Reciprocity maps with restricted ramification

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Let be an $p$ odd prime and $f$ a newform of level $N$.
Suppose that $f$ is ordinary at $p$, i.e., its $p$th Fourier coefficient is a unit at $p$.
Let $\mathcal{O}$ be a finite $\mathbb{Z}_p$-algebra containing the coefficients and $T_p$-eigenvalues of $f$.

**Notation (Galois representation attached to $f$)**

1. $V$ twist of the Galois representation attached to $f$ by its inverse determinant
2. $T$ a Galois stable $\mathcal{O}$-lattice in $V$
3. $T_{\text{quo}}$ image of $T$ in the 1-dimensional unramified quotient of $V$

**Definition (Selmer group of $f$)**

$\text{Sel}(\mathbb{Q}_\infty, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ is the subgroup of classes in $H^1(\mathbb{Q}_\infty, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ that are trivial in $H^1(\mathbb{Q}_\infty, p, T_{\text{quo}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ and unramified at all other places, for $\mathbb{Q}_\infty$ the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.
The main conjecture for modular forms

Notation \((p\text{-adic } L\text{-function of } f)\)

\(L_f \in \mathcal{O}[X]\) is the “usual” power series interpolating special values of \(L\)-functions of twists of \(f\)

Conjecture (Iwasawa Main Conjecture for Modular Forms)

The characteristic ideal of the Pontryagin dual \(\text{Sel}(\mathbb{Q}_\infty, T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p)\) of the Selmer group in \(\mathcal{O}[X] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\) is generated by \(L_f\), and the same is true in \(\mathcal{O}[T]\) if \(V\) is residually irreducible.

Under various hypotheses including good reduction \((p \nmid N)\), trivial Nebentypus, weight congruent to 2 modulo \(p - 1\), and residual irreducibility, the conjecture has been proven by Skinner and Urban, after work of Kato proving one divisibility.
Suppose that $p \geq 5$ and $p \nmid N\varphi(N)$.

**Notation**

$\Lambda = \mathbb{Z}_p[\mathbb{Z}_{p,N}^\times / \langle -1 \rangle]$, where $\mathbb{Z}_{p,N} = \varprojlim_r \mathbb{Z}/Np^r \mathbb{Z}$

**Notation**

1. $\mathfrak{h}$ denotes Hida’s ordinary cuspidal $\mathbb{Z}_p$-Hecke algebra of tame level $N$, which is a finite projective module over $\Lambda$ via diamond operators
2. $S$ denotes the $\mathfrak{h}$-module of $\Lambda$-adic cusp forms

$S \cong \text{Hom}_\Lambda(\mathfrak{h}, \Lambda)$, and $\mathfrak{h}$ is Gorenstein if and only if $S \cong \mathfrak{h}$. 
The lattice from cohomology

Notation
Let $\mathcal{T}$ denote the ordinary part of the inverse limit of the $H^1_{et}(X_1(Np^r)_{/\overline{Q}}, \mathbb{Z}_p(1))$ under trace maps. This is an $\mathfrak{h}$-module via the adjoint action of Hecke operators.

Any ordinary newform $f$ gives rise to a maximal ideal $m$ of $\mathfrak{h}$, which depends only on $f$ modulo a prime over $p$, and $\mathcal{T}_m$ has $T_f$ as a quotient.

Fact (Ordinariness of $\mathcal{T}$)
As $\mathfrak{h}[G_{Q_p}]$-modules, we have an exact sequence

$$0 \to \mathcal{T}_{\text{sub}} \to \mathcal{T} \to \mathcal{T}_{\text{quo}} \to 0,$$

where $\mathcal{T}_{\text{sub}} \cong \mathfrak{h}$ and $\mathcal{T}_{\text{quo}} \cong S$ as $\mathfrak{h}$-modules, and $\mathcal{T}_{\text{quo}}$ is unramified.
Ohta’s pairing

**Theorem (Ohta)**

There is a perfect pairing

\[ T \times T \to S(1) \]

of \( \mathfrak{h} \)-modules that is equivariant for the \( G_\mathbb{Q} \)-action on \( S \) for which a Galois element \( \sigma \) acts by the diamond operator \( \langle \bar{\sigma} \rangle \), where \( \bar{\sigma} \) is the image of \( \sigma \) in \( \mathbb{Z}^\times_{p,N} \).

Ohta’s pairing and Poitou-Tate duality allow one to relate the Selmer group of \( T \otimes_{\mathfrak{h}} S^\vee \) to the Selmer group of \( T^\vee \). We will focus on the latter.

**Notation**

Let \( K = \mathbb{Q}(\mu_{Np^\infty}) \), and note that \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}^\times_{p,N} \).

**Definition (Selmer group)**

The Selmer group \( \text{Sel}(K, T^\vee) \) is the subgroup of classes in \( H^1(K, T^\vee) \) that are trivial in \( H^1(K_v, T^\vee_{\text{sub}}) \) for all \( v \mid p \) and unramified at all other places.
Characters

Notation

Let $R$ be the $\mathbb{Z}_p$-algebra generated by the values of characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$. We require two characters $(\mathbb{Z}/Np\mathbb{Z})^\times \to R^\times$:

1. for diamond operators: $\theta$ primitive, even, and such that $\chi = \theta \omega^{-1}$ satisfies $\chi|_{(\mathbb{Z}/p\mathbb{Z})^\times} \neq 1$ or $\chi|_{(\mathbb{Z}/N\mathbb{Z})^\times} (p) \neq 1$, where $\omega$ is projection to $\mu_{p-1}(R)$
2. for Galois elements: $\epsilon$. Define $\pm$ by $\epsilon(-1) = \pm 1$.

Notation

1. $\mathfrak{h}_\theta = \mathfrak{h} \otimes_{\mathbb{Z}_p[\mathbb{Z}/Np\mathbb{Z}^\times]} R$ with the map to $R$ induced by $\theta$ and the map to $\mathfrak{h}$ be given by inverse diamond operators
2. $\Lambda_\epsilon = R[X]$ viewed as a quotient of $\Lambda \otimes_{\mathbb{Z}_p} R$ via $\epsilon$, where $X = [1 + p] - 1$

We also use superscripts to denote $(\mathbb{Z}/Np\mathbb{Z})^\times$-eigenspaces of modules over $\mathfrak{h}$ and $\Lambda$. 
Conjecture (Two-variable main conjecture)

The class of the dual Selmer group \((\text{Sel}(K, T^\vee_\theta)^\vee_\epsilon)\) in the Grothendieck group of the quotient of the category of finitely generated (torsion) \(\Lambda\hat{\otimes}_R \mathfrak{h}\)-modules by the category of pseudo-null (i.e., codimension 2) modules is equal to that of

\[
\frac{\Lambda_\epsilon \hat{\otimes}_R T^{\mp}_\theta}{(\Lambda_\epsilon \hat{\otimes}_R \mathfrak{h}_\theta) L_{\theta, \epsilon}},
\]

where \(T^{\mp}_\theta\) denotes the \((\mp1)\)-eigenspace under complex conjugation and \(L_{\theta, \epsilon}\) is a modified Mazur-Kitagawa two-variable \(p\)-adic \(L\)-function.

If \(\mathfrak{h}_\theta\) is Gorenstein or \(\epsilon\) is even, then \(T^{\mp}_\theta\) is free of rank 1 over \(\mathfrak{h}_\theta\), so we may view \(L_{\theta, \epsilon}\) as an element of \(\mathfrak{h}_\theta[\![X]\!]\) up to unit. The above conjecture says that \(L_{\theta, \epsilon}\) is a characteristic element for \((\text{Sel}(K, T^\vee_\theta)^\vee_\epsilon)\).
We can (and do) replace $\mathfrak{h}$, $S$, and $\mathcal{T}$ with their localizations at a maximal ideal $\mathfrak{m}$ of $\mathfrak{h}$ arising from a newform $f$ of tame level $N$. As direct summands of the original objects, the main conjecture respects this.

1. Under hypotheses that include $\epsilon = \theta = 1$ and $f$ is residually irreducible, then the main conjecture should follow from the work of Kato and Skinner-Urban after a duality argument.

2. Our interest is in the setting in which $f$ is congruent to an Eisenstein series, in which case $\mathcal{T}/\mathfrak{m}\mathcal{T}$ is reducible. We are particularly interested in the residual representation itself. For even $\epsilon$, this has been studied by Greenberg-Vatsal in the one-variable setting.
Definition (Eisenstein ideal)

Let $I$ be the Eisenstein ideal of $\mathfrak{h}$ is generated by $T_\ell - 1 - \ell \langle \ell \rangle$ (resp., $U_\ell - 1$) for primes $\ell \nmid Np$ (resp., $\ell \mid Np$).

Suppose $I\mathfrak{h}_\theta \neq \mathfrak{h}_\theta$, and let $m$ be the maximal ideal of $\mathfrak{h}_\theta$ containing $I$.

Notation

Set $T = \mathcal{T}_\theta / I\mathcal{T}_\theta$, $P = \mathcal{T}_\theta^+ / I\mathcal{T}_\theta^+$, and $Q = \mathcal{T}_\theta^- / I\mathcal{T}_\theta^-$.

Facts

1. There is an exact sequence of global Galois modules

$$0 \to P \to T \to Q \to 0$$

that is canonically locally split at places over $Np$. In particular, the maps $\mathcal{T}_{\text{sub}} / I\mathcal{T}_{\text{sub}} \to Q$ and $P \to \mathcal{T}_{\text{quo}} / I\mathcal{T}_{\text{quo}}$ are isomorphisms.

2. $Q$ is canonically isomorphic to $\mathfrak{h}/I$ as an $\mathfrak{h}$-module.
Question

What can we say about $\mathcal{G} = \text{Sel}(K, T^\vee)^\vee$?

Terminology

Let $S$ denote the set of primes over $p$ in $K$.

1. \textit{S-ramified}: unramified outside of the primes in $S$
2. \textit{S-split}: unramified and completely split at all primes in $S$
3. By an Iwasawa module over $K$ with a given property, we mean the Galois group of the maximal abelian, pro-$p$ extension of $K$ with that property.

Notation

1. $G_{K,S}$ Galois group of the maximal $S$-ramified extension of $K$
2. $\mathcal{U}$ norm compatible seq. in $p$-completions of $p$-units of number fields in $K$
3. $\mathcal{X}$ $S$-ramified Iwasawa module over $K$
4. $Y$ $S$-split Iwasawa module over $K$
Iwasawa cohomology

**Definition (Iwasawa cohomology)**

For a compact $S$-ramified Galois module $M$, $H^i_{Iw}(K, M)$ is the inverse limit of $i$th $S$-ramified continuous cohomology groups of $M$ under corestriction.

**Terminology (Compactly-supported cohomology)**

Compactly supported Iwasawa cohomology groups of $M$ fit in an exact sequence

$$
\cdots \rightarrow H^i_{c,Iw}(K, M) \rightarrow H^i_{Iw}(K, M) \rightarrow H^i_{p,Iw}(K, M) \rightarrow \cdots,
$$

for $H^i_{p,Iw}(K, M)$ the direct sum of local Iwasawa cohomology groups at primes over $p$. By Poitou-Tate duality, they satisfy

$$
H^i_{c,Iw}(K, M) \cong H^{2-i}(G_{K,S}, M^\vee(1))^\vee.
$$

**Examples**

1. $H^1_{Iw}(K, \mathbb{Z}_p(1)) \cong \mathcal{U}$, and there is an exact sequence

$$
0 \rightarrow Y \rightarrow H^2_{Iw}(K, \mathbb{Z}_p(1)) \rightarrow \bigoplus_{v \in S} \mathbb{Z}_p \rightarrow \mathbb{Z}_p \rightarrow 0
$$

2. $H^2_{c,Iw}(K, \mathbb{Z}_p(1)) \cong \mathcal{X}$ and $H^3_{c,Iw}(K, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p$
Using the local splittings $P \to T$ and restriction, we may define a cone with cohomology groups $H^i_{f,Iw}(K, T(1))$ fitting in long exact sequences

\[ \cdots \to H^i_{f,Iw}(K, T(1)) \to H^i_{Iw}(K, T(1)) \to H^i_{p,Iw}(K, P(1)) \to \cdots \]
\[ \cdots \to H^i_{f,Iw}(K, T(1)) \to H^i_{Iw}(K, Q(1)) \to H^{i+1}_{c,Iw}(K, P(1)) \to \cdots . \]

The second sequence reduces to

\[
0 \to H^1_{f,Iw}(K, T(1)) \to \mathcal{U} \otimes_{\mathbb{Z}_p} Q \xrightarrow{\kappa} \mathcal{X} \otimes_{\mathbb{Z}_p} P \to H^2_{f,Iw}(K, T(1))
\]
\[
\to H^2_{Iw}(K, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} Q \to P \to H^3_{f,Iw}(K, T(1)) \to 0.
\]

**Lemma (Comparison with Selmer)**

*There is a canonical exact sequence*

\[
0 \to \text{coker } \kappa \to \mathcal{S} \to Y \otimes_{\mathbb{Z}_p} Q \to P.
\]

**Question**

What is the cokernel of $\kappa : \mathcal{U} \otimes_{\mathbb{Z}_p} Q \to \mathcal{X} \otimes_{\mathbb{Z}_p} P$ on $\epsilon$-eigenspaces?
The case of even parity

Fact

$P$ has trivial $G_Q$-action, but $Q = Q_{\chi^{-1}}$. It follows that

$$(U \otimes_{\mathbb{Z}_p} Q)_\epsilon \cong U_{\chi \epsilon} \otimes_R Q \quad \text{and} \quad (X \otimes_{\mathbb{Z}_p} P)_\epsilon \cong X_{\epsilon} \otimes_R P.$$  

If $\epsilon$ is even, then $U_{\chi \epsilon}$ is trivial unless $\chi \epsilon = \omega$, in which case it is $R(1)$.

This implies the following 2-variable analogue of a result of Greenberg-Vatsal.

Corollary

If $\epsilon$ is even with $\epsilon \neq 1$ and $\chi \epsilon \neq \omega$, then there is a canonical exact sequence

$$0 \to X_{\epsilon} \otimes_R P \to \mathcal{G}_\epsilon \to Y_{\chi \epsilon} \otimes_R Q \to 0.$$  

The $R[[X]]$-characteristic ideals of $X_{\epsilon}$ and $Y_{\chi \epsilon}$ are generated by Kubota-Leopoldt $p$-adic $L$-functions by the classical Iwasawa main conjecture.
The case of odd parity

Suppose from now on that \( \epsilon \) is odd.

**Conjecture (Greenberg)**

\( Y^+ \) is finite, i.e., \( Y_\rho \) is finite for every even character \( \rho \).

**Facts**

1. If \( Y_{\chi\epsilon} \) is finite, then \( U_{\chi\epsilon} \) is generated by sequences of cyclotomic \( p \)-units.
2. There is a canonical homomorphism

   \[
   \Phi_\epsilon : \mathcal{X}_\epsilon \rightarrow \Lambda_\epsilon
   \]

   determined by the action of \( \mathcal{X} \) on cyclotomic \( p \)-units with the property that if \( Y_{\omega\epsilon-1} \) is finite, then \( \Phi_\epsilon \) is injective with finite cokernel \( (Y_{\omega\epsilon-1})^\vee(1) \).

The cocycle \( G_Q \rightarrow \text{Hom}_\Gamma(Q, P) \) attached to the exact sequence gives rise to a homomorphism \( \Upsilon_\theta : Y_\chi \rightarrow P \) (conjecturally an isomorphism) by composition with evaluation at the canonical generator of \( Q \).
The $S$-reciprocity map

**Definition ($S$-reciprocity map)**

Let $\mathcal{X}$ be the quotient of $\mathbb{Z}_p[\mathcal{X}]$ by the square of its augmentation ideal. The $S$-reciprocity map

$$\Psi : \mathcal{U} \rightarrow H^2_{Iw}(K, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathcal{X}.$$ 

is the first connecting map in the Iwasawa cohomology of the Tate twist of

$$0 \rightarrow \mathcal{X} \xrightarrow{\sigma \mapsto \sigma^{-1}} \mathcal{X} \xrightarrow{\tau \mapsto 1} \mathbb{Z}_p \rightarrow 0.$$ 

The analogous exact sequence for $Y$ in place of $\mathcal{X}$ is locally split at $p$. In place of usual cohomology, we again use that of a Selmer complex from Iwasawa cohomology to compactly-supported Iwasawa cohomology with connecting maps

$$\Theta : \mathcal{U} \rightarrow \mathcal{X} \otimes_{\mathbb{Z}_p} Y \quad \text{and} \quad q : H^2_{Iw}(K, \mathbb{Z}_p(1)) \rightarrow Y.$$ 

**Theorem (S.)**

The map $q$ splits the canonical injection, and $(q \otimes 1) \circ \Psi$ and $-\Theta$ are equal after switching the order of the tensor product.
Variant of a conjecture regarding cup products

**Notation**

1. $\Psi_{\epsilon,\chi} : U_{\chi \epsilon} \to \mathcal{X}_\epsilon \otimes Y_\chi$ is the map induced by $\Psi$ via commutativity of the tensor product

2. $u_{\chi \epsilon} \in U_{\chi \epsilon}$ is the image of the norm compatible system $1 - \zeta_{fpr}$ of elements $\mathbb{Q}(\mu_{Np^r})$, where $f$ is the tame conductor of $\chi \epsilon$.

**Conjecture (S.)**

For odd $\epsilon$, the $\Lambda_{\epsilon} \otimes_R (\mathfrak{h}/I)_\theta$ submodules of $\Lambda_{\epsilon} \otimes_R P$ generated by the image $\bar{L}_{\epsilon,\theta}$ of $L_{\epsilon,\theta}$ and $(\Phi_\epsilon \otimes \Upsilon_\theta)(\Psi_{\epsilon,\chi}(u_{\chi \epsilon}))$ are equal.

In fact, we expect that $(\Phi_\epsilon \otimes \Upsilon_\theta)(\Psi_{\epsilon,\chi}(u_{\chi \epsilon})) = \bar{L}_{\epsilon,\theta}$.

**Theorem (Wake-Wang Erickson, Fukaya-Kato)**

The conjecture holds if $Y_\theta$ and $Y_{\omega^2 \theta - 1}$ are finite and $P$ is $p$-torsion free.

These hypotheses are actually stronger than needed.
The proof of the following lemma uses the earlier theorem relating $\Theta$ and $\Psi$.

**Lemma**

**The first connecting homomorphism**

\[
\kappa_\epsilon : U_{\chi\epsilon} \otimes_R Q \rightarrow \mathcal{X}_\epsilon \otimes_R P
\]

in the sequence for $H^1_{f, Iw}(K, T(1))_\epsilon$ is equal to the composition

\[
U_{\chi\epsilon} \otimes_R Q \xrightarrow{-\Psi_{\epsilon, \chi} \otimes 1} \mathcal{X}_\epsilon \otimes_R Y_\chi \otimes_R Q \xrightarrow{1 \otimes \Upsilon_\theta \otimes 1} \mathcal{X}_\epsilon \otimes_R P \otimes_R Q \\
\rightarrow \mathcal{X}_\epsilon \otimes_R P \otimes_{\Lambda_\theta} Q \xrightarrow{\sim} \mathcal{X}_\epsilon \otimes_R P.
\]

**Theorem (S.)**

Let $\epsilon$ be odd. Suppose that $Y_\theta$, $Y_{\omega \chi^{-1}}$, $Y_{\chi \epsilon}$, and $Y_{\omega \epsilon^{-1}}$ are finite and that $P$ is $p$-torsion free. Then $\mathcal{G}_\epsilon$ and

\[
\frac{\Lambda_\epsilon \otimes_R P}{(\Lambda_\epsilon \otimes_R (\mathfrak{h}/I)_\theta) \cdot \tilde{L}_{\epsilon, \theta}}.
\]

are pseudo-isomorphic $\Lambda_\epsilon \otimes (\mathfrak{h}/I)_\theta$-modules.
Proposition

The canonical map \((\text{Sel}(K, T^\wedge))^\wedge \otimes_R h/I \rightarrow \mathcal{C}_\epsilon\) is an isomorphism.

Theorem (S.)

Suppose the conditions in the above theorem and that \(\mathfrak{p}\) is a prime ideal of \(\Lambda_\epsilon \hat{\otimes}_R h_\theta\) such that \(\mathfrak{p} = \mathfrak{P} \cap h_\theta\) is properly contained in the maximal ideal \(\mathfrak{m}\) of \(h_\theta\) containing \(I\). Then the main conjecture implies that the localizations of

\[
(\text{Sel}(K, T^\wedge))^\wedge \quad \text{and} \quad \frac{\Lambda_\epsilon \hat{\otimes}_R T^\wedge_\theta}{(\Lambda_\epsilon \hat{\otimes}_R h_\theta) \cdot \mathcal{L},\theta}
\]

at \(\mathfrak{p}\) are pseudo-isomorphic \((\Lambda_\epsilon \hat{\otimes}_R h_\theta)_{\mathfrak{p}}\)-modules.

Question

What of the two-variable residually reducible main conjecture can one obtain (supposing Greenberg’s conjecture) in cases where one divisibility in the two-variable main conjecture can be proven (e.g., via the work of Kato)?