q \to 0 \tag{4.2}$$

of $(\mathfrak{h}/I)_\theta[G_{\mathbf{Q},S'}]$-modules (see [Sh3, Section 4] and [FK2, Proposition 6.3.2]). The restriction of the $G_{\mathbf{Q}}$-action on $T$ to $G_K$ is everywhere unramified. Moreover, for any fixed decomposition group over a prime $\ell$ in $S'$, the exact sequence (4.2) of $\mathfrak{h}[G_{\mathbf{Q}_\ell}]$-modules is uniquely split [FK2, Proposition 6.3.4].

The connecting map $\kappa_{f,T}^2$ on $G_{\mathbf{Q},S'}$-cohomology induces a homomorphism

$$\Upsilon_\theta \colon Y_K(1)^\theta \cong H^2_{S'}(K, \mathbf{Z}_p(2))^\theta \cong H^2(G_{\mathbf{Q},S'}, Q(1)) \xrightarrow{\kappa_{f,T}^2} H^3_c(G_{\mathbf{Q},S'}, P(1)) \cong P$$

of $\Lambda_\theta$-modules, the action on the left being the Galois action and that on the right the Hecke action. Here, the second isomorphism is induced by the quotient map $\Lambda_\theta^{\iota}(1) \to Q$ on coefficients, as in [FK2, §9.1]. From now on, we set $\rho = \theta\omega^{-1}$ for brevity of notation. We have the following alternate description (see also [FK2, Proposition 9.4.3]).

**Proposition 4.1.1.** *The restriction to $G_{K,S'}$ of the cocycle $b\colon G_{\mathbf{Q},S'} \to \mathrm{Hom}_{\mathfrak{h}}(Q,P)$ defined by* (4.2) *factors through $Y_K$ and induces a Galois-equivariant map $Y_K^{\rho} \to \mathrm{Hom}_{\mathfrak{h}}(Q,P)$ of $\mathbf{Z}_p[\![\mathbf{Z}_{p,N}^{\times}]\!]$-modules. The Tate twist of the composition of this map with evaluation at the image of* 1 *in $Q(-1) \cong \Lambda_{\theta}^{\iota}/(\xi_{\theta})$ is $\Upsilon_{\theta}$.*

*Proof.* The map $\Upsilon_{\theta}$ is the composition of the maps in the bottom row of the commutative diagram

$$\begin{tikzcd}
H^2_{S'}(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} Q \arrow[r, "\sim"] \arrow[d] & H^2_{S'}(K,Q(1)) \arrow[r, "\kappa^2_{f,T}"] \arrow[d, "\mathrm{cor}"] & H^3_{c,S'}(K,P(1)) \arrow[r, "\sim"] \arrow[d, "\mathrm{cor}"] & P \arrow[d, equal] \\
Y_K(1)^{\theta} \arrow[r, "\sim"] & H^2(G_{\mathbf{Q},S'},Q(1)) \arrow[r, "\kappa^2_{f,T}"] & H^3_c(G_{\mathbf{Q},S'},P(1)) \arrow[r, "\sim"] & P,
\end{tikzcd} \tag{4.3}$$

where the left-hand vertical map is the composition

$$H^2_{S'}(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} Q \xrightarrow{q_K \otimes 1} Y_K \otimes_{\mathbf{Z}_p} Q \twoheadrightarrow Y_K(1)^{\theta} \otimes_{\Lambda_{\theta}} \Lambda_{\theta}/(\xi_{\theta}) \xrightarrow{\sim} Y_K(1)^{\theta}.$$

Define $\upsilon\colon Y_K \otimes_{\mathbf{Z}_p} Q \to P$ by

$$\upsilon(\tau \otimes \bar{\lambda}) = b(\tau)(\bar{\lambda})$$

for $\tau \in Y_K$ and $\lambda \in \Lambda_{\theta}$, where $\bar{\lambda}$ is the image of $\lambda$ in $Q$. We have a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Y_K \otimes_{\mathbf{Z}_p} Q \arrow[r] \arrow[d, "\upsilon"] & \mathscr{Y} \otimes_{\mathbf{Z}_p} Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] & T \arrow[r] & Q \arrow[r] & 0,
\end{tikzcd} \tag{4.4}$$

and we know that $\kappa^2_{f,\mathscr{Y}\otimes Q}$ agrees with $q_K \otimes 1$ by Proposition 3.1.3. It follows that the upper composition in (4.3) agrees with the composition

$$H^2_{S'}(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} Q \xrightarrow{q_K \otimes 1} Y_K \otimes_{\mathbf{Z}_p} Q \xrightarrow{\upsilon} P.$$

As $q_K$ induces the identity map on $H^2_{S'}(K,\mathbf{Z}_p(1))^{\rho} \cong Y_K^{\rho}$, the commutativity of the diagram then implies that the map $\Upsilon_{\theta}$ is as stated. □

We remark that it is $\Upsilon_{\theta}$, rather than $\Psi_K$, that Fukaya and Kato refer to as a reciprocity map in [FK2]. Conjectures 5.2 and 5.4 of [Sh3] imply that $\Upsilon_{\theta}$ is an isomorphism. Consider the following weaker version that appears in the work of Fukaya and Kato, which allows for instance that $P$ could have $p$-torsion, unlike $Y_K^{\rho}$.

**Conjecture 4.1.2.** *The map $\Upsilon_{\theta}$ is a pseudo-isomorphism of $(\Lambda \otimes_R \mathfrak{h}/I)$-modules.*

Recall that a pseudo-isomorphism of modules is one with pseudo-null (i.e., of annihilator of height at least 2) kernel and cokernel. A finitely generated module for $\Lambda \otimes_R \mathfrak{h}/I$ is pseudo-null if and only if it is finite, since $\mathfrak{h}/I$ has finite $R$-rank. Thus, a map of finitely generated $(\Lambda \otimes_R \mathfrak{h}/I)$-modules is a pseudo-isomorphism if and only if its kernel and cokernel are both finite.

Fukaya and Kato have proven that this form of the conjecture holds under various hypotheses, in particular under the assumption that $\xi_\theta$ has no multiple zeros when viewed as a function in $s$ (see [FK2] for the precise statement and [FKS, WWE] for additional results). Note that if $\Upsilon_\theta$ is a pseudo-isomorphism, then it is injective.

## 4.2 The Selmer complex

We form the Selmer complex $C_{f,S}(K,T(1))$ and compute eigenspaces of its cohomology. To start, we have the long exact sequence

$$\cdots \to H^i_{c,S}(K,P(1)) \to H^i_{f,S}(K,T(1)) \to H^i_S(K,Q(1)) \xrightarrow{\kappa^i_{f,T}} H^{i+1}_{c,S}(K,P(1)) \to \cdots \qquad (4.5)$$

of $\mathfrak{h}$-modules. Fix a character $\varepsilon\colon (\mathbf{Z}/Np\mathbf{Z})^\times \to R^\times$ (which we shall later assume to be odd). Let us set $\Delta = \mathrm{Gal}(F/\mathbf{Q}) \cong (\mathbf{Z}/Np\mathbf{Z})^\times$, and let $\Delta_p$ denote a decomposition group at $p$ in $\Delta$. We let $\mathrm{Gal}(K/\mathbf{Q}) \cong \mathbf{Z}_{p,N}^\times$ act diagonally on tensor products in the following.

**Lemma 4.2.1.** *The groups $H^i_{c,S}(K,P(1))^\varepsilon$ are trivial for $i \notin \{2,3\}$ and for $i = 3$ unless $\varepsilon$ is trivial. We have an isomorphism*

$$H^2_{c,S}(K,P(1))^\varepsilon \cong \mathfrak{X}_K^\varepsilon \otimes_R P$$

*of $\mathfrak{h}[\![\mathbf{Z}_{p,N}^\times]\!]$-modules, and we have $H^3_{c,S}(K,P(1))^\varepsilon \cong P$ if $\varepsilon = 1$.*

*Proof.* The statement for $i \notin \{1,2,3\}$ was remarked for general coefficients in Section 2.2. Lemma 2.5.1 tells us immediately that

$$H^3_{c,S}(K,P(1)) \cong P,$$

and $P^\varepsilon = 0$ unless $\varepsilon = 1$ since $P$ has trivial Galois action. Also, we have

$$H^2_{c,S}(K,P(1)) \cong \mathfrak{X}_K \otimes_{\mathbf{Z}_p} P$$

by part b of Lemma 2.5.1, from which the statement for $H^2$ follows. As for $i = 1$, an argument as in the proof of part b of Lemma 2.5.1, using the well-known consequence of triviality of the $\mu$-invariant that $\mathfrak{X}_K$ has no $p$-torsion (see [MW], [FW] and [NSW, Corollary 11.3.15]), provides the first isomorphism in

$$H^1_{c,S}(K,P(1)) \cong H^1_{c,S}(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} P \cong H^2(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p)^\vee \otimes_{\mathbf{Z}_p} P,$$

and then we need merely note that we have $H^2(G_{K,S},\mathbf{Q}_p/\mathbf{Z}_p) = 0$ by weak Leopoldt (see [NSW, Section 10.3]). □

As $Q = Q^{\rho^{-1}}$ is a free, finite rank $\mathbf{Z}_p$-module, we also have the following.

**Lemma 4.2.2.** *We have an isomorphism*

$$H^1_S(K,Q(1))^\varepsilon \cong \mathscr{U}_K^{\rho\varepsilon} \otimes_R Q$$

*and an exact sequence*

$$0 \to Y_K^{\rho\varepsilon} \otimes_R Q \to H^2_S(K,Q(1))^\varepsilon \to \ker(\mathbf{Z}_p[\Delta/\Delta_p] \to \mathbf{Z}_p)^{\rho\varepsilon} \otimes_R Q \to 0$$

*of* $\mathfrak{h}[\![\mathbf{Z}_{p,N}^\times]\!]$*-modules.*

Next, let us consider the connecting homomorphism $\kappa^1_{f,T}$. We define $\Psi_{\varepsilon,\rho}$ and $\Theta_{\varepsilon,\rho}$ as the $\rho\varepsilon$-components of the maps $\Psi_K$ and $\Theta_K$, respectively, followed by projection maps, as follows:

$$\Psi_{\varepsilon,\rho} \colon \mathscr{U}_K^{\rho\varepsilon} \xrightarrow{\Psi_K} (H^2_S(K,\mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathfrak{X}_K)^{\rho\varepsilon} \to Y_K^\rho \otimes_R \mathfrak{X}_K^\varepsilon \xrightarrow{\mathrm{sw}} \mathfrak{X}_K^\varepsilon \otimes_R Y_K^\rho$$

and

$$\Theta_{\varepsilon,\rho} \colon \mathscr{U}_K^{\rho\varepsilon} \xrightarrow{\Theta_K} (\mathfrak{X}_K \otimes_{\mathbf{Z}_p} Y_K)^{\rho\varepsilon} \to \mathfrak{X}_K^\varepsilon \otimes_R Y_K^\rho,$$

with the second-to-last map in the definition of $\Psi_{\varepsilon,\rho}$ employing the isomorphism

$$Y_K^\rho \xrightarrow{\sim} H^2_S(K,\mathbf{Z}_p(1))^\rho$$

that holds by our assumption on $\theta$, and the map sw being given by commutativity of the tensor product. Theorem 3.1.4 tells us that $\Psi_{\varepsilon,\rho} = -\Theta_{\varepsilon,\rho}$. In particular, $\Theta_{\varepsilon,\rho}$ is a map of $\Lambda$-modules.

Note in the following that we have $H^2_S(K,Q(1))^1 \cong Y_K^\rho \otimes_R Q$.

**Lemma 4.2.3.** *The* $\mathfrak{h}[\![\mathbf{Z}_{p,N}^\times]\!]$*-module homomorphism*

$$(\kappa^1_{f,T})^\varepsilon \colon \mathscr{U}_K^{\rho\varepsilon} \otimes_R Q \to \mathfrak{X}_K^\varepsilon \otimes_R P$$

*induced by* $\kappa^1_{f,T}$ *is given by the composition of maps*

$$\mathscr{U}_K^{\rho\varepsilon} \otimes_R Q \xrightarrow{-\Psi_{\varepsilon,\rho}\otimes 1} \mathfrak{X}_K^\varepsilon \otimes_R Y_K^\rho \otimes_R Q \xrightarrow{1\otimes\Upsilon_\theta\otimes 1} \mathfrak{X}_K^\varepsilon \otimes_R P(-1) \otimes_R Q \twoheadrightarrow \mathfrak{X}_K^\varepsilon \otimes_R P,$$

*where the final map is induced by the composition* $P(-1) \otimes_R Q \twoheadrightarrow P \otimes_{\Lambda_\theta} Q(-1) \xrightarrow{\sim} P$.

*Proof.* Using the map (4.4) of exact sequences, we see that $(\kappa^1_{f,T})^\varepsilon$ of (4.5) is given by the composition

$$\mathscr{U}_K^{\rho\varepsilon} \otimes_R Q \xrightarrow{\Theta_{\varepsilon,\rho}\otimes 1} \mathfrak{X}_K^\varepsilon \otimes_R Y_K^\rho \otimes_R Q \xrightarrow{1\otimes \upsilon} \mathfrak{X}_K^\varepsilon \otimes_R P,$$

where $\upsilon\colon Y_K^\rho \otimes_R Q \to P$ is induced by the map in said diagram. By Theorem 3.1.4, the first map is $-\Psi_{\varepsilon,\rho}\otimes 1$, and from Proposition 4.1.1 and its proof, the map $\upsilon$ is seen to be the composition of the remaining maps in the statement (which are not individually $\mathfrak{h}[\![\mathbf{Z}_{p,N}^\times]\!]$-equivariant). $\square$

For nontrivial $\varepsilon$, the long exact sequence attached to (4.2), together with Lemmas 4.2.1 and 4.2.2, gives rise to a complex

$$\begin{aligned} 0 \to H^1_{f,S}(K,T(1))^\varepsilon \to \mathscr{U}_K^{\rho\varepsilon} \otimes_R Q \xrightarrow{(\kappa^1_{f,T})^\varepsilon} \mathfrak{X}_K^\varepsilon \otimes_R P \\ \to H^2_{f,S}(K,T(1))^\varepsilon \to \ker(\mathbf{Z}_p[\Delta/\Delta_p] \to \mathbf{Z}_p)^{\rho\varepsilon} \otimes_R Q \to 0 \end{aligned} \tag{4.6}$$

which is exact except at $H^2_{f,S}(K,T(1))^\varepsilon$, where it has cohomology group $Y_K^{\rho\varepsilon} \otimes_R Q$.

If $\varepsilon$ is even and $\rho\varepsilon \neq \omega$, then $\mathscr{U}_K^{\rho\varepsilon} = 0$, so (4.6) reduces to a short exact sequence

$$0 \to \mathfrak{X}_K^\varepsilon \otimes_R P \to H^2_{f,S}(K,T(1))^\varepsilon \to H^2_S(K,Q(1))^\varepsilon \to 0,$$

and the $\Lambda$-characteristic ideals of $\mathfrak{X}_K^\varepsilon$ and $H^2_S(K,\mathbf{Z}_p(1))^{\rho\varepsilon}$ are well-known by the Iwasawa main conjecture [MW], while Conjecture 4.1.2 implies that $Q$ and $P$ have the same characteristic ideal as $\Lambda_\theta$-modules for inverse diamond operators. The reader should compare this with much earlier work of Greenberg and Vatsal: in particular, see [GV, Theorem 3.12], in which they sketched a proof of a main conjecture in such eigenspaces for cuspidal eigenforms congruent to Eisenstein series. We focus instead on the considerable additional subtleties that are introduced in the setting that $\varepsilon$ is odd.

From now on, we suppose that $\varepsilon$ is odd. In this case, $\mathscr{U}_K^{\rho\varepsilon}$ is $\Lambda$-torsion free and $\mathscr{U}_K^{\rho\varepsilon}$ and $\mathfrak{X}_K^\varepsilon$ both have $\Lambda$-rank one, with $\mathscr{U}_K^{\rho\varepsilon}$ free. Recall that there is a map $\Phi_\varepsilon\colon \mathfrak{X}_K^\varepsilon \to \Lambda_\varepsilon$ that satisfies

$$\Phi_\varepsilon(\sigma) = \varprojlim_r \sum_{k=0}^{p^{r-1}-1} ((\eta^\varepsilon_{k,r})^{1/p^r})^{\sigma-1}[u^k] \in \Lambda_\varepsilon,$$

where

$$\eta^\varepsilon_{k,r} = \prod_{\substack{i=1\\(i,f_\varepsilon p)=1}}^{Np-1} (1-\zeta_{Np^r}^{u^k\tilde{i}})^{\varepsilon\omega^{-1}(i)}$$

where $\tilde{i}$ denotes an integer congruent to $i$ modulo $Np$ such that $\tilde{i}^{p-1} \equiv 1 \bmod p^r$, where $f_\varepsilon$ denotes the prime-to-$p$ part of the conductor of $\varepsilon$, and where $\mathbf{Z}/p^r\mathbf{Z}$ and $\mu_{p^r}$ are identified in the sum. There is a canonical exact sequence of $\Lambda_\varepsilon$-modules

$$0 \to \alpha(Y_K^{\omega\varepsilon^{-1}})(1) \to \mathfrak{X}_K^\varepsilon \xrightarrow{\Phi_\varepsilon} \Lambda_\varepsilon \to ((Y_K^{\omega\varepsilon^{-1}})_{\mathrm{fin}})^\vee(1) \to 0,$$

where $\alpha(M)$ denotes the Iwasawa adjoint and $M_{\rm fin}$ denotes the maximal finite submodule of a finitely generated, torsion $\Lambda$-module $M$. Greenberg's conjecture for the even character $\omega\varepsilon^{-1}$ states that $Y_K^{\omega\varepsilon^{-1}}$ is finite, so $\alpha(Y_K^{\omega\varepsilon^{-1}})$ is trivial. It follows that $\mathfrak{X}_K^{\varepsilon}$ is pseudo-free of rank 1 over $\Lambda$, and $Y_K^{\varepsilon}$ is pseudo-cyclic over $\Lambda$.

Let $\mathscr{C}_K$ denote the $\Lambda$-module generated by norm compatible sequences of cyclotomic $p$-units so that $\mathscr{C}_K^{\rho\varepsilon}$ is generated by the image $u_{\rho\varepsilon}$ of the norm compatible system $(1-\zeta_{f_{\rho\varepsilon}p^r})_r$. We have an exact sequence

$$0 \to (\mathscr{U}_K/\mathscr{C}_K)^{\rho\varepsilon} \to (\Lambda/\xi_{\rho\varepsilon})^{\iota} \to \mathfrak{X}_K^{\rho\varepsilon} \to Y_K^{\rho\varepsilon} \to 0$$

without assumption. The group $\mathfrak{X}_K^{\rho\varepsilon}$ is canonically pseudo-isomorphic (i.e., isomorphic up to finite kernel and cokernel) to $\alpha(Y_K^{\omega\rho^{-1}\varepsilon^{-1}})(1)$, and Greenberg's conjecture for $\omega\rho^{-1}\varepsilon^{-1}$ implies that $\mathscr{U}_K^{\rho\varepsilon} = \mathscr{C}_K^{\rho\varepsilon}$.

Let $\mathscr{L}_{\varepsilon,\theta} \in \Lambda_\varepsilon \hat{\otimes}_R \mathscr{T}_\theta^+$ denote the image of the modified Mazur-Kitagawa-type $p$-adic $L$-function $\mathscr{L}^{\star}_{N,N/f_{\rho\varepsilon}}$ of [Sh3, (6.1)]. Here, we use the natural surjection $\mathbf{Z}_p[\![\mathbf{Z}_{p,N}]\!] \to \Lambda_\varepsilon$ of $\Lambda_N$-modules induced by $\varepsilon\omega^{-1}$ viewed as a Dirichlet character of modulus $f_\varepsilon p$. Let $\bar{\mathscr{L}}_{\varepsilon,\theta}$ denote the image of $\mathscr{L}_{\varepsilon,\theta}$ in $\Lambda_\varepsilon \otimes_R P$. The following stronger version of Conjecture 4.1.2 is equivalent to [Sh3, Conjecture 6.3].

**Conjecture 4.2.4.** *Let $\varepsilon$ be an odd character. The $\Lambda_\theta$-module $P$ is $p$-torsion free, and the $\Lambda_\varepsilon \otimes_R (\mathfrak{h}/I)_\theta$-submodules of $\Lambda_\varepsilon \otimes_R P$ generated by $\bar{\mathscr{L}}_{\varepsilon,\theta}$ and $(\Phi_\varepsilon \otimes \Upsilon_\theta)(\Psi_{\varepsilon,\rho}(u_{\rho\varepsilon}))$ are equal.*

*Remark* 4.2.5. We in fact expect that $(\Phi_\varepsilon \otimes \Upsilon_\theta)(\Psi_{\varepsilon,\rho}(u_{\rho\varepsilon})) = \bar{\mathscr{L}}_{\varepsilon,\theta}$.

The following result is a combination of the results of Wake and Wang Erickson [WWE, Corollary B] and Fukaya and Kato [FK2] (see [WWE, Corollary C]).

**Theorem 4.2.6** (Wake-Wang Erickson, Fukaya-Kato)**.** *Suppose that $Y_K^\theta/\xi_\theta Y_K^\theta$ is finite. Then Conjecture 4.1.2 holds. If moreover $Y_K^\rho$ is pseudo-cyclic and $P$ has trivial maximal finite $\Lambda_\theta$-submodule, then Conjecture 4.2.4 holds.*

*Proof.* It follows from Fukaya and Kato [FK2, Theorem 7.2.3] that

$$(\Phi_\varepsilon \otimes \xi'_\theta \Upsilon_\theta)(\Psi_{\varepsilon,\rho}(u_{\rho\varepsilon})) = (1 \otimes \xi'_\theta)\bar{\mathscr{L}}_{\varepsilon,\theta},$$

where $\xi'_\theta$ is power series corresponding to the derivative of $\xi_\theta$ as a function of $s$. Under the assumption that $Y_K^\theta/\xi_\theta Y_K^\theta$ is finite, Wake and Wang Erickson [WWE, Corollary B] prove that $\Upsilon_\theta$ is a pseudo-isomorphism. Since $Y_K^\rho$ is pseudo-cyclic, we then have that $P$ is pseudo-cyclic with characteristic ideal $(\xi_\theta)$. The submodules of $\Lambda_\varepsilon \otimes_R P/P_{\rm fin}$ generated by $(\Phi_\varepsilon \otimes \Upsilon_\theta)(\Psi_{\varepsilon,\rho}(u_{\rho\varepsilon}))$ and $\bar{\mathscr{L}}_{\varepsilon,\theta}$ are then forced to agree. So, under the further assumption that $P_{\rm fin} = 0$, the result holds. □

## 4.3 The residual Selmer group

In this subsection, we consider Greenberg's strict Selmer group

$$\mathrm{Sel}(K, T^\vee) = \ker\left( H^1(G_{K,S}, T^\vee) \to \bigoplus_{v \in S} H^1(G_{K_v}, Q^\vee) \right).$$

Observe that Tate and Poitou-Tate duality provide us with an isomorphism

$$\mathrm{Sel}(K, T^\vee)^\vee \cong \mathrm{coker}(H^1_{l,S}(K, Q(1)) \to H^2_{c,S}(K, T(1))).$$

**Lemma 4.3.1.** *We have an exact sequence of $\Lambda_N \otimes_{\mathbf{Z}_p} \mathfrak{h}/I$-modules*

$$0 \to \mathrm{coker}\, \kappa^1_{f,T} \to \mathrm{Sel}(K, T^\vee)^\vee \to Y_K \otimes_{\mathbf{Z}_p} Q \to P$$

*Proof.* The exact sequence in question is the sequence of maps on cokernels in the following map of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^1_S(K, Q(1)) \arrow[r] \arrow[d] & H^1_{l,S}(K, Q(1)) \arrow[r] \arrow[d] & \ker(\mathfrak{X}_K \to Y_K) \otimes_{\mathbf{Z}_p} Q \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & H^2_{c,S}(K, P(1)) \arrow[r] & H^2_{c,S}(K, T(1)) \arrow[r] & \mathfrak{X}_K \otimes_{\mathbf{Z}_p} Q \arrow[r] & P.
\end{tikzcd}$$

□

Note that $P^\varepsilon = 0$, as $\varepsilon \neq 1$ since $\varepsilon$ is odd, so on $\varepsilon$-eigenspaces the exact sequence of Lemma 4.3.1 reduces to a short exact sequence.

Let us set

$$S_\varepsilon = (\mathrm{Sel}(K, T^\vee)^\vee)^\varepsilon \quad \text{and} \quad L_\varepsilon = \frac{\Lambda_\varepsilon \otimes_R P}{(\Lambda_\varepsilon \otimes_R (\mathfrak{h}/I)_\theta) \cdot \bar{\mathscr{L}}_{\varepsilon,\theta}}.$$

We expect the following conjectures to hold. The first requires the pseudo-cyclicity of $P$ (which is a consequence of Conjecture 4.1.2 and the conjectural finiteness of $Y_K^{\omega^2\theta^{-1}}$).

**Conjecture 4.3.2.** *The $(\Lambda \otimes_R \mathfrak{h}/I)$-module $L_\varepsilon$ is torsion as a $\Lambda$-module.*

The second is a strengthening of a two-variable main conjecture for the dual Selmer group of $T^\vee$.

**Conjecture 4.3.3.** *The $(\Lambda \otimes_R \mathfrak{h}/I)$-modules $S_\varepsilon$ and $L_\varepsilon$ are pseudo-isomorphic.*

The following provides evidence for Conjecture 4.3.3.

**Theorem 4.3.4.** *If $L_\varepsilon$ is $\Lambda$-torsion, $Y_K^\theta/\xi_\theta Y_K^\theta$ and $Y_K^{\omega\varepsilon^{-1}}$ are finite, $Y_K^\rho$ is pseudo-cyclic, and $P$ has trivial maximal finite $\mathfrak{h}_\theta$-submodule, then there is an exact sequence*

$$0 \to (\mathscr{U}_K/\mathscr{C}_K)^{\rho\varepsilon} \otimes_R Q \to L'_\varepsilon \to S_\varepsilon \to Y_K^{\rho\varepsilon} \otimes_R Q \to 0,$$

*where $L'_\varepsilon$ is a finite index submodule of $L_\varepsilon$. In particular, $S_\varepsilon$ and $L_\varepsilon$ have the same length at all height one primes of $\Lambda_\varepsilon \otimes_R (\mathfrak{h}/I)_\theta$. If $Y_K^{\rho\varepsilon}$ is also finite, then $S_\varepsilon$ and $L_\varepsilon$ are pseudo-isomorphic.*

*Proof.* Recall from Lemmas 4.2.1 and 4.2.2 that $(\kappa^1_{f,T})^\varepsilon$ may be viewed as a map

$$(\kappa^1_{f,T})^\varepsilon \colon \mathscr{U}_K^{\rho\varepsilon} \otimes_R Q \to \mathfrak{X}_K^\varepsilon \otimes_R P.$$

A simple application of the snake lemma then yields an exact sequence

$$(\ker \kappa^1_{f,T})^\varepsilon \to (\mathscr{U}_K/\mathscr{C}_K)^{\rho\varepsilon} \otimes_R Q \to \frac{\mathfrak{X}_K^\varepsilon \otimes_R P}{(\kappa^1_{f,T})^\varepsilon(\mathscr{C}_K^{\rho\varepsilon} \otimes_R Q)} \to (\operatorname{coker} \kappa^1_{f,T})^\varepsilon \to 0. \tag{4.7}$$

Suppose that $Y_K^\theta/\xi_\theta Y_K^\theta$ is finite, $Y_K^\rho$ is pseudo-cyclic, and $P_{\mathrm{fin}} = 0$. By Theorem 4.2.6, the map $\Upsilon_\theta$ is pseudo-isomorphism and $P$ is pseudo-cyclic, hence pseudo-isomorphic to $Q$ as a $\Lambda_\theta$-module. By Theorem 4.2.6 and Lemma 4.2.3, we have

$$(\Lambda_\varepsilon \otimes_R (\mathfrak{h}/I)_\theta) \cdot \bar{\mathscr{L}}_{\varepsilon,\theta} = (\Phi_\varepsilon \otimes 1)(\kappa^1_{f,T})^\varepsilon(\mathscr{C}_K^{\rho\varepsilon} \otimes_R Q).$$

By Conjecture 4.3.2 (i.e., that $L_\varepsilon$ is $\Lambda$-torsion) and the fact that $\mathscr{C}_K^{\rho\varepsilon} \otimes_R Q$ is a free $\Lambda_\varepsilon$-module of the same rank as $\Lambda_\varepsilon \otimes_R P$, we conclude that $(\ker \kappa^1_{f,T})^\varepsilon = 0$. Suppose that $Y_K^{\omega\varepsilon^{-1}}$ is finite so that $\Phi_\varepsilon$ is an injective pseudo-isomorphism with finite cokernel isomorphic to $(Y_K^{\omega\varepsilon^{-1}})^\vee(1)$. The map

$$\frac{\mathfrak{X}_K^\varepsilon \otimes_R P}{(\kappa^1_{f,T})^\varepsilon(\mathscr{C}_K^{\rho\varepsilon} \otimes_R Q)} \xrightarrow{\Phi_\varepsilon \otimes 1} \frac{\Lambda_\varepsilon \otimes_R P}{(\Lambda_\varepsilon \otimes_R (\mathfrak{h}/I)_\theta) \cdot \bar{\mathscr{L}}_{\varepsilon,\theta}} = L_\varepsilon$$

is then also an injective pseudo-isomorphism, and we let $L'_\varepsilon$ be its image. We then have an exact sequence

$$0 \to (\mathscr{U}_K/\mathscr{C}_K)^{\rho\varepsilon} \otimes_R Q \to L'_\varepsilon \to (\operatorname{coker} \kappa^1_{f,T})^\varepsilon \to 0.$$

By Lemma 4.3.1, this yields the desired exact sequence. The rest follows as $(\mathscr{U}_K/\mathscr{C}_K)^{\rho\varepsilon}$ and $Y_K^{\rho\varepsilon}$ have the same $\Lambda_\varepsilon$-characteristic ideal, and $(\mathscr{U}_K/\mathscr{C}_K)^{\rho\varepsilon}$ is trivial if $Y_K^{\rho\varepsilon}$ is finite. $\square$

**Corollary 4.3.5.** *If $L_\varepsilon$ is $\Lambda$-torsion, $Y_K^+$ is finite, and $P$ is $p$-torsion free, then $S_\varepsilon$ and $L_\varepsilon$ are pseudo-isomorphic $\Lambda \otimes_R \mathfrak{h}/I$-modules.*

*Remark* 4.3.6. In the statement of Theorem 4.3.4, we may replace the finiteness of $Y_K^\theta/\xi_\theta Y_K^\theta$ with the assumption of Conjecture 4.2.4.

*Remark* 4.3.7. Since we are assuming in Theorem 4.3.4 that $\Upsilon_\theta$ is a pseudo-isomorphism, we have that $(\operatorname{coker}\kappa^1_{f,T})^\varepsilon$ is pseudo-isomorphic to $\operatorname{coker}\Psi_{\varepsilon,\rho}$ by Lemma 4.2.3. This object appears in [Sh1, Theorem 4.3], where it is related to an Iwasawa module over the extension of $K$ defined by $\mathfrak{X}_K^\varepsilon$.

## 4.4 The Selmer group modulo Eisenstein

In this subsection, we consider the pseudo-cyclicity of the Selmer group

$$\mathrm{Sel}(K,\mathscr{T}_\theta^\vee) = \ker\left(H^1(G_{K,S'},\mathscr{T}_\theta^\vee) \to \bigoplus_{v\in S} H^1(G_{K_v},\mathscr{T}_{\mathrm{sub}}^\vee) \oplus \bigoplus_{v\in S'-S} H^1(I_{K_v},\mathscr{T}_\theta^\vee)\right),$$

where $\mathscr{T}_{\mathrm{sub}}$ is the kernel of canonical projection from $\mathscr{T}_\theta$ to its maximal unramified at $p$ quotient $\mathscr{T}_{\mathrm{quo}}$, and $I_{K_v}$ denotes the inertia group in the absolute Galois group $G_{K_v}$ of $K_v$. (Here, we use $S'-S$ to denote $S'_K - S_K$.)

Let us set

$$C_{f,S'}(K,\mathscr{T}_\theta(1)) = \mathrm{Cone}(C_{S'}(K,\mathscr{T}_\theta(1)) \to C_{l,S}(K,\mathscr{T}_{\mathrm{quo}}(1)))[-1],$$

and

$$C_{f,S'}(K,T(1)) = \mathrm{Cone}(C_{S'}(K,T(1)) \to C_{l,S}(K,P(1)))[-1].$$

We require the following lemma. In it, since $S'-S$ does not contain the primes over $p$, we make sense of $C_{l,S'-S}(K,T(1))$ as a direct sum of local cochain complexes at primes over those in $S'-S$.

**Lemma 4.4.1.** *We have a canonical isomorphism*

$$\mathrm{Sel}(K,T^\vee)^\vee \cong \ker(H^2_{f,S'}(K,T(1)) \to H^2_{l,S}(K,Q(1)) \oplus H^2_{l,S'-S}(K,T(1))).$$

*Proof.* We first claim that the sequence

$$H^1_{l,S'-S}(K,T(1)) \to H^2_{c,S'}(K,T(1)) \to H^2_{c,S}(K,T(1)) \to 0,$$

where the last map is deflation (i.e., dual to inflation), is exact. To see this, set $\mathscr{G} = \ker(G_{K,S'} \to G_{K,S})$, and note that inflation-restriction provides an exact sequence

$$0 \to H^1(G_{K,S},T^\vee) \to H^1(G_{K,S'},T^\vee) \to \mathrm{Hom}_{G_{K,S}}(\mathscr{G},T^\vee).$$

As $\mathscr{G}$ is generated by inertia subgroups in $G_{K,S'}$ at primes over those in $S'-S$, restriction provides an injection

$$\mathrm{Hom}_{G_{K,S}}(\mathscr{G},T^\vee) \hookrightarrow \mathrm{Hom}\left(\bigoplus_{v\in S'-S} I_{K_v},T^\vee\right).$$

As the decomposition and inertia groups at primes over those in $S'-S$ in the Galois group of the maximal pro-$p$ extension of $K$ are equal, the latter group equals $\bigoplus_{v\in S'-S} H^1(G_{K_v}, T^\vee)$. Via Tate and Poitou-Tate duality, we have the claim.

Let

$$\mathscr{K} = \ker(H^2_{f,S'}(K,T(1)) \to H^2_{l,S}(K,Q(1)) \oplus H^2_{l,S'-S}(K,T(1))).$$

As in the discussion of (2.10), the group $\mathscr{K}$ fits in the first row of a map of right exact sequences

$$\begin{tikzcd} H^1_{l,S'-S}(K,T(1)) \oplus H^1_{l,S}(K,Q(1)) \arrow[r] \arrow[d, two heads] & H^2_{c,S'}(K,T(1)) \arrow[r] \arrow[d, two heads] & \mathscr{K} \arrow[r] \arrow[d] & 0 \\ H^1_{l,S}(K,Q(1)) \arrow[r] & H^2_{c,S}(K,T(1)) \arrow[r] & \mathrm{Sel}(K,T^\vee)^\vee \arrow[r] & 0. \end{tikzcd}$$

Employing our initial claim, a diagram chase tells us that the map $\mathscr{K} \to \mathrm{Sel}(K,T^\vee)^\vee$ is an isomorphism. $\square$

We now have the following rather standard result asserting a type of good control.

**Proposition 4.4.2.** *The canonical map*

$$(\mathrm{Sel}(K,\mathscr{T}_\theta^\vee)^\vee)^\varepsilon \otimes_{\mathfrak{h}} \mathfrak{h}/I \to S_\varepsilon$$

*is an isomorphism of* $(\Lambda \otimes_R \mathfrak{h}/I)$*-modules.*

*Proof.* Restriction from $H^1(G_{K_v}, \mathscr{T}_\theta^\vee)$ to $H^1(I_{K_v}, \mathscr{T}_\theta^\vee)$ is injective for $v \in S'-S$, so we may replace $I_{K_v}$ with $G_{K_v}$ in the definition of $\mathrm{Sel}(K,\mathscr{T}_\theta^\vee)$. We have an exact sequence

$$0 \to C_{l,S}(K,\mathscr{T}_{\mathrm{sub}}(1))[-1] \oplus C_{l,S'-S}(K,\mathscr{T}_\theta(1))[-1] \to C_{c,S'}(K,\mathscr{T}_\theta(1)) \to C_{f,S'}(K,\mathscr{T}_\theta(1)) \to 0.$$

By Tate and Poitou-Tate duality, we have

$$\mathrm{Sel}(K,\mathscr{T}_\theta^\vee)^\vee \cong \mathrm{coker}(H^1_{l,S}(K,\mathscr{T}_{\mathrm{sub}}(1)) \oplus H^1_{l,S'-S}(K,\mathscr{T}_\theta(1)) \to H^2_{c,S'}(K,\mathscr{T}_\theta(1))).$$

By this and Lemma 4.4.1, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Sel}(K,\mathscr{T}_\theta^\vee)^\vee \arrow[r] \arrow[d] & H^2_{f,S'}(K,\mathscr{T}_\theta(1)) \arrow[r] \arrow[d] & H^2_{l,S}(K,\mathscr{T}_{\mathrm{sub}}(1)) \oplus H^2_{l,S'-S}(K,\mathscr{T}_\theta(1)) \arrow[r] \arrow[d] & H^3_{c,S'}(K,\mathscr{T}_\theta(1)) \arrow[d] \\ 0 \arrow[r] & \mathrm{Sel}(K,T^\vee)^\vee \arrow[r] & H^2_{f,S'}(K,T(1)) \arrow[r] & H^2_{l,S}(K,Q(1)) \oplus H^2_{l,S'-S}(K,T(1)) \arrow[r] & H^3_{c,S'}(K,T(1)) \end{tikzcd} \tag{4.8}$$

in which the vertical arrows after the first are the reduction modulo $I$ maps on coefficients. Note that if $H^j_{f,S}(K,T(1))^\varepsilon = 0$ for all $j > i$ for some $i \geq 0$, then by Nakayama's lemma we have $H^j_{f,S}(K,\mathscr{T}_\theta(1))^\varepsilon = 0$ for all $j > i$ and

$$H^i_{f,S}(K,T(1))^\varepsilon \cong H^i_{f,S}(K,\mathscr{T}_\theta(1))^\varepsilon \otimes_{\mathfrak{h}} \mathfrak{h}/I. \tag{4.9}$$

Since $H^3_{f,S}(K,T(1))^\varepsilon = 0$ by the nontriviality of $\varepsilon$, we have (4.9) for $i = 2$. Similarly, the right two vertical arrows of (4.8) induce isomorphisms upon reduction modulo $I$, for instance as the local Galois groups have cohomological dimension 2. The result then follows from the five lemma. $\square$

We next state a form of the main conjecture for modular forms under somewhat weaker hypotheses than those of [FK2, Conjecture 11.2.9].

**Conjecture 4.4.3.** *The Selmer group $(\mathrm{Sel}(K,\mathscr{T}_\theta^\vee)^\vee)^\varepsilon$ is $(\Lambda\hat{\otimes}_R\mathfrak{h})$-torsion, and its class in the Grothendieck group of the quotient of the category of finitely generated torsion $(\Lambda\hat{\otimes}_R\mathfrak{h})$-modules by the category of pseudo-null modules is equal to the class of $(\Lambda_\varepsilon\hat{\otimes}_R\mathscr{T}_\theta^+)/\mathscr{L}_{\varepsilon,\theta}$.*

*Remark* 4.4.4. By the Galois equivariance of the pairing in (4.1), the discrete $\mathfrak{h}[G_{\mathbf{Q},S'}]$-modules $\mathscr{T}_\theta^\vee$ and $\mathscr{T}_\theta(-1)\otimes_{\mathfrak{h}}\mathscr{S}^\vee$ agree up to a twist, so their strict Selmer groups do as well. On the $p$-adic analytic side, the same twist is reflected in the functional equation for the Mazur-Kitagawa $p$-adic $L$-function.

Following the terminology of [WWE], let us say that $\mathfrak{h}_\theta$ is "weakly Gorenstein" if $(\mathfrak{h}_\theta)_\mathfrak{q}$ is Gorenstein for every height one prime ideal $\mathfrak{q}$ of $\mathfrak{h}_\theta$. If $\mathfrak{h}_\theta$ is weakly Gorenstein, then the above two conjectures imply a certain local cyclicity of Selmer groups.

**Theorem 4.4.5.** *Suppose that $L_\varepsilon$ is $\Lambda$-torsion, $Y_K^+$ is finite, and $P$ is $p$-torsion free. Let $\mathfrak{P}$ be a prime ideal of $\Lambda_\varepsilon\hat{\otimes}_R\mathfrak{h}_\theta$ such that $\mathfrak{p} = \mathfrak{P}\cap\mathfrak{h}_\theta$ is properly contained in the maximal ideal $\mathfrak{m}$ of $\mathfrak{h}_\theta$ containing $I$. Conjecture 4.4.3 implies that the localizations $(\mathrm{Sel}(K,\mathscr{T}_\theta^\vee)^\vee)^\varepsilon_\mathfrak{P}$ and $((\Lambda_\varepsilon\hat{\otimes}_R\mathscr{T}_\theta^+)/\mathscr{L}_{\varepsilon,\theta})_\mathfrak{P}$ are pseudo-isomorphic as $(\Lambda\hat{\otimes}_R\mathfrak{h}_\theta)_\mathfrak{P}$-modules.*

*Proof.* The Hecke algebra $\mathfrak{h}_\theta$ is weakly Gorenstein since $Y_K^\rho$ is pseudo-cyclic and $Y_K^\theta/\xi_\theta Y_K^\theta$ is finite [WWE, Theorem A]. Since $\mathscr{T}_\theta^+$ is noncanonically isomorphic to the dualizing module $\mathscr{S}_\theta$ of $\mathfrak{h}_\theta$ by [Sh3, Theorem 4.3], its localization at any height one prime is free of rank one. Assuming Conjecture 4.4.3, it then suffices to show that $(\mathrm{Sel}(K,\mathscr{T}_\theta^\vee)^\vee)^\varepsilon_\mathfrak{P}$ is pseudo-cyclic as a $(\Lambda\hat{\otimes}_R\mathfrak{h}_\theta)_\mathfrak{P}$-module. By Corollary 4.3.5, we have that $S_\varepsilon$ is pseudo-isomorphic to the $(\Lambda\otimes_R\mathfrak{h}/I)$-module $L_\varepsilon$ that is pseudo-cyclic by Theorem 4.2.6. By Proposition 4.4.2 and Nakayama's lemma, we see that $(\mathrm{Sel}(K,\mathscr{T}_\theta^\vee)^\vee)^\varepsilon_\mathfrak{P}$ itself is pseudo-cyclic. For this, note that if a $\Lambda\hat{\otimes}_R(\mathfrak{h}_\theta)_\mathfrak{m}$-module becomes finite upon reduction modulo $I$, then it is pseudo-null. $\square$

*Remark* 4.4.6. One may replace the assumptions of Corollary 4.3.5 in the first sentence of the statement of Theorem 4.4.5 with the weaker but more long-winded assumptions of Theorem 4.3.4 (including the finiteness of $Y_K^{\rho\varepsilon}$).

# A Continuous cohomology

In this appendix, we study the continuous cohomology of a profinite group $G$ with coefficients in topological $\Lambda[\![G]\!]$-modules for a profinite ring $\Lambda$. We want to view such cohomology groups themselves as topological $\Lambda$-modules. It seems probable that many of the non-cited results are not new, but we have not found convenient references.

## A.1 Topological modules over a profinite ring

Let $\Lambda$ be a profinite ring, which is to say a compact, Hausdorff, totally disconnected topological ring with a basis of open neighborhoods of zero consisting of ideals of finite index. We denote the category of Hausdorff $\Lambda$-modules with continuous $\Lambda$-module homomorphisms by $\mathscr{T}_\Lambda$, its full subcategory of locally compact (Hausdorff) $\Lambda$-modules by $\mathscr{L}_\Lambda$ and the full, abelian subcategories of compact $\Lambda$-modules and discrete $\Lambda$-modules by $\mathscr{C}_\Lambda$ and $\mathscr{D}_\Lambda$, respectively. The category of all $\Lambda$-modules, with $\Lambda$-module homomorphisms, will be denoted $\mathrm{Mod}_\Lambda$. We write $X \in \mathscr{A}$ to indicate that $X$ is an object of a category $\mathscr{A}$.

We begin with the following standard facts.

**Lemma A.1.1.** *Inverse (resp., direct) limits of objects in the category $\mathscr{T}_\Lambda$ exist, and they are endowed with the initial (resp., final) topology with respect to the inverse (resp., direct) system defining the limit.*

For a proof of the following, see [RZ, Lemma 5.1.1].

**Lemma A.1.2.** *Every finite object in $\mathscr{T}_\Lambda$ is discrete. Every object in $\mathscr{C}_\Lambda$ is an inverse limit of finite $\Lambda$-module quotients, and every object in $\mathscr{D}_\Lambda$ is a direct limit of finite $\Lambda$-submodules.*

For topological spaces $X$ and $Y$, we use $\mathrm{Maps}(X,Y)$ to denote the set of continuous maps from $X$ to $Y$, which we endow with the compact-open topology. This topology has a subbase of open sets

$$V(K,U) = \{f \in \mathrm{Maps}(X,Y) \mid f(K) \subseteq U\}$$

for $K \subseteq X$ compact and $U \subseteq Y$ open. We have the following easily-proven lemma (cf. [Fl, Proposition 3]).

**Lemma A.1.3.** *For $X$ a topological space and $N \in \mathscr{T}_\Lambda$, the abelian group* $\mathrm{Maps}(X,N)$ *is a topological left $\Lambda$-module under the action defined by $(\lambda \cdot f)(x) = \lambda \cdot f(x)$ for $\lambda \in \Lambda$, $f \in$* $\mathrm{Maps}(X,N)$*, and $x \in X$.*

**Lemma A.1.4.** *Let $X$ be a compact space. Let $(N_\alpha, \pi_{\alpha,\beta})$ be an inverse system of finite objects and surjective maps in $\mathscr{C}_\Lambda$, and let $(N, \pi_\alpha)$ be the inverse limit of the system. Then the isomorphism*

$$\phi \colon \mathrm{Maps}(X,N) \to \varprojlim_\alpha \mathrm{Maps}(X,N_\alpha)$$

*of $\Lambda$-modules induced by the inverse limit is also a homeomorphism.*

*Proof.* The isomorphism $\phi$ is continuous by the universal property of the inverse limit, since each map

$$\phi_\alpha \colon \mathrm{Maps}(X,N) \to \mathrm{Maps}(X,N_\alpha)$$

is continuous. For $K$ a compact subset of $X$ and $U$ an open subset of $N$, we have

$$\phi_\alpha(V(K,U)) = V(K,U_\alpha),$$

where $U_\alpha = \pi_\alpha(U)$, noting that any splitting of $\pi_\alpha$ as maps of sets is continuous.

We claim that the system $V(K,U_\alpha)$ of open neighborhoods in the $\mathrm{Maps}(X,N_\alpha)$ defines an open set in the inverse limit $\mathscr{M} = \varprojlim \mathrm{Maps}(X,N_\alpha)$. Let $(f_\alpha)_\alpha \in \mathscr{M}$ with $f_\alpha \in V(K,U_\alpha)$ for all $\alpha$. For $m \in K$, the sequence $(f_\alpha(m))_\alpha$ defines an element $u \in U$. Since $U$ is open in $N$, there exist $V_{u,\alpha} \subseteq N_\alpha$ with $\pi_\alpha(u) \in V_{u,\alpha}$ and such that almost all $V_{u,\alpha}$ equal $N_\alpha$ and $\prod_\alpha V_{u,\alpha} \cap (\varprojlim N_\alpha)$ is contained in $\prod_\alpha U_\alpha$. Then $Q_m = \bigcap_\alpha f_\alpha^{-1}(V_{u,\alpha})$ is closed and open in $X$, so $K_m = K \cap Q_m$ is compact, open in $K$, and contains $m$. In particular, we have

$$(f_\alpha)_\alpha \in \mathscr{M} \cap \prod_\alpha V(K_m, V_{u,\alpha}) \subseteq \mathscr{M} \cap \prod_\alpha V(K_m, U_\alpha). \tag{A.1}$$

Since $K$ is compact, there exist $d \geq 1$ and $m_1, \ldots, m_d \in K$ such that the $K_{m_i}$ cover $K$. The intersection of the open subsets $\mathscr{M} \cap \prod_\alpha V(K_{m_i}, V_{u_i,\alpha})$ of $\mathscr{M}$, where $u_i = (f_\alpha(m_i))_\alpha$, is then an open subset of $\mathscr{M}$ containing $(f_\alpha)$. □

**Lemma A.1.5.** *Let $(X_\alpha, \iota_{\alpha,\beta})$ be a direct system of finite objects and injective maps in the category of discrete spaces, and let $(X, \iota_\alpha)$ be the direct limit of the system. Let $N \in \mathscr{T}_\Lambda$. Then the isomorphism*

$$\theta \colon \mathrm{Maps}(X,N) \to \varprojlim_\alpha \mathrm{Maps}(X_\alpha,N)$$

*of $\Lambda$-modules is a homeomorphism.*

*Proof.* That $\theta$ is continuous follows from the universal property, since each restriction map

$$\theta_\alpha \colon \mathrm{Maps}(X,N) \to \mathrm{Maps}(X_\alpha,N)$$

is continuous. For $K$ a compact, hence finite, subset of $X$ and $U$ an open subset of $N$, we have

$$\theta_\alpha(V(K,U)) = V(K_\alpha,U),$$

where $K_\alpha = \iota_\alpha^{-1}(K)$.

We need to show that the system $V(K_\alpha,U)$ of open subsets in the $\mathrm{Maps}(X_\alpha,N)$ defines an open set in the inverse limit. Being finite, $K$ is contained in $\iota_\beta(X_\beta)$ for some $\beta$ and therefore equals $\iota_\beta(K_\beta)$. Setting $\mathscr{N} = \varprojlim \mathrm{Maps}(X_\alpha,N)$, we claim that

$$\mathscr{N} \cap \left( \prod_{\alpha \neq \beta} \mathrm{Maps}(X_\alpha,N) \times V(K_\beta,U) \right) = \mathscr{N} \cap \prod_\alpha V(K_\alpha,U). \tag{A.2}$$

To see this, let $(f_\alpha)_\alpha$ lie in the left-hand side of (A.2). For a given $\alpha$, choose $\gamma \geq \alpha$ such that $\gamma \geq \beta$. Since $\iota_{\beta,\gamma}$ is injective, we have $\iota_{\beta,\gamma}(K_\beta) = K_\gamma$, so $f_\gamma \in V(K_\gamma,U)$. Since $\iota_{\alpha,\gamma}(K_\alpha) \subseteq K_\gamma$, we have $f_\alpha \in V(K_\alpha,U)$ as well, as required. □

Fix a set $\mathscr{I}$ of open ideals of $\Lambda$ that forms a basis of neighborhoods of 0 in $\Lambda$. We recall the following lemma for convenience.

**Lemma A.1.6.**

a. *Let $M$ be a finitely generated, compact $\Lambda$-module. Then we have an isomorphism*

$$M \cong \varprojlim_{I \in \mathscr{I}} M/IM$$

*of topological $\Lambda$-modules.*

b. *Let $M$ be a finitely generated, compact $\Lambda$-module and $N$ be a compact or discrete $\Lambda$-module. Or, let $M$ and $N$ be compact $\Lambda$-modules endowed with the $\mathscr{I}$-adic topology. Then*

$$\mathrm{Hom}_{\Lambda,\mathrm{cts}}(M,N) = \mathrm{Hom}_\Lambda(M,N).$$

*Proof.* Part a is [Li, Proposition 3.1.7], and part b is then [Li, Lemma 3.1.4]. □

**Lemma A.1.7.** *Let $M,N \in \mathscr{C}_\Lambda$, and suppose that $N$ is an inverse limit in $\mathscr{C}_\Lambda$ of a system $(N_\alpha,\pi_{\alpha,\beta})$ of finite $\Lambda$-modules and surjective homomorphisms. Then the isomorphism*

$$\mathrm{Hom}_{\Lambda,\mathrm{cts}}(M,N) \xrightarrow{\sim} \varprojlim_\alpha \mathrm{Hom}_{\Lambda,\mathrm{cts}}(M,N_\alpha)$$

*of groups induced by the inverse limit is also a homeomorphism.*

*Proof.* Note that the compact-open topology on $\mathrm{Hom}_{\Lambda,\mathrm{cts}}(M,N)$ is the subspace topology from $\mathrm{Maps}(M,N)$. This is then a corollary of Lemma A.1.4, since the topology on the inverse limit of homomorphism groups agrees with the subspace topology of the inverse limit topology on the inverse limit of the spaces $\mathrm{Maps}(M,N_\alpha)$. □

We also have the analogue for homomorphisms of discrete $\Lambda$-modules, which is a corollary of Lemma A.1.5.

**Lemma A.1.8.** *Let $M,N \in \mathscr{D}_\Lambda$, and suppose that $M$ is a direct limit in $\mathscr{D}_\Lambda$ of a system $(M_\alpha, \iota_{\alpha,\beta})$ of finite $\Lambda$-modules and injective homomorphisms. Then the isomorphism*

$$\mathrm{Hom}_\Lambda(M,N) \xrightarrow{\sim} \varprojlim_\alpha \mathrm{Hom}_\Lambda(M_\alpha,N)$$

*of groups induced by the inverse limit is also a homeomorphism.*

We let $\Lambda^\circ$ denote the opposite ring to $\Lambda$. For a topological $\Lambda$-module $X$, we let

$$X^\vee = \mathrm{Hom}_{\mathrm{cts}}(X,\mathbf{R}/\mathbf{Z})$$

denote the Pontryagin dual $\Lambda^\circ$-module. We endow $X^\vee$ with the compact-open topology, and then $X^\vee$ becomes an object in $\mathscr{T}_{\Lambda^\circ}$. The Pontryagin dual preserves the property of local compactness. For any $X \in \mathscr{L}_\Lambda$, the natural map $X \to (X^\vee)^\vee$ is an isomorphism in $\mathscr{L}_\Lambda$. If $X$ is a compact (resp., discrete) $\Lambda$-module, then $X^\vee$ is a discrete (resp., compact) $\Lambda^\circ$-module. In fact, the Pontryagin dual provides contravariant, exact equivalences between the abelian categories $\mathscr{C}_\Lambda$ and $\mathscr{D}_{\Lambda^\circ}$.

In the following proposition, we suppose that $\Lambda$ is an $R$-algebra over a profinite commutative ring $R$, where the map from $R$ to the center of $\Lambda$ is continuous. One can always take $R = \widehat{\mathbf{Z}}$.

**Proposition A.1.9.** *Suppose that $T,U \in \mathscr{C}_\Lambda$ are endowed with the $\mathscr{I}$-adic topology (e.g., are finitely generated $\Lambda$-modules). Then the map*

$$\mathrm{Hom}_\Lambda(T,U) \to \mathrm{Hom}_{\Lambda^\circ}(U^\vee,T^\vee)$$

*that sends $\rho$ to the map $\rho^*$ with $\rho^*(\phi) = \phi \circ \rho$ for all $\phi \in U^\vee$ is an isomorphism in $\mathscr{C}_R$.*

*Proof.* Given $\psi \in \mathrm{Hom}_{\Lambda^\circ}(U^\vee,T^\vee)$, we can define $\psi^* \in \mathrm{Hom}_\Lambda(T,U)$ uniquely by $\phi(\psi^*(t)) = \psi(\phi)(t)$ for all $\phi \in U^\vee$. The resulting map is the inverse to the map in question, so we have an isomorphism of $R$-modules.

Since $\mathrm{Hom}_\Lambda(T,U) = \mathrm{Hom}_{\Lambda,\mathrm{cts}}(T,U)$ by Lemma A.1.6b, we can view $\mathrm{Hom}_\Lambda(T,U)$ as a topological $R$-module. Note that $U/IU$ is finite for all $I \in \mathscr{I}$, since $U$ has both the $\mathscr{I}$-adic topology and the topology an inverse limit of finite $\Lambda$-modules, at least on one of which must by definition have $U/IU$ as a quotient. For each such $I$, we then have a compatible isomorphism

$$\mathrm{Hom}_\Lambda(T,U/IU) \xrightarrow{\sim} \mathrm{Hom}_{\Lambda^\circ}((U/IU)^\vee,T^\vee)$$

of finite $R$-modules, and these are topological isomorphisms as finite objects are discrete. Together with Lemma A.1.6, Lemmas A.1.7 and A.1.8 imply that the inverse limits of these modules are isomorphic to $\mathrm{Hom}_\Lambda(T,U)$ and $\mathrm{Hom}_{\Lambda^\circ}(U^\vee,T^\vee)$, respectively, in $\mathscr{C}_R$. □

## A.2 Cochains and cohomology groups

Let $G$ be a profinite group. We set $\mathscr{T}_{\Lambda,G} = \mathscr{T}_{\Lambda[\![G]\!]}$, which is to say that $\mathscr{T}_{\Lambda,G}$ is the category of topological $\Lambda$-modules with a continuous $\Lambda$-linear action of $G$ with morphisms that are continuous $\Lambda[G]$-module homomorphisms. Similarly, we set $\mathscr{L}_{\Lambda,G} = \mathscr{L}_{\Lambda[\![G]\!]}$, $\mathscr{C}_{\Lambda,G} = \mathscr{C}_{\Lambda[\![G]\!]}$, and $\mathscr{D}_{\Lambda,G} = \mathscr{D}_{\Lambda[\![G]\!]}$.

The category of chain complexes over an additive category $\mathscr{A}$ that admits kernels and cokernels will be denoted by $\mathrm{Ch}(\mathscr{A})$, with $\mathrm{Ch}^+(\mathscr{A})$, $\mathrm{Ch}^-(\mathscr{A})$, and $\mathrm{Ch}^b(\mathscr{A})$ denoting the full subcategories of bounded below, bounded above, and bounded complexes in $\mathscr{A}$.

The complex of inhomogeneous continuous cochains provides a functor

$$C(G,\cdot)\colon \mathscr{T}_{\Lambda,G} \to \mathrm{Ch}^+(\mathrm{Mod}_\Lambda).$$

For $M \in \mathscr{T}_{\Lambda,G}$, we endow each $C^i(G,M)$ with the compact-open topology. The coboundary maps in the complex $C(G,M)$ are then continuous as $G$ is a topological group, $M$ is a continuous $G$-module, and the action of $\Lambda$ on $M$ commutes with the $G$-action. For $M \in \mathscr{T}_{\Lambda,G}$, we denote the $\Lambda$-modules that are the $i$th cocycle, coboundary, and cohomology groups for $C(G,M)$ by $Z^i(G,M)$, $B^i(G,M)$, and $H^i(G,M)$, respectively. Both $Z^i(G,M)$ and $B^i(G,M)$ may be viewed as objects in $\mathscr{T}_\Lambda$ under the subspace topologies from $C^i(G,M)$. We have the following easy lemma.

**Lemma A.2.1.** *For $A \in \mathscr{D}_{\Lambda,G}$, the topological $\Lambda$-modules $C^i(G,A)$ are discrete.*

*Proof.* Since $G$ is profinite and $A$ is discrete, the image of any element $f \in C^i(G,A)$ is finite. Moreover, for each $a$ in the image, $f^{-1}(a)$ must be open, so compact since $G$ is profinite. Thus $\{f\} = \bigcap_{a\in \mathrm{im} f} V(f^{-1}(a),\{a\})$ is open. □

If $A \in \mathscr{D}_{\Lambda,G}$, then $A$ is a direct limit of its finite $\Lambda[G]$-submodules $A_\alpha$, and we have

$$H^i(G,A) \cong \varinjlim_{\alpha} H^i(G,A_\alpha).$$

We endow $H^i(G,A)$ with the discrete topology, under which it is an object of $\mathscr{D}_\Lambda$. This of course agrees with the subquotient topology from $C^i(G,A)$, where we view $H^i(G,A)$ as $Z^i(G,A)/B^i(G,A)$. The compact case requires a bit more work, and the result is found in the following statement.

**Proposition A.2.2.** *Assume that $H^i(G,M)$ is finite for every finite $\mathbf{Z}[P^{-1}][G]$-module $M$ and every $i \geq 0$, where $P$ is a set of primes of $\mathbf{Z}$ that act invertibly on $\Lambda$. Suppose that $T \in \mathscr{C}_{\Lambda,G}$, and write $T = \varprojlim T_\alpha$ for some finite $\Lambda[G]$-module quotients $T_\alpha$. Then the natural map*

$$H^i(G,T) \to \varprojlim_{\alpha} H^i(G,T_\alpha)$$

*is an isomorphism of $\Lambda$-modules. Moreover, $B^i(G,T)$ and $Z^i(G,T)$ are closed subspaces of $C^i(G,T)$, and the subquotient topology on $H^i(G,T)$ induced by the isomorphism*

$$H^i(G,T) \cong \frac{Z^i(G,T)}{B^i(G,T)}$$

*agrees with the profinite topology induced by the above isomorphism.*

*Proof.* Since $C^i(G,T)$ is the inverse limit of the $C^i(G,T_\alpha)$ by Lemma A.1.4, diagram chasing tells us that

$$Z^i(G,T) \cong \varprojlim_{\alpha} Z^i(G,T_\alpha) \quad \text{and} \quad B^i(G,T) \cong \varprojlim_{\alpha} B^i(G,T_\alpha), \tag{A.3}$$

and similarly for $H^i(G,T)$ (see also [Li, Proposition 3.2.8]). A quick check of definitions tells us that the inverse limit topologies on $\varprojlim Z^i(G,T_\alpha)$ and $\varprojlim B^i(G,T_\alpha)$ agree with the subspace topologies from $\varprojlim C^i(G,T_\alpha)$, so the maps of (A.3) are homeomorphisms. In particular, since $B^i(G,T_\alpha)$ and $Z^i(G,T_\alpha)$ are closed subspaces of the discrete space $C^i(G,T_\alpha)$ (noting Lemma A.2.1), both $B^i(G,T)$ and $Z^i(G,T)$ are closed in $C^i(G,T)$.

The topology on $\varprojlim H^i(G,T_\alpha)$ is the subspace topology from the product $\prod_\alpha H^i(G,T_\alpha)$. Moreover, the topology on this inverse limit agrees with that of the quotient of the corresponding product of cocycles by that of coboundaries. Since the inverse limits of cocycles and coboundaries also have the subspace topology for the respective products, the canonical isomorphism between $Z^i(G,T)/B^i(G,T)$ and $H^i(G,T)$ is also a homeomorphism. $\square$

Cup products on continuous cohomology exist quite generally, as stated in the following lemma. The proof is immediate from the definitions. For this, if $\Omega$ denotes a profinite ring, then $\mathscr{T}_{\Omega-\Lambda,G}$ denotes the category of topological $\Omega$-$\Lambda$-bimodules with a continuous commuting action of $G$.

**Lemma A.2.3.** *Let $\Lambda$, $\Omega$, and $\Sigma$ be profinite rings. Let $M \in \mathscr{T}_{\Omega-\Lambda,G}$, $N \in \mathscr{T}_{\Lambda-\Sigma,G}$, and $L \in \mathscr{T}_{\Omega-\Sigma,G}$, and suppose that*

$$\phi : M \times N \to L,$$

*is a continuous, $\Lambda$-balanced, $G$-equivariant homomorphism of $\Omega$-$\Sigma$-bimodules. Then we have continuous, $\Lambda$-balanced cup products*

$$C^i(G,M) \times C^j(G,N) \xrightarrow{\cup} C^{i+j}(G,L)$$

*of $\Omega$-$\Sigma$-bimodules for each $i, j \geq 0$.*

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