Ralph Greenberg

Classical Iwasawa Theory

Fix an odd prime $p$.

Notation: $\mu_m$ = group of $m^{th}$ roots of unity in $\overline{\mathbb{Q}}$

$K_n = \mathbb{Q}(\mu_{p^{n+1}})$, $K_0 = \mathbb{Q}(\mu_p)$

$A_n = Cl(K_n)_p$

$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots$

$K_\infty = \bigcup_{n\geq 1}^{\infty} K_n = \mathbb{Q}(\mu_{p^\infty})$

Themes: Behavior of $A_n$'s $(n\geq 0)$ is related to the behavior of $\zeta(1-m)$, where $m\geq 1$ (even).

(Kummer, Herbrand-Ribet, Iwasawa, Mazur-Wiles, ...)

$\zeta(1-m) = -\frac{B_m}{m}$, where $B_m = m^{th}$ Bernoulli # $\quad \frac{xe^x}{e^x-1} = \sum \frac{B_m}{m!}x^m$.

$n=0$. $K_0 = \mathbb{Q}(\mu_p)$. Let $\Delta = G_0 = Gal(K_0/\mathbb{Q})$, $\omega: \Delta \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^\times$. For $\delta\in\Delta$, $\omega(\delta) = \delta|_{\mu_p} \in Aut(\mu_p) = (\mathbb{Z}/p\mathbb{Z})^\times$

$(\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z})$.

$\mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^\times$

There exists a homomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$. We regard $\omega$ as a homom. $\Delta \to \mathbb{Z}_p^\times$.

$A_0 = Cl(K_0)_p$ (usually exponent $p$). $\Delta$ acts on $A_0$: $A_0 = A_0^{(\omega^i)}$, where $A_0^{(\omega^i)} = \{a\in A_0 \mid \delta(a) = \omega^i(\delta)a$ for all $\delta\in\Delta\}$

Chars. of $\Delta$ are $\{\omega^i : 0\leq i < p-1\}$

This decomposition is valid if $A_0$ = a $\mathbb{Z}_p[\Delta]$-module.

$A_0^{(\omega^0)} = 0$, $A_0^{(\omega^1)} = 0$.

Herbrand-Ribet: Assume $i$ odd, $j$ even, $i\not\equiv 1$ mod $p-1$ & $i+j\equiv 1$ mod $p-1$, $j\geq 2$ (i.e., $\omega^i\omega^j = \omega$). Then $A^{(\omega^i)} \neq 0 \iff \zeta(1-j) \equiv 0 \pmod{p\mathbb{Z}_p}$

Kummer Congruences: If $j_1, j_2 \geq 2$ even & $j_1 \equiv j_2 \pmod{p-1}$ & $j_1 \not\equiv 0$ mod $p-1$, then $\zeta(1-j_1) \equiv \zeta(1-j_2) \pmod{p\mathbb{Z}_p}$.

Greenberg 2

__Example__ $p=37$. $p \mid B_{32}$ $\zeta(1-32) \equiv 0 \bmod 37\mathbb{Z}_{37} \Rightarrow A_0^{(\omega^5)} \neq 0$.

$\zeta(1-j) \equiv 0 \bmod p\mathbb{Z}_p$ iff $j \equiv 32 \bmod 36$.

$A_0^{(\omega^i)} = 0$ for odd $i \not\equiv 5 \bmod (p-1)$. $A_0^{(\omega^i)} = 0$ for even $i$.

$A_0 \cong A_0^{(\omega^5)} \cong \mathbb{Z}/p\mathbb{Z}$.

interpretation in terms of Galois extns. of $K_0$.

Let $L_0$ = p-Hilbert class field of $K_0$.

$\mathrm{Gal}(L_0/K_0) \cong A_0$ (CFT)

$$\begin{tikzcd}
L_0 \arrow[d, dash] \\
K_0 \arrow[d, dash, "\Delta"'] \\
\mathbb{Q}
\end{tikzcd}$$

$1 \to \mathrm{Gal}(L_0/K_0) \to \mathrm{Gal}(L_0/\mathbb{Q}) \to \Delta \to 1$

$\Delta$ acts on $\mathrm{Gal}(L_0/K_0)$ by inner autom.

$\Delta$ acts on $\mathrm{Gal}(L_0/K_0) \cong \mathbb{Z}/p\mathbb{Z}$ by $\omega^5$ (in ex.)

Kummer theory interpretation. Let $c \in A_0$, $c \in Cl(I)$, where $I$ = fractional ideal of $\mathcal{O}_{K_0}$.

$c\bar{c} = c_0$. We can chose $I$ so that $I\bar{I} = (1)$.

$I^p = (\alpha)$ where $\alpha \in K_0$. We can choose $\alpha$ so that $\alpha\bar{\alpha} = 1$. We can even chose $\alpha$ so that $\alpha(K_0^\times)^p \in \left(K_0^\times/(K_0^\times)^p\right)^{(\omega^5)}$

Let $M_0 = K_0(\sqrt[p]{\alpha})$. $\mathrm{Gal}(M_0/K_0) \cong \mathbb{Z}/p\mathbb{Z}$. $M_0/\mathbb{Q}$ is Galois

$$\begin{tikzcd}
M_0 \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] \\
K_0 \arrow[d, dash, "\Delta"] \\
\mathbb{Q}
\end{tikzcd}$$

$\Delta$ acts on $\mathrm{Gal}(M_0/K_0)$ by $\omega^{32}$.

$$\begin{tikzcd}
L_0 \arrow[dr, dash, "\omega^5"'] & & M_0 \arrow[dl, dash, "\omega^{32}"] \arrow[dr, dash] & \\
& K_0 \arrow[dr, dash] \arrow[d, dash, "\Delta"'] & & M_0^+ \arrow[dl, dash] \\
& \mathbb{Q} \arrow[r, dash] & K_0^+ &
\end{tikzcd}$$

$K_0^+ = \mathbb{Q}(\cos(\frac{2\pi}{p}))$

$\mathrm{Gal}(M_0^+/K_0^+) \cong \mathbb{Z}/p\mathbb{Z}$ $M_0 = M_0^+ K_0$ $M_0/K_0$ is unramified outside of $p$

$p = 37$ divides $\zeta(1-32)$.

$\omega^{32}$ factors through $\mathrm{Gal}(K_0^+/\mathbb{Q})$

$p=691$, $p \mid B_{12}$ and $p \mid B_{200}$.

$K_0 = \mathbb{Q}(\mu_p)$, $A_0 = Cl(K_0)_p \cong A_0^{(\omega^{679})} \oplus A_0^{(\omega^{491})} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$

$$\begin{tikzcd}
L_0^{(\omega^{679})} \arrow[dr, dash] & L_0^{(\omega^{491})} \arrow[d, dash] & M_0^{(\omega^{200})} \arrow[dl, dash] & M_0^{(\omega^{12})} \arrow[dll, dash] \\
& K_0 \arrow[d, dash] & & \\
& \mathbb{Q} & &
\end{tikzcd}$$

Greenberg 3

Galois cohomological interpretation

$p=37$. Consider $\mu_p^{\otimes i} = \mathbb{Z}/p\mathbb{Z}$ with $G_\mathbb{Q} \twoheadrightarrow \Delta$, $G_\mathbb{Q}$ acts on $\mu_p^{\otimes i}$

by $G_\mathbb{Q} \to \Delta \xrightarrow{\omega^i} (\mathbb{Z}/p\mathbb{Z})^\times$.

$$H^1(G_\mathbb{Q}, \mu_p^{\otimes i}) \xrightarrow[\sim]{\text{res}} H^1(G_{K_0}, \mu_p^{\otimes i})^\Delta = \mathrm{Hom}(G_{K_0}, \mu_p^{\otimes i})^\Delta$$

$$H^1_{unr}(\mathbb{Q}, \mu_p^{\otimes i}) \underset{\text{res}}{\xrightarrow{\sim}} H^1_{unr}(K, \mu_p^{\otimes i})^\Delta = \mathrm{Hom}_\Delta(\mathrm{Gal}(L_0/K_0), \mu_p^{\otimes i})$$

trivial on all inertia groups

$$H^1_{unr}(\mathbb{Q}, \mu_p^{\otimes i}) \cong \mathrm{Hom}_\Delta(A_0, \mu_p^{\otimes i}) \cong \mathrm{Hom}(A_0^{\omega^{i}}, \mathbb{Z}/p\mathbb{Z})$$

$$p=37: \quad \cong \mathrm{Hom}_\Delta(\mu_p^{\otimes 5}, \mu_p^{\otimes i}) \cong \begin{cases} 0 & \omega^i \neq \omega^5 \\ \mathbb{Z}/p\mathbb{Z} & \omega^i = \omega^5 \ (i \equiv 5 \bmod p-1). \end{cases}$$

Let $\Sigma = \{p, \infty\}$. $j$ even

$$H^1_{\Sigma\text{-ram}}(\mathbb{Q}, \mu_p^{\otimes j}) = \begin{cases} 0 & \text{if } \omega^j \neq \omega^{32}, \omega^0 \\ \mathbb{Z}/p\mathbb{Z} & \text{otherwise} \end{cases}$$

trivial on inertia except $p$

# Ralph Greenberg II

$K_n = \mathbb{Q}(\mu_{p^{n+1}})$ $\quad$ $p$ odd

$G_n = \mathrm{Gal}(K_n/\mathbb{Q})$

If $g \in G_n$, then $g(\zeta) = \zeta^{a_g}$ for all $\zeta \in \mu_{p^{n+1}}$.

Define $\chi_n : G_n \xrightarrow{\sim} (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ by $\chi_n(g) = a_g + p^{n+1}\mathbb{Z}$, or $\chi_n(g) = g|_{\mu_{p^{n+1}}} \in \mathrm{Aut}(\mu_{p^{n+1}})$.
$\cong GL_1(\mathbb{Z}/p^{n+1}\mathbb{Z})$
$= (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$

$\mathrm{Hom}(G_n, (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times) = \{\chi_n^i \mid 0 \le i < p^n(p-1)\}$

$p = 37$ $\quad$ $A_n = Cl(K_n)_p \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$, $\quad |A_n| = p^{n+1}$.

$G_n$ acts on $A_n$ by a homom. $\varphi_n : G_n \to (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$, $\quad \varphi_n = \chi^{i_n}$.

($n = 0$, $\chi_0 = \omega$, $\varphi_0 = \omega^5$; $G_0 = \Delta$, $p \mid B_{32}$)

$n = 1$, $\varphi_1 = \chi_1^{1049}$, $\quad \chi_1 \varphi_1^{-1} = \chi_1^{284}$, $\quad p^2 \mid B_{284}$

Suppose $m \ge n \ge 0$. We have two maps ① $A_n \to A_m$, $\quad cl(I) \to cl(I\mathcal{O}_{K_m})$.

This turns out to be injective; $\{\varepsilon \in \mathcal{O}_{K_n}^\times \mid \varepsilon\bar{\varepsilon} = 1\}$ = roots of unity in $K_n$.

② $N_{m/n} : A_m \to A_n$, $\quad cl(I) \to cl(N_{K_m/K_n}(I))$ $\quad$ surjective.

Action of $G_m$ on $A_m$ determines action of $G_n$ on $A_n$.

$i_m \equiv i_n \bmod p^n(p-1)$ $\qquad \equiv 5 \bmod 36$ $\quad$ ($i_n$) converges $p$-adically

$\lim_{n\to\infty} i_n = 13 + 20\cdot 37 + 30 \cdot 37^2 + \cdots$ $\qquad$ (Kevin Buzzard)

↖ the unique zero of $L_p(\omega^{32}, s)$

$$\begin{array}{ccc} L_n & & M_n \\ & \diagdown \; K_n \; \diagup & \\ & | & \\ & \mathbb{Q} & \end{array}$$

$L_n$ = $p$-Hilbert class field of $K_n$.

$\mathrm{Gal}(L_n/K_n) \cong A_n \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$

$G_n$ acts on $\mathrm{Gal}(L_n/K_n)$ by $\chi_n^{i_n}$.

Have isomorphisms $\mathrm{Gal}(L_n/K_n) \cong A_n$, $\quad \mathrm{Gal}(M_n/K_n) \cong \mathbb{Z}/p^{n+1}\mathbb{Z}$

$\therefore M_n$ $\Sigma$-ramified, where $\Sigma = \{p, \infty\}$

$G_n$ acts on $\mathrm{Gal}(M_n/K_n)$ by $\chi_n \chi_n^{-i_n} = \chi_n^{1-i_n}$.

$K_\infty = \bigcup K_n = \mathbb{Q}(\mu_{p^\infty})$, $G_\infty = \mathrm{Gal}(K_\infty/\mathbb{Q}) = \varprojlim G_n \cong \mathbb{Z}_p^\times$ $\qquad \chi_\infty : G_\infty \xrightarrow{\sim} \mathbb{Z}_p^\times$

$\chi_\infty(g) = g|_{\mu_{p^\infty}} \in \mathrm{Aut}(\mu_{p^\infty})$ $\qquad \mu_{p^\infty} = \bigcup \mu_{p^{n+1}}$, $\quad \mathbb{Z}_p(1) = \varprojlim_n \mu_{p^{n+1}}$.

$\mathbb{Z}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p)$.

Apply $\chi_\infty^{-1}$. $\qquad G_\infty = \Delta \times \Gamma$.

$$\begin{array}{ccccc} & & K_\infty & & \\ & \overset{\Delta}{\diagup} & \Big| {\scriptstyle G_\infty} & \overset{\Gamma}{\diagdown} & \\ \mathbb{Q}_\infty & & & & K_0 \\ & \underset{\Gamma}{\diagdown} & & \underset{\Delta}{\diagup} & \\ & & \mathbb{Q} & & \end{array}$$

Greenberg II 2

How does $G_\infty$ act on $A_\infty = \varinjlim A_n \cong \mathbb{Q}_p/\mathbb{Z}_p$ or $X_\infty = \varprojlim A_n \cong \mathbb{Z}_p$?

Acts by $\varphi_\infty : G_\infty \longrightarrow \mathbb{Z}_p^\times$, $\varphi_\infty = \lim_{n\to\infty} \chi^{i_n}$. $\chi = \chi|_\Delta \chi|_\Gamma$.

$\chi|_\Delta = \omega$, $\chi|_\Gamma = \kappa$, $\kappa : \Gamma \longrightarrow 1+p\mathbb{Z}_p$.

$\kappa^s$ can be defined for any $s \in \mathbb{Z}_p$

$\varphi_\infty = \lim_{n\to\infty} \chi^{i_n} = \lim_{n\to\infty} \omega^{i_n}\kappa^{i_n} = \omega^s \kappa^t$ $\quad t = \lim_{n\to\infty} i_n \in \mathbb{Z}_p$

$X_\infty = \varprojlim A_n$ is isomorphic to a Galois group

$L_\infty = \bigcup_{n\geq 0} L_n$, $K_\infty = \bigcup_{n\geq 0} K_n$ $\quad Gal(L_\infty/K_\infty) \cong \varprojlim Gal(L_n/K_n)$.

$$\begin{tikzcd}[column sep=small]
L_m \arrow[d, dash] \arrow[dr, dash] & \\
L_n \arrow[dr, dash] & K_m \arrow[d, dash] \\
& K_n
\end{tikzcd}$$

Suppose that $m \geq n \geq 0$.

$$\begin{tikzcd}
Gal(L_m/K_m) \arrow[r, "Res_{m/n}"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & Gal(L_n/K_n) \arrow[d, "\wr"] \\
A_m \arrow[r, "N_{m/n}"] & A_n
\end{tikzcd}$$

$Res_{m/n}$ surj. since $L_n \cap K_m = K_n$.

$X_\infty \cong Gal(L_\infty/K_\infty) \cong \mathbb{Z}_p$.

$L_\infty$ = maximal abelian, everywhere unram. pro-p extn. of $K_\infty$.

$G_\infty$ acts on $X_\infty$ by $\omega^s\kappa^t$.

$$\begin{tikzcd}[column sep=small]
L_\infty \arrow[dr, dash] & & M_\infty \arrow[dl, dash] \\
& K_\infty \arrow[d, dash, "G_\infty \cong \Gamma\times\Delta"] & \\
& \mathbb{Q} &
\end{tikzcd}$$

$M_\infty^{(\omega^{1-s})}$ = maximal abelian $\Sigma$-ramified pro-p extn. of $K_\infty$ s.t. $\Delta$ acts on $Gal(M_\infty^{(\omega^{1-s})}/K_\infty)$ by $\omega^{1-s}$.

$Gal(M_\infty^{(\omega^{1-s})}/K_\infty) \cong \mathbb{Z}_p$. $\quad \Gamma$ acts by $\kappa^{1-t}$.

$\Lambda = \mathbb{Z}_p[[\Gamma]] = \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$, $\quad \Gamma/\Gamma^{p^n} = Gal(K_n/K_0)$.

$X_\infty$ is a $\Lambda$-module.

our case: $X_\infty = \Lambda/(\gamma - \kappa(\gamma)^t)$ $\quad \Gamma = \overline{\langle\gamma\rangle}$, $\gamma\in\Gamma$, $\gamma|_{K_1} \neq id$.

# Ralph Greenberg III

$p$ prime

$$\begin{tikzcd} L_\infty \arrow[d, dash, "X"'] \\ K_\infty \arrow[d, dash, "\Gamma"] \\ K \end{tikzcd}$$

$\Gamma \cong \mathbb{Z}_p$ $\qquad K_\infty = \bigcup_n K_n$

$\Gamma \supset \Gamma^p \supset \Gamma^{p^2} \supset \dots \qquad Gal(K_n/K) = \Gamma/\Gamma^{p^n}$

$L_\infty$ = max. abelian pro-$p$ unram. ext of $K_\infty$

$X = Gal(L_\infty/K_\infty) \cong \varprojlim_n Gal(L_n/K_n)$

Assumption: There is only one prime of $K$ which is ramified in $K_\infty/K$ & that prime is totally ramified in $K_\infty/K$ (enough for the prime to be ramified in $K_1/K$).

Examples $K = \mathbb{Q}(\mu_p)$, $K_\infty = \mathbb{Q}(\mu_{p^\infty})$, $\Gamma \cong 1 + p\mathbb{Z}_p$, $p$ tot. ram.

$$\begin{tikzcd} K_\infty \arrow[d, dash, "\Delta"] \\ \mathbb{Q}_\infty \arrow[d, dash, "\Gamma"] \\ \mathbb{Q} \end{tikzcd}$$

$$\begin{tikzcd} L_\infty \arrow[d, dash, "X"] \arrow[dd, dash, bend right, "\mathcal{G}"'] \\ K_\infty \arrow[d, dash, "\Gamma"] \\ K \end{tikzcd}$$

$1 \to X \to \mathcal{G} \to \Gamma \to 1$

Let $\gamma \in \Gamma$, topological generator.

$\Gamma = \overline{\langle \gamma \rangle}$. Pick $\tilde{\gamma} \in \mathcal{G}$ lifting $\gamma$.

If $x \in X$, then $\gamma(x) = \tilde{\gamma} x \tilde{\gamma}^{-1}$.

$\mathcal{G}' = X^{\gamma - 1}$ $\quad$ Pf: $\gamma(x)x^{-1} = \tilde{\gamma} x \tilde{\gamma}^{-1} x^{-1} \in \mathcal{G}'$. $\qquad \mathcal{G}/X^{\gamma-1}$

$1 \to X/X^{\gamma-1} \to \mathcal{G}/X^{\gamma-1} \to \Gamma \to 1$ $\qquad \Gamma$ acts triv. on $X/X^{\gamma-1} \Rightarrow \mathcal{G}/X^{\gamma-1}$ abelian. //

This implies $\mathcal{G}/X^{\gamma-1}$ is abelian; i.e., $\mathcal{G}' \subseteq X^{\gamma-1} \Rightarrow \mathcal{G}' = X^{\gamma-1}$.

$$\begin{tikzcd} & L_\infty \arrow[dl, dash, "\mathcal{G}'"'] \arrow[d, dash, "X"] \\ L_\infty^{(0)} \arrow[dr, dash, "\mathcal{G}/\mathcal{G}'"'] & K_\infty \arrow[d, dash, "\Gamma"] \\ & K \end{tikzcd}$$

$L_\infty^{(0)}$ = max. ab. ext. of $K$ in $L_\infty$

$$\begin{tikzcd} L_\infty^{(0)} \arrow[dr, dash, "I"] \arrow[dd, dash, "\text{ab.}"'] & \\ & L_0 \arrow[dl, dash] \\ K & \end{tikzcd}$$

$L_0/K$ abelian, pro-$p$ unram. ext. of $K$

Hence, $L_0 \subseteq$ $p$-Hilbert class field of $K$.

$\Rightarrow L_0 =$ $p$-Hilbert class field of $K$.

$$\begin{tikzcd} & L_\infty^{(0)} \arrow[dl, dash, "I"'] \arrow[dr, dash, "H"] & \\ L_0 \arrow[dr, dash] & & K_\infty \arrow[dl, dash] \\ & K & \end{tikzcd}$$

$I \cap H = 1 \Rightarrow L_\infty^{(0)} = L_0 K_\infty$. $\qquad L_0 \cap K_\infty = K$.

Greenberg III 2

$\mathrm{Gal}(L_\infty^{(0)}/K) = \mathrm{Gal}(L_0/K) \times \Gamma$

$H \cong \mathrm{Gal}(L_0/K) \cong Cl(K)_p$, $\quad H \cong X/X^{\gamma-1}$.

$Cl(K_n)_p \cong \mathrm{Gal}(L_n/K_n) = X/X^{\gamma^{p^n}-1}$

Additive notation. $T = \gamma - 1$. $\quad Tx = \gamma x - x$.

$A_n = Cl(K_n)_p = X/((T+1)^{p^n}-1)X$

Special case: Assume $Cl(K)_p = 0$. This means $X/TX = 0$. Then $X = 0$, so $Cl(K_n)_p = 0 \ \forall n$. ($U \subseteq X$ open $\Rightarrow X/U$ fin. $\Rightarrow T^n X \subseteq U$ for some $n \gg 0$ so $X = TX \Rightarrow X = 0$).

$\Lambda = \mathbb{Z}_p[[T]]$ $\quad \mathfrak{m} = (p, T)$ maximal ideal of this compl., noeth., local ring

$X$ is a $\mathbb{Z}_p[T]$-module, obviously

$T$ top. nilpotent: $T^n x \to 0$ in $X$ $\Rightarrow X$ $\Lambda$-module

$X$ fin. gen $\Lambda$-module since $X/TX$ finite: Suppose $x_1, \dots, x_n \in X$ are chosen so that images in $X/TX$ generate it as a $\mathbb{Z}_p$-module. $Y = \Lambda x_1 + \dots + \Lambda x_n$. Apply Nakayama's Lemma to $X/Y$. //

$X$ torsion $\Lambda$-module: $\Lambda/T \cong \mathbb{Z}_p$ infinite $\mathrm{rk}_\Lambda(X) \le \mathrm{rank}_{\Lambda/T}(X/TX) = 0$. //

Ralph Greenberg IV

**Theorem** (Iwasawa) $K_\infty/K$ $\mathbb{Z}_p$-extn. Then theorem is that there exist integers $\lambda, \mu, \nu$ satisfying $|Cl(K_n)_p| = p^{\lambda n + \mu p^n + \nu}$ for $n >> 0$.

Def of $\lambda$ and $\mu$: Let $X = Gal(L_\infty/K_\infty)$, $L_\infty$ = max abelian unram. pro-p ext. of $K_\infty$
(or $X = \varprojlim_n A_n$, where $A_n = Cl(K_n)_p$).
$X$ is a f.g., torsion $\Lambda$-module. Let $Y = \mathbb{Z}_p$-torsion in $X$. Then $Y$ has bounded exponent: $Y = X[p^t]$ for some $t \geq 0$, and $X/Y \cong \mathbb{Z}_p^\lambda$.
Then $\lambda$ = Iwasawa's $\lambda$-invar.
$\Lambda/p\Lambda$ $\Lambda = \mathbb{Z}_p[[T]]$, $T = \gamma - 1$, $\Gamma = \overline{\langle\gamma\rangle} = Gal(K_\infty/K)$.
$\Lambda/p\Lambda = \mathbb{F}_p[[T]]$, $X[p] \cong (\Lambda/p\Lambda)^{\mu_1} \times (\text{finite})$,
$X[p^2]/X[p] \cong (\Lambda/p\Lambda)^{\mu_2} \times (\text{finite})$, $\mu = \sum_{i=1}^{t} \mu_i$.
$\lambda = \dim_{\mathbb{Q}_p}(X \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$.
$A_\infty = \varinjlim_{n\to\infty} A_n$. If $\mu = 0$, then $A_\infty \cong (\mathbb{Q}_p/\mathbb{Z}_p)^\lambda$, $X \cong \mathbb{Z}_p^\lambda \times (\text{finite})$.

Conjecture (Iwasawa) Let $K_\infty = K^{cyc}$. Then $\mu = \mu(K_\infty/K) = 0$.

Theorem (Ferrero-Washington) If $K_\infty/K$ is abelian, then $\mu = 0$.

If $\mu = 0$, then $A_n \approx (\mathbb{Z}/p^n\mathbb{Z})^\lambda$.
up to fin. gps. of bd. order

$\mu > 0 \Leftrightarrow \dim_{\mathbb{Z}/p\mathbb{Z}}(A_n[p]) \geq \mu_1 p^n$ - const.

Remark $\mu(K_\infty/K) > 0$ is possible ($K_\infty \neq K^{cyc}$) if $\exists$ $\infty$ many primes $v$ of $K$ which split completely in $K_\infty/K$. e.g.

Let $K$ be any # field, and let $\tilde{K}$ = compositum of all $\mathbb{Z}_p$-extns. of $K$.
$Gal(\tilde{K}/K) \cong \mathbb{Z}_p^{r_2+1+\delta}$, where $r_2$ = # complex primes of $K$ & $\delta \geq 0$.
Conj (Leopoldt) $\delta = 0$.

Greenberg IV 2

Suppose $r_2 > 0$ & $p$ splits completely in $K/\mathbb{Q}$.

$$\begin{array}{l} L_\infty - \tilde{K} \\ \quad \big| \ \mathbb{Z}_p^{r_2+\delta} \\ \quad K_\infty \\ \quad \big| \ \Gamma \cong \mathbb{Z}_p \\ \quad K \end{array}$$

Let $K_\infty = K_\infty^{cyc}$. <u>Claim:</u> $\tilde{K} \subseteq L_\infty$.

Hence $\lambda = \mathrm{rank}_{\mathbb{Z}_p}(X) \geq r_2$.

Pf: Let $\mathfrak{P}$ be a prime of $K$ over $p$.

$$\begin{array}{l} \tilde{K}_\mathfrak{P} \\ \quad \big| \\ (K_\infty)_\mathfrak{P} = \mathbb{Q}_p^{cyc} \\ \quad \big| \\ K_\mathfrak{P} = \mathbb{Q}_p \end{array}$$

$\mathrm{Gal}(\tilde{K}_\mathfrak{P}/K_\mathfrak{P}) \cong \mathbb{Z}_p^?$

$\tilde{K}_\mathfrak{P} \subset \mathbb{Q}_p^{unr}\mathbb{Q}_p^{cyc}$

$\Rightarrow \tilde{K}_\mathfrak{P}/\mathbb{Q}_p^{cyc}$ unram

$\Rightarrow \tilde{K} \subseteq L_\infty$. //

\begin{tikzcd}
 & \mathbb{Q}_p^{unr}\mathbb{Q}_p^{cyc} \arrow[dl, dash] \arrow[dr, dash] \arrow[dd, dash, "\mathbb{Z}_p^2"] & \\
\mathbb{Q}_p^{unr} \arrow[dr, dash, "\mathbb{Z}_p"'] & & \mathbb{Q}_p^{cyc} \arrow[dl, dash, "\mathbb{Z}_p"] \\
 & \mathbb{Q}_p &
\end{tikzcd}

$\leftarrow$ composition of all $\mathbb{Z}_p$-extns.

$r_2 = 0$ ($K$ totally real)

Leopoldt. $\tilde{K} = K_\infty^{cyc}$.

<u>Conjecture</u> (Greenberg) $\lambda(K_\infty/K) = 0$ (and $\mu = 0$ too).

I.e., $X$ fin group, $X = \varprojlim_{n\to\infty} A_n$, $|A_n|$ is bounded.

<u>Ex</u> $K = \mathbb{Q}(\sqrt{254})$, $p = 3$, $A_0 \cong \mathbb{Z}/3\mathbb{Z}$, $A_1 \cong \mathbb{Z}/9\mathbb{Z}$, $A_2 \cong \mathbb{Z}/27\mathbb{Z}$, ..., $A_4 \cong \mathbb{Z}/243\mathbb{Z}$,

$A_5 \cong \mathbb{Z}/243\mathbb{Z}$, $A_6 \cong \mathbb{Z}/243\mathbb{Z}$, ..., $A_n \cong \mathbb{Z}/3^5\mathbb{Z}$ $\forall n \geq 4$.

$A_0 \not\to A_5$ $\varinjlim A_n = 0$, $X \cong \mathbb{Z}/3^5\mathbb{Z}$.

$$\begin{array}{l} K_\infty \\ \big| \\ K \end{array}$$

Assume only one prime is ram in $K_\infty/K$ & that it is totally ramified.

Assume $X \cong \mathbb{Z}_p^\lambda$. $A_n \cong X/(\gamma^{p^n}-1)X$ $\forall n$.

$\left|X/(\gamma^{p^n}-1)X\right| \underset{\uparrow}{\sim} \det((\gamma^{p^n}-1) : X \to X) \sim$ product of eigenvalues of $\gamma^{p^n}-1$.

up to $p$-adic unit

Let $\alpha_1, \ldots, \alpha_\lambda$ be the eigenvalues of $\gamma$. The e.v. of $\gamma - 1$ are $\alpha_1 - 1, \ldots, \alpha_\lambda - 1$.

$|\alpha_i - 1|_p < 1 \Rightarrow \det(\gamma^{p^n}-1) = \prod_{i=1}^{\lambda}(\alpha_i^{p^n}-1) \sim \prod_{i=1}^{\lambda} \log_p(\alpha_i^{p^n}) = (p^n)^\lambda \prod \log(\alpha_i) = p^{n\lambda+\nu}$.

Here: $\nu = 0$.

$\mu$ invariant $X = \mathbb{F}_p[[T]]$ $X/(\gamma^{p^n}-1)X$, $\gamma^{p^n}-1 = (1+T)^{p^n}-1$

Then image in $\mathbb{F}_p[[T]]$ is $T^{p^n}$ $\Rightarrow$ $A_n \cong \mathbb{F}_p[[T]]/T^{p^n}$ $|A_n| = p^{p^n}$

Greenberg IV 3

For some $n_0 \geq 0$, we have $A_n \cong X/\theta_{n/n_0} Y$, where $Y = \mathrm{Gal}(L_\infty/K_\infty L_{n_0})$, $n \geq n_0$,

and $\theta_{n/n_0} = \dfrac{(1+T)^{p^n}-1}{(1+T)^{p^{n_0}}-1}$. //

Robert Pollack

Iwasawa Theory of Elliptic Curves

$F/\mathbb{Q}$ finite extn.

$E/F$ elliptic curve

$L/F$ finite $\nearrow$ $E(L)$ f.g. abelian gp. $(\leftrightarrow \mathcal{O}_L^\times)$

$\searrow$ Ш$(E/L)$ Tate-Shafarevich group

(conj.) finite abelian group $(\leftrightarrow Cl(L))$

towers of number fields

Fix a prime $p$ (odd).

$$\begin{array}{lll} & K_\infty & \\ & \mid & \\ \mathbb{Z}_p^\times & K_n = \mathbb{Q}(\mu_{p^{n+1}}) & 1+p\mathbb{Z}_p \ (\cong \mathbb{Z}_p) \\ & \mid p^n & \\ & K_0 & \\ & \mid & \\ & \mathbb{Q} & \end{array} \qquad \begin{array}{ll} F_\infty & \\ \mid & \\ F_n & \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p \\ \mid p^n & \mathbb{Z}_p\text{-extension (arbitrary)} \\ F & \\ \mid & \\ \mathbb{Q} & \end{array}$$

Theorem (Mazur '73) Assume all primes of $F$ over $p$ are ordinary for $E/\mathbb{Q}$ $p \nmid a_p$.
If $E(F)$ finite & $\underbrace{\text{Ш}(E/F)[p^\infty] \text{ finite}}_{\text{should always be true}}$, then $\mathrm{rank}(E/F_n)$ bounded as $n \to \infty$.

- $F=\mathbb{Q}$, $p$ supersingular, same conclusion holds (Perrin-Riou '90).

non-examples

$$\begin{array}{ccccc} & & M & & \\ & F_\infty^{ac} & \mathbb{Z}_p^2 & F_\infty^{cycl} = F(\mu_{p^\infty}) & \\ \text{dihedral} & & F & & \\ & & \mid 2 & & \\ & & \mathbb{Q} & & \end{array}$$

- $F/\mathbb{Q}$ quadratic imaginary, $F_\infty$ = anticyclotomic $\mathbb{Z}_p$-extn.

1) $E$ does not have CM by $F$ (...)

sign of FE for $E/F = -1 \Rightarrow \mathrm{rk}(E/F_n) \to \infty$ as $n \to \infty$.

Moreover $\mathrm{rk}(E/F_n) = p^n + O(1)$.

2) $E$ has CM by $F$.

$p$ ordinary $\mathrm{rk}(E/F_n) = \begin{cases} 2p^n + O(1) & \text{sign of FE}/\mathbb{Q} = -1 \\ O(1) & \quad '' \quad = 1 \end{cases}$

$p$ supersingular $\mathrm{rk}(E/F_n) = 2\sum_{k=1}^{n} \varphi(p^k) + O(1)$ $\quad \in$ sign of FE.

Pollack 2

Selmer group

$Sel_p(E/F) \subseteq H^1(Gal(\bar{F}/F), E[p^\infty](\bar{F}))$

cut out by local conditions

$v$ place of $F$

$E(F_v)/p^n E(F_v) \hookrightarrow H^1(F_v, E[p^n])$ Kummer map

$n \to \infty$

$E(F_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(F_v, E[p^\infty])$ comes from pts.

local cond. at $v$

$$0 \to E(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to Sel_p(E/F) \longrightarrow Ш(E/F)[p^\infty] \to 0$$

$\cong (\mathbb{Q}_p/\mathbb{Z}_p)^{rk(E/F)}$

finite gp., conjecturally

$F_\infty/F$ $\mathbb{Z}_p$-extension

$\varinjlim_n Sel_p(E/F_n) = Sel_p(E/F_\infty)$ $\Gamma = Gal(F_\infty/F)$

$\mathbb{Z}_p[Gal(F_n/F)]$ $\varprojlim \mathbb{Z}_p[Gal(F_n/F)] \cong \mathbb{Z}_p[[\Gamma]] = \Lambda$ ← Iwasawa algebra

$X_\infty = X(E/F_\infty) = Sel_p(E/F_\infty)^\vee = Hom(Sel_p(E/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$

compact

Conjecture (Mazur) $F_\infty$ cyclotomic $\mathbb{Z}_p$-extn., $p$ ordinary $\Rightarrow$ $X_\infty$ f.g. torsion $\Lambda$-module

Remarks 1) conj. $\Rightarrow$ rank$(E/F_n)$ bounded $\because$ finite $\to E(F_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to Sel_p(E/F_\infty)$

$X_\infty \to \mathbb{Z}_p^{rk(E/F_n)} \to$ finite $X_\infty/(X_\infty)_{\mathbb{Z}_p\text{-torsion}} \to \mathbb{Z}_p^{rk(E/F_n)}$

any f.g. torsion $\Lambda$-module $\to$ $\cong$ $\mathbb{Z}_p^\lambda$, $\lambda \in \mathbb{Z}^{\geq 0}$. $rk(E/F_n) \leq \lambda$

bad case $X_\infty = \Lambda \cong \mathbb{Z}_p[[T]]$ (noncanonically)

2) $F = \mathbb{Q}$, BSD $\chi : Gal(\mathbb{Q}_n/\mathbb{Q}) \to \bar{\mathbb{Q}}^\times$. $L(E,\chi,1) = 0 \Leftrightarrow E(\mathbb{Q}_n)^\chi$ is infinite

Pollack 3.

Mazur's conj. + BSD $\Rightarrow L(E,\chi,1) \neq 0$ for almost all $\chi$.

- signs of F.E. don't hurt . proven by D. Rohrlich (~88)

3) Rohrlich's thm. works for supersingular $p$, but analogue of Mazur's conj. is false.

Reason: even though Ш$(E/F_n)$ finite, can grow into $(\mathbb{Q}_p/\mathbb{Z}_p)^4$ or even $\infty$ many copies.

Thm (Mazur) $F_\infty/F$ is any $\mathbb{Z}_p$-extn. Assume $X_\infty$ is f.g. $\Lambda$-torsion $\Rightarrow p$ ordinary

& $|Ш(E/F_n)[p^\infty]| < \infty \quad \forall n \geq 0$.

Then $\exists \mu, \lambda, \nu \in \mathbb{Z}^{\geq 0}$ s.t. $\mathrm{ord}_p(|Ш(E/F_n)|) = \mu p^n + \lambda n + \nu$ for $n \gg 0$.

Thm (Perrin-Riou/Kurihara) $F = \mathbb{Q}$, $p$ good supersingular prime. Assume $|Ш(E/\mathbb{Q}_n)[p^\infty]| < \infty$ $\forall n \geq 0$. Then $\exists \mu^+, \mu^-, \lambda^+, \lambda^- \in \mathbb{Z}^{\geq 0}$ s.t. $\mathrm{ord}_p\left(\dfrac{|Ш(E/\mathbb{Q}_n)|}{|Ш(E/\mathbb{Q}_{n-1})|}\right) = \mu^\varepsilon(p^n - p^{n-1}) + \lambda^\varepsilon$

$\varepsilon \Leftrightarrow n$ odd/even

$$+ p^{n-1} - p^{n-2} + p^{n-3} - p^{n-4} + \begin{cases} p^2 - p \\ p - 1 \end{cases}.$$

Control theorem (Mazur) $p$ good ordinary. For each $n \geq 0$, there is a map
$$\mathrm{Sel}_p(E/F_n) \longrightarrow \mathrm{Sel}_p(E/F_\infty)^{\Gamma_n} \qquad \Gamma_n = \mathrm{Gal}(F_\infty/F_n)$$
has finite kernel & cokernel bdd. indep. of size $n$.

# Robert Pollack II

<u>Thm</u> (Mazur) $p$ good ordinary,
$F_\infty/F$ arbitrary $\mathbb{Z}_p$-extension $\mathrm{Sel}_p(E/F)$ finite $\Rightarrow \mathrm{rk}(E/F_n)$ is bounded as $n \to \infty$.

<u>Conj</u> $F_\infty/F$ cyclotomic $\mathbb{Z}_p$-extn., $p$ good ordinary $\Rightarrow X_\infty = \mathrm{Sel}_p(E/F_\infty)^\vee$ $\Lambda$-torsion

<u>Control Thm</u> (Mazur) $p$ good ordinary, $F_\infty/F$ arbitrary $\mathbb{Z}_p$-extn.
The natural map $\mathrm{Sel}_p(E/F_n) \to \mathrm{Sel}_p(E/F_\infty)^{\Gamma_n}$, $\Gamma_n = \mathrm{Gal}(F_\infty/F_n)$, has finite kernel & cokernel bdd. indep. of $n$.

<u>Remarks</u>

1) C.T. false in supersingular case
2) C.T. + $\Lambda$-module theory $\Rightarrow$ growth formulas for $Ш(E/F_n)[p^\infty]$.
   ($X_\infty$ is $\Lambda$-torsion)
3) C.T. $\Rightarrow$ first theorem

<u>Prop</u> $p$ good ordinary, $\mathrm{Sel}_p(E/F)$ finite $\Rightarrow X_\infty$ $\Lambda$-torsion
(Prop $\Rightarrow$ first thm.)

<u>Pf of Prop</u>: $n=0$, CT $\Rightarrow \mathrm{Sel}_p(E/F_\infty)^\Gamma$ is finite $\rightsquigarrow \mathrm{Sel}_p(E/F_\infty)[\gamma-1]$ is finite $\Rightarrow X_\infty/(\gamma-1)X_\infty$ is finite $\Rightarrow X_\infty$ torsion f.g. $\Lambda$-mod.

<u>Thm</u> (Kato) Conj. is true when $F/\mathbb{Q}$ is abelian & $E/\mathbb{Q}$.

Structure Thm: $X$ f.g. $\Lambda$-module $\Rightarrow \exists\ X \xrightarrow{\varphi} \Lambda^r \oplus \left(\bigoplus \Lambda/f_i\right)$ w/ fin. kernel & cokernel.

characteristic power series of a torsion $\Lambda$-module $X$: $\mathrm{char}_\Lambda X = (\prod f_i)\Lambda \subseteq \Lambda$.

<u>Thm</u> (Mazur/Swinnerton-Dyer) $F=\mathbb{Q}$, $p$ good ordinary. $E/\mathbb{Q}$ modular (known) $\Rightarrow$
$\exists!\ L_p(E/\mathbb{Q}_\infty) \in \Lambda \otimes \mathbb{Q}_p$ s.t. if $\chi : \Gamma \to \mathbb{C}_p^\times$ of order $p^n \rightsquigarrow \Lambda \xrightarrow{\chi} \mathbb{C}_p$.

$$\chi(L_p(E/\mathbb{Q}_\infty)) = \begin{cases} \frac{1}{\alpha^{n+1}} \tau(\chi) \frac{L(E,\bar\chi,1)}{\Omega_E} & n \geq 1 \\ (1-\frac{1}{\alpha})^2 \frac{L(E,1)}{\Omega_E} & n=0 \end{cases}$$

$\alpha$ unique unit root of $x^2 - a_p x + p$ $(p \nmid a_p)$.

$\tau(\chi)$ Gauss sum, $\Omega_E$ Neron period, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

Pollack II 2

MC $\mathrm{char}_\Lambda X_\infty = L_p(E/\mathbb{Q}_\infty)\cdot\Lambda$.

Use BSD to compute Ш$(E/\mathbb{Q}_n)[p^\infty]$

BSD $L(E/\mathbb{Q}_n,1) \neq 0$ then

$$\frac{L(E/\mathbb{Q}_n,1)}{\Omega_{E/\mathbb{Q}_n}} = \frac{|Ш(E/\mathbb{Q}_n)|\cdot|\mathrm{Tam}(E/\mathbb{Q}_n)|}{|E^{tor}(\mathbb{Q}_n)|^2\sqrt{D(\mathbb{Q}_n)}}.$$

so no reg. term

Assume $E(\mathbb{Q}_n)$ fin $\forall n$. Examine $\frac{|Ш(E/\mathbb{Q}_n)|}{|Ш(E/\mathbb{Q}_{n-1})|}$.

Choose $n$ large enough so that

$\mathrm{Tam}(E/\mathbb{Q}_n)$ satisfies.

Imai: $E(\mathbb{Q}_\infty)^{tor}$ finite $\Rightarrow$ $E^{tor}(\mathbb{Q}_n)$ stabilizes

$$L(E/\mathbb{Q}_n,1) = \prod_{\chi\in\widehat{\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})}} L(E,\chi,1)$$

$$\frac{|Ш(E(\mathbb{Q}_n)|}{|Ш(E(\mathbb{Q}_{n-1})|} \overset{BSD}{=} \prod_{\chi \text{ order } p^n} \frac{L(E,\chi,1)}{\Omega_E}\cdot\left(\frac{D(\mathbb{Q}_n)}{D(\mathbb{Q}_{n-1})}\right)^{1/2}$$

$$\mathrm{ord}_p(\,''\,) = \tfrac{1}{2}\mathrm{ord}_p(D(\mathbb{Q}_n)/D(\mathbb{Q}_{n-1})) + \sum \mathrm{ord}_p\left(\frac{L(E,\chi,1)}{\Omega_E}\right)$$

$$= \cancel{''} + \sum_\chi \mathrm{ord}_p(\chi(L_p(E/\mathbb{Q}_\infty))) - \sum_\chi \cancel{\mathrm{ord}_p(\tau(\chi))} = \sum_\chi \mathrm{ord}_p(\chi(L_p(E/\mathbb{Q}_\infty)))$$

p-adic Weierstrass preparation thm.

$$f\in\Lambda \quad f = p^\mu(T^\lambda + a_{\lambda-1}T^{\lambda-1} + \dots + a_1T + a_0)\,\underset{}{\overset{\text{unit}}{\lambda(T)}}$$

Now $\chi(f) = f(\zeta_{p^n}-1)$ $\zeta_{p^n} = p^n$ th root of 1.

$\mathrm{ord}_p(\zeta_{p^n}-1) = \frac{1}{p^{n-1}(p-1)}$, so $\mathrm{ord}_p(\chi(f)) = \mu + \frac{\lambda}{p^{n-1}(p-1)}$, $n>>0$.

ratio of Ш $= \mu(p^n - p^{n-1}) + \lambda$.

Constructing $L_p(E/\mathbb{Q}_\infty)$

$\mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \simeq (\mathbb{Z}/p^n\mathbb{Z})^\times$ analogous to Stickelberger elts. $\widetilde{\Theta}_n \in \mathbb{Q}[\mathcal{G}_n]$.

$\sigma_a \leftarrow\!\!\shortmid a$

Pollack II 3

$\tilde{\theta}_n \in \mathbb{Q}[g_n] \quad (\frac{1}{d}\mathbb{Z}[g_n])$ s.t. $\chi(\tilde{\theta}_n) = \tau(\chi) \dfrac{L(E,\chi,1)}{\Omega_E}$

$\chi \in \hat{g}_n \quad \tilde{\theta}_n = \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} c_a \sigma_a, \quad c_a \in \mathbb{Q}.$

$E/\mathbb{Q} \underset{\text{modularity}}{\longleftrightarrow} f_E$ normalized newform of level $N_E$ & wt. 2

$c_a = \frac{1}{2}\left(\int_{a/p^n}^{\infty} f_E(z)dz + \int_{-a/p^n}^{\infty} f_E(z)dz\right) / \Omega_E \in \mathbb{Q}.$

$\exists\, d$ s.t. $c_a \in \frac{1}{d}\mathbb{Z}$ $(\forall\, a, n, p)$.

$n=0$ $\quad \dfrac{\int_0^\infty f_E(z)}{\Omega_E} = \dfrac{L(f_E,1)}{\Omega_E} = L(E,1)/\Omega_E$

$\pi : \mathbb{Q}[g_n] \to \mathbb{Q}[g_{n-1}] \qquad \upsilon : \mathbb{Q}[g_{n-1}] \to \mathbb{Q}[g_n] \qquad \sigma \longmapsto \sum_{\tau \mapsto \sigma} \tau$

Prop $\pi(\tilde{\theta}_n) = a_p \tilde{\theta}_{n-1} - \upsilon(\tilde{\theta}_{n-2})$ $\qquad$ Pf: $f|T_p = a_p f$

Trick: $\tilde{\Psi}_n = \tilde{\theta}_n - \frac{1}{\alpha}\, \upsilon(\tilde{\theta}_{n-1}) \Rightarrow \pi \tilde{\Psi}_n = \alpha \tilde{\Psi}_{n-1}$.

$\Psi_n =$ proj. of $\tilde{\Psi}_{n+1}$ to $\frac{1}{d}\mathbb{Z}_p[G_n]$ $\qquad G_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$

$L_p(E/\mathbb{Q}_\infty) = \varprojlim_n \frac{1}{\alpha^n} \Psi_n \in \Lambda \otimes \mathbb{Q}_p.$

# Pollack III

- $\tilde{\Theta}_n(E) \in \mathbb{Q}[\mathcal{G}_n]$ $\quad(\frac{1}{d}\mathbb{Z}[\mathcal{G}_n])$

$$\tilde{\Psi}_n = \tilde{\Theta}_n(E) - \frac{1}{\alpha}\nu(\tilde{\Theta}_{n-1}(E)) \qquad \pi\Psi_n = \alpha\Psi_{n-1}$$

$$L_p(E/\mathbb{Q}_\infty) = \varprojlim \frac{1}{\alpha^n}\Psi_n \in \Lambda\otimes\mathbb{Q}_p$$

$E \leftrightarrow f \in S_2(\Gamma_0(N))$ $\quad p\nmid N \quad f|T_p = a_p f$

$\quad\quad \cap|$

$S_2(\Gamma_0(Np)) \overset{\text{Want}}{-} f_\alpha \in S_2(\Gamma_0(Np)) \quad f_\alpha|U_p = \alpha f_\alpha$

$x^2 - a_p x + p \qquad f_\alpha = f(z) - \frac{1}{\alpha}f(pz)$.

Remarks If $p$ is supersingular, $\alpha$ not a unit, but you can still construct $L_p(E/\mathbb{Q}_\infty)$ in $\mathbb{Q}_p[[T]]$. Better, it's in the subring of overconvergent power series on the open unit disk.

Algebraic side

$$0 \to \mathrm{Sel}_p(E/\mathbb{Q}_\infty) \to H^1(\mathbb{Q}_\infty, E[p^\infty]) \to \prod_{\substack{v \text{ prime} \\ \text{of } \mathbb{Q}_\infty}} \frac{H^1(\mathbb{Q}_{\infty,v}, E[p^\infty])}{E(\mathbb{Q}_{\infty,v})\otimes\mathbb{Q}_p/\mathbb{Z}_p}.$$

$v|\ell$, $E(\mathbb{Q}_{n,v}) \cong \mathbb{Z}_\ell^{[\mathbb{Q}_{n,v}:\mathbb{Q}_\ell]} \times$ fin. group.

$\ell\neq p$, $E(\mathbb{Q}_{\infty,v})\otimes\mathbb{Q}_p/\mathbb{Z}_p = 0$.

$\Sigma$ = primes of bad red. for $E, p, \infty$,
$\mathbb{Q}_\Sigma$ = max. extn. of $\mathbb{Q}$ unram. out $\Sigma$.

$$0 \to \mathrm{Sel}_p(E/\mathbb{Q}_\infty) \to H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, E[p^\infty]) \to \bigoplus_{v|\ell\in\Sigma} H^1(\mathbb{Q}_{\infty,v}, E[p^\infty]) \times \frac{H^1(\mathbb{Q}_{\infty,p}, E[p^\infty])}{E(\mathbb{Q}_{\infty,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p}.$$

$\mathbb{Q}_\Sigma$ = max ext of $\mathbb{Q}$ unram out $\Sigma$

Facts
- $v\nmid p$ $H^1(\mathbb{Q}_{\infty,v}, E[p^\infty])$ is a $\Lambda$-cotorsion module.

Greenberg: Iwasawa theory p-adic reps.

- $H^1(\mathbb{Q}_{\infty,p}, E[p^\infty])$ has $\Lambda$-corank 2.

(local Euler char.)

- $\mathrm{corank}_\Lambda H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, E[p^\infty]) - \mathrm{corank}_\Lambda H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, E[p^\infty]) = 1 \Rightarrow$

  $\mathrm{Corank}_\Lambda H^1(-) \geq 1$.

- $\mathrm{corank}_\Lambda(E(\mathbb{Q}_{\infty,p})\otimes\mathbb{Q}_p/\mathbb{Z}_p) = \begin{cases} 1 & p \text{ ordinary} \\ 2 & p \text{ s.s.} \end{cases}$

Pollack III 2

If $p$ s.s., $\mathrm{corank}_\Lambda \mathrm{Sel}_p(E/\mathbb{Q}_\infty) \geq 1$ ($=1$ if some $H^2$ is small).

Tate local duality

$$H^1(\mathbb{Q}_{n,p}, E[p^\infty]) \times H^1(\mathbb{Q}_{n,p}, T_p E) \to H^2(\mathbb{Q}_{n,p}, \mu_{p^\infty}) \cong \mathbb{Q}_p/\mathbb{Z}_p. \quad \text{perfect pairing}$$

$$\begin{array}{cc} \cup\!\mathrm{I} & \cup\!\mathrm{I} \\ E(\mathbb{Q}_{n,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p & E(\mathbb{Q}_{n,p}) \otimes \mathbb{Z}_p \end{array} \leftarrow \text{exact annihilators of each other}$$

$$\rightsquigarrow \frac{H^1(\mathbb{Q}_{n,p}, E[p^\infty])}{E(\mathbb{Q}_{n,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \cong E(\mathbb{Q}_{n,p}) \otimes \mathbb{Z}_p$$

$= 0$ if $a_p \not\equiv 1 \bmod p$

$$0 \to \hat{E}(\mathbb{Q}_{n,p}) \otimes \mathbb{Z}_p \to E(\mathbb{Q}_{n,p}) \otimes \mathbb{Z}_p \to \tilde{E}(\mathbb{F}_p) \otimes \mathbb{Z}_p \to 0$$

$$\frac{H^1(\mathbb{Q}_{\infty,p}, E[p^\infty])}{E(\mathbb{Q}_{\infty,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} = \varinjlim \frac{H^1(\mathbb{Q}_{n,p}, E[p^\infty])}{E(\mathbb{Q}_{n,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}. \qquad (\quad)^\vee \cong \varprojlim_n \hat{E}(\mathbb{Q}_{n,p}) \quad \text{(trace)}$$

<u>Thm</u> For $n \geq 0$, $\exists\, c_n \in \hat{E}(\mathbb{Q}_{n,p})$ st. $\langle c_n, c_{n-1}, \ldots, c_1, c_0 \rangle$ span $\hat{E}(\mathbb{Q}_{n,p})$ over $\mathbb{Z}_p[\underbrace{\mathrm{Gal}(\mathbb{Q}_{n,p}/\mathbb{Q}_p)}_{G_n}]$ st. if $p$ ordinary, $\mathrm{Tr}^n_{n-1}(c_n) = \alpha c_{n-1}$ & $\mathrm{Tr}^1_0(c_1) = (\alpha - 1) c_0$,

$p$ s.s. $\mathrm{Tr}^n_{n-1}(c_n) = a_p c_{n-1} - c_{n-2}$, $\mathrm{Tr}^1_0(c_1) = u c_0$, $u \in \mathbb{Z}_p^\times$, $n \geq 2$ ) Kobayashi

<u>Consequences</u>

① $p \nmid a_p$, $a_p \not\equiv 1 \pmod p \Rightarrow \hat{E}(\mathbb{Q}_{n,p})$ is a free $\mathbb{Z}_p[G_n]$-mod. gen. by $c_n$.
from trace relation

$$\mathbb{Z}_p[G_n] \xrightarrow{\sim} \hat{E}(\mathbb{Q}_{n,p}) \quad \leftarrow \mathbb{Z}_p\text{-ranks are both } p!$$
$$1 \longmapsto c_n$$

$$\varprojlim_n \hat{E}(\mathbb{Q}_{n,p}) \cong \Lambda \Rightarrow \mathrm{rk}_\Lambda \left( \frac{H^1(\mathbb{Q}_{\infty,p}, E[p^\infty])}{E(\mathbb{Q}_{\infty,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \right) = 1 \quad (\text{still true if } a_p \equiv 1 \bmod p).$$

② $p \mid a_p$. Assume $a_p = 0$ ($p > 3$).
$\hat{E}(\mathbb{Q}_{n,p})$ needs 2 generators as a Gal. module $\mathrm{Tr}^n_{n-1}(c_n) = -c_{n-2}$, $\mathrm{Tr}^n_{n-2}(c_n) = -p c_{n-2}$.
$p^{\lfloor \frac{n}{2} \rfloor} \mid \mathrm{Tr}^n_0 c_n \Rightarrow \varprojlim \hat{E}(\mathbb{Q}_{n,p}) = 0 \to E(\mathbb{Q}_{\infty,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = H^1(\mathbb{Q}_{\infty,p}, E[p^\infty])$

Pollack III 3

<u>Control theorem</u> $\mathcal{H}_{n,\ell} = \bigoplus_{\substack{v \mid \ell \\ \text{in } \mathbb{Q}_n}} H^1(\mathbb{Q}_{n,v}, E[p^\infty])$

$$\mathcal{H}_{n,p} = \frac{H^1(\mathbb{Q}_{n,p}, E[p^\infty])}{E(\mathbb{Q}_{n,p}) \otimes \mathbb{Q}_p/\mathbb{Z}_p}$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & \ker s_n \arrow[r, hook] \arrow[d] & H^1(\mathbb{Q}_\infty/\mathbb{Q}_n, E[p^\infty](\mathbb{Q}_\infty)) \arrow[d] & \\
0 \arrow[r] & \mathrm{Sel}_p(E/\mathbb{Q}_n) \arrow[r] \arrow[d, "s_n"] & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_n, E[p^\infty]) \arrow[r] \arrow[d, "h_n"] & \bigoplus_{\ell \in \Sigma} \mathcal{H}_{n,\ell} \arrow[d, "\oplus r_{n,\ell}"] \\
0 \arrow[r] & \mathrm{Sel}_p(E/\mathbb{Q}_\infty)^{\Gamma_n} \arrow[r] \arrow[d] & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, E[p^\infty])^{\Gamma_n} \arrow[r] \arrow[d] & \bigoplus_{\ell \in \Sigma} \mathcal{H}_{\infty,\ell}^{\Gamma_n} \\
 & \mathrm{coker}\, s_n & H^2(\Gamma_n, E[p^\infty]) = 0 &
\end{tikzcd}$$

finite

$\Gamma_n$ cohom dim 1

$$H^1(\Gamma_n, E[p^\infty](\mathbb{Q}_\infty)) \cong E[p^\infty](\mathbb{Q}_\infty)_{\Gamma_n}$$

$$0 \to E[p^\infty](\mathbb{Q}_n) \to E[p^\infty](\mathbb{Q}_\infty) \xrightarrow{\gamma^{p^n}-1} E[p^\infty](\mathbb{Q}_\infty) \to E[p^\infty](\mathbb{Q}_\infty)_{\Gamma_n} \to 0$$

finite $\Longrightarrow$ finite

(still need bounded indep. of $n$, but true)
exercise

$\ker(r_{n,\ell})$ fin., bounded indep. $n$.

$\mathcal{H}_{n,p} \xrightarrow{r_{n,p}} \mathcal{H}_{\infty,p}$ $\quad a_p \not\equiv 1 \mod p$

$\varprojlim_m \hat{E}(\mathbb{Q}_{m,p}) \twoheadrightarrow \hat{E}(\mathbb{Q}_{n,p}) \Rightarrow \ker(r_{n,p}) = 0$

exercise

# Pollack IV

$$\ker(s_n) \hookrightarrow (E[p^\infty](\mathbb{Q}_\infty))_{\Gamma} \longrightarrow \bigoplus \ker(r_{n,v}) \cap \operatorname{im}(\alpha)$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_p(E/\mathbb{Q}_n) \arrow[r] \arrow[d, "s_n"] & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_n, E[p^\infty]) \arrow[r, "\alpha"] \arrow[d, "h_n"] & \bigoplus_v H_{n,v} \arrow[d, "\oplus r_{n,v}"] \\
0 \arrow[r] & \mathrm{Sel}_p(E/\mathbb{Q}_\infty)^{\Gamma_n} \arrow[r] & H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, E[p^n])^{\Gamma_n} \arrow[r] & \bigoplus_w \mathcal{H}_{\infty,w}^{\Gamma_n}
\end{tikzcd}$$

$\operatorname{coker}(s_n) \longrightarrow 0 = H^2$

- $\ker(s_n)$ finite bounded indep. of $n$, $E(\mathbb{Q})[p]=0 \Rightarrow \ker(s_n)=0$

- $|\ker(r_{n,v})| \sim \mathrm{Tam}_v(E/\mathbb{Q}_n)$ $v \nmid p$
- $\ker(r_{n,p}) = 0$ if $a_p \not\equiv 1 \bmod p$ & fin. bdd indep. of $n$ otherwise, ptap.

$\varprojlim_n \hat{E}(\mathbb{Q}_{m,p}) \twoheadrightarrow \hat{E}(\mathbb{Q}_{n,p})$

$\uparrow a_p \not\equiv 1 \bmod p$ (look at the explicit $c_n$'s)

$\rightsquigarrow$ Control

<u>Cor</u> ($E(\mathbb{Q})$ finite, $Ш(E/\mathbb{Q})[p]=0$, $p \nmid \mathrm{Tam}(E/\mathbb{Q})$, $a_p \not\equiv 0,1 \pmod p$) $\Rightarrow$
$E(\mathbb{Q}_n)$ finite for all $n \geq 0$, $Ш(E/\mathbb{Q}_n)[p]=0$ $\forall n \geq 0$ $(\mu=\lambda=0)$.

<u>Rmk</u> $\frac{L(E,1)}{\Omega_E}$ p-unit, $(1-\frac{1}{\alpha})^2$ p-unit under BSD

<u>Pf</u>: $\mathrm{Sel}_p(E/\mathbb{Q}) = 0 \overset{CT}{\Longrightarrow} \mathrm{Sel}_p(E/\mathbb{Q}_\infty)^{\Gamma} = 0 \Rightarrow X_\infty/TX_\infty = 0 \Rightarrow X_\infty = 0$
$\overset{CT}{\Rightarrow} \mathrm{Sel}_p(E/\mathbb{Q}_n) = 0$ $\forall n \geq 0$

supersingular plap: $\mathcal{H}_{\infty,p} = 0$. $\ker(r_{n,p}) = \mathcal{H}_{n,p}$. $\ker(r_{n,p})^\vee = \hat{E}(\mathbb{Q}_{n,p}) \cong \mathbb{Z}_p^{p^n}$.

$S_n = \mathrm{Sel}_p(E/\mathbb{Q}_n)$ $n \in \mathbb{N}$, $X_n = S_n^\vee$.

$0 \to S_n \to S_\infty^{\Gamma_n} \longrightarrow \bigoplus \ker(r_{n,v}) \cap \operatorname{im}(\alpha)$

$\wr$

$\bigoplus \ker(r_{n,v})$

Pollack IV 2

$\hat{E}(\mathbb{Q}_{n,p}) \times (\text{finite}) \longrightarrow (X_\infty)_{\Gamma_n} \longrightarrow X_n \longrightarrow 0.$

<u>Simplest case</u> $(p \mid a_p)$

Assume: $\mathrm{Sel}_p(E/\mathbb{Q}) = 0$. $p \nmid \mathrm{Tam}(E/\mathbb{Q})$.

$n=0$: $\mathbb{Z}_p \twoheadrightarrow (X_\infty)_\Gamma \longrightarrow X_0 = 0$

We know $\mathrm{rk}_\Lambda X_\infty \geq 1$. So $\mathrm{rk}_{\mathbb{Z}_p} (X_\infty)_\Gamma \geq 1 \Rightarrow = \Rightarrow X_\infty \cong \Lambda$.

$n \geq 0$ $\quad \underset{\text{gen by } c_n, c_n}{\hat{E}(\mathbb{Q}_{n,p})} \xrightarrow{P_n} \mathbb{Z}_p[G_n] \twoheadrightarrow \underset{\text{finite but large}}{X_n} \to 0 \qquad G_n = \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \qquad \Lambda_{\Gamma_n} = \mathbb{Z}_p[G_n]$

$X_n \simeq \mathbb{Z}_p[G_n] / (P_n(c_n), P_n(c_{n-1}))$

<u>Facts</u> ① One can define $\mu, \lambda$-invariants for elements of $\mathbb{Z}_p[G_n]$.

② $\exists$ <u>exact formulas</u> for $\lambda(P_n(c_n))$ $\forall n \geq 0$, $\mu(P_n(c_n)) = 0$ $\forall n \geq 0$.

tend to $\infty$ in $n$ in a <u>very regular</u> way

③ $\mathrm{ord}_p(|Ш_n|) = \sum_{k=0}^{n} \lambda(P_n(c_n)) < \infty$.

$\hat{E}(\mathbb{Q}_{n,p}) \times (\text{fin.}) \to (X_\infty)_{\Gamma_n} \to X_n \to 0.$

Kato $\Rightarrow$ $X_\infty$ has $\Lambda$-rank $= 1$.

$0 \to \underset{\Lambda\text{-torsion}}{Y} \to X_\infty \to \underset{\Lambda\text{-torsion free}}{Z} \to 0 \qquad Z \hookrightarrow \Lambda \to \underset{\text{fin.}}{H} \to 0$

$$\begin{tikzcd}
\hat{E}(\mathbb{Q}_{n,p}) \arrow[r] \arrow[d, shift right, "\mathrm{Tr}"'] \arrow[rrr, bend right, "P_n"] & (X_\infty)_{\Gamma_n} \arrow[r] & Z_{\Gamma_n} \arrow[r] & \Lambda_{\Gamma_n} = \mathbb{Z}_p[G_n] \arrow[d, shift right, "\pi"'] \\
\hat{E}(\mathbb{Q}_{n-1,p}) \arrow[u, shift right, "i"'] \arrow[rrr, "P_{n-1}"] & & & \mathbb{Z}_p[G_{n-1}] \arrow[u, shift right, "\nu"']
\end{tikzcd}$$

Define $\varphi_n = P_n(c_n) \in \mathbb{Z}_p[G_n]$. $\mathrm{Tr}^n_{n-1} c_n = a_p c_{n-1} - c_{n-2} \Rightarrow \pi \varphi_n = a_p \varphi_{n-1} - \nu \varphi_{n-2}$

$\tau = \mathrm{char}_\Lambda Y \qquad \Theta_n^{alg} = \varphi_n \cdot \tau.$

<u>Main Conjecture</u> $\underset{\text{Mazur-Tate elts}}{\Theta_n^{an}} = \Theta_n^{alg} u, \quad u \in \mathbb{Z}_p[G_n]^\times \qquad n \geq 0. \qquad$ (Perrin-Riou '90)

Pollack IV 3

Kato $\Rightarrow$ $\Theta_n^{alg} \mid \Theta_n^{an}$ in $\mathbb{Z}_p[G_n]$.

- $\lambda(\Theta_n^*)$ $\leftarrow$ $\exists$ "exact" formulas for these invariants which go to $\infty$ as $n \to \infty$.

* analytic or alg.

<u>Prop</u> $\chi$ char. of order $p^n$ on $G_n$. $\chi(\Theta_n^{alg}) = 0 \iff \mathrm{Sel}_p(E/\mathbb{Q}_n)^\chi$ infinite.

BSD + MC: $\chi(\Theta_n^{alg}) = 0 \overset{MC}{\iff} \chi(\Theta_n^{an}) = 0 \iff L(E,\chi,1) = 0 \overset{BSD}{\iff} E(\mathbb{Q}_n)^\chi$ infinite.

<u>Prop</u> If $\varphi_0 \neq 0$, then $\chi(\Theta_n^{alg}) = 0$ for only fin. many $\chi$ (order $p^n$).

<u>Cor</u> $\mathrm{Sel}_p(E/\mathbb{Q})$ finite $\Rightarrow$ $\mathrm{cork}_{\mathbb{Z}_p} \mathrm{Sel}(E/\mathbb{Q}_n)$ is bounded indep. of $n$ (Perrin-Riou)

<u>Open questions</u> (<u>supersingular</u>)

1) $F = \mathbb{Q}$. $\rightsquigarrow$ replace w/ $F/\mathbb{Q}$ fin. extn.
2) $F_\infty = \mathbb{Q}_\infty$ $\rightsquigarrow$ replace this w/ arbitrary $\mathbb{Z}_p$-extns. (B D Kim)
3) Replace $E/\mathbb{Q}$ w/ a modular form of higher weight (non-ordinary).
4) nonabelian case ??
5) main conjecture is open

John Coates

# Noncommutative Iwasawa Theory of Elliptic Curves

What is Iwasawa theory?

i) A motive over $\mathbb{Q}$ ($E/\mathbb{Q}$ elliptic curve, Tate motive, $\mathrm{Sym}^n E$, ...)

ii) $p$

iii) Given a Galois extn. $F_\infty$ of $\mathbb{Q}$, whose Galois group is a $p$-adic Lie group (e.g. a closed sgp. of $GL_n(\mathbb{Z}_p)$, $n \geq 1$), always assume that $F_\infty/\mathbb{Q}$ is unramified outside a finite set of primes in $\mathbb{Q}$.

e.g., $F_\infty = \mathbb{Q}(\mu_{p^\infty})$, $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})$, $m > 1$, $F_\infty = \mathbb{Q}(E_{p^\infty})$

Notation: $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$, $\Lambda(G) = \varprojlim_U \mathbb{Z}_p[G/U]$ $U$ runs over all open normal subgroups of $G$.

$\Lambda(G)$ Iwasawa algebra of $G$. noncommutative $\Leftrightarrow$ $G$ nonabelian.

Basic Aim: To study patterns in the behavior of $E(L)$ & $Ш(E/L)(p)$ as $L$ ranges over families of finite extensions of $\mathbb{Q}$ in $F_\infty$.

Ex $E = X_1(11)$: $y^2 + y = x^3 - x^2$. $E \leftrightarrow f_E(\tau) = \sum_{n=1}^{\infty} a_n q^n$ cusp form of weight 2 on $\Gamma_0(11)$.

Take $p = 3$, $F_\infty = \mathbb{Q}(\mu_{3^\infty}, \sqrt[3^\infty]{m})$. $\mathcal{M} = \{m > 1 : m \text{ is cube free}\}$

$\cup$

$\mathcal{M}' = \{m \in \mathcal{M} :$ if a prime $q$ divides $m$, then $q \neq 11$ & $q = 3$ or $N_q = q + 1 - a_q \not\equiv 0 \mod 3\}$

$m = 2, 3, 5, 6, 7, \ldots \in \mathcal{M}'$

Case 1 $m \in \mathcal{M}$ but $(m, 11) = 1$.

Theorem Assume $m \in \mathcal{M}'$. Then $E(F) = \mathbb{Z}/5\mathbb{Z}$ & $Ш(E/F)(3) = 0$ $\forall$ finite ext. $F$ of $\mathbb{Q}$ in $F_\infty$.

Conj. Assume $m \in \mathcal{M}$ but $11 \nmid m$. Then $E(F_\infty)$ is a f.g. ab. group.

Case 2 $m \in \mathcal{M}$ & $11 | m$. $L_n = \mathbb{Q}(\sqrt[3^n]{m})$

Theorem Assume $m \in \mathcal{M}$ & $11 | m$. Then $\mathrm{rank}_{\mathbb{Z}} E(L_n) + \mathrm{corank}_{\mathbb{Z}_3} Ш(E/L_n)(3) \geq n$ $(n = 1, 2, \ldots)$.

## Coates 2

**Theorem** Assume that $m = 11^a m'$, where $a \geq 1$ and $m' \in \mathcal{M}'$.
Then $\mathrm{rank}_{\mathbb{Z}} E(L_n) + \mathrm{corank}_{\mathbb{Z}_3} Ш(E/L_n)(3) = n \quad (n = 1, 2, \ldots)$

Add: when $11 \mid m$, $E(\mathbb{Q}(\sqrt[3]{m}))$, difficult to find a single one!

**The relevant complex L-functions**

$L(E,s)$ = complex L-function of $E$

$T_\ell(E) = \varprojlim E_{\ell^n}$, $V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, $H^1_\ell(E) = \mathrm{Hom}(V_\ell(E), \mathbb{Q}_\ell)$.

$L(E,s)$ Euler product attached to $\{H^1_\ell(E)\}$

**Theorem** $E$ is modular $\Rightarrow$ $L(E,s)$ is entire & satisfies the standard fnl. equation.

Twists of $L(E,s)$ by Artin representations of $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$.

**Def** An Artin representation of $G$ is a homomorphism $\rho : G \to \mathrm{Aut}(W_\rho)$, where $W_\rho$ is a f.d. vector space$/\mathbb{Q}$, which factors through a finite quotient of $G$.

**Ex** $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})$, $F_n = \mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{m})$. $\quad \chi_n : \mathrm{Gal}(F_n/\mathbb{Q}(\mu_{p^n})) \longrightarrow \mathbb{C}^\times$ exact order $p^n$.

$$\begin{array}{c} F_n \\ \big| {\scriptstyle p^n} \\ \mathbb{Q}(\mu_{p^n}) \\ \big| \\ \mathbb{Q} \end{array}$$

**Def** $\rho_n = \mathrm{Ind}(\chi_n)$ (to $\mathrm{Gal}(F_n/\mathbb{Q})$) Artin character of $G$.
irreducible & $\rho_n = \hat{\rho}_n$. $\dim \rho_n = \varphi(p^n)$.

Fix $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. $H^1_\ell(E)_{\mathbb{C}} \otimes_{\mathbb{Q}} W_\rho$ $\dim = 2d$, $d = \dim(\rho)$.
$L(E, \rho, s)$ = Euler product (attached to $H^1_\ell(E)_{\mathbb{C}} \otimes_{\mathbb{Q}} W_\rho$).
$N(E,\rho)$ = conductor of $H^1_\ell(E)_{\mathbb{C}} \otimes W_\rho$.

**Def** $\Lambda(E, \rho, s) = \left(\dfrac{N(E,\rho)}{\pi^{2d}}\right)^{s/2} \left(\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\right)^d L(E, \rho, s)$.

Coates 3

Conjecture $\Lambda(E,\rho,s)$ entire & satisfies FE

$\Lambda(E,\rho,s) = w(E,\hat{\rho},2-s)$ where $w(E,\rho) \in \mathbb{C}^\times$ is Deligne's root #.

This is proven when $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})$.

$L(E,\rho,1)$ (T. & V. Dokchitser)

$w(E,\rho) = \prod_q w_q(E,\rho)$.

p-adic world: $F_\infty/\mathbb{Q}$, $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$

module over $\Lambda(G) \leftrightarrow \{L(E,\rho,1)$ as $\rho$ runs over Artin reps. of $G\}$

$\mathcal{S}(E/F_\infty) \subset H^1(F_\infty, E_{p^\infty})$

$G$ acts on $\mathcal{S}(E/F_\infty)$ naturally on the left

$X(E/F_\infty) = \mathrm{Hom}(\mathcal{S}(E/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$ $\quad (\sigma f)(a) = f(\sigma^{-1}a)$

Continuous actions $\Rightarrow$ left actions of $\Lambda(G)$

Basic question: How big a module is $X(E/F_\infty)$?

$X(E/F_\infty)$ is always f.g. over $\Lambda(G)$.

Definition Let $M$ be a f.g. left $\Lambda(G)$-module. We say that $M$ is $\Lambda(G)$-torsion if, for each $x \in M$, there exists $f \in \Lambda(G)$, not a divisor of 0, such that $f \cdot x = 0$.

Ardakov & Zhang

expect (true if $\mathbb{Q}^{cyc} \subset F_\infty$) $X(E/F_\infty)$ not $\Lambda(G)$-torsion

Def We say $E$ is potentially supersingular at $p$ if $E$ achieves good supersingular red. over a finite extension of $\mathbb{Q}_p$. We say $E$ is potentially ordinary at $p$ if $E$ achieves good ordinary or split mult. red. over a fin. extn. of $\mathbb{Q}_p$. expect $\Lambda(G)$-torsion (true often when $F_\infty \supset \mathbb{Q}^{cyc}$)

General Conjecture Given $F_\infty/\mathbb{Q}$ p-adic Lie extension, is it true that $\mathrm{ord}_{s=1}(L(E,\rho,s)) \le c$, where $\rho$ ranges over all Artin reps. of $G$ ($c$ indep. of $\rho$)?

# Coates II

## Difficult algebra (Lazard, Serre)

$G$ compact $p$-adic Lie group of dim $d$, $\Lambda(G)$ Iwasawa algebra

Fact: $\Lambda(G)$ is left or right noetherian

Hypothesis $G$ has no elt. of order $p$, e.g. $G = \mathbb{Z}_p \times \mathbb{Z}_p^\times$, $p > 2$,
$G \subset GL_n(\mathbb{Z}_p)$, $p > n+1$.

Theorem $\Lambda(G)$ has finite global dimension, and for every f.g. $\Lambda(G)$-module $M$,
$H_i(G, M) = 0$ if $i > d = \dim(G)$.

Theorem If $G$ is pro-$p$, then $\Lambda(G)$ is a local ring w/out zero divisors.

Nakayama's Lemma Assume $G$ is pro-$p$. Let $M$ be a compact $\Lambda(G)$-module. Then $M$ is f.g. over $\Lambda(G)$ $\Leftrightarrow$ $M_G$ is a f.g. $\mathbb{Z}_p$-module

(i) $E$ elliptic curve/$\mathbb{Q}$, and $p$ is a prime s.t. $E$ pot. ordinary at $p$.
(ii) $F_\infty/\mathbb{Q}$ $p$-adic Lie extension w/ $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$.

Hypothesis $G$ has no elts. of order $p$. e.g. $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})$, $p > 2$
$F_\infty = \mathbb{Q}(E_{p^\infty})$, $p \geq 5$.

(iii) $F_\infty \supset \mathbb{Q}^{cyc}$

$$G \left( \begin{array}{c} F_\infty \\ \mid H \\ \mathbb{Q}^{cyc} \\ \mid \Gamma \\ \mathbb{Q} \end{array} \right.$$

$G/H = \Gamma \xrightarrow{\sim} \mathbb{Z}_p$
$\Lambda(H)$ subring of $\Lambda(G)$

Def $\mathfrak{M}_H(G)$ = category of all f.g. $\Lambda(G)$-modules $M$ s.t. $M/M(p)$ is a f.g. $\Lambda(H)$-mod.
$M(p)$ = $p$-primary submodule of $M$.

Remark If $H = 1$, then $\mathfrak{M}_H(G)$ is the category of all f.g. torsion $\Lambda(\Gamma)$-modules.

joint w/ F.K.S.V.

Conjecture Assume $E$ is pot. ordinary at $p$. Then $X(E/F_\infty) \in \mathfrak{M}_H(G)$.

Coates II 2

**Theorem** Assume $\exists$ a finite extn. $F$ of $\mathbb{Q}$ in $F_\infty$ and an elliptic curve $E'$ $(/\mathbb{Q})$ ← not needed
s.t. (i) $E$ isogenous to $E'/F$ (ii) $X(E'/F^{cyc})$ is a f.g. $\mathbb{Z}_p$-module,
(iii) $F_\infty/F$ is pro-$p$
Then $X(E/F_\infty) \in \mathfrak{M}_H(G)$.

$$\begin{array}{l} F_\infty \\ \;|\; \Big) H_F \\ F^{cyc} \\ \;| \\ F \\ \;| \\ \mathbb{Q} \end{array} \qquad G_F \text{ spanning } F_\infty / F$$

case. $E = E'$

$$\mathcal{S}(E/F^{cyc}) \to \mathcal{S}(E/F_\infty)^{H_F}$$

$$X(E/F^{cyc}) \leftarrow X(E/F_\infty)_{H_F} \leftarrow \text{kernel f.g. } \mathbb{Z}_p\text{-module } (F_\infty/F \text{ unram. out fin. set})$$

So $X(E/F_\infty)_{H_F}$ f.g. $\mathbb{Z}_p$-mod $\underset{\text{pro-}p}{\Rightarrow}$ $X(E/F_\infty)$ f.g. $\Lambda(H_F)$-mod.

**Ex** $y^2 + xy = x^3 - x - 1$ $\quad N_E = 2 \cdot 3 \cdot 7^2$
$p = 7$ achieves good ordinary reduction over $\mathbb{Q}_7(\mu_7)$.
**Fact** $\mathcal{S}(E/\mathbb{Q}(\mu_{7^\infty})) = 0$. So $X(E/F_\infty) \in \mathfrak{M}_H(G)$ provided $F_\infty/\mathbb{Q}(\mu_{7^\infty})$ is pro-7.

**More algebra** $G$ compact $p$-adic Lie group, no elt. order $p$, and $H$ closed normal sgp. s.t. $G/H = \Gamma \cong \mathbb{Z}_p$.

**Ore set**

**Def** $S = \{ f \in \Lambda(G) \mid \Lambda(G)/\Lambda(G)f \text{ is a f.g. } \Lambda(H)\text{-module} \} \subset \Lambda(G)$

$M$ left $\Lambda(G)$-module $\Rightarrow$ $M$ is S-torsion if for each $m$ in $M$, $\exists$ $s \in S$ such that
$s \cdot m = 0$.

**Lemma** Let $M$ be any f.g. left $\Lambda(G)$-module. Then $M$ is S-torsion $\Leftrightarrow$
$M$ is f.g. over $\Lambda(H)$.

**Localization in $\Lambda(G)$** Let $W$ be a subset of $\Lambda(G)$. We say $W$ is an Ore set if it is mult. closed, it consists of nonzero divisors, and it satisfies Ore condition: For each $w \in W$ & $x \in \Lambda(G)$, $\exists$ $v_1, v_2 \in W$ & $y_1, y_2 \in \Lambda(G)$ s.t.
$w y_1 = x v_1$, $\quad y_2 w = v_2 x$.

Coates II 3

$W$ Ore set $\quad \Lambda(G)_W = W \times \Lambda(G)/\sim \qquad (w,x) \sim (w',x')$ if $\exists$ $v$ & $v'$ in $W$
w/ $vw = v'w'$ & $vx = v'x'$.

$$\updownarrow$$
$$w^{-1}x$$

<u>Theorem</u> $S$ is an Ore set in $\Lambda(G)$.

Proof: $s \in S$, $x \in \Lambda(G)$. $\quad \Lambda(G)/\Lambda(G)s$ f.g. over $\Lambda(H) \Rightarrow$ $S$-torsion.
So $\exists\, v \in S$ s.t. $v(x + \Lambda(G)s) = \Lambda(G)s$, $\quad vx = ys$, $\quad y \in \Lambda(G)$. //

<u>Def</u> $S^* = \bigcup_{n \geq 0} p^n S$, again an Ore set
<u>Lemma</u> $M \in \mathfrak{M}_H(G) \Leftrightarrow M$ is $S^*$-torsion.

<u>K-theory localization sequence</u>

$K_0(\mathfrak{M}_H(G))$ = Grothendieck group of $\mathfrak{M}_H(G)$

$R$ any ring, $\quad K_1(R) = \lim\limits_{n \to \infty} GL_n(R)/[GL_n(R), GL_n(R)]$.

<u>Fundamental exact sequence</u>

$$K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial} K_0(\mathfrak{M}_H(G)) \to 0$$

mult. on right
w/ $\quad \Lambda(G)^n \to \Lambda(G)^n$.

$x = [A] \in K_1(\Lambda(G)_{S^*})$, $\quad A \in GL_n(\Lambda(G)_{S^*}) \qquad A = (sI_n)^{-1}B$, $\quad$ where $s \in S^*$.

<u>Def</u> $\partial(x) = [\operatorname{coker}(B)] - [\operatorname{coker}(sI_n)]$.

<u>Lemma</u> $\partial$ is surjective.

<u>Def</u> If $M \in \mathfrak{M}_H(G)$, we define its char. elt. $\xi_M \in K_1(\Lambda(G)_{S^*})$ to be any elt. s.t. $\partial(\xi_M) = [M]$.

# John Coates III

**Algebra** $G$ compact $p$-adic Lie group, no elt. of order $p$

$H$ closed normal subgroup $G/H \cong \Gamma \xrightarrow{\sim} \mathbb{Z}_p$

$S, S^* \subset \Lambda(G)$

$$K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial} K_0(\mathfrak{M}_H(G)) \to 0$$

**Def** $M \in \mathfrak{M}_H(G)$, char. elt. of $M$ is any elt. $\xi_M \in K_1(\Lambda(G)_{S^*})$ s.t. $\partial(\xi_M) = [M]$.

**Theorem** $\Lambda(G)_{S^*}^{\times} \to K_1(\Lambda(G)_{S^*})$ is surjective.

$\rho : G \to GL_n(\mathcal{O})$, $\mathcal{O}$ = ring of integers of a finite extension $L$ of $\mathbb{Q}_p$.

What is $\xi_M(\rho)$? $\rho : \Lambda(G) \to M_n(\mathcal{O})$ ring homom.

$\xi_M$ = image of $s^{-1}f$, $s \in S^*$, $f \in \Lambda(G)$

**Idea** To define $\xi_M(\rho) = \det(\rho(f)) / \det(\rho(s))$.

$\xi_M(\rho) \in L \cup \{\infty\}$

**Ex** $E = X_1(11)$ $y^2 + y = x^3 - x^2$ $p = 5$, $F_\infty = \mathbb{Q}(E_{5^\infty})$

$E_2$ = unique ell. curve of cond. 11 w/ no rat. pts. other than 0

$E_2(\mathbb{Q}) = 0$

$$G\left(\begin{array}{c} F_\infty \\ | H \\ \mathbb{Q}^{cyc} \\ | \\ \mathbb{Q} \end{array}\right)$$

$$\begin{array}{c} \mathbb{Q}(E_5) \\ | 5 \\ \mathbb{Q}(\mu_5) \\ | 4 \\ \mathbb{Q} \end{array} \Big) \chi_1 \qquad \begin{array}{l} \rho_1 = \operatorname{Ind} \chi_1 \\ \text{irred. of dim } 4 \end{array} \qquad \chi_2 \Big( \begin{array}{c} \mathbb{Q}(E_{2,5}) \\ | 5 \\ \mathbb{Q}(\mu_5) \\ | \\ \mathbb{Q} \end{array} \qquad \rho_2$$

**Fact:** $\xi(E/F_\infty)(\rho_1) = 5^3 \cdot \text{unit}$ $\xi(E/F_\infty)(\rho_2) = 5 \cdot \text{unit}$

$$G\left(\begin{array}{c} F_\infty \\ | H \\ \mathbb{Q}^{cyc} \\ | \\ \mathbb{Q} \end{array}\right)$$

$E$ has pot. ordinary reduction at $p$

$H_i(G, X(E/F_\infty))$ $i = 0, \dots, d$ $d = \dim(G)$

Coates III 2

**Conjecture** Assume $X(E/F_\infty) \in \mathfrak{M}_H(G)$. Then $H_i(H_F, X(E/F_\infty))$ is finite for all $i \geq 1$ & all $F$ in $F_\infty$.

$$G_F\left(\begin{array}{l} F_\infty \\ \mid\ H_F \\ F^{cyc} \\ \mid\ \Gamma_F \\ F \end{array}\right.$$
$$\begin{array}{l} \mid \\ \mathbb{Q} \end{array}$$

**Remark** take $F_\infty = \mathbb{Q}(E_{p^\infty})$ or $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})$. Then $X(E/F_\infty) \in \mathfrak{M}_H(G)$ $\Rightarrow H_i(H_F, X(E/F_\infty))$ are finite for all $i \geq 1$ & all $F$.

**Ingredients** $H^i(H_F, E_{p^\infty})$ are finite for all $i \geq 0 \Rightarrow \prod_{i\geq 0} \#(H^i(G_F, E_{p^\infty}))^{(-1)^i} = 1$.

**Remark & pure algebra**

$M$ f.g. $\Lambda(H)$-module

$H'$ open in $H$ s.t. $H'$ is pro-$p$ $\quad \Lambda(H') \rightsquigarrow$ skew field $Q(H')$ of fractions

**Def** $\mathrm{rank}_{\Lambda(H')} M = \dim Q(H') \otimes_{\Lambda(H')} M$.

**Thm** $\mathrm{rank}_{\Lambda(H')} M = \sum_{i=0}^{\infty} (-1)^i \mathrm{rank}_{\mathbb{Z}_p} H_i(H', M) \rightsquigarrow \mathrm{rank}_{\mathbb{Z}_p} (M)_{H'}$ if $H_i(H', M)$ is finite $\forall i \geq 1$.

$$G_F\left(\begin{array}{l} F_\infty \\ \mid \\ F^{cyc} \\ \mid \\ F \end{array}\right.$$
$$\begin{array}{l} \mid \\ \mathbb{Q} \end{array}$$

Choose $F$ s.t. $G_F$ pro-$p$.

$X(E/F_\infty) \in \mathfrak{M}_H(G)$ & vanishing higher $H$ homology

$Y(E/F_\infty) = X(E/F_\infty)/X(E/F_\infty)(p)$ f.g. over $\Lambda(H)$

$\mathrm{rank}_{\Lambda(H_F)} Y(E/F_\infty)$

$E: y^2 + xy = x^3 - x - 1$

$p = 7$, $F_\infty = \mathbb{Q}(E_{7^\infty})$

$\quad \mid$

$\quad F = \mathbb{Q}(\mu_7)$

$H_F$ is pro-7

$\mathrm{rank}_{\Lambda(H_F)} X(E/F_\infty) = 3$

$X(E/F_\infty)(7) = 0$.

From now on, $F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})$, $E$ has pot. good ordinary reduction at $p$ odd.

Coates III 3

$$\begin{tikzcd} F_\infty \arrow[d, dash, "H_K"] \arrow[dd, dash, bend right, "G_K"'] \\ K^{cyc} = \mathbb{Q}(\mu_{p^\infty}) \arrow[d, dash, "\Gamma_K"] \\ K = \mathbb{Q}(\mu_p) \arrow[d, dash] \\ \mathbb{Q} \end{tikzcd}$$

$G_K$ pro-p, $H_K \cong \mathbb{Z}_p$, so $\Lambda(H_K) \cong \mathbb{Z}_p[[T]]$.

Theorem (Kato) $X(E/K^{cyc})$ is a torsion $\Lambda(\Gamma_K)$-module.

Theorem $H_i(H_K, X(E/F_\infty)) = 0$ for all $i \geq 1$.

Theorem $X(E/F_\infty)$ is $\Lambda(G)$-torsion.

What about $X(E/F_\infty) \in \mathfrak{M}_H(G)$?

$\mu$-invariants $M$ f.g. $\Lambda(\Gamma_K)$-torsion module

$M(p) \longrightarrow \bigoplus \Lambda(\Gamma_K)/\Lambda(\Gamma_K)p^{m_i}$ pseudo-isomorphism

$\mu_{\Gamma_K}(M) = \sum m_i$

$M$ f.g. torsion $\Lambda(G_K)$-module

$M(p) \longrightarrow \bigoplus \Lambda(G_K)/\Lambda(G_K)p^{m_i}$ $\mu_{G_K}(M) = \sum m_i$

$\uparrow \exists$ pseudo-isom.

Theorem We always have $\mu_{G_K}(X(E/F_\infty)) \leq \mu_{\Gamma_K}(X(E/K^{cyc}))$. Moreover, they are equal $\Leftrightarrow$ $X(E/F_\infty) \in \mathfrak{M}_H(G)$.

Assume $X(E/F_\infty) \in \mathfrak{M}_H(G)$; $\operatorname{rank}_{\Lambda(H_K)} Y(E/F_\infty)$?

Def Let $r_s(E/K^{cyc})$ = # of places $v$ of $K^{cyc}$ s.t. $v \mid m$ & $E$ has split mult. reduction at $v$.

Let $r_g(E/K^{cyc})$ = # of places $v \mid m$, $v \nmid p$ & $E$ has good red at $v$ w/ $\tilde{E}_v(k_v)(p) \neq 0$.

Theorem Assume $X(E/F_\infty) \in \mathfrak{M}_H(G)$. Then $\operatorname{rank}_{\Lambda(H_K)} Y(E/F_\infty) = \operatorname{rank}_{\mathbb{Z}_p} Y(E/K^{cyc}) + r_s(E/K^{cyc}) + 2r_g(E/K^{cyc})$.

root #s all $-1$ $\Leftrightarrow$ this rank is odd

# Coates IV

$p > 2$

$F_\infty = \mathbb{Q}(\mu_{p^\infty}, \sqrt[p^\infty]{m})$

$| \; H_K$

$K^{cyc}$

$|$

$K = \mathbb{Q}(\mu_p)$

$|$

$\mathbb{Q}$

($G_K$, $G$)

$E$ pot. good ordinary reduction at $p$.

$r_s(E/K^{cyc})$ = # of places $v$ of $K^{cyc}$ w/ $v | m$ & $E$ having split multiplicative reduction at $v$.

$r_g(E/K^{cyc})$ = # of places $v$ of $K^{cyc}$ w/ $v|m$ & $v \nmid p$ & $E$ has good reduction at $v$ and $\tilde{E}_v(k_v)(p) \neq 0$.

$Y(E/L) = X(E/L) / X(E/L)(p)$

dual Sel

**Def** $\tau(E/F_\infty) = \mathrm{rank}_{\mathbb{Z}_p} Y(E/K^{cyc}) + r_s(E/K^{cyc}) + 2r_g(E/K^{cyc})$

**Thm** Assume $X(E/F_\infty) \in \mathfrak{M}_H(G)$. Then $\mathrm{rank}_{\Lambda(H_K)} Y(E/F_\infty) = \tau(E/F_\infty)$.

**Ex** $E = X_1(11)$

(1) $p=3$, $X(E/K^{cyc}) = 0$, $\tau(E/F_\infty) = 0 \Leftrightarrow$ every prime factor $q$ of $m$ satisfies $q \neq 11$ & $N_q \not\equiv 0 \bmod 3$.

$p = 7$, $X(E/K^{cyc}) \cong \mathbb{Z}_7$, $P = (1+2, -2)$, $z = \zeta + \zeta^2 + \zeta^4$.

$m = 5$, $\tau(E/F_\infty) = 3$ $\quad m = 2$, $\tau(E/F_\infty) = 1$.

$L(E, \rho, s)$ $\rho$ Artin repn. of $G$ $\quad w(E,\rho)$ = root number

$\hat{\rho} = \rho \Rightarrow w(E,\rho) = \pm 1$.

irreducible self-dual Artin repns. of $G$ of dim $>1$ are $\{\rho_n\}$.

$\mathbb{Q}(\mu_{p^n}, \sqrt[p^n]{m})$

$|$

$\mathbb{Q}(\mu_{p^n})$

$|$

$\mathbb{Q}$

**Thm** Assume $X(E/F_\infty) \in \mathfrak{M}_H(G)$. Then $\tau(E/F_\infty)$ is odd $\Leftrightarrow w(E, \rho_n) = -1 \quad \forall n \geq 1$.

**Theorem** Assume $X(E/F_\infty) \in \mathfrak{M}_H(G)$. Then it has no nonzero pseudonull submodule. In particular, $Y(E/F_\infty)$ has no $\Lambda(H_K)$-torsion.

$L_n = \mathbb{Q}(\sqrt[p^n]{m})$ $\quad (n = 1, 2, \ldots)$ $\quad s_{E/L} = \mathbb{Z}_p$-corank of $Y(E/L)$

**Corollary** Assume $\tau(E/F_\infty)$ is odd. Then $s_{E/L_n} \geq n + s_{E/\mathbb{Q}} \quad \forall n \geq 1$.

Coates IV 2

**Corollary** $Y(E/F_\infty) = 0 \iff \tau(E/F_\infty) = 0$.

**Conjecture** Let $\rho$ be any irreducible Artin char. of $G$ & let $\Phi$ be the set of all 1-d chars. of $\mathrm{Gal}(K^{cyc}/\mathbb{Q})$. Then $\sum_{\psi \in \Phi} \mathrm{ord}_{s=1}(L(E, \rho\psi, s)) \leq \frac{p}{p-1} \tau(E/F_\infty)$.

**Case $\tau(E/F_\infty) = 0$.**

**Theorem** Assume that $X(E/K^{cyc}) = 0$ & that $r_s(E/K^{cyc}) = r_g(E/K^{cyc}) = 0$. ($\Rightarrow X(E/F_\infty) = 0$). For each fin. ext $F$ of $\mathbb{Q}$ in $F_\infty$, we $E(F)$ is finite & $Ш(E/F)(p)$ is finite.

*Proof:* $\mathfrak{S}(E/F) \longrightarrow \underset{\overset{\|}{0}}{\mathfrak{S}(E/F_\infty)^{G_F}}$ kernel is contained in $H^1(G_F, E_{p^\infty}(F_\infty))$.

Ex $E = X_1(11)$, $p = 3$, $m = 11$
$p = 7$, $m = 2$
$p = 5$, $m = 5$

**Problem** Assume $\tau(E/F_\infty) = 0$. Prove that $L(E, \rho, 1) \neq 0$ for all Artin repns. $\rho$ of $G$.

**Case $\tau(E/F_\infty) = 1$.** $\tau(E/F_\infty) = 1 \iff \begin{cases} Y(E/K^{cyc}) = 0 \ \& \ r_s(E/K^{cyc}) = 1, \text{ or} \\ Y(E/K^{cyc}) \cong \mathbb{Z}_p \ \& \ r_s(E/K^{cyc}) = 0. \end{cases}$

**Theorem** Assume that $\tau(E/F_\infty) = 1$. Then $s_{E/L_n} = n + s_{E/\mathbb{Q}}$ $\forall n \geq 1$.

Fukaya, Kato, Sujatha, Coates.

$L(E/L_n, s) = L(E, \rho_n, s) L(E, \rho_{n-1}, s) \cdots L(E, \rho_1, s) L(E/\mathbb{Q}, s)$

**Conj** Assume $\tau(E/F_\infty) = 1$. Then $\mathrm{ord}_{s=1} L(E, \rho_n, s) = 1$ $\forall n$.
(work of Darmon - Tian)

Coates IV 3

$$\begin{array}{ccc} K_1(\Lambda(G)_{S^*}) & \xrightarrow{\partial} & K_0(\mathfrak{M}_H(G)) \to 0 \\ \Psi & & \Psi \\ \underline{\text{Def}}\ \xi(E/F_\infty) & \longmapsto & [X(E/F_\infty)] \end{array}$$

G. Zabradi Found $\xi(E/F_\infty)$ when $\tau(E/F_\infty)=1$.

$\eta$ topological generator of $H_K$. $\sigma\in G$ w/ image in $G(K^{cyc}/\mathbb{Q})$ top. generator

Case 1 $X(E/K^{cyc})\cong\mathbb{Z}_p$ & $\tau(E/F_\infty)=1$.

Thm $\xi(E/F_\infty)=$ class of $\sigma-\varepsilon$, $\varepsilon=\begin{cases}-1 & X(E/\mathbb{Q}^{cyc})=0\\ 1 & X(E/\mathbb{Q}^{cyc})\cong\mathbb{Z}_p.\end{cases}$

Case 2 $X(E/K^{cyc})=0$ & $\tau(E/F_\infty)=1$.

$\xi(E/F_\infty)=$ class of $\left(\sigma-\frac{\eta^{\chi(\sigma)}-1}{\eta-1}\right)$, $\chi=$ cycl. char.

$\#:\Lambda(G)\longrightarrow\Lambda(G)$
$g\longmapsto g^{-1}$

$\#(\xi(E/F_\infty))=\xi(E/F_\infty)\,u(E/F_\infty)\prod_{q\mid m}d_q$ ← coming from complex fnl. equation (~ Euler factors)

$u(E/F_\infty)$: unit of $K_1(\Lambda(G))$

Definition of p-adic L-function

$\mathcal{L}(E/F_\infty)\in K_1(\Lambda(G)_{S^*})$

$\mathcal{L}(E/F_\infty)(\rho)=$

$R=$ set of prime divisors of $pm$

$L_R(E,\rho,s)$: omit Euler factors at primes in $R$.

In CFKSV, MC only correct if one replaces $\rho$ by contragredient.

$$\mathcal{L}(E/F_\infty)(\rho)=\underbrace{\frac{L_R(E,\hat\rho,1)}{(\Omega^+)^{d^+(\rho)}(\Omega^-)^{d^-(\rho)}}}_{\text{alg. (Dokchitser-D.)}}\ e_p(\hat\rho)\ \frac{P_p(\rho,u^{-1})}{P_p(\hat\rho,w^{-1})}\ u^{-f_p}$$

← exponent of conductor

$e_p(\hat\rho)$: local compon at p of epsilon factor of L fn.

$P_p$: Euler factors

$(1-uX)(1-wX)=X^2-a_pX+p$

$u$: unit, $w$: nonunit

Main Conjecture $\exists\ \mathcal{L}(E/F_\infty)(\rho)$ w/ this property & $\mathcal{L}(E/F_\infty)(\rho)$ and $\xi(E/F_\infty)$ have

Eric Urban

$p$ odd prime. Fix $\overline{\mathbb{Q}} \overset{i_\infty}{\hookrightarrow} \mathbb{C}$, $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p$. $\omega$ = Teichmüller char.

$\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \overline{\mathbb{Q}}$, $p^n$ = conductor of $\chi$.

$k \geq 2$. $M_k(\chi, \mathbb{C})$ = modular forms of weight $k$ & char. $\chi$

$f : \mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{im}(z) > 0\} \to \mathbb{C}$ holomorphic

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ $(f|_k \gamma)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$

$\Gamma_0(p^n) \subset SL_2(\mathbb{Z})$

$\gamma \in \Gamma_0(p^n) \Leftrightarrow \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod p^n$

$f \in M_k(\chi, \mathbb{C})$ iff $f|_k \gamma = \chi(\gamma) f$ $\forall \gamma \in \Gamma_0(p^n)$ $\chi(-1) = (-1)^k$.

$f$ holomorphic + holomorphy at the cusps

$f(z+1) = f(z)$ for all $z \in \mathfrak{h}$, $f(z) = \sum_{n \in \mathbb{Z}} a_n q^n$, $q = e^{2\pi i z}$.

$a_n = 0$ if $n < 0$.

$f|_k \gamma$ for $\gamma \in SL_2(\mathbb{Z})$ $f$ is a cusp form if $a_0(f|\gamma) = 0$ $\forall \gamma \in SL_2(\mathbb{Z})$

$S_k(\chi, \mathbb{C}) \subset M_k(\chi, \mathbb{C})$ f.d.

Hecke operators act on these spaces: $\forall \ell \neq p$, $T_\ell$.

$(f|T_\ell)(z) = \sum_{a=0}^{\ell-1} f\left(\frac{z+a}{\ell}\right) + \chi(\ell) f(\ell z)$,

$(f|U_p)(z) = \sum_{a=0}^{p-1} f\left(\frac{z+a}{p}\right)$.

$A \subset \mathbb{C}$ containing the values of $\chi$ $\begin{matrix} S_k(\chi, A) \\ M_k(\chi, A) \end{matrix} \ni f$; $a(n, f) \in A$ $\forall n$

If $f$ is an eigenform for all Hecke operators, and $a_1(f) = 1$ $\leftarrow$ normalized, then $f|T_\ell = a_\ell(f) f$

& $f|U_p = a_p(f) f$

$S_k(\chi, \overline{\mathbb{Q}}) \otimes \mathbb{C} = S_k(\chi, \mathbb{C})$

Example of modular forms:

Eisenstein series $E_{k,\chi}(z) = \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_0(p^n)} (cz+d)^{-k} \chi(d)$ $\Gamma_\infty = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right\}$ $k > 2$ (or $k = 2$ & $\chi \neq 1$)

After a suitable normalization,

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$$E_{k,\chi}(z) = \frac{L(1-k,\chi)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n.$$

$$\sigma_{k-1,\chi}(n) = \sum_{(d,p)=1} d^{k-1}\chi(d)$$

$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ $\quad L(1-k,\chi) \in \overline{\mathbb{Q}}$ if $k \geq 2$, $(-1)^k = \chi(-1)$.

if $k \not\equiv 0 \bmod (p-1)$ $(\chi \not\equiv 1 \bmod p)$ $\Rightarrow$ $L(1-k,\chi)$ is a $p$-adic integer.

<u>Hida theory</u> $\Lambda = \mathbb{Z}_p[[\Gamma]]$, $\Gamma = 1+p\mathbb{Z}_p$, $\Delta = \mu_{p-1} \subset \mathbb{Z}_p^\times$

$\cong$ $\quad$ $\Gamma \ni u$ topological generator

$\mathbb{Z}_p[[T]]$ $\quad 1+T \leftrightarrow u \in \Gamma$.

$\psi$ finite order character of $\Gamma = 1+p\mathbb{Z}_p$

$k \geq 2$, $\quad \psi_k : \Lambda \to \overline{\mathbb{Q}}_p$

$u \longmapsto \psi(u) u^k$.

<u>$\Lambda$-adic forms</u> $\quad \mathbb{F} \in \Lambda[[q]]$, $\mathbb{F}(q) = \sum_{n=0}^{\infty} a_n(\mathbb{F}) q^n$

$$\psi_k(\mathbb{F}) = \sum_{n=0}^{\infty} \psi_k(a_n) q^n \in \overline{\mathbb{Q}}_p[[q]]$$

$\mathbb{F}$ $\Lambda$-adic form $\Leftrightarrow$ $\forall k \geq 2$, $\psi$, $\psi_k(\mathbb{F}) = \iota_p \circ \iota_\infty^{-1}(\mathbb{F}_{k,\psi})$

with $\mathbb{F}_{k,\psi} \in M_k(\psi\omega^{-k}\chi_0, \mathbb{C})$ where $\chi_0 : \Delta \to \mu_{p-1}$.

$\chi_0$ nebentypus of $\mathbb{F}$

$T_\ell$, $U_p$ act on $\Lambda$-adic forms. $\quad F \in M_k(\chi, \overline{\mathbb{Q}}_p) \circlearrowleft U_p$

$e = \lim_{n\to\infty} U_p^{n!}$ $\quad f$ ordinary if $fe = f$.

$M_k^{ord}(\chi, \_) = e M_k(\chi, \_)$

$\cup$

$S_k^{ord}(\chi, \_)$

$\mathbb{F}$ $\Lambda$-adic form $\Rightarrow$ $\mathbb{F}$ ordinary if $F_{k,\psi}$ ordinary $\forall k, \psi$

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$M^{ord}_{\chi_0}(\Lambda)$ = ordinary $\Lambda$-adic forms of nebentypus $\chi_0$.

$\cup$

$S^{ord}_{\chi_0}(\Lambda) = \{ \mathbb{F} \in M^{ord}_{\chi_0}(\Lambda) \mid a_0(\mathbb{F}) = 0 \}$ (enough when no tame level)

<u>Theorem</u> (Hida 80's) (1) $M^{ord}_{\chi_0}(\Lambda) \otimes_{\psi,k} \overline{\mathbb{Q}_p} \xrightarrow{\sim} M^{ord}_k(\psi\omega^{-k}\chi_0, \overline{\mathbb{Q}_p})$ $\forall k \geq 2, \psi$

Similarly for cusp forms.

. Moreover, $S^{ord}_{\chi_0}$ & $M^{ord}_{\chi_0}$ free of fin. type $/\Lambda$.

(2) $0 \to S^{ord}_{\chi_0} \to M^{ord}_{\chi_0} \to \Lambda \to 0$ $\quad$ $\underline{\chi_0 \text{ even}}$

$\mathbb{F} \longmapsto a_0(\mathbb{F})$

<u>Example</u> Kubota-Leopoldt p-adic L-function

$\chi_0 : \Delta \to \mu_{p-1}$, $\mathcal{L}_{\chi_0} \in \mathrm{Frac}(\Lambda)$

If $\chi_0 \neq \omega^{-1}$, then $\mathcal{L}_{\chi_0} \in \underset{\mathbb{Z}_p[[T]]}{\underset{\text{IS}}{\Lambda}}$. If $\chi_0 = \omega^{-1}$, $T \cdot \mathcal{L}_{\chi_0} \in \Lambda \cong \mathbb{Z}_p[[T]]$.

$\psi_k(\mathcal{L}_{\chi_0}) = L^{(p)}(1-k, \chi_0\psi\omega^{1-k})$ $\quad$ (p) $\rightsquigarrow$ remove Euler factor at p.

d prime to p $\quad \langle d\rangle_p = \omega^{-1}(d)d \in 1+p\mathbb{Z}_p = \Gamma \subset \Lambda$

In this case, $\Lambda = \mathbb{Z}_p[[T]]$, $\langle d\rangle_T = (1+T)^{\log_p d\omega^{-1}(d)/\log_p u}$.

$\sigma_{T,\chi_0}(n) = \sum_{\substack{d\mid n \\ (d,p)=1}} \langle d\rangle_T d^{-1}\chi_0(d)$

$\chi_0 \neq \omega^{-1}$ $\quad \mathbb{E}_{\chi_0} = \frac{1}{2}\cdot\mathcal{L}_{\chi_0} + \sum_{n=1}^{\infty} \sigma_{T,\chi_0}(n) q^n \in \Lambda[[q]]$

$\psi_k(\mathbb{E}_{\chi_0}) = E_{k,\chi_0\psi\omega^{1-k}}$ $\rightsquigarrow$ $\Lambda$-adic form of nebentypus $\chi_0\omega$

$E_{k,\chi}$ ordinary eigenforms $E_{k,\chi}|U_p = E_{k,\chi}$ $\quad E_{k,\chi}|T_\ell = (1+\ell^{k-1}\chi(\ell))E_{k,\chi}$

$\Rightarrow \mathbb{E}_{\chi_0}$ is an eigenform $U_p$-eigenvalue 1, $T_\ell$: $1 + \langle\ell\rangle_T \ell^{-1}\chi_0(\ell)$.

$h^{ord}_{\chi_0}$ $\Lambda$-subalg. of $\mathrm{End}_\Lambda(S^{ord}_{\chi_0})$ gen. by images of $T_\ell, U_p$.

$h^{ord}_{\chi_0} \otimes_{\psi_k} \overline{\mathbb{Z}}_p = h^{ord}_k(\psi\omega^{-k}\chi_0) \circlearrowright S^{ord}_k(\chi_0\omega^{-k}\psi, \overline{\mathbb{Q}}_p)$.

$\downarrow\lambda$

$\overline{\mathbb{Z}}_p$

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$\sum_{n=1}^{\infty} \lambda(T_n) q^n \qquad T_m \leftrightarrow \Gamma_0(p^n) \begin{pmatrix} 1 & \\ & m \end{pmatrix} \Gamma_0(p^n)$

$\parallel$

q-expansion of an eigenform in $S_k^{ord}(\chi_0 \omega^{-k} \psi, \overline{\mathbb{Q}}_p)$

$$\begin{tikzcd} h_{\chi_0}^{ord} \arrow[r, "\lambda"] & \overline{\mathbb{Z}}_p \\ \Lambda \arrow[u] \arrow[ur, "\psi_k"'] & \end{tikzcd} \quad \text{arithmetic}$$

$\mathrm{Spec}(h_{\chi_0}^{ord})(\overline{\mathbb{Z}}_p) = \mathrm{Hom}(h_{\chi_0}^{ord}, \overline{\mathbb{Z}}_p) \qquad \lambda$

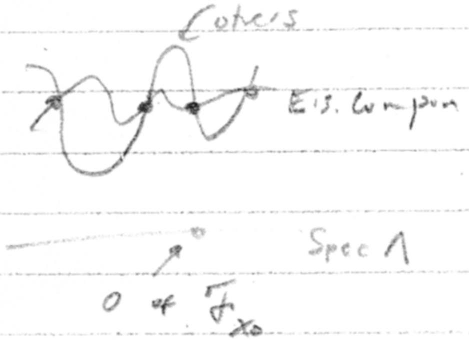
$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$\mathrm{Spec}(\Lambda) \qquad\qquad\qquad\qquad\qquad \psi_k$.

$H_{\chi_0}^{ord} \longrightarrow \Lambda \qquad T_\ell \longmapsto 1 + \sigma_T(\ell)\,\ell^{-1}\chi_0(\ell)$

↑ acting on all ord Λ-adic m.f.

irred. component of Spec(H) E.is. compon.



# Urban II

$\chi_0$ odd char. of $\Delta = \mu_{p-1} \subset \mathbb{Z}_p^\times$

$\chi_0 \neq \omega^{-1} \rightsquigarrow \mathcal{J}_{\chi_0} \in \Lambda$

$\mathbb{E}_{\chi_0} \in M^{ord}_{\chi_0\omega}(\Lambda)$

$\frac{\mathcal{J}_{\chi_0}}{2} + \sum_{n=1}^{\infty} \sigma_{T,\chi_0}(n) q^{n-1}$

$\sigma_{T,\chi_0}(n) = \sum_{\substack{d\mid n \\ (d,p)=1}} \langle d \rangle_T d^{-1} \omega \chi_0(d)$

$H^{ord}_{\chi_0\omega}$ = Hecke acting on $M^{ord}_{\chi_0\omega}$

$h^{ord}_{\chi_0\omega}$ = " " " $S^{ord}_{\chi_0\omega}$

$\lambda_{Eis} : H^{ord}_{\chi_0\omega} \to \Lambda$

$\uparrow$ eigenvalues of $\mathbb{E}_{\chi_0}$

$$\begin{array}{ccc} H^{ord}_{\chi_0\omega} & \hookrightarrow & h^{ord}_{\chi_0\omega} \times \Lambda \\ \cap\!\! \mid & & \cap\!\! \mid \\ \prod I_i \times \Lambda & = & \prod_i I_i \times \Lambda \end{array}$$

Eisenstein component $\mathrm{Spec}(\Lambda) \xleftrightarrow{\lambda_{Eis}} \mathrm{Spec}(H^{ord}_{\chi_0\omega})$

$$\begin{array}{c} \cup \\ \mathrm{Spec}(h^{ord}_{\chi_0\omega}) \end{array}$$

$\mathrm{Spec}(h^{ord}_{\chi_0\omega} \otimes_{H^{ord}_{\chi_0\omega}} \Lambda) = h^{ord}_{\chi_0\omega} / I_{\chi_0\omega}$ w/ $I_{\chi_0\omega}$ = ideal of $h^{ord}_{\chi_0\omega}$ generated by $T_\ell - \lambda_{Eis}(T_\ell)$ $\ell \nmid p$, $U_p - 1$

$\uparrow$ Eis. ideal $\qquad \lambda_{Eis}(T_\ell) = \sigma_{T,\chi_0}(\ell)$

Thm $h^{ord}_{\chi_0\omega} / I_{\chi_0\omega} \twoheadrightarrow \Lambda / \mathcal{J}_{\chi_0}$ $\quad \chi_0 \neq \omega^{-1}$

Proof: $0 \to S^{ord}_{\chi_0\omega} \to M^{ord}_{\chi_0\omega} \to \Lambda \to 0$ $\qquad$ Take $g \in M^{ord}_{\chi_0\omega}$ s.t. $a_0(g) = 1$,

$\mathbb{F} \mapsto a_0(\mathbb{F})$ $\qquad \mathbb{H} = \mathbb{E}_{\chi_0} - \mathcal{J}_{\chi_0} g$

(Assume $\mathcal{J}_{\chi_0} \notin \Lambda^\times$.) $\qquad a(1, \mathbb{H}) \equiv 1 \mod m_\Lambda$

Urban # 2

$$h^{ord}_{\chi_0\omega} \longrightarrow \Lambda/(\sigma_{f_{\chi_0}})$$

$$T \longmapsto \frac{a(1,\mathbb{H}|T)}{a(1,\mathbb{H})}$$ well-defined because $\mathbb{H}$ is a cusp form

$$\frac{a(1,\mathbb{H}|T)}{a(1,\mathbb{H})} = \lambda_{Eis}(T)$$

$$h^{ord}_{\chi_0\omega}/I_{\chi_0\omega} \simeq \Lambda/\sigma_f \twoheadrightarrow \Lambda/f_{\chi_0}$$

Galois representations $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$

$\forall \ell$, $Frob_\ell$ geometric Frobenius $\in G_{\mathbb{Q}}$.

Thm (Eichler-Shimura, Deligne)

$f$ Hecke eigenform of wt. $k$ & nebentypus $\chi$ ($F \in M_k(\chi, \mathbb{C})$).

Then $\exists$ $\rho_f : G_{\mathbb{Q}} \to GL_2(\bar{\mathbb{Q}}_p)$ s.t. $\rho_f$ unram. away from $p$ &

$\det(1-\rho_p(Frob_\ell)X) = 1 - a_\ell(f)X + \chi(\ell)\ell^{k-1}X^2$ $\forall \ell \neq p$.

If, moreover, $f$ is ordinary, then $\rho_F|_{D_p} \sim \begin{pmatrix} \delta_f & * \\ 0 & \delta_f \chi \varepsilon^{1-k} \end{pmatrix}$

Tcycl. char.

$\delta_f$ unram. char s.t. $\delta_f(Frob_p) = \alpha_p$ where

$\alpha_p = a_p$ if $\chi \neq 1$ & $\alpha_p$ unit root of $1 - a_p(f)X + p^{k-1}$ otherwise.

Example $f = E_{k,\chi}$ $\rho \sim \begin{pmatrix} 1 & 0 \\ 0 & \chi\varepsilon^{1-k} \end{pmatrix}$

Thm If $f$ is cuspidal, then $\rho_F$ is absolutely irred.

Sketch: $L(f\otimes\chi, s)$ = entire function

If $\rho_f$ were reducible $\rho_f \sim \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$

$\to L(f\otimes\chi_1^{-1}, s) = \zeta(s) L(\chi_2\chi_1^{-1}, s)$ has a pole.

When we have a Hida family $H^{ord}_{\chi_0\omega} \xrightarrow[\lambda_{\mathbb{I}}]{} \mathbb{I}$ ← irred. component $\quad F_{\mathbb{I}}$ = f.f. of $\mathbb{I}$

$\rightsquigarrow \rho_{\mathbb{I}} : G_{\mathbb{Q}} \to GL_2(F_{\mathbb{I}})$.

$\det(1-\rho_{\mathbb{I}}(Frob_\ell)X) = 1 - \lambda_{\mathbb{I}}(T_\ell)X + \chi_0(\ell)\langle\ell\rangle_T^{-1}\ell$, $\ell \neq p$.

$$\rho_{\mathbb{I}}|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & \omega\chi_0\varepsilon^{-1}\langle\varepsilon\rangle_T \end{pmatrix}$$

Urban II 3

$\rho_{\mathbb{E}_{\chi_0}} \simeq \begin{pmatrix} 1 & 0 \\ 0 & \langle \varepsilon \rangle_T \kappa \chi_0 \end{pmatrix}$ $\kappa = \varepsilon\omega^{-1}$

$R_{Eis}$ local compon. of $h^{ord}_{\chi_0\omega}$ corr. to max. ideal of $h^{ord}_{\chi_0\omega}$ cont. $I_{\chi_0\omega}$

$R_{Eis}/I_{\chi_0\omega} \hookrightarrow \Lambda/\mathcal{F}_{\chi_0}$.

For each irred compon $\mathbb{I}$ of $R_{Eis}$, we have $\rho_{\mathbb{I}}$.

$\rho_{R_{Eis}} : G_{\mathbb{Q}} \longrightarrow GL_2(\tilde{R}_{Eis})$ $\tilde{R}_{Eis}$ = Frac. ring of $R_{Eis}$

$tr(\rho_{R_{Eis}}) \equiv tr(\rho_{\mathbb{E}_{\chi_0}}) \mod I_{\chi_0\omega}$

We are going to construct a lattice $\mathcal{L} \subset V_{\tilde{R}_{Eis}} = \tilde{R}^2_{Eis}$ w/ Galois action given by $\rho_{R_{Eis}}$ s.t. $0 \to N(\chi_0\langle\varepsilon\rangle\kappa) \to \mathcal{L}/I_{\chi_0\omega}\mathcal{L} \to R_{Eis}/I_{\chi_0\omega} \to 0$
w/ quotient w/ action given by $\chi_0\langle\varepsilon\rangle\kappa^{-1}$,
where $N$ some torsion module over $\Lambda$ s.t. $\mathrm{char}_\Lambda N$ is divisible by $\mathcal{F}_{\chi_0}$.

Choose in $V_{\tilde{R}_{Eis}} = (\tilde{R}_{Eis})^2 = (\prod_{\mathbb{I}} F_{\mathbb{I}})^2$ an elt. $v^+$ s.t. $\rho_{R_{Eis}}(c)\, v^+ = v^+$
with nontrivial projection on any component.

$\mathcal{L} = R_{Eis}[G_{\mathbb{Q}}]$-module gen. by $v^+ \Rightarrow \mathcal{L}$ lattice w/ $\mathcal{L} \otimes \mathcal{F}_\Lambda \cong V_{\tilde{R}_{Eis}}$

$\mathcal{L} = \mathcal{L}^- \oplus \mathcal{L}^+$ s.t. $\mathcal{L}^+$ = fixed part of $\mathcal{L}$ by $\rho_{\tilde{R}_{Eis}}(c)$

$\mathcal{L}^-$ = $(-1)$-eigenspace, $\sigma \in R_{Eis}[G_{\mathbb{Q}}]$ $\rho_{\mathcal{L}}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$

$a_\sigma \in Hom(\mathcal{L}^-, \mathcal{L}^-)$, $b_\sigma \in Hom(\mathcal{L}^+, \mathcal{L}^-)$

$a_\sigma + d_\sigma = tr(\rho_{\mathcal{L}}(\sigma)) \in R_{Eis}$ $\forall \sigma \in R_{Eis}[G_{\mathbb{Q}}]$

$$\underset{\text{"}}{tr(\rho_{\mathcal{L}}(\sigma))}$$

$1 + \chi_0\kappa^{-1}\langle\varepsilon\rangle_T(\sigma) \mod I_{\chi_0\omega}$

$a_\sigma \equiv \chi_0\kappa^{-1}\langle\varepsilon\rangle(\sigma)$, $d_\sigma \equiv 1 \mod I_{\chi_0\omega}$.

$a_{\sigma\tau} = a_\sigma a_\tau + b_\sigma c_\tau$ $b_\sigma c_\tau \in I_{\chi_0\omega}$ $\forall \sigma, \tau$

$\rho_{\mathcal{L}}(\sigma) \cdot v^+ = b_\sigma v^+ + d_\sigma v^+$ $\mathcal{L}^+ = R_{Eis} v^+$

$\hookrightarrow$ these generate $\mathcal{L}^+$

$c_\tau b_\sigma \in I_{\chi_0\omega}$

$c_\tau(\mathcal{L}^-) \subset I_{\chi_0\omega} \cdot v^+ = I_{\chi_0\omega}\mathcal{L}^+$

The action of $\sigma$ on $\mathcal{L}/I\mathcal{L}$ fits in an exact seq.

$$0 \to \mathcal{L}^-/_{I_{\chi_0\omega}}\mathcal{L}^-(\chi_0\kappa\langle\varepsilon\rangle) \to \mathcal{L}/I\mathcal{L} \to \mathcal{L}^+/_{I}\mathcal{L}^+ \to 0 \qquad (*)$$

$\nearrow$ Mazur-Wiles char ideal divis by ${}^{\sigma}\mathcal{J}_{\chi_0}$ $\qquad\qquad$ $\mathcal{L}^+/I\mathcal{L}^+ \cong \Lambda/{}^{\sigma}\mathcal{J}_{\chi_0}$

$Hom_{\mathbb{Z}_p}(\mathcal{N}, \mathbb{Q}_p/\mathbb{Z}_p) = \mathcal{N}^* = Hom_\Lambda(\mathcal{N}, \Lambda^*) \qquad \Lambda^* = Hom_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$

$$\begin{array}{ccc} \mathcal{N}^* & \longrightarrow & H^1(G_{\mathbb{Q}}, \Lambda^*(\chi_0\kappa^{-1}\langle\varepsilon\rangle_T)) \\ \psi & & \psi \\ \varphi & \longmapsto & \varphi\circ c_{\mathcal{N}} \end{array}$$

Remark $M$ $G_{\mathbb{Q}}$-module. $H^1(G_{\mathbb{Q}}, M)$ classifies the extensions

$0 \to M \to E \to \Lambda \to 0$

$c_{\mathcal{N}} : G_{\mathbb{Q}} \to \mathcal{N}$ cocycle corr. to $\mathcal{N}$

$\forall \ell \neq p$, $\rho_{\mathcal{L}}$ is unram. at $\ell$

$\ell = p$, $\rho_{\mathcal{L}} \sim \begin{pmatrix} 1 & * \\ 0 & \langle\varepsilon\rangle\chi_0\kappa \end{pmatrix}$ $\qquad \chi_0 \not\equiv 1 \mod m_{R_{Eis}}$

so $(*)$ splits after res. to $I_p$.

$Sel(\chi_0) = \ker\left(H^1(G_{\mathbb{Q}}, \Lambda^*(\chi_0\langle\varepsilon\rangle_T\kappa)) \to \bigoplus_{\ell} H^1(I_\ell, -)\right)$

So $Sel(\chi_0)^\vee \twoheadrightarrow \mathcal{N} \to char(Sel(\chi_0)^*)$ is divisible by $\mathcal{J}_{\chi_0}$.

$\updownarrow$

$X_\infty(\chi_0) \qquad X_\infty = Gal(L_\infty/K_\infty) \quad L_\infty$ max. pro-p unram. ab. ext. of $K_\infty = \mathbb{Q}(\mu_{p^\infty})$

$\forall \chi_0 \neq \omega^{-1} \quad {}^{\sigma}\mathcal{J}_{\chi_0} \mid g_{\chi_0}$

on the other hand, $\prod_{\chi_0 \text{ odd}} \mathcal{J}_{\chi_0} \sim \prod_{\chi_0 \text{ odd}} g_{\chi_0}$ $\qquad$ so $=$. //

(analytic class # formula)

# Urban III

$F$ number field

$G_F$ absolute Galois group

$\mathcal{O}/\mathbb{Z}_p$ $\quad T = T_p$ free submodule of f.t.

$\rho : G_F \longrightarrow GL(T_p)$

**Def** If $E/\mathbb{Q}_p$, $V =$ $L$-v.s. $L = \mathrm{Frac}(\mathcal{O})$

$\rho : G_E \longrightarrow GL(V)$

$\rho$ is ordinary iff $\exists$ a filtration $F^i V$ of $V$ s.t.

$F^{i+1} V \subset F^i V \qquad F^n V = 0 \quad n \gg 0$

s.t. $\underset{\mathrm{Gr}^i V}{\overset{\shortparallel}{F^i V / F^{i+1} V}} \circlearrowleft I_E$ by $\varepsilon^i$ $\qquad \varepsilon =$ cycl. char.

If $\mathrm{Gr}^i V \neq 0$, we say that $-i$ is a Hodge-Tate weight of $V$

$\rho : G_F \longrightarrow GL(T_p)$ is ordinary if $\forall v | p$ place of $F$, $\rho|_{D_v}$ is ordinary.

$\forall v | p$ $\quad F^i_v T_p$ s.t. $F^i_v T_p / F^{i+1}_v T_p \underset{\varepsilon^i}{\circlearrowleft} I_v$

pot. ordinary: $\rho|G_{F'}$ ordinary for some $F'/F$ finite

Example $f$ mod form of wt $k \geq 2$, level prime to $p$

s.t. $p \nmid a(p, f)$, then $\rho_f|_{I_p} \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon^{1-k} \end{pmatrix}$

$F^i = V_f$ if $i \leq 1-k$

$F^i =$ unram. line if $0 \geq i > 1-k$ $\qquad F^i = 0$ if $i > 0$.

$0, k-1$ are the HT weights.

$V_p = T_p \otimes L$ $\quad H^1(F, V_p)$ classifies the extns. of the form

$0 \to V_p \to E \to L \to 0$

$\mathrm{Sel}(F, \rho) \subset H^1(F, V_p)$ classifies the extensions which are ordinary repns.

$\Sigma$ fin. set of places of $F$, $\rho$ ordinary

$\mathrm{Sel}^{\Sigma}(F, V_p/T_p) \subset H^1(F, V_p/T_p)$ $\qquad$ classifies $0 \to p^{-n}T_p/T_p \to E \to p^{-n}\mathcal{O}/\mathcal{O} \to 0$

$\quad \shortparallel \; \varinjlim_{n \to \infty} \mathrm{Sel}(F, p^{-n}T_p/T_p) \subset H^1(F, p^{-n}T_p/T_p)$

Urban III 2

such that $\forall v \nmid p$, $v \notin \Sigma$

$$0 \to (p^{-n}T_p/T_p)^{I_v} \to E^{I_v} \to p^{-n}\mathcal{O}/\mathcal{O} \to 0$$

If $v|p$, $E|_{I_v} \in H^1_{ord}(F_v, p^{-n}T_p/T_p)$ classifies extns. which are obtained as reductions mod $p^n$ of ordinary repns.

$$0 \to p^{-n}T_p/T_p \to E \to p^{-n}\mathcal{O}/\mathcal{O} \to 0$$

$F_v^i E \rightsquigarrow F_v$ acts on $F_v^i E/F_v^{i+1}E$ by $\varepsilon^i$

$\mathcal{O}/_{p^n}\mathcal{O}\, e + F^0E$ is stable by $F_v$

$\quad\quad ||$

$\mathcal{O}/_{p^n}\mathcal{O}\, e + F^0 p^{-n}T_p/T_p$ " " " " $\Rightarrow$ the image of the class of $E$ in $H^1(I_v, (p^{-n}T_p/T_p)/p^{-n}F^0T_p/F^0T_p)$ is trivial.

We replace this by saying that the class in $H^1(I_v, T_p/F^0T_p \otimes \mathbb{Q}_p/\mathbb{Z}_p)$ is trivial.

Ex $\rho = \varepsilon^{-n}$ if $n > 0 \Rightarrow F^0 = 0 \Rightarrow$ the condition for the cocycle in $H^1(F, \mathbb{Q}_p/\mathbb{Z}_p(\varepsilon^n))$ is unramified for all $v|p$.

if $n < 0$ no condition.

$K_\infty/K$ $\mathbb{Z}_p$-ext.

$$\mathrm{Sel}^\Sigma(K_\infty, V_p/T_p) = \varinjlim_n \mathrm{Sel}(K_n, V_p/T_p)$$

$\mathrm{Sel}^\Sigma(F, T_p \otimes \Lambda^*)$

Using Shapiro's Lemma $\subset H^1(F, T_p \otimes \Lambda^*)$ $\quad \Lambda = \mathbb{Z}_p[[T]]$

$\Lambda^* = \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$

Any $\mathbb{I} = \mathbb{Z}_p[[T_1, \ldots, T_r]]$ $\quad V_{\mathbb{I}}$ repn of $G_F$ ordinary

$\mathrm{Spec}(\mathbb{I})$ + filtration on $V_{\mathbb{I}}$

Construction of elts. in general Selmer groups

Main idea: deform reducible representations

$\mathcal{O}$ local noetherian reduced ring

Urban III 3

$\rho_0 = \rho_1 \oplus \cdots \oplus \rho_s$ $\quad \rho_i : G_F \to GL_{n_i}(\mathcal{O})$.

Assume that $\rho_i$ mod $m_{\mathcal{O}}$ is abs irred. $\bar{\rho}_i \not\cong \bar{\rho}_j$ for $i \neq j$

A "big" deformation of $\rho_0$ is given by $(\rho, R, I)$ where

(i) $R$ is a local noetherian $\mathcal{O}$-algebra

(ii) $I \subsetneq R$ ideal $\quad$ (iii) $\rho : G_F \to GL_n(\mathrm{Frac}(R))$ $\quad n = n_1 + \cdots + n_s$

(iv) $\mathrm{tr}(\rho) \subset R$ $\quad \mathrm{tr}(\rho) \equiv \mathrm{tr}(\rho_0) \bmod I$.

(v) $\mathrm{tr}\,\rho \neq \mathrm{tr}(\rho_1') + \cdots + \mathrm{tr}(\rho_s')$ where the $\rho_i'$ is a deformation of $\rho_i$.

In particular, if $s=2$, big means irred.; $s=3$ big means $\rho$ is not the sum of 3 reprs.

Char of $R/m_R > n$.

<u>Exs</u> $\quad s=2 \quad \rho_0 = \rho_1 \oplus \rho_2 \quad (\rho, R, I) \rightsquigarrow$ one can construct a Gal stab. lattice $\mathcal{L} \subset V_\rho = \mathrm{Frac}(R)^n$

$$0 \to \rho_1 \otimes N \to \mathcal{L}/I\mathcal{L} \to \rho_2 \otimes R/I \to 0 \rightsquigarrow \text{elt. } c_N \text{ of } H^1(F, \rho_1 \otimes \rho_2^\vee \otimes N)$$

w/ $N$ some torsion $R$-mod. s.t. $I \mid \mathrm{Fitt}_R(N)$ $\quad N^* \hookrightarrow H^1(F, \rho_1 \otimes \rho_2^\vee \otimes \mathcal{O}^*)$

$\phi \mapsto \phi \circ c_N$

If $\rho_1, \rho_2, \rho$ ord, then

the image of $c$ falls in the ordinary classes.

If also unram at $\Sigma$, then $\mathrm{Im}(c) \subset \mathrm{Sel}^{\Sigma}(F, \rho_1 \otimes \rho_2^\vee \otimes \mathcal{O}^*)$.

$s=3 \quad \rho_0 = \rho_1 \oplus \rho_2 \oplus \rho_3 \quad (\rho, R, I)$

We can construct a Galois stable lattice $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ (not Gal stab.)

s.t.

$$0 \to N_1 \oplus N_2 \to \mathcal{L}/I\mathcal{L} \to \rho_3 \otimes R/I \to 0$$

$N_i = \mathcal{L}_i \otimes R/I \quad N_1, N_2$ not indep. Gal stable in gen.

$$\rho_{\mathcal{L}/I\mathcal{L}} \sim \begin{pmatrix} \rho_1 & B & D \\ C & \rho_2 & E \\ 0 & 0 & \rho_3 \end{pmatrix} \text{ s.t. } B \cdot C \equiv 0 \bmod I$$

If we have information about nonexistence of elts. in $\mathrm{Sel}(\rho_i \otimes \rho_3^\vee)$ for $i = 1, 2$ $\Rightarrow$ information about $\xi_C$.

## Eric Urban IV

*unitary groups*

$a \geq b \geq 0$, $K$ imaginary quadratic field

$G_{a,b}$ = unitary group defined by the skew-Hermitian matrix

$$T_{a,b} = \begin{pmatrix} & & -1_b \\ & \theta & \\ 1_b & & \end{pmatrix} \quad \theta \text{ skew-Hermitian definite}$$

$G_{a,b}(\mathbb{R}) \cong U(a,b) \supset U(a) \times U(b)$

$c_1 \geq c_2 \geq \dots \geq c_b \quad c_{b+1} \geq \dots \geq c_d$

$c_b \geq c_{b+1} + d \qquad d = a+b$

$\tau = (c_{b+1}, \dots, c_d, c_1, \dots, c_b)$

$\rightsquigarrow$ automorphic factor on the Hermitian domain $\mathfrak{h}_{a,b} = G_{a,b}(\mathbb{R}) / U(a) \times U(b)$

holomorphic automorphic form of weight $\tau$.

- $\Pi$ automorphic representation of $G_{a,b}(\mathbb{A})$

  $\pi = \pi_f \otimes \pi_\infty$ w/ $\pi_\infty$ a holomorphic discrete series of weight $\tau$

- $\chi$ Hecke character of $K$

  $\chi_\infty(z) = \left(\frac{z}{\bar{z}}\right)^{\kappa} (z\bar{z})^{\kappa'} \quad \kappa, \kappa' \in \frac{1}{2}\mathbb{Z} \quad 2\kappa \equiv 2\kappa' \bmod 2.$

- (Reg) $c_b \geq \kappa + \frac{d}{2} - 1, \quad \kappa - \frac{d}{2} - 1 \geq c_{b+1}$

  $\rightsquigarrow$ Galois representation
  conjectural

Rep(a,b): Conjecture $\exists\ R_p(\pi) : G_K \longrightarrow GL_d(\bar{\mathbb{Q}}_p)$ s.t. $^{(1)} R_p(\pi)^\vee(1-d) \cong R_p(\pi)^c$

(2) $L(R_p(\pi), s) = L(\pi^\vee, s + \frac{1-d}{2})$

(weaker: $L \rightsquigarrow L^S \quad (2)^{(weak)}\ S \neq \emptyset \quad (2) = (2)^{(strong)}\ S = \emptyset$)

$G_{a+1,b+1}$ $P$ parabolic, stabilizing anistropic line

$P = MN \quad M \cong G_{a,b} \times \mathbb{G}_m/K. \quad \pi \times \chi$

$\varphi_s \in \mathrm{Ind}_{P(\mathbb{A})}^{G_{a+1,b+1}(\mathbb{A})} (\pi \times \chi)\, \delta^s, \quad s \in \mathbb{C}$

Urban IV 2

$$E(\phi, s, g) = \sum_{\gamma \in P \backslash G_{a+1,b+1}(\mathbb{Q})} \phi_s(\gamma g) \quad Re(s) >> 0$$

$g \in G_{a+1,b+1}(\mathbb{A})$

$\chi' = \chi|_{\mathbb{A}_{\mathbb{Q}}^\times}$ $\begin{cases} \chi' = |\,|_{\mathbb{Q}}^{2k'} \\ \chi' \neq |\,|_{\mathbb{Q}}^{2k'} \end{cases}$

Under this condition, the valuation at $s = \frac{1}{2} - k'$ defines a holomorphic form on $G_{a+1,b+1}$: $E(\pi, \chi, \phi)$ holomorphic form of weight $(k - \frac{d}{2} - 1, c_{b+1}, -, c_d; c_1, -, c_b, k + \frac{d}{2} - 1)$.

Moreover, the L-function of this automorphic form is given by

$L^S(E(\pi, \chi, \phi), s) = L(\pi, s) L(\chi, s - k' - \frac{1}{2}) L(\chi^{-c}, s + k' + \frac{1}{2})$.

Therefore, the corresponding Galois repn (assuming Rep(a, b)) is given by $R_p(\pi)(-1) \oplus \chi_p^c \varepsilon^{-k' - \frac{d}{2}} \oplus \chi_p^{-1} \varepsilon^{k' - d/2 - 1} = R_p(E(\pi, \chi, \phi))$

We choose $\phi$ so that $E(\pi, \chi, \phi)$ is "ordinary at p". "$R_p(E(\pi, \chi))$"

(remark: $\pi_p$ ordinary $\Rightarrow$ $R_p(\pi)$ ordinary)

$\leadsto$ we study the Eisenstein ideal for this ES.

We deform the repn $R(E(\pi, \chi))$ $\leadsto$ Eisenstein ideal, and we can try to perform the strategy used for the Iwasawa main conjecture.

$\leadsto$ lattice construction mod Eisenstein ideal

$$\begin{pmatrix} \chi_p^{-1} \varepsilon^{k - \frac{d}{2} - 1} & C & D \\ B & R_p(\pi)(-1) & E \\ 0 & 0 & \chi_p^c \varepsilon^{-k' - \frac{d}{2}} \end{pmatrix}$$

(assuming Rep(a+1, b+1))

that allows us to construct two types of cocycles

$\mathrm{Sel}_K(\chi_p'^{-1} \varepsilon^{2k'-1})^{\overset{c}{\uparrow}} = \mathrm{Sel}_{\mathbb{Q}}(\chi_p' \varepsilon^{2k'-1})$

fixed part

$\mathrm{Sel}_K(R_p(\pi) \otimes \chi_p^{-c} \varepsilon^{-k' - \frac{d}{2} - 1})$

In the case $\mathrm{Sel}_{\mathbb{Q}}(\chi_p'^{-1} \varepsilon^{2k'-1}) = 0$ $\leadsto$ Eisenstein Ideal | $\# \mathrm{Sel}_K(R_p(\pi) \otimes \chi_p^{-1} \varepsilon^{-k' - d/2 - 1})$.

In the $U(2,2)$ situation, we assume $R(2,2)$.

$\mathbb{F}$ Hida family for $GL(2)$, $\mathbb{F} \in \mathbb{I}[[q]]$ w/ $\mathbb{I}/\mathbb{Z}_p[[W]]$

$\mathbb{F}$ integrally closed

$\uparrow$ weight var.

Urban IV 3

$\Gamma_K = \mathrm{Gal}(K_\infty/K)$  $K_\infty$ $\mathbb{Z}_p^2$-max. ext. of $K$
p splits in $K$

$\Lambda = \mathbb{I}[[\Gamma_K]]$

$* \rho_{\mathbb{F}} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{I}) \Leftarrow \bar{\rho}_{\mathbb{F}}$ is absolutely irred.

$Sel_{K_\infty}(\rho_{\mathbb{F}}) = Sel_K(\rho_{\mathbb{F}} \otimes_{\mathbb{I}} \Lambda^*)$  $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$

$\mathcal{J}_{\mathbb{F},K}(s_+, s_-) \in \mathbb{I}[[s_+, s_-]] \cong \Lambda$,  char power series of $X_{K_\infty}(\rho_{\mathbb{F}})$

Thm A  $\mathcal{L}^S_{\mathbb{F},K}((1+s_+)u^{-1}-1, s_-) \times \mathcal{L}^S_{\chi_{\mathbb{F}}^{-1}\omega}((1+s_+)^{-1}(1+w)^{-1}u^3-1) \mid$
Eisenstein ideal of the universal ordinary cuspidal Hecke algebras for $U(2,2) = \mathrm{Eis}$
(some technical conds. $\rightsquigarrow$ can apply thm of Vatsal)
$\chi_{\mathbb{F}}$ = nebentypus of the Hida family

Thm B  If $\mathcal{L}_{\chi_f^{-1}\omega}$ (under $\mathrm{Rep}(2,2)$) is a unit, then $\mathrm{Eis} \mid \mathcal{J}_{\mathbb{F},K}((1+s_+)u^{-1}-1, s_-)$
char. ideal of Selmer gp. attached to $\chi_f^{-1}\omega$

Cor  If $\mathbb{F}$ is the Hida family lifting the form of wt. 2 attached to an ordinary elliptic curve, then $\mathcal{L}^S_{\mathbb{F},K} \mid \mathcal{J}^S_{\mathbb{F},K}$
S-Selmer gp.

Case $\chi' = |\ |^{2k'}$

$R = R_p(\pi) \otimes \chi_p \varepsilon^{d/2-k'}$  $R^c \cong R^\vee(1)$

$L(R,s) = \varepsilon(R,s) L(R,-s)$

Theorem (Skinner-Urban) ① Assume $\mathrm{Rep}(a+1, b+1)$ (strong form).
Assume $L(R,0)=0$. Then rank $Sel_K(R^\vee(1)) \geq 1$
② Assume $R(a+2, b+2)$. If $L(R,s)$ vanishes at $s=0$ to even order, then rank $Sel_K(R^\vee(1)) \geq 2$.

$\mathcal{L}_E \cdot \mathcal{L}_{E(K/\mathbb{Q})} \mid \mathcal{J}_E \, \mathcal{J}_{E(K/\mathbb{Q})}$ need $K$ for $U(2,2)$

Kato: $\mathcal{J}_{E(K/\mathbb{Q})} \mid \mathcal{L}_{E(K/\mathbb{Q})}$, $\mathcal{J}_E \mid \mathcal{L}_E$.