A comparison of Poitou-Tate and Kummer maps

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Let p be a prime number. Let F be a number field, and let S denote a finite set of primes of F including those above p and any real infinite places. If $p = 2$, we assume that F is totally imaginary. We use $G_{F,S}$ to denote the Galois group of the maximal unramified outside S extension of F.

Let $\mathfrak{X}_F$ denote the maximal abelian pro-p quotient of $G_{F,S}$, which is quite easily identified with the Pontryagin dual $H^1(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p)^\vee$ of $H^1(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p)$. On the other hand, let Y_F denote he maximal unramified quotient of $\mathfrak{X}_F$ in which all primes above those in S split completely. There is a standard identification of Y_F with a subgroup of the continuous cohomology group $H^2_{\text{cts}}(G_{F,S}, \mathbf{Z}_p(1))$ that arises from Kummer theory, as we shall describe in the proof of the proposition below.

Now, Poitou-Tate duality provides us with a canonically defined homomorphism

$$H^1(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p)^\vee \to H^2_{\text{cts}}(G_{F,S}, \mathbf{Z}_p(1)),$$

and hence, via our identifications, a homomorphism $\mathfrak{X}_F \to Y_F$. The natural question to ask is whether or not this map is the restriction map on Galois groups, and the answer is that in fact it is. However, as we have been unable to find a proof of this non-obivous but useful fact in the literature, we provide one here.

Proposition 0.1. *The Poitou-Tate map* $H^1(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p)^\vee \to H^2_{\text{cts}}(G_{F,S}, \mathbf{Z}_p(1))$ *induces the canonical restriction* $\mathfrak{X}_F \to Y_F$ *on Galois groups.*

Proof. Let us begin with notation. Let F_S denote the Galois group of the maximal unramified outside S-extension of F, and let $\mathcal{O}_S$ denote its ring of S-integers. Moreover, as in [NSW], let I_S and C_S be the S-idèle group and S-idèle class group of F_S, respectively, and let $I_S(F)$ and $C_S(F)$ denote the S-idéle group and S-idéle class group of F, respectively. Finally, let $\mathrm{Cl}_{F,S}$ denote the S-class group of F.

We first recall the definition of the Poitou-Tate map

$$H^1(G_{F,S}, \mathbf{Q}_p/\mathbf{Z}_p)^\vee \to H^2_{\text{cts}}(G_{F,S}, \mathbf{Z}_p(1)).$$

We find it most convenient to work modulo p^n throughout and then take inverse limits. Modulo p^n, the Poitou-Tate map arises simply as the composition

$$H^1(G_{F,S}, \mathbf{Z}/p^n\mathbf{Z})^\vee \to H^0(G_{F,S}, \mathrm{Hom}(\mu_{p^n}, C_S))^\vee \to H^2(G_{F,S}, \mu_{p^n}),$$

where the first map is the dual of the connecting homomorphism arising from the sequence

$$0 \to \mu_{p^n} \to \mathrm{Hom}(\mu_{p^n}, I_S) \to \mathrm{Hom}(\mu_{p^n}, C_S) \to 0$$

and the second map arises from the duality

$$H^0(G_{F,S}, \mathrm{Hom}(\mu_{p^n}, C_S)) \times H^2(G_{F,S}, \mu_{p^n}) \xrightarrow{\cup} H^2(F, C_S) \xrightarrow{\text{inv}} \mathbf{Q}/\mathbf{Z},$$

where "inv" denotes the invariant map.

Next, we explain the injection $Y_F \hookrightarrow H^2_{\text{cts}}(G_{F,S}, \mathbf{Z}_p(1))$, again working modulo p^n. Recall that the reciprocity homomorphism

$$\mathrm{rec}\colon C_S(F)/p^n \to \mathfrak{X}_F/p^n$$

is dual to the connecting homomorphism

$$\delta\colon H^1(G_{F,S}, \mathbf{Z}/p^n\mathbf{Z}) \to H^2(G_{F,S}, \mathbf{Z})[p^n]$$

under the cup product

$$H^2(G_{F,S}, \mathbf{Z})[p^n] \times H^0(G_{F,S}, C_S)/p^n \xrightarrow{\cup} H^2(G_{F,S}, C_S)[p^n] \xrightarrow{\text{inv}} \mathbf{Z}/p^n\mathbf{Z}$$

in the sense that

$$\mathrm{inv}(\delta\phi \cup a) = \phi(\mathrm{rec}(a))$$

for $\phi \in H^1(G_{F,S}, \mathbf{Z}/p^n\mathbf{Z})$ and $a \in C_S(F)/p^n$.

From the long exact sequence attached to

$$0 \to \mathcal{O}_S^\times \to I_S \to C_S \to 0$$

and the fact that the cokernel of $I_S(F) \to C_S(F)$ is isomorphic to $\mathrm{Cl}_{F,S}$, we have an isomorphism $H^1(G_{F,S}, \mathcal{O}_S^\times) \cong \mathrm{Cl}_{F,S}$, and an induced reciprocity map

$$\mathrm{rec}\colon \mathrm{Cl}_{F,S}/p^n \to Y_F/p^n.$$

From the long exact sequence attached to

$$0 \to \mu_{p^n} \to \mathcal{O}_S^\times \xrightarrow{p^n} \mathcal{O}_S^\times \to 0,$$

we obtain an injection that is the composite

$$Y_F/p^n \xrightarrow{\mathrm{rec}^{-1}} \mathrm{Cl}_{F,S}/p^n \xrightarrow{\sim} H^1(G_{F,S}, \mathcal{O}_S^\times)/p^n \hookrightarrow H^2(G_{F,S}, \mu_{p^n}).$$

This gives the identification of Y_F/p^n with a subgroup of $H^2(G_{F,S}, \mu_{p^n})$ arising from Kummer theory and class field theory.

Putting all of the definitions together, the proposition is reduced to the commutativity of the diagram

$$\begin{array}{ccccc}
H^0(G_{F,S}, \mathrm{Hom}(\mu_{p^n}, C_S)) & \times & H^2(G_{F,S}, \mu_{p^n}) & \xrightarrow{\cup} & \mathbf{Q}/\mathbf{Z} \\
\downarrow & & \uparrow & & \| \\
H^1(G_{F,S}, \mathbf{Z}/p^n\mathbf{Z}) & \times & H^1(G_{F,S}, \mathcal{O}_S^\times) & & \| \\
\downarrow & & \uparrow & & \| \\
H^2(G_{F,S}, \mathbf{Z}) & \times & H^0(G_{F,S}, C_S) & \xrightarrow{\cup} & \mathbf{Q}/\mathbf{Z}
\end{array}$$

in the obvious sense. This diagram can be found (without proof of its commutativity) in [NSW, (8.4.6)], though not in the second edition of the book.

To show its commutativity, we replace the right-hand composition with the composition

$$H^0(G_{F,S}, C_S) \to H^1(G_{F,S}, C_S[p^n]) \to H^2(G_{F,S}, \mu_{p^n})$$

which is in fact its negative by a standard lemma (e.g., [NSW, (1.3.4)]), since we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_{p^n} & \longrightarrow & I_S[p^n] & \longrightarrow & C_S[p^n] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_S^\times & \longrightarrow & I_S & \longrightarrow & C_S & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & & \\
0 & \longrightarrow & \mathcal{O}_S^\times & \longrightarrow & I_S & \longrightarrow & C_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

noting the p-divisibility of C_S [NSW, (10.9.5)]. We therefore have a new diagram

$$\begin{array}{ccccc}
H^0(G_{F,S}, \mathrm{Hom}(\mu_{p^n}, C_S)) & \times & H^2(G_{F,S}, \mu_{p^n}) & \xrightarrow{\cup} & \mathbf{Q}/\mathbf{Z} \\
\downarrow & & \uparrow & & \uparrow{\scriptstyle -1} \\
H^1(G_{F,S}, \mathbf{Z}/p^n\mathbf{Z}) & \times & H^1(G_{F,S}, C_S[p^n]) & \xrightarrow{\cup} & \mathbf{Q}/\mathbf{Z} \\
\downarrow & & \uparrow & & \| \\
H^2(G_{F,S}, \mathbf{Z}) & \times & H^0(G_{F,S}, C_S) & \xrightarrow{\cup} & \mathbf{Q}/\mathbf{Z}
\end{array}$$

which commutes by two applications of [La, Theorem III.2.1]. That is, for the lower rectangle, take the pairing $\mathbf{Z} \times C_S \to C_S$, and for the upper, take the Galois-equivariant pairing

$$I_S[p^n] \times \mathrm{Hom}(\mu_{p^n}, I_S[p^n]) \to C_S[p^n]$$

given by multiplication on $I_S[p^n]$ followed by projection, noting that

$$I_S[p^n] = \varinjlim_{E \subset F_S} \bigoplus_{v \in S_E} \mu_{p^n}(E_v).$$

The result follows. $\square$

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References

[La] S. Lang, Topics in the cohomology of groups, Lecture Notes in Mathematics, vol. 1625, Springer, Berlin, 1996.

[NSW] J. Neurkich, A. Schmidt, K. Wingberg, Cohomology of Number Fields, Springer, Berlin, 2000.