Reciprocity maps with restricted ramification

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Let be an p odd prime and f a newform of level N.
Suppose that f is ordinary at p, i.e., its pth Fourier coefficient is a unit at p
Let $\mathcal{O}$ be a finite $\mathbb{Z}_p$-algebra containing the coefficients and T_p-eigenvalues of f.

Notation (Galois representation attached to f)

1. V twist of the Galois representation attached to f by its inverse determinant
2. T a Galois stable $\mathcal{O}$-lattice in V
3. T_{quo} image of T in the 1-dimensional unramified quotient of V

Definition (Selmer group of f)

$\mathrm{Sel}(\mathbb{Q}_\infty, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ is the subgroup of classes in $H^1(\mathbb{Q}_\infty, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ that are trivial in $H^1(\mathbb{Q}_{\infty,p}, T_{\text{quo}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$ and unramified at all other places, for $\mathbb{Q}_\infty$ the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$

The main conjecture for modular forms

Notation (p-adic L-function of f)

$\mathcal{L}_f \in \mathcal{O}[\![X]\!]$ is the "usual" power series interpolating special values of L-functions of twists of f

Conjecture (Iwasawa Main Conjecture for Modular Forms)

The characteristic ideal of the Pontryagin dual $\mathrm{Sel}(\mathbb{Q}_\infty, T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ *of the Selmer group in* $\mathcal{O}[\![X]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is generated by* $\mathcal{L}_f$*, and the same is true in* $\mathcal{O}[\![T]\!]$ *if* V *is residually irreducible.*

Under various hypotheses including good reduction ($p \nmid N$), trivial Nebentypus, weight congruent to 2 modulo $p-1$, and residual irreducibility, the conjecture has been proven by Skinner and Urban, after work of Kato proving one divisibility.

Suppose that $p \geq 5$ and $p \nmid N\varphi(N)$.

Notation

$\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_{p,N}^{\times}/\langle -1 \rangle]\!]$, where $\mathbb{Z}_{p,N} = \varprojlim_r \mathbb{Z}/Np^r\mathbb{Z}$

Notation

1. $\mathfrak{h}$ denotes Hida's ordinary cuspidal $\mathbb{Z}_p$-Hecke algebra of tame level N, which is a finite projective module over Λ via diamond operators
2. $\mathcal{S}$ denotes the $\mathfrak{h}$-module of Λ-adic cusp forms

$\mathcal{S} \cong \mathrm{Hom}_\Lambda(\mathfrak{h}, \Lambda)$, and $\mathfrak{h}$ is Gorenstein if and only if $\mathcal{S} \cong \mathfrak{h}$.

The lattice from cohomology

Notation

Let $\mathcal{T}$ denote the ordinary part of the inverse limit of the $H^1_{\text{ét}}(X_1(Np^r)_{/\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))$ under trace maps. This is an $\mathfrak{h}$-module via the adjoint action of Hecke operators.

Any ordinary newform f gives rise to a maximal ideal $\mathfrak{m}$ of $\mathfrak{h}$, which depends only on f modulo a prime over p, and $\mathcal{T}_{\mathfrak{m}}$ has T_f as a quotient.

Fact (Ordinariness of $\mathcal{T}$)

As $\mathfrak{h}[G_{\mathbb{Q}_p}]$-modules, we have an exact sequence

$$0 \to \mathcal{T}_{\text{sub}} \to \mathcal{T} \to \mathcal{T}_{\text{quo}} \to 0,$$

where $\mathcal{T}_{\text{sub}} \cong \mathfrak{h}$ and $\mathcal{T}_{\text{quo}} \cong \mathcal{S}$ as $\mathfrak{h}$-modules, and $\mathcal{T}_{\text{quo}}$ is unramified.

Ohta's pairing

Theorem (Ohta)

There is a perfect pairing

$$\mathcal{T} \times \mathcal{T} \to \mathcal{S}(1)$$

of $\mathfrak{h}$-modules that is equivariant for the $G_{\mathbb{Q}}$-action on $\mathcal{S}$ for which a Galois element σ acts by the diamond operator $\langle \bar{\sigma} \rangle$, where $\bar{\sigma}$ is the image of σ in $\mathbb{Z}_{p,N}^{\times}$.

Ohta's pairing and Poitou-Tate duality allow one to relate the Selmer group of $\mathcal{T} \otimes_{\mathfrak{h}} \mathcal{S}^{\vee}$ to the Selmer group of $\mathcal{T}^{\vee}$. We will focus on the latter.

Notation

Let $K = \mathbb{Q}(\mu_{Np^{\infty}})$, and note that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_{p,N}^{\times}$.

Definition (Selmer group)

The Selmer group $\mathrm{Sel}(K, \mathcal{T}^{\vee})$ is the subgroup of classes in $H^1(K, \mathcal{T}^{\vee})$ that are trivial in $H^1(K_v, \mathcal{T}_{\mathrm{sub}}^{\vee})$ for all $v \mid p$ and unramified at all other places.

Notation

Let R be the $\mathbb{Z}_p$-algebra generated by the values of characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$. We require two characters $(\mathbb{Z}/Np\mathbb{Z})^\times \to R^\times$:

1. for diamond operators: θ primitive, even, and such that $\chi = \theta\omega^{-1}$ satisfies $\chi|_{(\mathbb{Z}/p\mathbb{Z})^\times} \neq 1$ or $\chi|_{(\mathbb{Z}/N\mathbb{Z})^\times}(p) \neq 1$, where ω is projection to $\mu_{p-1}(R)$
2. for Galois elements: ϵ. Define $\pm$ by $\epsilon(-1) = \pm 1$.

Notation

1. $\mathfrak{h}_\theta = \mathfrak{h} \otimes_{\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times]} R$ with the map to R induced by θ and the map to $\mathfrak{h}$ be given by inverse diamond operators
2. $\Lambda_\epsilon = R[\![X]\!]$ viewed as a quotient of $\Lambda \otimes_{\mathbb{Z}_p} R$ via ϵ, where $X = [1+p] - 1$

We also use superscripts to denote $(\mathbb{Z}/Np\mathbb{Z})^\times$-eigenspaces of modules over $\mathfrak{h}$ and Λ.

Conjecture (Two-variable main conjecture)

The class of the dual Selmer group $(\mathrm{Sel}(K, \mathcal{T}_\theta^\vee)^\vee)_\epsilon$ in the Grothendieck group of the quotient of the category of finitely generated (torsion) $\Lambda \hat{\otimes}_R \mathfrak{h}$-modules by the category of pseudo-null (i.e., codimension 2) modules is equal to that of

$$\frac{\Lambda_\epsilon \hat{\otimes}_R \mathcal{T}_\theta^\mp}{(\Lambda_\epsilon \hat{\otimes}_R \mathfrak{h}_\theta)\mathcal{L}_{\theta,\epsilon}},$$

where $\mathcal{T}_\theta^\mp$ denotes the (∓ 1)-eigenspace under complex conjugation and $\mathcal{L}_{\theta,\epsilon}$ is a modified Mazur-Kitagawa two-variable p-adic L-function.

If $\mathfrak{h}_\theta$ is Gorenstein or ϵ is even, then $\mathcal{T}_\theta^\mp$ is free of rank 1 over $\mathfrak{h}_\theta$, so we may view $\mathcal{L}_{\theta,\epsilon}$ as an element of $\mathfrak{h}_\theta[\![X]\!]$ up to unit. The above conjecture says that $\mathcal{L}_{\theta,\epsilon}$ is a characteristic element for $(\mathrm{Sel}(K, \mathcal{T}_\theta^\vee)^\vee)_\epsilon$.

Remarks

We can (and do) replace $\mathfrak{h}$, $\mathcal{S}$, and $\mathcal{T}$ with their localizations at a maximal ideal $\mathfrak{m}$ of $\mathfrak{h}$ arising from a newform f of tame level N. As direct summands of the original objects, the main conjecture respects this.

1. Under hypotheses that include $\epsilon = \theta = 1$ and f is residually irreducible, then the main conjecture should follow from the work of Kato and Skinner-Urban after a duality argument.
2. Our interest is in the setting in which f is congruent to an Eisenstein series, in which case $\mathcal{T}/\mathfrak{m}\mathcal{T}$ is reducible. We are particularly interested in the residual representation itself. For even ϵ, this has been studied by Greenberg-Vatsal in the one-variable setting.

Definition (Eisenstein ideal)

Let I be the Eisenstein ideal of $\mathfrak{h}$ is generated by $T_\ell - 1 - \ell\langle\ell\rangle$ (resp., $U_\ell - 1$) for primes $\ell \nmid Np$ (resp., $\ell \mid Np$).

Suppose $I\mathfrak{h}_\theta \neq \mathfrak{h}_\theta$, and let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{h}_\theta$ containing I.

Notation

Set $T = \mathcal{T}_\theta / I\mathcal{T}_\theta$, $P = \mathcal{T}_\theta^+ / I\mathcal{T}_\theta^+$, and $Q = \mathcal{T}_\theta^- / I\mathcal{T}_\theta^-$.

Facts

1. *There is an exact sequence of global Galois modules*

$$0 \to P \to T \to Q \to 0$$

   that is canonically locally split at places over Np. In particular, the maps $\mathcal{T}_{\mathrm{sub}} / I\mathcal{T}_{\mathrm{sub}} \to Q$ and $P \to \mathcal{T}_{\mathrm{quo}} / I\mathcal{T}_{\mathrm{quo}}$ are isomorphisms.
2. *Q is canonically isomorphic to $\mathfrak{h}/I$ as an $\mathfrak{h}$-module.*

Iwasawa-theoretic objects

Question

What can we say about $\mathfrak{S} = \mathrm{Sel}(K, T^\vee)^\vee$?

Terminology

Let S denote the set of primes over p in K.

1. S-ramified: unramified outside of the primes in S
2. S-split: unramified and completely split at all primes in S
3. By an Iwasawa module over K with a given property, we mean the Galois group of the maximal abelian, pro-p extension of K with that property.

Notation

1. $G_{K,S}$ Galois group of the maximal S-ramified extension of K
2. $\mathcal{U}$ norm compatible seq. in p-completions of p-units of number fields in K
3. $\mathfrak{X}$ S-ramified Iwasawa module over K
4. Y S-split Iwasawa module over K

Definition (Iwasawa cohomology)

For a compact S-ramified Galois module M, $H^i_{\mathrm{Iw}}(K,M)$ is the inverse limit of ith S-ramified continuous cohomology groups of M under corestriction.

Terminology (Compactly-supported cohomology)

Compactly supported Iwasawa cohomology groups of M fit in an exact sequence

$$\cdots \to H^i_{c,\mathrm{Iw}}(K,M) \to H^i_{\mathrm{Iw}}(K,M) \to H^i_{p,\mathrm{Iw}}(K,M) \to \cdots,$$

for $H^i_{p,\mathrm{Iw}}(K,M)$ the direct sum of local Iwasawa cohomology groups at primes over p. By Poitou-Tate duality, they satisfy

$$H^i_{c,\mathrm{Iw}}(K,M) \cong H^{2-i}(G_{K,S}, M^\vee(1))^\vee.$$

Examples

1. $H^1_{\mathrm{Iw}}(K,\mathbb{Z}_p(1)) \cong \mathcal{U}$, and there is an exact sequence
$$0 \to Y \to H^2_{\mathrm{Iw}}(K,\mathbb{Z}_p(1)) \to \bigoplus_{v\in S} \mathbb{Z}_p \to \mathbb{Z}_p \to 0$$
2. $H^2_{c,\mathrm{Iw}}(K,\mathbb{Z}_p(1)) \cong \mathfrak{X}$ and $H^3_{c,\mathrm{Iw}}(K,\mathbb{Z}_p(1)) \cong \mathbb{Z}_p$

Using the local splittings $P \to T$ and restriction, we may define a cone with cohomology groups $H^i_{f,\mathrm{Iw}}(K, T(1))$ fitting in long exact sequences

$$\cdots \to H^i_{f,\mathrm{Iw}}(K,T(1)) \to H^i_{\mathrm{Iw}}(K,T(1)) \to H^i_{p,\mathrm{Iw}}(K,P(1)) \to \cdots$$
$$\cdots \to H^i_{f,\mathrm{Iw}}(K,T(1)) \to H^i_{\mathrm{Iw}}(K,Q(1)) \to H^{i+1}_{c,\mathrm{Iw}}(K,P(1)) \to \cdots .$$

The second sequence reduces to

$$\begin{aligned} 0 \to H^1_{f,\mathrm{Iw}}(K,T(1)) \to \mathcal{U} \otimes_{\mathbb{Z}_p} Q \xrightarrow{\kappa} \mathfrak{X} \otimes_{\mathbb{Z}_p} P \to H^2_{f,\mathrm{Iw}}(K,T(1)) \\ \to H^2_{\mathrm{Iw}}(K,\mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} Q \to P \to H^3_{f,\mathrm{Iw}}(K,T(1)) \to 0. \end{aligned}$$

Lemma (Comparison with Selmer)

There is a canonical exact sequence

$$0 \to \operatorname{coker} \kappa \to \mathfrak{S} \to Y \otimes_{\mathbb{Z}_p} Q \to P.$$

Question

What is the cokernel of $\kappa \colon \mathcal{U} \otimes_{\mathbb{Z}_p} Q \to \mathfrak{X} \otimes_{\mathbb{Z}_p} P$ on ϵ-eigenspaces?

Fact

P has trivial $G_{\mathbb{Q}}$-action, but $Q = Q_{\chi^{-1}}$. It follows that

$$(\mathcal{U} \otimes_{\mathbb{Z}_p} Q)_\epsilon \cong \mathcal{U}_{\chi\epsilon} \otimes_R Q \quad \text{and} \quad (\mathfrak{X} \otimes_{\mathbb{Z}_p} P)_\epsilon \cong \mathfrak{X}_\epsilon \otimes_R P.$$

If ϵ is even, then $\mathcal{U}_{\chi\epsilon}$ is trivial unless $\chi\epsilon = \omega$, in which case it is $R(1)$.

This implies the following 2-variable analogue of a result of Greenberg-Vatsal.

Corollary

If ϵ is even with $\epsilon \neq 1$ and $\chi\epsilon \neq \omega$, then there is a canonical exact sequence

$$0 \to \mathfrak{X}_\epsilon \otimes_R P \to \mathfrak{S}_\epsilon \to Y_{\chi\epsilon} \otimes_R Q \to 0.$$

The $R[\![X]\!]$-characteristic ideals of $\mathfrak{X}_\epsilon$ and $Y_{\chi\epsilon}$ are generated by Kubota-Leopoldt p-adic L-functions by the classical Iwasawa main conjecture.

Suppose from now on that ϵ is odd.

Conjecture (Greenberg)

Y^+ is finite, i.e., Y_ρ is finite for every even character ρ.

Facts

1. *If $Y_{\chi\epsilon}$ is finite, then $\mathcal{U}_{\chi\epsilon}$ is generated by sequences of cyclotomic p-units.*
2. *There is a canonical homomorphism*

   $$\Phi_\epsilon \colon \mathfrak{X}_\epsilon \to \Lambda_\epsilon$$

   determined by the action of $\mathfrak{X}$ on cyclotomic p-units with the property that if $Y_{\omega\epsilon^{-1}}$ is finite, then Φ_ϵ is injective with finite cokernel $(Y_{\omega\epsilon^{-1}})^\vee(1)$.

The cocycle $G_{\mathbb{Q}} \to \mathrm{Hom}_{\mathfrak{h}}(Q, P)$ attached to the exact sequence gives rise to a homomorphism $\Upsilon_\theta \colon Y_\chi \to P$ (conjecturally an isomorphism) by composition with evaluation at the canonical generator of Q.

Definition (S-reciprocity map)

Let $\mathcal{X}$ be the quotient of $\mathbb{Z}_p[\mathfrak{X}]$ by the square of its augmentation ideal. The S-reciprocity map

$$\Psi\colon \mathcal{U} \to H^2_{\mathrm{Iw}}(K, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathfrak{X}.$$

is the first connecting map in the Iwasawa cohomology of the Tate twist of

$$0 \to \mathfrak{X} \xrightarrow{\sigma \mapsto \sigma - 1} \mathcal{X} \xrightarrow{\tau \mapsto 1} \mathbb{Z}_p \to 0.$$

The analogous exact sequence for Y in place of $\mathfrak{X}$ is locally split at p. In place of usual cohomology, we again use that of a Selmer complex from Iwasawa cohomology to compactly-supported Iwasawa cohomology with connecting maps

$$\Theta\colon \mathcal{U} \to \mathfrak{X} \otimes_{\mathbb{Z}_p} Y \quad \text{and} \quad q\colon H^2_{\mathrm{Iw}}(K, \mathbb{Z}_p(1)) \to Y.$$

Theorem (S.)

The map q splits the canonical injection, and $(q \otimes 1) \circ \Psi$ and $-\Theta$ are equal after switching the order of the tensor product.

Variant of a conjecture regarding cup products

Notation

1. $\Psi_{\epsilon,\chi} \colon \mathcal{U}_{\chi\epsilon} \to \mathfrak{X}_\epsilon \otimes Y_\chi$ is the map induced by Ψ via commutativity of the tensor product
2. $u_{\chi\epsilon} \in \mathcal{U}_{\chi\epsilon}$ is the image of the norm compatible system $1 - \zeta_{fp^r}$ of elements $\mathbb{Q}(\mu_{Np^r})$, where f is the tame conductor of $\chi\epsilon$.

Conjecture (S.)

For odd ϵ, the $\Lambda_\epsilon \otimes_R (\mathfrak{h}/I)_\theta$ submodules of $\Lambda_\epsilon \otimes_R P$ generated by the image $\bar{\mathcal{L}}_{\epsilon,\theta}$ of $\mathcal{L}_{\epsilon,\theta}$ and $(\Phi_\epsilon \otimes \Upsilon_\theta)(\Psi_{\epsilon,\chi}(u_{\chi\epsilon}))$ are equal.

In fact, we expect that $(\Phi_\epsilon \otimes \Upsilon_\theta)(\Psi_{\epsilon,\chi}(u_{\chi\epsilon})) = \bar{\mathcal{L}}_{\epsilon,\theta}$.

Theorem (Wake-Wang Erickson, Fukaya-Kato)

The conjecture holds if Y_θ and $Y_{\omega^2\theta^{-1}}$ are finite and P is p-torsion free.

These hypotheses are actually stronger than needed.

Structure of the dual Selmer group

The proof of the following lemma uses the earlier theorem relating Θ and Ψ.

Lemma

The first connecting homomorphism

$$\kappa_\epsilon : \mathcal{U}_{\chi\epsilon} \otimes_R Q \to \mathfrak{X}_\epsilon \otimes_R P$$

in the sequence for $H^1_{f,\mathrm{Iw}}(K, T(1))_\epsilon$ is equal to the composition

$$\mathcal{U}_{\chi\epsilon} \otimes_R Q \xrightarrow{-\Psi_{\epsilon,\chi} \otimes 1} \mathfrak{X}_\epsilon \otimes_R Y_\chi \otimes_R Q \xrightarrow{1 \otimes \Upsilon_\theta \otimes 1} \mathfrak{X}_\epsilon \otimes_R P \otimes_R Q$$
$$\twoheadrightarrow \mathfrak{X}_\epsilon \otimes_R P \otimes_{\Lambda_\theta} Q \xrightarrow{\sim} \mathfrak{X}_\epsilon \otimes_R P.$$

Theorem (S.)

Let ϵ be odd. Suppose that Y_θ, $Y_{\omega\chi^{-1}}$, $Y_{\chi\epsilon}$, and $Y_{\omega\epsilon^{-1}}$ are finite and that P is p-torsion free. Then $\mathfrak{S}_\epsilon$ and

$$\frac{\Lambda_\epsilon \otimes_R P}{(\Lambda_\epsilon \otimes_R (\mathfrak{h}/I)_\theta) \cdot \bar{\mathcal{L}}_{\epsilon,\theta}}.$$

are pseudo-isomorphic $\Lambda_\epsilon \otimes (\mathfrak{h}/I)_\theta$-modules.

Consequences for the unreduced lattice

Proposition

The canonical map $(\mathrm{Sel}(K, \mathcal{T}_\theta^\vee)^\vee)_\epsilon \otimes_{\mathfrak{h}} \mathfrak{h}/I \to \mathfrak{S}_\epsilon$ *is an isomorphism.*

Theorem (S.)

Suppose the conditions in the above theorem and that $\mathfrak{P}$ *is a prime ideal of* $\Lambda_\epsilon \hat{\otimes}_R \mathfrak{h}_\theta$ *such that* $\mathfrak{p} = \mathfrak{P} \cap \mathfrak{h}_\theta$ *is properly contained in the maximal ideal* $\mathfrak{m}$ *of* $\mathfrak{h}_\theta$ *containing* I*. Then the main conjecture implies that the localizations of*

$$(\mathrm{Sel}(K, \mathcal{T}_\theta^\vee)^\vee)_\epsilon \quad \text{and} \quad \frac{\Lambda_\epsilon \hat{\otimes}_R \mathcal{T}_\theta^+}{(\Lambda_\epsilon \otimes_R \mathfrak{h}_\theta) \cdot \mathcal{L}_{\epsilon,\theta}}$$

at $\mathfrak{P}$ *are pseudo-isomorphic* $(\Lambda_\epsilon \hat{\otimes}_R \mathfrak{h}_\theta)_{\mathfrak{P}}$*-modules.*

Question

What of the two-variable residually reducible main conjecture can one obtain (supposing Greenberg's conjecture) in cases where one divisibility in the two-variable main conjecture can be proven (e.g., via the work of Kato)?