Homological Algebra

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Introduction

These notes provide an introduction to homological algebra and the category theory that underpins its modern structure. Central to homological algebra is the notion of a chain complex, which is to say a collection of abelian groups C_i, with C_i said to be "in degree i", and differentials $d_i \colon C_i \to C_{i-1}$ for $i \in \mathbb{Z}$, with the property that the composition of two consecutive differentials is trivial, i.e., $d_{i-1} \circ d_i = 0$. We may visualize this as

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_2 \to \cdots .$$

For $j \in \mathbb{Z}$, the jth homology group of the complex $C. = (C_i, d_i)$ is the abelian group

$$H_j(C) = \frac{\ker d_j}{\operatorname{im} d_{j+1}}.$$

We do not feel it hyperbole to say that the introduction of these simple definitions led to a revolution in how much of mathematics is done.

REMARKS 0.0.1.

a. Often, it is more convenient to have the differentials increase, rather than decrease, the degree by 1. In this case, we use subscripts instead of superscripts and obtain what are known as cochain complexes and cohomology groups.

b. In many cases of interest, chain and cochain complexes are trivial in negative degree or sufficiently large positive degree, and are taken to be 0 without explanation.

c. More generally, chain complexes of abelian groups can be replaced by chain complexes of objects in an abelian category, which is a sort of category modeled on that of abelian groups. As well shall later discuss in depth, a category is a collection of objects and morphisms between them, such as abelian groups and homomorphisms.

The notion of homology first arose in topology. In a sense, the homology of a manifold had long been studied through invariants, prior to the introduction of homology groups in the mid-1920s. For instance, take the Betti number, introduced in 1871, but now defined as the rank of the first homology group of a manifold. Or take the work of Poincaré in 1899, in which he introduced a duality theorem that we now view as an isomorphism between homology and cohomology groups of closed oriented manifolds. After a suggestion of Emmy Noether in 1925, mathematicians began to study the homology groups themselves, and the names of the mathematicians who embarked upon this study in the ensuing decade now decorate some of the most fundamental theorems in algebraic topology: Hopf, Mayer, Vietoris, Alexander, Alexandroff, Čech, Lefschetz, and so on. Many different chain complexes were developed with homology groups that are isomorphic to what we now consider the homology of a topological space (with $\mathbb{Z}$-coefficients). These provide interesting topological information: for instance, the 0th homology group of a manifold is the number of its connected components and the first homology group is the abelianization of its fundamental group.

EXAMPLE 0.0.2. We imagine the sphere as divided into two closed disks b_1 and b_2, connected along the equator. Each of these disks is given an orientation such that the orientations of their boundary circles are opposite to each other. We next cover the circle with two 1-disks, or edges e_1 and e_2, and again we give each an orientation agreeing with the orientation arising from b_1. Finally, we have two vertices v_1 and v_2 that are the boundary of the edges. We think of each disk, edge, and vertex as contributing one generator to a free abelian group in degree the dimension that yields a complex C:

$$0 \to \mathbb{Z}b_1 \oplus \mathbb{Z}b_2 \xrightarrow{d_1} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \xrightarrow{d_0} \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \to 0.$$

The boundary maps are given by the orientations of the boundaries of the corresponding generators. That is, we have

$$d_1(b_1) = e_1 - e_2 = -d_1(b_2) \quad \text{and} \quad d_0(e_1) = v_1 - v_2 = -d_0(e_2).$$

One computes easily that $H_0(C) = \mathbb{Z}$, $H_1(C) = 0$, and $H_2(C) = \mathbb{Z}$, and these are the homology groups of the sphere.

Homological algebra quickly passed from algebraic topology to the realm of differential topology through the work of de Rham in 1931, who introduced a cohomology theory computed by a complex of differential forms. Group cohomology, again long-studied in low degrees, was introduced in its full generality by Eilenberg and MacLane in the early 1940s. We relate its combinatorial definition.

EXAMPLE 0.0.3. For $i \geq 0$, the ith (homogeneous) cochain group $C^i(G, A)$ of a group G with coefficients in a module A over its group ring $\mathbb{Z}[G]$ is the set of maps $F \colon G^{i+1} \to A$ such that

$$h \cdot F(g_0, \dots, g_i) = F(hg_0, \cdots, hg_i)$$

for $h \in G$, and the ith differential applied to F satisfies

$$d^i(F)(g_0, \dots, g_{i+1}) = \sum_{j=0}^{i+1} (-1)^j F(g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_{i+1}).$$

The 0th cohomology group of this complex is isomorphic to the set of elements of A fixed by G, and if the action of G on A is trivial, then $H^1(G, A)$ is isomorphic to the group of homomorphisms from G to A.

Group cohomology quickly found its application in field theory in the form of Galois cohomology, and through that in algebraic number theory. The main theorems of class field theory were reworked in terms of Galois cohomology by Artin and Tate in the early 1950s. Cohomology also became a crucial tool in algebraic geometry and commutative algebra through the work of many preeminent mathematicians, such as Zariski, Serre, and Grothendieck.

In the mid-1950s, a book of Cartan and Eilenberg advanced the field of homological algebra, and set it in line with category theory, with their introduction of derived functors through the use of projective and injective resolutions. The work of Leray had already introduced spectral sequences, which could be used to relate compositions of two derived functors to a third derived functor. In the late 1950s and 1960s, Grothendieck introduced abelian categories, and his more general viewpoint cemented category theory as the the foundation of homological algebra. Grothendieck also introduced the notion of a derived category, in which the chain complexes themselves, rather than the cohomology groups, play the central role.

Today, techniques in homological algebra continue to be developed and refined. Homological algebra stands an essential tool for mathematicians working in most any area of algebra, geometry, or topology. It would be a far-too-arduous and not necessarily enlightening task for this author to enumerate all of the various homology and cohomology theories in use at this time.

These notes will focus on the abstract foundation of homological algebra. We begin with the basics of category theory and abelian categories. We then develop the basic notions of chain complexes, injective and projective resolutions, and derived functors, following that with a treatment of spectral sequences. Finally, we will turn to derived categories. We hope to provide plenty of examples throughout, though we do not expect to develop specific cohomology theories in great depth.

REMARK 0.0.4. The current version is missing important topics in several places. It is also unproofread and so likely far from error-free.

CHAPTER 1

Category Theory

1.1. Categories

The extremely broad concept of a "category" allows us to deal with many of the constructions in mathematics in an abstract context. We begin with the definition. We will mostly ignore set-theoretical considerations that can be used to put what follows on a firmer basis, but note that a *class* is a collection of objects that can be larger than a set, e.g., the class of all sets, in order that we might avoid Russell's paradox.

DEFINITION 1.1.1. A *category* $\mathscr{C}$ is

(1) a class of *objects* $\mathrm{Obj}(\mathscr{C})$,

(2) for every $A, B \in \mathrm{Obj}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of *morphisms* from A to B, where we often use the notation $f: A \to B$ to indicate that f is an element of $\mathrm{Hom}_{\mathscr{C}}(A, B)$, and

(3) a map
$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$$
for each $A, B, C \in \mathrm{Obj}(\mathscr{C})$, called *composition*, where, for $f: A \to B$ and $g: B \to C$, their composition is denoted $g \circ f$,

such that

i. for each $A \in \mathrm{Obj}(\mathscr{C})$, there exists an *identity morphism* $\mathrm{id}_A: A \to A$ such that, for all $f: A \to B$ and $g: B \to A$ with $B \in \mathrm{Obj}(\mathscr{C})$, we have
$$f \circ \mathrm{id}_A = f \quad \text{and} \quad \mathrm{id}_A \circ g = g,$$
and

ii. composition is *associative*, i.e.,
$$h \circ (g \circ f) = (h \circ g) \circ f$$
for any three morphisms $h: C \to D$, $g: B \to C$, and $f: A \to B$ between objects $A, B, C, D \in \mathrm{Obj}(\mathscr{C})$.

DEFINITION 1.1.2. We say that a category is *small* if its objects form a set.

Categories are plentiful.

EXAMPLES 1.1.3.

a. The category **Sets** which has sets as its objects and maps of sets as its morphisms.

b. The category **Gps** which has groups as its objects and group homomorphisms as it morphisms.

c. Similarly, we have categories **Rings**, the objects of which we take to be the (possibly zero) rings with 1 and with morphisms the ring homomorphisms that preserve 1, and **Fields**.

d. If R is a ring, then the category R-**mod** has objects the left R-modules and morphisms the left R-module homomorphisms.

e. The category **Top** which has topological spaces as its objects and continuous maps as its morphisms.

DEFINITION 1.1.4. A *directed graph* $\mathscr{G}$ is a collection consisting of

(1) a set $V_{\mathscr{G}}$ of *vertices* of $\mathscr{G}$ and,

(2) for every $v, w \in V_{\mathscr{G}}$, a set $E_{\mathscr{G}}(v,w)$ of *edges* from v to w in $\mathscr{G}$.

TERMINOLOGY 1.1.5. In category theory, we often refer to the vertices of a directed graph as *dots* and the edges as *arrows*.

EXAMPLE 1.1.6. The following picture provides the data of a directed graph with 4 vertices and edge sets with between 0 and 2 elements each:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[r, bend left] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[u, bend right] \end{tikzcd}$$

DEFINITION 1.1.7. The *category (freely) generated by a directed graph* $\mathscr{G}$ is the category I with $\mathrm{Obj}(I) = V_{\mathscr{G}}$ and, for $v, w \in \mathrm{Obj}(I)$, with $\mathrm{Hom}_I(v,w)$ equal to the set of all words $e_n e_{n-1} \cdots e_1$ for some $n \geq 0$ (with $n = 0$ providing the empty word) with $e_i \in E_{\mathscr{G}}(v_{i-1}, v_i)$ for $v_i \in V_{\mathscr{G}}$ for $1 \leq i \leq n$, with $v_0 = v$ and $v_n = w$, together with the composition given by concatenation of words.

EXAMPLE 1.1.8. Consider the directed graph $\mathscr{G}$ given by

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3.$$

The category I generated by $\mathscr{G}$ has three objects v_1, v_2, v_3 and morphism sets

$$\begin{aligned} &\mathrm{Hom}_I(v_i,v_i) = \{\mathrm{id}_{v_i}\}, && \mathrm{Hom}_I(v_i,v_{i+1}) = \{e_i\}, \\ &\mathrm{Hom}_I(v_1,v_3) = \{e_2e_1\}, \quad \text{and} && \mathrm{Hom}_I(v_i,v_j) = \varnothing \text{ if } j < i. \end{aligned}$$

EXAMPLE 1.1.9. Consider the directed graph $\mathscr{G}$ given by

$$\begin{tikzcd} v_1 \arrow[r, bend left, "e_1"] & v_2 \arrow[l, bend left, "e_2"] \end{tikzcd}.$$

Let I be the category generated $\mathscr{G}$. For $i, j \in \{1,2\}$ the set $\mathrm{Hom}_I(v_i,v_j)$ consists of the words with alternating letters e_1 and e_2 that start with e_i and end with e_j (including the empty word if $i = j$).

We may construct new categories out of old. The following provides a useful example.

DEFINITION 1.1.10. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The *product category* $\mathscr{C} \times \mathscr{D}$ is the category with objects the pairs (C, D) with $C \in \mathrm{Obj}(\mathscr{C})$ and $D \in \mathrm{Obj}(\mathscr{D})$ and morphisms $(f, g)\colon (C, D) \to (C', D')$ for any $f\colon C \to C'$ in $\mathscr{C}$ and $g\colon D \to D'$ in $\mathscr{D}$.

DEFINITION 1.1.11. Given a category $\mathscr{C}$, we define the *opposite category* $\mathscr{C}^{\mathrm{op}}$ to have the same class of objects as $\mathscr{C}$ and

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(B, A)$$

for $A, B \in \mathrm{Obj}(\mathscr{C})$.

DEFINITION 1.1.12. Given a category $\mathscr{C}$, its *morphism category* $\mathrm{Mor}(\mathscr{C})$ is the category with objects the morphisms in $\mathscr{C}$ and morphisms $f \to g$ for $f\colon A \to B$ and $g\colon A' \to B'$ in $\mathscr{C}$ the pairs (α, β) of morphisms $\alpha\colon A \to A'$ and $\beta\colon B \to B'$ in $\mathscr{C}$ with $\beta \circ f = g \circ \alpha$.

EXAMPLE 1.1.13. Let G be a monoid, which is to set with an associative binary operation and a two-sided identity element with respect to the operation. Then G is a category with one object, morphisms equal to its elements, and composition law given by multiplication. Then G^{op} is again a group with the same elements but the multiplication reversed. A category with one object is also called a *monoid*, and we have a one-to-one correspondence between monoids and these categories.

We will often have cause to single out a particular class of morphisms in a category known as isomorphisms.

DEFINITION 1.1.14. Let $\mathscr{C}$ be a category.

a. A morphism $f\colon A \to B$ in $\mathscr{C}$ is an *isomorphism* if there exists morphism $g\colon B \to A$ in $\mathscr{C}$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

b. Two objects A and B in $\mathscr{C}$ are said to be *isomorphic* if there exists an isomorphism $f\colon A \to B$ in $\mathscr{C}$.

c. If $f\colon A \to B$ is a morphism and $g \circ f = \mathrm{id}_A$ (resp., $f \circ g = \mathrm{id}_B$), then we say that g is a *right inverse* (resp., a *left inverse*) to f. If both $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, then we say that g is (an) *inverse* to f, or that f and g are *inverse to each other* (or *mutually inverse*, or *inverses*).

EXAMPLES 1.1.15.

a. The isomorphisms in **Sets** are the bijections.

b. The isomorphisms in **Gps** are the isomorphisms of groups.

c. The isomorphisms in Top are the homeomorphisms.

d. Every morphism in a monoid is an isomorphism.

DEFINITION 1.1.16.

a. A morphism $f\colon A \to B$ in a category $\mathscr{C}$ is a *monomorphism* if for any $g, h\colon C \to A$ with $C \in \mathrm{Obj}(\mathscr{C})$, the property that $f \circ g = f \circ h$ implies $g = h$.

b. A morphism $f\colon A \to B$ in a category $\mathscr{C}$ is an *epimorphism* if for any $g, h\colon B \to C$ with $C \in \mathrm{Obj}(\mathscr{C})$, the property that $g \circ f = h \circ f$ implies $g = h$.

EXAMPLES 1.1.17.

a. In **Sets** and R-**mod**, a morphism is a monomorphism (resp., epimorphism) if and only if it is injective (resp., surjective).

b. The natural injection $\mathbb{Z} \to \mathbb{Q}$ in **Rings** is an epimorphism, since a ring homomorphism $\mathbb{Q} \to R$ is completely determined by its value on 1.

REMARK 1.1.18. A morphism $f: A \to B$ in a category $\mathscr{C}$ is a monomorphism if and only if the opposite morphism $f^{\mathrm{op}}: B \to A$ in $\mathscr{C}^{\mathrm{op}}$ is an epimorphism.

We have the following.

LEMMA 1.1.19. *Let $f: A \to B$ and $g: B \to A$ be morphisms in a category $\mathscr{C}$ such that $g \circ f = \mathrm{id}_A$. Then f is a monomorphism and g is an epimorphism.*

PROOF. Let $h, k: C \to A$ be morphisms such that $f \circ h = f \circ k$. Then

$$k = g \circ f \circ k = g \circ f \circ h = h.$$

Thus f is a monomorphism. Similarly, g is an epimorphism, or apply Remark 1.1.18. □

In other words, right inverses are monomorphisms and left inverses are epimorphisms.

DEFINITION 1.1.20. Let $\mathscr{C}$ be a category and $C \in \mathrm{Obj}(\mathscr{C})$.

a. A *subobject* of C is a pair (A, ι) consisting of an object A and a monomorphism $\iota: A \to C$.

b. A *quotient* of C is a pair (B, π) consisting of an object B and an epimorphism $\pi: C \to B$.

1.2. Functors and natural transformations

To compare two categories, we need some notion of a map between them. Such maps are referred to as functors. There are two basic types.

DEFINITION 1.2.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories.

a. A *covariant functor* (or simply *functor*) $F: \mathscr{C} \to \mathscr{D}$ between two categories $\mathscr{C}$ and $\mathscr{D}$ is a map of objects $F: \mathrm{Obj}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{D})$ and a map of morphisms

$$F: \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

for each $A, B \in \mathrm{Obj}(\mathscr{C})$ such that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(g) \circ F(f)$ for all $f: A \to B$ and $g: B \to C$ for each $A, B, C \in \mathrm{Obj}(\mathscr{C})$.

b. As with a covariant functor, a *contravariant functor* $F: \mathscr{C} \to \mathscr{D}$ is again a map on objects, but with maps between sets of morphisms of the form

$$F: \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(B), F(A))$$

that satisfies $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(g \circ f) = F(f) \circ F(g)$.

We give some examples of functors.

EXAMPLES 1.2.2.

a. We have the *forgetful functors* **Gps** $\to$ **Sets**, **Rings** $\to$ **Sets**, and **Top** $\to$ **Sets**, which take objects to their underlying sets and morphisms to the corresponding set-theoretic maps.

b. We have another forgetful functor from R-**mod** to the category **Ab** of abelian groups.

c. A homomorphism of monoids $G \to G'$ induces a functor of the corresponding categories, and conversely.

d. For any category $\mathscr{C}$, we have a contravariant functor op: $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ (as well as $\mathscr{C} \to \mathscr{C}^{\mathrm{op}}$) given by the identity on objects and the map on morphisms that takes $f : B \to A$ to $f^{\mathrm{op}} : A \to B$. The composition $F \circ \mathrm{op}$ for a contravariant functor F is a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

e. Given an object $A \in \mathrm{Obj}(\mathscr{C})$ for some category $\mathscr{C}$, we can define a functor $h_A : \mathscr{C} \to \mathrm{Sets}$ by

$$h_A(B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$$

and, for $g : B \to C$,

$$h_A(g)(f) = g \circ f$$

for all $f : A \to B$.

f. We have a contravariant functor $h^A : \mathscr{C} \to$ **Sets** with

$$h^A(B) = \mathrm{Hom}_{\mathscr{C}}(B, A) \quad \text{and} \quad h^A(g)(f) = f \circ g$$

for $B, C \in \mathrm{Obj}(\mathscr{C})$, $g : B \to C$, and $f : C \to A$.

DEFINITION 1.2.3. A *directed graph* $\mathscr{G}$ is a collection consisting of

(1) a set $V_{\mathscr{G}}$ of *vertices* of $\mathscr{G}$ and,

(2) for every $v, w \in V_{\mathscr{G}}$, a set $E_{\mathscr{G}}(v, w)$ of *edges* from v to w in $\mathscr{G}$.

TERMINOLOGY 1.2.4. In category theory, we often refer to the vertices of a directed graph as *dots* and the edges as *arrows.*

EXAMPLE 1.2.5. The following picture provides the data of a directed graph with 4 vertices and edge sets with between 0 and 2 elements each:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[r, bend left] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[u, bend right]
\end{tikzcd}$$

DEFINITION 1.2.6. A *diagram* in $\mathscr{C}$ is a functor from a category generated by a graph to $\mathscr{C}$.

REMARK 1.2.7. Let $\mathscr{G}$ be a directed graph, let I be the category generated by $\mathscr{G}$, and let $\mathscr{C}$ be a category. Given a map $F : V_{\mathscr{G}} \to \mathscr{C}$ and functions $F : E_{\mathscr{G}}(v, w) \to \mathrm{Hom}_{\mathscr{C}}(F(v), F(w))$ for each $v, w, \in V_{\mathscr{G}}$, there exists a unique functor $F : I \to \mathscr{C}$ that agrees with F on $V_{\mathscr{G}}$ and on $E_{\mathscr{G}}(v, w) \subseteq \mathrm{Hom}_I(v, w)$ for every $v, w \in V_{\mathscr{G}}$.

REMARK 1.2.8. Often, we consider finite graphs, in which every collection of vertices and edges is finite. The resulting diagrams are known as finite diagrams.

DEFINITION 1.2.9. A *commutative diagram* in $\mathscr{C}$ is a diagram $F: I \to \mathscr{C}$, where I is the category generated by a graph, which is a constant function on every set of morphisms.

EXAMPLE 1.2.10. To give a functor from I as in Example 1.1.8 to a category $\mathscr{C}$ is to proscribe three objects A, B, C in $\mathscr{C}$ and two morphisms $f: A \to B$ and $g: B \to C$. Thus, such a diagram may be represented by

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

and it is automatically commutative.

EXAMPLE 1.2.11. To give a functor from I as in Example 1.1.9 to a category $\mathscr{C}$ is to proscribe two objects A, B in $\mathscr{C}$ and two morphisms $f: A \to B$ and $g: B \to A$. The diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

is commutative if and only if $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

REMARK 1.2.12. A morphism $f \to g$ in $\mathrm{Mor}(\mathscr{C})$ for a category $\mathscr{C}$ may be thought of as a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}$$

in $\mathscr{C}$.

DEFINITION 1.2.13. A *subcategory* $\mathscr{C}$ of a category $\mathscr{D}$ is a category with objects consisting of a subclass of $\mathrm{Obj}(\mathscr{D})$ and morphisms $\mathrm{Hom}_{\mathscr{C}}(A, B)$ for $A, B \in \mathrm{Obj}(\mathscr{C})$ consisting of a subset of $\mathrm{Hom}_{\mathscr{D}}(A, B)$ containing id_A for $A = B$ and such that composition maps in $\mathscr{C}$ agree with the restriction of the composition maps in $\mathscr{D}$ between the same objects.

REMARK 1.2.14. A subcategory $\mathscr{C}$ of a category $\mathscr{D}$ is endowed with a canonical inclusion functor that takes an object of $\mathscr{C}$ to the same object of $\mathscr{D}$ and is the identity map on morphism.

DEFINITION 1.2.15. Let $F: \mathscr{C} \to \mathscr{D}$ be a functor.

a. The functor F is called *faithful* if it is one-to-one on morphisms.

b. The functor F is called *full* if it is onto on morphisms.

c. A functor F is *fully faithful* if it is both faithful and full.

d. A subcategory is called a *full subcategory* if it the corresponding inclusion functor is full.

REMARK 1.2.16. The inclusion functor attached to a subcategory is always faithful.

REMARK 1.2.17. A fully faithful functor is sometimes referred to as an embedding of categories, or sometimes a full embedding (and when so, a faithful but not necessarily full functor might instead be referred to as an embedding).

EXAMPLES 1.2.18.

a. The category **Ab** is a full subcategory of **Gps**.

b. The category **Fields** is a full subcategory of **Rings**.

c. The above-described forgetful functors to sets are faithful but not full.

d. The category in which the objects are sets but the morphisms are bijections of sets is a subcategory of **Sets** that has the same objects but is not full.

Clearly, a functor always takes isomorphisms to isomorphisms. Of course, a fully faithful functor F preserves notions of monomorphism, epimorphism, and isomorphism (in that f has one of these properties if and only if $F(f)$ has the same property). The reader may also quickly check the following.

LEMMA 1.2.19. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor, and let $f\colon A \to B$ be a morphism in $\mathscr{C}$. If F is faithful and $F(f)$ is a monomorphism (resp., epimorphism) then f is a monomorphism (resp., epimorphism).*

DEFINITION 1.2.20. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be two (covariant) functors. A *natural transformation* $\eta\colon F \rightsquigarrow G$ is a class of morphisms $\eta_A\colon F(A) \to G(A)$ for each $A \in \mathrm{Obj}(\mathscr{C})$ subject to the condition that

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

commutes for every $f\colon A \to B$ and $A, B \in \mathrm{Obj}(\mathscr{C})$. If instead F and G are contravariant functors, then the direction of the vertical arrows in the diagram are reversed.

REMARK 1.2.21. Given two categories $\mathscr{C}$ and $\mathscr{D}$ with $\mathscr{C}$ small and defining composition of natural transformations in the obvious way, we get a *functor category* $\mathbf{Func}(\mathscr{C}, \mathscr{D})$ with objects the functors $\mathscr{C} \to \mathscr{D}$ and morphisms the natural transformations between functors.

EXAMPLE 1.2.22. If we think of groups G and N as monoids, so that functors $G \to N$ are homomorphisms, then a natural transformation $f \rightsquigarrow f'$ between two homomorphisms $f, f'\colon G \to N$ is given simply by an element $x \in N$ such that $f'(g) = x f(g) x^{-1}$ for all $g \in G$.

DEFINITION 1.2.23. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be functors. A natural transformation $\eta\colon F \rightsquigarrow G$ is said to be a natural isomorphism if each η_A for $A \in \mathscr{C}$ is an isomorphism.

REMARK 1.2.24. Every natural isomorphism $\eta\colon F \rightsquigarrow G$ has an inverse $\eta^{-1}\colon G \rightsquigarrow F$ with $\eta_A^{-1} = (\eta_A)^{-1}$ for $A \in \mathrm{Obj}(\mathscr{C})$.

DEFINITION 1.2.25. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be functors, and let $\eta_A\colon F(A) \to G(A)$ be morphisms for each $A \in \mathrm{Obj}(\mathscr{C})$. We say that these morphisms are *natural* if the η_A form a natural transformation $\eta\colon F \to G$.

1.3. The Yoneda embedding

DEFINITION 1.3.1. Let $\mathscr{C}$ be a small category. The *Yoneda embedding* is the functor

$$h^{\mathscr{C}} : \mathscr{C} \to \mathbf{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$$

defined by $h^{\mathscr{C}}(A) = h^A$ for $A \in \mathrm{Obj}(\mathscr{C})$ and $h^{\mathscr{C}}(f)\colon h^A \rightsquigarrow h^B$ for $f\colon A \to B$ in $\mathscr{C}$ given by

$$h^{\mathscr{C}}(f)_C(g) = f \circ g$$

for each $g\colon C \to A$ in $\mathscr{C}$ and any $C \in \mathrm{Obj}(\mathscr{C})$.

REMARK 1.3.2. The reader should check that the Yoneda embedding is a well-defined functor.

THEOREM 1.3.3. *Let $\mathscr{C}$ be a small category. The Yoneda embedding $h^{\mathscr{C}}$ is fully faithful.*

PROOF. We first show faithfulness. Let $f, g\colon A \to B$ be two morphisms with $h^{\mathscr{C}}(f) = h^{\mathscr{C}}(g)$. Then

$$f = f \circ \mathrm{id}_A = h^{\mathscr{C}}(f)_A(\mathrm{id}_A) = h^{\mathscr{C}}(g)_A(\mathrm{id}_A) = g \circ \mathrm{id}_A = g.$$

As for fullness, suppose that $\eta\colon h^A \rightsquigarrow h^B$ for some $A, B \in \mathrm{Obj}(\mathscr{C})$. We claim that $\eta = h(e)$, where $e = \eta_A(\mathrm{id}_A)$. To see this, note that the fact that η is a natural transformation means, in particular, that the diagram

$$\begin{tikzcd} h^A(A) \arrow[r, "\eta_A"] \arrow[d, "h^A(f)"'] & h^B(A) \arrow[d, "h^B(f)"] \\ h^A(C) \arrow[r, "\eta_C"] & h^B(C) \end{tikzcd}$$

commutes for any $f\colon C \to A$. Applying both compositions to the identity morphism of A, we get the two equal terms

$$h^B(f) \circ \eta_A(\mathrm{id}_A) = h^B(f)(e) = f \circ e = h(e)_C(f)$$

and

$$\eta_C \circ h^A(f)(\mathrm{id}_A) = \eta_C(\mathrm{id}_A \circ f) = \eta_C(f),$$

and therefore, the desired equality. □

REMARK 1.3.4. Similarly, we have a fully faithful contravariant functor

$$h_{\mathscr{C}} : \mathscr{C} \to \mathbf{Func}(\mathscr{C}, \mathbf{Sets})$$

given by the h_A for $A \in \mathrm{Obj}(\mathscr{C})$ and natural transformations between them. This is just the Yoneda embedding for the category $\mathscr{C}^{\mathrm{op}}$.

Theorem 1.3.3 can be thought of as a more general version of the following standard theorem of group theory.

COROLLARY 1.3.5 (Cayley's theorem). *Every group G is isomorphic to a subgroup of the symmetric group S_G on G.*

PROOF. Consider the monoid $\mathbb{G}$ formed by G. Recall that in $\mathbb{G}$, morphisms are elements of G. As $h\colon \mathbb{G} \to \mathbf{Func}(\mathbb{G}^{\mathrm{op}}, \mathbf{Sets})$ is a functor, Yoneda's lemma provides an injective function

$$h\colon G \to \mathrm{Hom}_{\mathbf{Func}(\mathbb{G}^{\mathrm{op}},\mathbf{Sets})}(h^G, h^G)$$

on morphisms with the properties that $h(e) = \mathrm{id}_{h^G}$ and $h(xy) = h(x) \circ h(y)$ for $x, y \in G$. Since $\mathbb{G}$ has only the object G, and $h^G(G) = G$, this induces a one-to-one function $\rho\colon G \to \mathrm{Maps}(G,G)$ with $\rho(x) = h(x)_G$ and satisfying $\rho(xy) = \rho(x) \circ \rho(y)$ and $\rho(e) = \mathrm{id}_G$. In particular, we have $\rho(x^{-1}) \circ \rho(x) = \mathrm{id}_G$ for every $x \in G$, so its image lands in S_G, and the resulting map $G \to S_G$ is an injective homomorphism. □

We shall actually require the following strengthening of Theorem 1.3.3.

THEOREM 1.3.6 (Yoneda's lemma). *For any object A of a small category $\mathscr{C}$ and contravariant functor $F\colon \mathscr{C} \to \mathbf{Sets}$, there is a bijection*

$$\mathrm{Hom}_{\mathbf{Func}(\mathscr{C}^{\mathrm{op}},\mathbf{Sets})}(h^A, F) \xrightarrow{\sim} F(A)$$

given by $\eta \mapsto \eta_A(\mathrm{id}_A)$ that is natural in A and F.

PROOF. Let $B \in \mathrm{Obj}(\mathscr{C})$. Given $x \in F(B)$, consider the composition

$$\mathrm{Hom}_{\mathscr{C}}(B,A) \xrightarrow{F} \mathrm{Hom}_{\mathbf{Sets}}(F(B), F(A)) \xrightarrow{\mathrm{ev}_x} F(A),$$

where ev_x is evaluation at x. This defines a natural transformation $\xi^x\colon h^A \rightsquigarrow F$. If $\eta\colon h^A \rightsquigarrow F$ and $f\colon B \to A$, then

$$F(f) \circ \eta_A(\mathrm{id}_A) = \eta_B(\mathrm{id}_A \circ f) = \eta_B(f)$$

by the naturality of η. On the other hand, if $x \in F(A)$, then

$$\xi^x_A(\mathrm{id}_A) = \mathrm{ev}_x(F(\mathrm{id}_A)) = \mathrm{ev}_x(\mathrm{id}_{F(A)}) = x.$$

Hence the maps $\eta \mapsto \eta_A(\mathrm{id}_A)$ and $x \mapsto \xi^x$ are inverse to each other. □

1.4. Limits and colimits

In this section, $\mathscr{C}$ denotes a category, and I denotes a small category.

NOTATION 1.4.1. We write $i \in I$ to denote, more simply, that $i \in \mathrm{Obj}(I)$.

DEFINITION 1.4.2. Let $F\colon I \to \mathscr{C}$ be a functor. When it exists, the *limit* of F is a pair $(\lim F, (\phi_i)_{i \in I})$ consisting of an object $\lim F$ in $\mathscr{C}$ and morphisms

$$\phi_i\colon \lim F \to F(i)$$

for each $i \in I$ such that $\phi_j = F(\kappa) \circ \phi_i$ for all morphisms $\kappa\colon i \to j$ in I and with the *universal property* that if X is any object of $\mathscr{C}$ together with morphisms $\psi_i\colon X \to F(i)$ for which $\psi_j = \kappa \circ \psi_i$ for all morphisms $\kappa\colon i \to j$, then there exists a unique morphism $f\colon X \to \lim F$ such that $\psi_i = \phi_i \circ f$ for all $i \in I$.

NOTATION 1.4.3. We usually use $\lim F$ to refer more simply to a pair $(\lim F, (\phi_i)_{i \in I})$ that is a limit of $F\colon I \to \mathscr{C}$, with the maps understood.

REMARK 1.4.4. The universal property of the limit of a functor F as in Definition 1.4.2 may be visualized by commutative diagrams

$$\begin{tikzcd}
& X \arrow[d, "f"] \arrow[ddl, bend right, "\psi_i"'] \arrow[ddr, bend left, "\psi_j"] & \\
& \lim F \arrow[dl, "\phi_i"'] \arrow[dr, "\phi_j"] & \\
F(i) \arrow[rr, "F(\kappa)"] & & F(j).
\end{tikzcd}$$

LEMMA 1.4.5. *If* $(X, (\psi_i)_{i\in I})$ *and* $(\lim F, (\phi_i)_{i\in I})$ *are limits of a functor* $F\colon I \to \mathscr{C}$*, then there is a unique isomorphism* $f\colon X \to \lim F$ *such that* $\psi_i = \phi_i \circ f$ *for all* $i \in I$.

PROOF. There are morphisms $f\colon X \to \lim F$ and $g\colon \lim F \to X$ that are unique with the respective properties that $\psi_i = \phi_i \circ f$ and $\phi_i = \psi_i \circ g$ for all $i \in I$. Note that we have $\phi_i = \phi_i \circ f \circ g$ for all $i \in I$. On the other hand, the universal property of X implies that the identity id_X is the unique morphism h such that $\psi_i \circ h = \psi_i$ for all $i \in I$, so $f \circ g = \mathrm{id}_X$. Similarly, $g \circ f$ is the identity of $\lim F$ by its universal property. Therefore, the unique map f is an isomorphism. □

REMARK 1.4.6. Lemma 1.4.5 says that a limit, when it exists, is unique up to unique isomorphism (respecting the universal property) and for that reason, we refer to "the", rather than "a", limit.

If I has only identity morphisms, then the limit of a functor $F\colon I \to \mathscr{C}$ is determined entirely by the image objects $A_i = F(i)$ for all $i \in I$. Hence the notation in the following definition makes sense.

DEFINITION 1.4.7. Let I be a category with only identity morphisms, and let $F\colon I \to \mathscr{C}$ be a functor. Set $A_i = F(i)$ for each $i \in I$.

a. The limit $\prod_{i\in I} A_i$ of F, when it exists, is called the *product* of the A_i.

b. The maps
$$p_i\colon \prod_{i\in I} A_i \to A_i$$
resulting from the universal property of the product are known as *projection maps*.

EXAMPLES 1.4.8. The product coincides with direct product in the categories **Sets**, **Gps**, **Top**, **Rings** and R-**mod**. Products of more than one object do not exist in the category **Fields**.

REMARK 1.4.9. Given a commutative diagram in a category $\mathscr{C}$, it arises by definition from a functor $F\colon I \to \mathscr{C}$, where I is a category generated by a directed graph. Therefore, we may speak of the limit of the diagram.

DEFINITION 1.4.10. The limit $A_1 \times_B A_2$ of a diagram

(1.4.1)
$$\begin{tikzcd}
& A_1 \arrow[d, "f_1"] \\
A_2 \arrow[r, "f_2"] & B
\end{tikzcd}$$

in $\mathscr{C}$, when it exists, is called the *pullback* of the diagram.

REMARK 1.4.11. The pullback of (1.4.1) is endowed with morphisms p_1 and p_2 that make

$$\begin{tikzcd} A_1 \times_B A_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & B \end{tikzcd}$$

commute.

EXAMPLE 1.4.12. In **Sets**, **Gps**, **Top**, and R-**mod**, the pullback is the subobject (i.e., subset, subgroup, subspace, or submodule) with underlying set

$$\{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\}.$$

We also have the dual notion to limits:

DEFINITION 1.4.13. Let $F\colon I \to \mathscr{C}$ be a functor. When it exists, the *colimit* of F is a pair $(\operatorname{colim} F, (\alpha_i)_{i \in I})$ consisting of an object $\operatorname{colim} F \in \operatorname{Obj}(\mathscr{C})$ together with morphisms

$$\alpha_i\colon F(i) \to \operatorname{colim} F$$

for each $i \in I$ such that $\alpha_j \circ F(\kappa) = \alpha_i$ for all morphisms $\kappa\colon i \to j$ and with the *universal property* that if X is any object of $\mathscr{C}$ together with morphisms $\beta_i\colon X \to F(i)$ for which $\psi_j \circ \kappa = \psi_i$ for all morphisms $\kappa\colon i \to j$, then there exists a unique morphism $f\colon \operatorname{colim} F \to X$ such that $\beta_i = f \circ \alpha_i$ for all $i \in I$.

NOTATION 1.4.14. An colimit of a functor $F\colon I \to \mathscr{C}$ is usually denoted simply by the object $\operatorname{colim} F$, with the morphisms omitted.

REMARK 1.4.15. The properties of the colimit expressed in Definition 1.4.13 may be summarized by the commutativity of the diagrams

$$\begin{tikzcd} F(i) \arrow[rr, "F(\kappa)"] \arrow[dr, "\alpha_i"] \arrow[ddr, bend right, "\beta_i"'] & & F(j) \arrow[dl, "\alpha_j"'] \arrow[ddl, bend left, "\beta_j"] \\ & \operatorname{colim} F \arrow[d, "f"] & \\ & X & \end{tikzcd}$$

for all $\kappa\colon i \to j$ in I.

We have the obvious analogue of Lemma 1.4.5, which again tells us that we may speak of "the" colimit.

LEMMA 1.4.16. *If* $(X, (\beta_i)_{i \in I})$ *and* $(\operatorname{colim} F, (\alpha_i)_{i \in I})$ *are colimits of a functor* $F\colon I \to \mathscr{C}$, *then there is a unique isomorphism* $f\colon \operatorname{colim} F \to X$ *such that* $\alpha_i = f \circ \beta_i$ *for all* $i \in I$.

REMARK 1.4.17. When it exists, the colimit of $F\colon I \to \mathscr{C}$ in $\mathscr{C}$ satisfies

$$\operatorname{colim} F = \mathrm{op} \circ \lim(\mathrm{op} \circ F),$$

so its underlying object is an limit in $\mathscr{C}^{\mathrm{op}}$.

DEFINITION 1.4.18. The colimit of a functor $F\colon I \to \mathscr{C}$ from a category I with only identity morphisms is called a *coproduct*, and it is denoted $\coprod_{i\in I} F(i)$.

EXAMPLES 1.4.19.

a. The coproduct in **Sets** and **Top** of two objects X_1 and X_2 is the disjoint union $X_1 \amalg X_2$.

b. The coproduct in **Gps** of two groups G_1 and G_2 is the free product $G_1 * G_2$.

c. The coproduct in R-**mod** (and in particular **Ab**) of two R-modules A_1 and A_2 is the direct sum $A_1 \oplus A_2$.

d. The coproduct in the category **CRings** of commutative rings R_1 and R_2 is the tensor product $R_1 \otimes R_2$.

REMARK 1.4.20. Examples 1.4.19(a-d) generalize directly to arbitrary collections of objects.

REMARK 1.4.21. Much as with limits, we may speak of a colimit of a diagram in a category.

DEFINITION 1.4.22. The colimit of a diagram

(1.4.2)
$$\begin{tikzcd} B \arrow[r, "g_1"] \arrow[d, "g_2"] & A_1 \\ A_2, & \end{tikzcd}$$

in $\mathscr{C}$ is called the *pushout* $A_1 \amalg_B A_2$.

REMARK 1.4.23. The pushout of the diagram (1.4.2) fits into a diagram

$$\begin{tikzcd} B \arrow[r, "g_1"] \arrow[d, "g_2"] & A_1 \arrow[d, "\iota_1"] \\ A_2 \arrow[r, "\iota_2"] & A_1 \amalg_B A_2, \end{tikzcd}$$

where ι_1 and ι_2 are induced by the universal property of the colimit.

EXAMPLE 1.4.24. In **Sets** and **Top**, the pushout is the quotient (set or topological space) of the disjoint union of A_1 and A_2 under the equivalence relation identifying $g_1(b)$ with $g_2(b)$ for all $b \in B$.

DEFINITION 1.4.25. We say that a category $\mathscr{C}$ *admits* the limit (resp., colimit) of a functor $F\colon I \to \mathscr{C}$ if the limit (resp., colimit) exists in $\mathscr{C}$.

REMARK 1.4.26. More generally, we may speak of $\mathscr{C}$ *admitting* the limits (or colimits) of any collection of functors from small categories to $\mathscr{C}$.

DEFINITION 1.4.27. A category is called *complete* if it admits all limits.

EXAMPLE 1.4.28. The category of finite sets is not complete.

DEFINITION 1.4.29. A category is called *cocomplete* if it admits all colimits.

REMARK 1.4.30. To say that $\mathscr{C}$ is complete is to say that $\mathscr{C}^{\mathrm{op}}$ is cocomplete.

PROPOSITION 1.4.31. *The category* **Sets** *is both complete and cocomplete.*

PROOF. Let $F\colon I \to \mathbf{Sets}$ be a functor. We merely describe the limit and colimit of F and leave the rest to the reader. The limit is

$$\lim F = \left\{ (a_i)_i \in \prod_{i \in I} F(i) \mid F(\phi)(a_i) = a_j \text{ if } \phi\colon i \to j \text{ in } I \right\},$$

and the colimit is

$$\operatorname{colim} F = \amalg_{i \in I} F(i)/\sim$$

where $\sim$ is the minimal equivalence relation satisfying $a_i \sim a_j$ for $a_i \in F(i)$ and $a_j \in F(j)$ if there exists $\phi\colon i \to j$ with $F(\phi)(a_i) = a_j$. $\square$

The reader will easily verify the following.

PROPOSITION 1.4.32. *Let I be a small category and $\mathscr{C}$ a (co)complete category. Then the category* $\mathbf{Func}(I, \mathscr{C})$ *is (co)complete.*

COROLLARY 1.4.33. *Let $F\colon I \to \mathscr{C}$ be a functor, and supposing that $\mathscr{C}$ is small, consider the Yoneda embedding $h^{\mathscr{C}}\colon \mathscr{C} \to$* $\mathbf{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$. *Then $h^{\mathscr{C}} \circ F$ has a limit and colimit in* $\mathbf{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$.

We explain some restricted and successively more general notions of limit and colimit.

DEFINITION 1.4.34.

a. An *initial object* A in $\mathscr{C}$ is an object such that for each $B \in \mathrm{Obj}(\mathscr{C})$, there is a unique morphism $A \to B$ in $\mathscr{C}$.

b. A *terminal object* X in $\mathscr{C}$ is an object such that for each $B \in \mathrm{Obj}(\mathscr{C})$, there is a unique morphism $B \to X$ in $\mathscr{C}$.

c. An object which is both initial and terminal is called a *zero object*.

REMARKS 1.4.35.

a. Consider the empty functor $F\colon \varnothing \to \mathscr{C}$. Then $\lim F$ is a terminal object, and $\operatorname{colim} F$ is an initial object, if they exist.

b. Terminal and initial objects are unique up to unique isomorphism when defined.

We provide some examples.

EXAMPLES 1.4.36.

a. The empty set $\varnothing$ is the initial object in the category **Sets**, while any set with one element is a terminal object.

b. The trivial group is a zero object in the category **Gps**.

c. The zero ring is a terminal object and $\mathbb{Z}$ is an initial object in **Rings**.

We omit the proof of the following easy lemma.

LEMMA 1.4.37. *Let I be a small category and $F\colon I \to \mathscr{C}$ a functor.*

a. Suppose that I has an initial object i. Then

$$\lim F = F(i).$$

b. Suppose that I has a terminal object j. Then

$$\operatorname{colim} F = F(j).$$

DEFINITION 1.4.38.

a. The limit of a diagram

$$\cdots \to A_3 \to A_2 \to A_1$$

in a category $\mathscr{C}$ is referred to as the *sequential limit* of the objects A_i.

b. The colimit of a diagram

$$A_1 \to A_2 \to A_3 \to \cdots$$

in a category $\mathscr{C}$ is referred to as the *sequential colimit* of the objects A_i.

EXAMPLE 1.4.39. In **Ab**, the sequential limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$ with respect to homomorphisms $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ given by reduction modulo p^n is the group $\mathbb{Z}_p$ of p-adic integers. The sequential colimit of these same groups with respect to the maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z}$ induced by multiplication modulo p is the group $\mathbb{Q}_p/\mathbb{Z}_p$, equal to the p-power torsion in $\mathbb{Q}/\mathbb{Z}$.

The sequential limit (resp., sequential colimit) is just a special case of the notion of an inverse limit (resp., direct limit), which is a more usual terminology.

Recall that a directed set I is a set with a partial ordering $\leq$ such that for any $i, j \in I$, there exists $k \in I$ with $i \leq k$ and $j \leq k$.

DEFINITION 1.4.40.

a. A *directed category* I is a category with a nonempty directed set I of objects and at most one morphism $i \to j$ for any $i, j \in I$, which exists if and only if $i \leq j$.

b. A *codirected category* is a category I such that I^{op} is directed.

DEFINITION 1.4.41. Let I be a category such that I^{op} is directed. The limit of a functor $F\colon I \to \mathscr{C}$ is referred to the *inverse limit* of the objects $F(i)$ over the inverse system of objects $F(i)$ for $i \in I$ and morphisms $F(\kappa)$ for $\kappa\colon i \to j$ in I, and it is denoted $\varprojlim_{i \in I} F(i)$.

DEFINITION 1.4.42. Let I be a directed category. The colimit of a functor $F\colon I \to \mathscr{C}$ is the *direct limit* of the objects $F(i)$ over the directed system of objects $F(i)$ for $i \in I$ and morphisms $F(\kappa)$ for $\kappa\colon i \to j$ in I and is denoted $\varinjlim_{i \in I} F(i)$ (or sometimes just $\varinjlim F$).

It turns out that it is useful to generalize the notion of a directed system slightly in the context of category theory, keeping the requirement of being directed but weakening the requirement of the category arising from a partial ordering.

DEFINITION 1.4.43. A small category I is said to be *filtered* if it is nonempty and the following hold.

i. For every pair (i, j) of objects in I, there exists an object k and morphisms $i \to k$ and $j \to k$ in I.

ii. For every two morphisms $\kappa, \kappa' \colon i \to j$ in I, there exists a morphism $\lambda \colon j \to k$ in I such that $\lambda \circ \kappa = \lambda \circ \kappa'$.

DEFINITION 1.4.44. A small category is said to be *cofiltered* if its opposite category is filtered.

EXAMPLE 1.4.45. Any small category with a terminal (resp., initial) object is filtered (resp., cofiltered).

We are typically interested in cofiltered limits and filtered colimits, of which inverse limits and direct limits are respectively special cases. We denote such limits as we do inverse and direct limits. The axiom of choice, together with Lemmas 1.4.5 and 1.4.16, allows us to define functors as in the following definition.

DEFINITION 1.4.46. Let $\mathscr{C}$ be a category and I a small category. Suppose that $\mathscr{C}$ admits limits (resp., colimits) from I. Then the limit (resp., colimit) functor

$$\lim \colon \mathbf{Func}(I, \mathscr{C}) \to \mathscr{C} \qquad (\text{resp., } \operatorname{colim} \colon \mathbf{Func}(I, \mathscr{C}) \to \mathscr{C})$$

is any functor that takes a functor $F \colon I \to \mathscr{C}$ to a limit (resp., colimit) of F and a natural transformation $\eta \colon F \rightsquigarrow G$ to the unique morphism

$$\lim \eta \colon \lim F \to \lim G \qquad (\text{resp., } \operatorname{colim} \eta \colon \operatorname{colim} F \to \operatorname{colim} G)$$

given by the universal property. When I is cofiltered (resp., filtered), the limit (resp., colimit) functor is denoted $\varprojlim$ (resp., $\varinjlim$).

1.5. Adjoint functors

DEFINITION 1.5.1. We say that $F \colon \mathscr{C} \to \mathscr{D}$ is *left adjoint* to $G \colon \mathscr{D} \to \mathscr{C}$ if there exist bijections

$$\eta_{(C,D)} \colon \operatorname{Hom}_{\mathscr{D}}(F(C), D) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{C}}(C, G(D))$$

for each $C \in \operatorname{Obj}(\mathscr{C})$ and $D \in \operatorname{Obj}(\mathscr{D})$ such that the $\eta_{(C,D)}$ form a natural transformation of functors

$$\eta \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Sets}.$$

We also say that G is *right adjoint* to F, and we say that F and G are *adjoint functors.*

REMARK 1.5.2. To say that $\eta \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Sets}$ is a natural transformation in Definition 1.5.1 is a fancier way of saying that given morphisms $f \colon C' \to C$ in $\mathscr{C}$ and $g \colon D \to D'$ in $\mathscr{D}$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{D}}(F(C), D) \arrow[r, "\eta_{(C,D)}"] \arrow[d, "t \mapsto g \circ t \circ F(f)"] & \operatorname{Hom}_{\mathscr{C}}(C, G(D)) \arrow[d, "u \mapsto G(g) \circ u \circ f"] \\
\operatorname{Hom}_{\mathscr{D}}(F(C'), D') \arrow[r, "\eta_{(C',D')}"] & \operatorname{Hom}_{\mathscr{C}}(C', G(D')).
\end{tikzcd}$$

REMARK 1.5.3. Adjointness is a weak form of the property of being mutually inverse. I.e., if G is a left and right inverse to F (on objects and morphisms), then we have

$$\mathrm{Hom}_{\mathscr{D}}(F(C), D) \cong \mathrm{Hom}_{\mathscr{C}}(G \circ F(C), G(D)) = \mathrm{Hom}_{\mathscr{C}}(C, G(D)),$$

so F is left adjoint to G, and similarly, F is right adjoint to G.

The following provides a standard example of adjoint functors.

EXAMPLE 1.5.4. The forgetful functor **Gps** $\to$ **Sets** is right adjoint to the functor **Sets** $\to$ **Gps** taking a set to the free group on it. That is, for any set X and group $\mathscr{G}$, there is a bijection of sets

$$\mathrm{Hom}(F_X, \mathscr{G}) \to \mathrm{Maps}(X, \mathscr{G}),$$

where F_X denotes the free group on X.

PROPOSITION 1.5.5. *Let S be a set. The functor $h_S\colon$ **Sets** $\to$ **Sets** is right adjoint to the functor $t_S\colon$ **Sets** $\to$ **Sets** given by $t_S(T) = T \times S$ and $t_S(f) = f \times \mathrm{id}_S$ for any sets T, T' and $f\colon T \to T'$.*

PROOF. We define our bijections by

$$\tau_{T,U}\colon \mathrm{Maps}(T \times S, U) \to \mathrm{Maps}(T, \mathrm{Maps}(S, U))$$
$$\tau_{T,U}(f) = (t \mapsto (s \mapsto f(t, s))).$$

We leave the verification of naturality to the reader. □

Later, we will treat what is perhaps the most standard example of adjointness: that of Hom and $\otimes$ in categories of modules.

PROPOSITION 1.5.6. *Fix categories I and $\mathscr{C}$, and suppose that all limits $F\colon I \to \mathscr{C}$ exist. The functor* lim *has a left adjoint Δ given by taking $A \in \mathrm{Obj}(\mathscr{C})$ to the constant functor c_A, where $c_A(i) = A$ for all $i \in I$, and taking $g\colon A \to B$ for $A, B \in \mathrm{Obj}(\mathscr{C})$ to the natural transformation $c_A \rightsquigarrow c_B$ given by $g\colon c_A(i) = A \to c_B(i) = B$ for all $i \in I$.*

PROOF. We must describe natural isomorphisms

$$\mathrm{Hom}_{\mathbf{Func}(I,\mathscr{C})}(c_A, F) \cong \mathrm{Hom}_{\mathscr{C}}(A, \lim F)$$

for $A \in \mathrm{Obj}(\mathscr{C})$ and $F\colon I \to \mathscr{C}$. I.e., given a natural tranformation $\eta\colon c_A \rightsquigarrow F$, we must associate a map $f\colon A \to \lim F$, and conversely. Such a natural transformation η consists of maps

$$\eta_i\colon c_A(i) = A \to F(i)$$

that are compatible in the sense that $\eta_j = \kappa \circ \eta_i$ for all $\kappa\colon i \to j$. Thus, the existence of a unique f is simply the universal property of the limit. On the other hand, if we have f, then we have maps

$$\phi_i \circ f\colon A \to F(i),$$

where ϕ_i is the map $\lim F \to F(i)$ arising in the definition of the limit. These maps then define the universal transformation η. □

We now see exactly how adjointness weakens inverseness.

DEFINITION 1.5.7. Two categories $\mathscr{C}$ and $\mathscr{D}$ are said to be *equivalent* if there exist functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $\eta\colon G \circ F \rightsquigarrow \mathrm{id}_{\mathscr{C}}$ and $\eta'\colon F \circ G \rightsquigarrow \mathrm{id}_{\mathscr{D}}$. Two such functors F and G are said to be *quasi-inverse*, and F and G are said to be *equivalences of categories*.

EXAMPLE 1.5.8. A category $\mathscr{C}$ with one object 0 and one morphism is equivalent to the category $\mathscr{D}$ with two objects 1, 2 and four morphisms, the identity morphisms of 1 and 2 and isomorphisms $1 \to 2$ and $2 \to 1$. We have quasi-inverse functors F and G with $F(0) = 1$ and $F(\mathrm{id}_0) = \mathrm{id}_1$ and $G(1) = G(2) = 0$ and $G(f) = \mathrm{id}_0$ for all f. To see naturality, note that every morphism between two objects in either category is unique.

NOTATION 1.5.9. Let $\eta\colon F \rightsquigarrow F'$ be a natural transformation between functors $F, F'\colon \mathscr{C} \to \mathscr{D}$.

a. If $G\colon \mathscr{D} \to \mathscr{E}$ is a functor, then we define a natural transformation $G(\eta)\colon G \circ F \rightsquigarrow G \circ F'$ by

$$G(\eta)_C = G(\eta_C)\colon G(F(C)) \to G(F'(C))$$

for all objects C of $\mathscr{C}$.

b. If $H\colon \mathscr{B} \to \mathscr{C}$ is a functor, then we define a natural transformation $\eta(H)\colon F \circ H \rightsquigarrow F' \circ H$ by

$$\eta(H)_B = \eta_{H(B)}\colon F(H(B)) \to F'(H(B))$$

for all objects B of $\mathscr{C}$.

DEFINITION 1.5.10. Let $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ be functors.

a. A *unit* for the pair (F, G) is a natural transformation $\mathrm{id}_{\mathscr{D}} \rightsquigarrow G \circ F$.

b. A *counit* for the pair (F, G) is a natural transformation $F \circ G \rightsquigarrow \mathrm{id}_{\mathscr{C}}$.

c. A *unit-counit adjunction* is a pair (F, G), a unit η for (F, G), and a counit η' for (F, G) satisfying

$$\mathrm{id}_F = \eta'(F) \circ F(\eta)\colon F \rightsquigarrow F$$

as morphisms in $\mathbf{Func}(\mathscr{C}, \mathscr{D})$ and

$$\mathrm{id}_G = G(\eta') \circ \eta(G)\colon G \rightsquigarrow G$$

as morphisms in $\mathbf{Func}(\mathscr{D}, \mathscr{C})$.

PROPOSITION 1.5.11. *A functor $F\colon \mathscr{C} \to \mathscr{D}$ is left adjoint to a functor $G\colon \mathscr{D} \to \mathscr{C}$ if and only if there exists a unit-counit adjunction for the pair (F, G).*

PROOF. Suppose that F is left adjoint to G. We define $\eta\colon \mathrm{id}_{\mathscr{C}} \rightsquigarrow G \circ F$ as follows. For $C \in \mathrm{Obj}(\mathscr{C})$, we have bijections

$$\mathrm{Hom}_{\mathscr{D}}(F(C), F(C)) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(C, G \circ F(C))$$

by adjointness, and we define η_C to be the image of $\mathrm{id}_{F(C)}$. For $D \in \mathrm{Obj}(\mathscr{D})$, we also have

$$\mathrm{Hom}_{\mathscr{D}}(F \circ G(D), D) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(G(D), G(D))$$

and define $\eta' \colon F \circ G \rightsquigarrow \mathrm{id}_{\mathscr{D}}$ by taking η'_D to be the image of $\mathrm{id}_{G(D)}$ under the inverse of this map. We leave it to the reader to check that these are natural and form a unit-counit adjunction. The converse is left to the reader as well. □

1.6. Representable functors

DEFINITION 1.6.1. Let $F \colon \mathscr{C} \to \mathbf{Sets}$ be a contravariant functor. Then F is said to be *representable* if there exists a natural isomorphism $h^B \rightsquigarrow F$ for some $B \in \mathrm{Obj}(\mathscr{C})$. (In other words, we have natural bijections

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \xrightarrow{\sim} F(A)$$

for all objects A of $\mathscr{C}$.) We then say that B *represents* F.

Using Yoneda's lemma and assuming $\mathscr{C}$ to be small, we can reword Definition 1.6.1 as saying that there exists $B \in \mathrm{Obj}(\mathscr{C})$ such that there are compatible bijections between the set of natural transformations $h^A \rightsquigarrow F$ and the set of morphisms $A \to B$ for each $A \in \mathrm{Obj}(\mathscr{C})$.

EXAMPLE 1.6.2. Consider the contravariant functor $P \colon \mathbf{Sets} \to \mathbf{Sets}$ which takes a set S to its *power set* $P(S)$, the set of all subsets of S and a map $f \colon S \to T$ to the map $P(f) \colon P(T) \to P(S)$ by mapping $U \subset T$ to $f^{-1}(U)$. Then P is represented by the set $\{0,1\}$ via the isomorphism

$$\mathrm{Maps}(S, \{0,1\}) \xrightarrow{\sim} P(S)$$

by $\phi \mapsto \phi^{-1}(\{1\})$. These isomorphisms form a natural transformation:

$$\begin{tikzcd}
\mathrm{Maps}(T, \{0,1\}) \arrow[r, "\sim"] \arrow[d, "h^{\{0,1\}}(f)"] & P(T) \arrow[d, "P(f)"] \\
\mathrm{Maps}(S, \{0,1\}) \arrow[r, "\sim"] & P(S)
\end{tikzcd}$$

for $f \colon S \to T$. Here, the lefthand vertical map takes ϕ to $\phi \circ f$ and the righthand vertical map takes a subset X of T to $q^{-1}(X)$. We check that

$$(\phi \circ f)^{-1}(\{1\}) = f^{-1}(\phi^{-1}(\{1\})).$$

The following is a corollary of Yoneda's lemma.

LEMMA 1.6.3. *A representable functor is represented by a unique object up to isomorphism. If B and C represent a contravariant functor $F \colon \mathscr{C} \to$ **Sets**, then such an isomorphism $f \colon B \to C$ is unique making the diagrams*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(A, B) \arrow[r, "\sim"] \arrow[d, "h^A(f)"] & F(A) \arrow[d, equal] \\
\mathrm{Hom}_{\mathscr{C}}(A, C) \arrow[r, "\sim"] & F(A)
\end{tikzcd}$$

commute for all $A \in \mathrm{Obj}(\mathscr{C})$.

PROOF. Let $F\colon \mathscr{C} \to \mathbf{Sets}$ be a representable (contravariant) functor represented by $B \in \mathrm{Obj}(\mathscr{C})$ and $C \in \mathrm{Obj}(\mathscr{C})$. Then we have natural isomorphisms $\xi\colon h^B \rightsquigarrow F$ and $\xi'\colon h^C \rightsquigarrow F$. The composition $\xi' \circ \xi^{-1}\colon h^B \to h^C$ is equal to $h^{\mathscr{C}}(f)$ for a unique $f\colon B \to C$ by the weak form of Yoneda's lemma. □

THEOREM 1.6.4. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor between small categories.*

a. The functor F has a right adjoint if and only if the functor $h^D \circ F$ is representable for each $D \in \mathrm{Obj}(\mathscr{D})$. If G is right adjoint to F, then $h^D \circ F$ is representable by $G(D)$.

b. If F has a right adjoints G and G', then there exists a unique natural isomorphism $\xi\colon G \rightsquigarrow G'$ such that diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{D}}(F(C), D) & \xrightarrow{\eta_{(C,D)}} & \mathrm{Hom}_{\mathscr{C}}(C, G(D)) \\ \Big\| & & \Big\downarrow{\scriptstyle f \mapsto \xi_D \circ f} \\ \mathrm{Hom}_{\mathscr{D}}(F(C), D) & \xrightarrow{\eta'_{(C,D)}} & \mathrm{Hom}_{\mathscr{C}}(C, G'(D)) \end{array}$$

commute for all $C \in \mathrm{Obj}(\mathscr{C})$ and $D \in \mathrm{Obj}(\mathscr{D})$, where the horizontal morphisms are the adjunction isomorphisms.

PROOF. Assume that F has a right adjoint G, and consider the adjunction morphisms

$$\eta_{(C,D)}\colon \mathrm{Hom}_{\mathscr{D}}(F(C), D) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(C, G(D)).$$

In other words,

$$h^D \circ F(C) \cong h^{G(D)}(C),$$

so $G(D)$ represents $h^D \circ F$. In this case, the uniqueness in part b is an immediate consequence of Lemma 1.6.3.

Now suppose that $h^D \circ F$ is representable for each D by some object $G(D)$ (chosen using the axiom of choice). Then there exist isomorphisms $\eta_{(C,D)}$ that are natural in C. We must also define G on morphisms $f\colon D \to D'$ in $\mathscr{D}$. Such an f induces a natural transformation $h^D \rightsquigarrow h^{D'}$ which provides morphisms

$$h^D \circ F(C) \to h^{D'} \circ F(C)$$

for all $C \in \mathrm{Obj}(\mathscr{C})$ and thus induces $h^{G(D)}(C) \to h^{G(D')}(C)$, and these are natural in C. Thus, we have a natural transformation $h^{G(D)} \rightsquigarrow h^{G(D')}$. Since the Yoneda embedding is fully faithful, we have a unique morphism $G(D) \to G(D')$ inducing this natural transformation, which we define to be $G(f)$. We leave to the reader the check that G as defined is a functor. □

DEFINITION 1.6.5. Let $F\colon \mathscr{C} \to \mathbf{Sets}$ be a covariant functor. We say that F is *representable* if there exists a natural isomorphism $h_A \rightsquigarrow F$ for some $A \in \mathrm{Obj}(\mathscr{C})$. (That is, there are natural isomorphisms

$$F(B) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(A, B)$$

in $B \in \mathrm{Obj}(\mathscr{C})$.) In this case, we say that A *represents* F.

REMARK 1.6.6. A covariant functor $F\colon \mathscr{C} \to \mathbf{Sets}$ is representable if and only if the contravariant functor $F \circ \mathrm{op}\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ is representable. The same object of $\mathscr{C}$ will represent both objects.

EXAMPLE 1.6.7. Let $F\colon \mathbf{Gps} \to \mathbf{Sets}$ be the forgetful functor. Then F can be represented by $\mathbb{Z}$. To see this, we define the set map

$$G \to \mathrm{Hom}_{\mathbf{Gps}}(\mathbb{Z}, G)$$

by $a \mapsto (1 \mapsto a)$ for $a \in G$. Naturality is clear.

EXAMPLE 1.6.8. Let $F\colon \mathbf{Gps} \to \mathbf{Sets}$ be the functor which sends a group to its subset $G[n]$ of elements of order dividing n. Then F can be represented by $\mathbb{Z}/n\mathbb{Z}$.

EXAMPLE 1.6.9. Consider a functor $F\colon I \to \mathscr{C}$ between small categories. To say that the contravariant functor $\lim h^{\mathscr{C}} \circ F\colon \mathscr{C} \to \mathbf{Sets}$ is representable is exactly to say that there exists an object X in $\mathscr{C}$ such that one has natural isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}(A, X) \xrightarrow{\sim} (\lim h^{\mathscr{C}} \circ F)(A) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathscr{C}}(A, F(\cdot))$$

for $A \in \mathscr{C}$. In other words, $\lim h^{\mathscr{C}} \circ F$ is representable if and only if $\lim F$ exists in $\mathscr{C}$.

EXAMPLE 1.6.10. Consider a functor $F\colon I \to \mathscr{C}$ between small categories. View $h_{\mathscr{C}}$ as a covariant functor $\mathscr{C} \to \mathrm{Hom}(\mathscr{C}, \mathbf{Sets})$. To say that the functor $\lim(h_{\mathscr{C}} \circ F)\colon \mathscr{C} \to \mathbf{Sets}$ is representable is exactly to say that there exists an object $X \in \mathscr{C}$ such that one has natural isomorphisms

$$\mathrm{Hom}_{\mathscr{C}}(X, A) \xrightarrow{\sim} (\lim h_{\mathscr{C}} \circ F)(A) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathscr{C}}(F(\cdot), A)$$

for $A \in \mathscr{C}$. In other words, $\lim h_{\mathscr{C}} \circ F$ is representable if and only if $\operatorname{colim} F$ exists in $\mathscr{C}$.

1.7. Equalizers and images

DEFINITION 1.7.1. Let $\mathscr{C}$ be a category, and let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B. \tag{1.7.1}$$

be a diagram in $\mathscr{C}$.

a. The limit $\mathrm{eq}(f, g)$ of the diagram (1.7.1), when it exists, is called its *equalizer*.

b. The colimit $\mathrm{coeq}(f, g)$ of (1.7.1) is called its *coequalizer*.

We have a commutative diagram:

$$\mathrm{eq}(f, g) \longrightarrow A \underset{g}{\overset{f}{\rightrightarrows}} B \longrightarrow \mathrm{coeq}(f, g).$$

EXAMPLES 1.7.2.

a. Let X, Y be sets, and consider maps $f, g\colon X \to Y$. In $\mathbf{Sets}$, we have

$$\mathrm{eq}(f, g) = \{x \in X \mid f(x) = g(x)\}$$

and $\mathrm{coeq}(f, g)$ is the quotient of Y by the minimal equivalence relation $\sim$ generated by $f(x) \sim g(x)$ for all $x \in X$.

b. In R-**mod**, the equalizer is expressed as in **Sets**. For an R-module homomorphism $f\colon A \to B$, we have

$$\operatorname{coeq}(f,g) = B/\{(f-g)(a) \mid a \in A\}.$$

LEMMA 1.7.3. *Let $f, g\colon A \to B$ be morphisms in a category $\mathscr{C}$.*

a. Suppose that $\operatorname{eq}(f,g)$ *exists. Then the induced map $h\colon \operatorname{eq}(f,g) \to A$ is a monomorphism.*

b. Suppose that $\operatorname{coeq}(f,g)$ *exists. Then the induced map $k\colon B \to \operatorname{coeq}(f,g)$ is an epimorphism.*

PROOF. Suppose that $\alpha, \beta\colon C \to \operatorname{eq}(f,g)$ are morphisms in $\mathscr{C}$ such that $h \circ \alpha = h \circ \beta$. Let $h' = h \circ \alpha$, and note that $f \circ h' = g \circ h'$. But then $\alpha\colon C \to \operatorname{eq}(f,g)$ is unique such that $h' = h \circ \alpha$ by the universal property of $\operatorname{eq}(f,g)$. Since $h' = h \circ \beta$ as well, we have $\alpha = \beta$. Part b follows from part a by working in the additive category. $\square$

We begin with the notions of image and coimage in an arbitrary category.

DEFINITION 1.7.4. Let $f\colon A \to B$ be a morphism in a category that has finite products, finite coproducts, equalizers, and coequalizers.

a. The *image* of f is the equalizer of the two morphisms $\iota_i\colon B \to B \amalg_A B$.

b. The *coimage* of f is the coequalizer of the two projection morphisms $p_i\colon A \times_B A \to A$.

REMARK 1.7.5. The image of a morphism $f\colon A \to B$ fits in a diagram

(1.7.2)
$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] \arrow[dd, "f"'] & & B \arrow[dd, "\iota_1"] \\ & \operatorname{im} f \arrow[ur] \arrow[dl] & \\ B \arrow[rr, "\iota_2"] & & B \amalg_A B, \end{tikzcd}$$

where the morphism $A \to \operatorname{im} f$ is induced by the universal property of $\operatorname{im} f$ and the morphism $\operatorname{im} f \to B$ are identical to each other. Note that $\operatorname{im} f \to B$ is a monomorphism, since it is an equalizer. Dually, we also have an induced epimorphism $\operatorname{coim} f \to B$.

EXAMPLE 1.7.6. We check that the definition of $\operatorname{im} f$ agrees with the usual notion in the category of R-modules. We claim that $\{f(a) \mid a \in A\}$ and its inclusion in B is the equalizer of the diagram

$$B \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} B \amalg_A B.$$

Since

$$B \amalg_A B \cong \frac{B \oplus B}{\{(f(a),0)-(0,f(a)) \mid a \in A\}},$$

the claim follows from Example 1.7.2(b).

LEMMA 1.7.7. *For any $f\colon A \to B$ in a category that admits equalizers and coequalizers there is a unique morphism $u\colon \operatorname{coim} f \to \operatorname{im} f$ such that the composition*

$$A \xrightarrow{s} \operatorname{coim} f \xrightarrow{u} \operatorname{im} f \xrightarrow{t} B$$

of induced morphisms is f.

PROOF. Consider the diagram

$$\begin{tikzcd}
A \times_B A \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[rr, "f"] \arrow[d, "s"'] \arrow[drr, "r"] & & B \arrow[r, shift left, "\iota_1"] \arrow[r, shift right, "\iota_2"'] & B \amalg_A B \\
& \operatorname{coim} f & & \operatorname{im} f \arrow[u, "t"] &
\end{tikzcd} \tag{1.7.3}$$

where the diagonal map r is from (1.7.2). We have that $f \circ p_1 = f \circ p_2$, so

$$t \circ r \circ p_1 = t \circ r \circ p_2$$

We know that t is a monomorphism since the image is an equalizer, so $r \circ p_1 = r \circ p_2$. By the universal property of the coimage, it follows that there exists a unique map $u\colon \operatorname{coim} f \to \operatorname{im} f$ making the diagram (1.7.3) commute. □

DEFINITION 1.7.8. We say that a morphism $f\colon A \to B$ in a category that admits equalizers and coequalizers is *strict* if the induced morphism $\operatorname{coim} f \to \operatorname{im} f$ is an isomorphism.

EXAMPLE 1.7.9. Every morphism in the category of R-modules is strict.

CHAPTER 2

Abelian Categories

2.1. Additive categories

DEFINITION 2.1.1. An *additive category* $\mathscr{C}$ is a category with the following properties:

i. for $A, B \in \mathrm{Obj}(\mathscr{C})$, the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(A, B)$ in $\mathscr{C}$ has an abelian group law (addition) with the property that for any diagram

$$A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \xrightarrow{h} D,$$

in $\mathscr{C}$, we have

$$h \circ (g_1 + g_2) \circ f = h \circ g_1 \circ f + h \circ g_2 \circ f,$$

ii. $\mathscr{C}$ has a zero object 0,

iii. $\mathscr{C}$ admits finite coproducts.

Note that between any two objects A and B in an additive category $\mathscr{C}$, there always exists the zero morphism $0 \colon A \to B$, the unique map factoring through 0. This is the identity element in the abelian group $\mathrm{Hom}_{\mathscr{C}}(A, B)$.

EXAMPLES 2.1.2.

a. The categories **Ab** and R-**mod** are additive categories, with the usual addition of homomorphisms.

b. The full subcategory R-**mod** of finitely generated R-modules is an additive category.

c. The category of topological Hausdorff abelian groups (with continuous group homomorphisms) is an additive category.

In an additive category, we denote the coproduct of two objects A_1 and A_2 by $A_1 \oplus A_2$.

LEMMA 2.1.3. *Finite products exist in an additive category, and there are natural isomorphisms $A_1 \amalg A_2 \cong A_1 \times A_2$ for $A_1, A_2 \in \mathrm{Obj}(\mathscr{C})$. The resulting inclusion morphisms $\iota_i \colon A_i \to A_1 \oplus A_i$ and projection morphisms and $p_i \colon A_1 \oplus A_2 \to A_i$ obtained by viewing $A_1 \amalg A_2$ as a product and coproduct, respectively, satisfy $p_i \circ \iota_i = \mathrm{id}_{A_i}$ and $p_i \circ \iota_j = 0$ for $i \neq j$, while $\iota_1 \circ p_1 + \iota_2 \circ p_2 = \mathrm{id}_{A_1 \amalg A_2}$.*

PROOF. We have maps $\iota_i \colon A_i \to A_1 \oplus A_2$ by definition of the direct sum. We also have maps $p_i \colon A_1 \oplus A_2 \to A_i$ defined by

$$p_i \circ \iota_j = \begin{cases} \mathrm{id}_{A_i} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and the universal property of the coproduct.

We then have

$$(\iota_1 \circ p_1 + \iota_2 \circ p_2) \circ \iota_i = \iota_i,$$

and hence

$$\iota_1 \circ p_1 + \iota_2 \circ p_2 = \mathrm{id}_{A_1 \amalg A_2}.$$

Given an object $B \in \mathrm{Obj}(\mathscr{C})$ and maps $g_i \colon B \to A_i$, we then have a map

$$\psi = \iota_1 \circ g_1 + \iota_2 \circ g_2 \colon B \to A_1 \oplus A_2,$$

which is unique such that

$$p_i \circ \psi = g_i.$$

Hence $A_1 \oplus A_2$ satisfies the universal property of the product. □

DEFINITION 2.1.4. An object A in an additive category $\mathscr{C}$ together with objects A_i, inclusion morphisms $\iota_i \colon A_i \to A$, and projection morphisms $p_i \colon A \to A_i$ for $i \in \{1,2\}$ for which $p_i \circ \iota_j$ is zero if $i \neq j$ and id_{A_i} if $i = j$ and for which $\iota_1 \circ p_1 + \iota_2 \circ p_2 = \mathrm{id}_A$ is called a *biproduct* of the objects A_1 and A_2, and we write it as $A_1 \oplus A_2$.

The reader will verify the following.

LEMMA 2.1.5. *In an additive category $\mathscr{C}$, the biproduct $A_1 \oplus A_2$ is a coproduct of the A_i via the inclusion morphisms ι_i and to the product of the A_i via the morphisms p_i.*

The notion of a biproduct allows us to reinterpret addition in an additive category. First, note the following definitions.

DEFINITION 2.1.6. Let A be an object in an additive category $\mathscr{C}$.

a. The *diagonal morphism* $\Delta_A \colon A \to A \oplus A$ in $\mathscr{C}$ is the unique morphism induced by two copies of $\mathrm{id}_A \colon A \to A$ and the universal property of the product.

b. The *codiagonal morphism* $\nabla_A \colon A \oplus A \to A$ in $\mathscr{C}$ is the unique morphisms induced by two copies of $\mathrm{id}_A \colon A \to A$ and the universal property of the coproduct.

DEFINITION 2.1.7. Let $\mathscr{C}$ be an additive category, and let $f_1 \colon A_1 \to B_1$ and $f_2 \colon A_2 \to B_2$ be morphisms in $\mathscr{C}$. The biproduct $f \oplus f_2$ of the maps f_1 and f_2 is the morphism $A_1 \oplus A_2 \to B_1 \oplus B_2$ induced as the (morphism defined by the universal property of the) coproduct of the composite maps $A_i \to B_i \to B_1 \oplus B_2$, the latter morphisms being inclusions.

REMARK 2.1.8. Equivalently, the direct sum of f_1 and f_2 as in Definition 2.1.7 is induced as the product of the composite maps $A_1 \oplus A_2 \to A_i \to B_i$, the initial morphisms being projections.

Of course, we could make these definitions in an arbitrary category using products and coproducts.

LEMMA 2.1.9. *Let $f, g \colon A \to B$ be two morphisms in an additive category $\mathscr{C}$. Then we have*

$$f + g = \nabla_B \circ (f \oplus g) \circ \Delta_A.$$

PROOF. Let ι_i^A and p_i^A respectively denote the inclusion maps and projection maps for the biproduct $A \oplus A$, and similarly for B. We have

$$\begin{aligned}\nabla_B \circ (f \oplus g) \circ \Delta_A &= \nabla \circ (f \oplus g) \circ (\iota_1^A \circ p_1^A + \iota_2^A \circ p_2^A) \circ \Delta_A \\ &= \nabla_B \circ (f \oplus g) \circ \iota_1^A \circ p_1^A \circ \Delta_A + \nabla_B \circ (f \oplus g) \circ \iota_2^A \circ p_2^A \circ \Delta_A.\end{aligned}$$

Taking the first term without loss of generality, we have

$$\nabla_B \circ ((f \oplus g) \circ \iota_1^A) \circ (p_1^A \circ \Delta_A) = \nabla_B \circ (\iota_1^B \circ f) \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f.$$

□

DEFINITION 2.1.10. A functor $F\colon \mathscr{C} \to \mathscr{D}$ between additive categories is called *additive* if for each $A, B \in \mathrm{Obj}(\mathscr{C})$, the map

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$$

is a group homomorphism.

EXAMPLE 2.1.11. Let $\mathscr{C}$ be an additive category. Then for any $A \in \mathrm{Obj}(\mathscr{C})$, the functors h^A and h_A may be considered as functors to **Ab**, rather than **Sets**. The resulting functors are additive.

EXAMPLE 2.1.12. Let R be a ring, and let A be a right R-module. Then the functor

$$t_A\colon R\text{-}\mathbf{mod} \to \mathbf{Ab}$$

given by $t_A(B) = A \otimes_R B$ and $t_A(f) = \mathrm{id}_A \otimes f$ for $f\colon B \to C$ is additive.

LEMMA 2.1.13. *A functor $F\colon \mathscr{C} \to \mathscr{D}$ of additive categories is additive if and only if F preserves biproducts, which is to say that the natural morphisms $F(A_1) \oplus F(A_2) \to F(A_1 \oplus A_2)$ and $F(A_1 \oplus A_2) \to F(A_1) \oplus F(A_2)$ are inverse isomorphisms for all objects A_1, A_2 in $\mathscr{C}$.*

PROOF. Suppose first that F is an additive functor. Note that $F(\iota_i \circ p_i) = \mathrm{id}_{F(A_i)}$ and $F(\iota_i \circ p_j) = F(0)$ for $i \neq j$, but $F(0) = 0$ by additivity of F. Again by additivity of F, we have

$$F(\iota_1) \circ F(p_1) + F(\iota_2) \circ F(p_2) = F(\mathrm{id}_{A_1 \oplus A_2}) = \mathrm{id}_{F(A_1 \oplus A_2)}.$$

It follows that $F(A_1 \oplus A_2)$ is a biproduct of $F(A_1)$ and $F(A_2)$ in $\mathscr{D}$, so in particular it is a coproduct.

On the other hand, if F preserves biproducts and $f, g\colon A \to B$ are morphisms in $\mathscr{C}$, then it is easy to see that $F(f \oplus g) = F(f) \oplus F(g)$, and Lemma 2.1.9 tells us that

$$F(f + g) = F(\nabla_B \circ (f \oplus g) \circ \Delta_A) = \nabla_{F(B)} \circ (F(f) \oplus F(g)) \circ \Delta_{F(A)} = F(f) + F(g).$$

□

COROLLARY 2.1.14. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a fully faithful functor of additive categories. Then F is an additive functor.*

For additive functors, we may consider a finer notion of representability.

REMARK 2.1.15. If $F\colon \mathscr{C} \to \mathbf{Ab}$ is an additive contravariant (resp., covariant) functor of additive categories, then we may consider it to be representable if there exists an object $X \in \mathrm{Obj}(\mathscr{C})$ and a natural isomorphism $\eta\colon h^X \rightsquigarrow F$ (resp., $\eta\colon h_X \rightsquigarrow F$). In this case, the morphisms η_A for $A \in \mathrm{Obj}(\mathscr{C})$ will be isomorphisms of groups.

2.2. Kernels and cokernels

REMARK 2.2.1. In an additive category, any morphism to 0 is an epimorphism, and any morphism from 0 is a monomorphism.

DEFINITION 2.2.2. Let $\mathscr{C}$ be an additive category, and let $f\colon A \to B$ be a morphism in $\mathscr{C}$.

a. The *kernel* $\ker f$ of f is the equalizer $\mathrm{eq}(f,0)$, when it exists.

b. The *cokernel* $\mathrm{coker}\, f$ is the coequalizer $\mathrm{coeq}(f,0)$, when it exists.

EXAMPLE 2.2.3. In the category R-**mod**, these definitions agree with the classical ones.

LEMMA 2.2.4. *A morphism in an additive category that admits kernels is a monomorphism if and only if it has zero kernel. A morphism in an additive category that admits cokernels is an epimorphism if and only if it has zero cokernel.*

PROOF. Let $f\colon A \to B$ be a monomorphism, and let $h\colon \ker f \to A$ be the induced morphism. Since $f \circ h = 0$ by definition of the kernel, we have $h = 0$, as f is a monomorphism. This forces $\ker f$ to be 0, since h factors through 0. (Or, one could just apply Lemma 1.7.3.) On the other hand, suppose that f has trivial kernel, and let $g, h\colon C \to A$ be maps with $f \circ g = f \circ h$. Then $f \circ (g - h) = 0$, and by universal property of the kernel, $g - h$ factors through 0, i.e., is 0.

The proof for cokernels is similar, or is the result on kernels in the opposite (additive) category. □

REMARK 2.2.5. Let $\mathscr{C}$ be an additive category that admits kernels (resp., cokernels). Then there is a functor

$$\ker\colon \mathrm{Mor}(\mathscr{C}) \to \mathscr{C} \qquad (\text{resp., } \mathrm{coker}\colon \mathrm{Mor}(\mathscr{C}) \to \mathscr{C})$$

which takes an object in $\mathrm{Mor}(\mathscr{C})$ to its kernel (resp., cokernel) and a morphism in $\mathrm{Mor}(\mathscr{C})$ to the natural morphism between kernels (resp., cokernels).

PROPOSITION 2.2.6. *Let $\mathscr{C}$ be an additive category that admits kernels and cokernels. Let $f\colon A \to B$ be a morphism in $\mathscr{C}$. Then*

$$\mathrm{im}\, f \cong \ker(B \to \mathrm{coker}\, f)$$

and

$$\mathrm{coim}\, f \cong \mathrm{coker}(\ker f \to A).$$

PROOF. We prove the first isomorphism. Let $g\colon B \to \mathrm{coker}\, f$. By Yoneda's lemma, it suffices to show that $h^{\mathrm{im}\, f}$ and $h^{\ker g}$ are naturally isomorphic. For $C \in \mathrm{Obj}(\mathscr{C})$, we have a map

$$\mathrm{Hom}_{\mathscr{C}}(C, \mathrm{im}\, f) \xrightarrow{\sim} \{\alpha\colon C \to B \mid \iota_1 \circ \alpha = \iota_2 \circ \alpha\}$$

that takes a morphism $C \to \operatorname{im} f$ and composes it with the morphism $\operatorname{im} f \to B$ given by definition of the equalizer of the maps $\iota_i \colon B \to B \amalg_A B$. It is a bijection by the universal property of the equalizer.

For any $D \in \operatorname{Obj}(\mathscr{C})$ and morphisms $\phi_1, \phi_2 \colon B \to D$ such that $\phi_1 \circ f = \phi_2 \circ f$, note that there exists a unique morphism $k \colon B \amalg_A B \to D$ with $\phi_i = k \circ \iota_i$. Any $\alpha \colon C \to B$ such that $\iota_1 \circ \alpha = \iota_2 \circ \alpha$ then satisfies $\phi_1 \circ \alpha = \phi_2 \circ \alpha$ for any such $\phi_i \colon B \to D$ and any D. On the other hand, note that the ι_i themselves satisfy the property that $\iota_1 \circ f = \iota_2 \circ f$ and are morphisms $\iota_i \colon B \to D$ with $D = B \amalg_A B$. In other words, we have

$$\begin{aligned}
&\{\alpha \colon C \to B \mid \iota_1 \circ \alpha = \iota_2 \circ \alpha\} \\
&= \{\alpha \colon C \to B \mid \phi_1 \circ \alpha = \phi_2 \circ \alpha \text{ if } \phi_1 \circ f = \phi_2 \circ f \text{ for some } \phi_1, \phi_2 \colon B \to D \text{ (for some } D \in \operatorname{Obj}(\mathscr{C}))\}.
\end{aligned}$$

Now, we are in an additive category, so this equals

$$\{\alpha \colon C \to B \mid \phi \circ \alpha = 0 \text{ if } \phi \circ f = 0 \text{ for some } \phi \colon B \to D\}. \tag{2.2.1}$$

By the universal property of $\operatorname{coker} f$, for any $\phi \colon B \to D$ with $\phi \circ f = 0$, there is a morphism $j \colon \operatorname{coker} f \to D$ with $j \circ g = \phi$. If $g \circ \alpha = 0$, then $\phi \circ \alpha = j \circ g \circ \alpha = 0$, and this works for any $\phi \colon B \to D$ with $\phi \circ f = 0$. On the other hand, g itself satisfies $g \circ f = 0$ so is such a ϕ. It follows that the set in (2.2.1) equals

$$\{\alpha \colon C \to B \mid g \circ \alpha = 0\}.$$

By the universal property of $\ker g$, this is in bijection with $\operatorname{Hom}_{\mathscr{C}}(C, \ker g)$, taking an α in the set to the unique morphism to $\ker g$ through which it factors. Clearly, the composition of these bijections is natural in C, so we have the desired natural isomorphism. □

2.3. Abelian categories

DEFINITION 2.3.1. An *abelian* category is an additive category $\mathscr{C}$ in which

i. every morphism in $\mathscr{C}$ admits a kernel and a cokernel and

ii. every morphism in $\mathscr{C}$ is strict.

EXAMPLES 2.3.2.

a. The category R-**mod** is abelian.

b. The full subcategory $\mathscr{C}$ of R-**mod** of finitely generated R-submodules is not abelian in all cases. E.g., when R is commutative and non-Noetherian, we can take I to be an ideal of R that is not finitely generated, and so the kernel of $R \to R/I$ is not in $\mathscr{C}$.

c. The category of topological Hausdorff abelian groups is not abelian, but it is additive and admits kernels and cokernels. For instance, consider the inclusion map $\iota \colon \mathbb{Q} \to \mathbb{R}$ with $\mathbb{R}$ having its usual topology and $\mathbb{Q}$ having the subspace topology. Then $\ker \iota = 0$ and $\operatorname{coker} \iota = 0$ (since $\mathbb{Q}$ is dense in $\mathbb{R}$, and thus every continuous map from $\mathbb{R}$ is determined by its values on $\mathbb{Q}$). By Proposition 2.2.6, we have $\operatorname{im} \iota \cong \mathbb{R}$ but $\operatorname{coim} \iota \cong \mathbb{Q}$.

REMARK 2.3.3. Note that if $\mathscr{C}$ is an abelian category, then so is $\mathscr{C}^{\mathrm{op}}$. The roles of mono- and epimorphisms, kernels and cokernels, and images and coimages switch in $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$.

PROPOSITION 2.3.4. *The functor category* **Func**$(\mathscr{C},\mathscr{D})$ *from a small category* $\mathscr{C}$ *to an abelian category* $\mathscr{D}$ *is abelian.*

PROOF. We sketch the proof. First, note that it is additive: we have the zero functor which sends all objects to the zero object and all morphisms to the zero (identity) morphism of the zero object, and if $F, G\colon \mathscr{C} \to \mathscr{D}$ are functors, then $F \oplus G$ is given by $(F \oplus G)(C) = F(C) \oplus G(C)$ and $(F \oplus G)(f) = F(f) \oplus G(f)$ for $f\colon A \to B$ in C. This can be used to define the addition on morphisms (i.e., natural transformations) as before.

Next, the kernel of a natural transformation $\eta\colon F \rightsquigarrow G$ is defined by $(\ker\eta)(C) = \ker\eta_C$ and $(\ker\eta)(f)$ for $f\colon A \to B$ is the kernel of the induced morphism $\ker\eta_A \to \ker\eta_B$. The cokernel is defined similarly. Note that

$$(\operatorname{coim}\eta)_A \cong \operatorname{coker}(\ker\eta \rightsquigarrow \eta)_A \cong \operatorname{coker}(\ker\eta_A \rightsquigarrow \eta_A) \cong \operatorname{coim}\eta_A,$$

and similarly for images. Finally, since $\mathscr{D}$ is abelian, the natural map $\operatorname{coim}\eta \rightsquigarrow \operatorname{im}\eta$ is an isomorphism $\operatorname{coim}\eta_A \to \operatorname{im}\eta_A$ on objects A in $\mathscr{C}$, hence has a natural inverse determined by the inverses of these morphisms. □

We discuss a related class of examples.

DEFINITION 2.3.5. Let X be a topological space and $\mathscr{C}$ an abelian category. Consider the category $\mathscr{U}_X$ with objects the open sets in X and morphisms the inclusion maps between subspaces to the category $\mathscr{C}$.

a. A *presheaf* F on X with values in $\mathscr{C}$ is a contrvariant functor $F\colon \mathscr{U}_X \to \mathscr{C}$. The category of presheaves is the functor category **Func**$(\mathscr{U},\mathscr{C})$.

b. Suppose that $\mathscr{C}$ admits arbitrary products. A *sheaf* F on X with values in $\mathscr{C}$ is a presheaf $F\colon \mathscr{U}_X \to \mathscr{C}$ such that each $F(U)$ for $U \in \mathscr{U}_X$ is the equalizer of the diagram

$$\prod_{V\in\mathscr{V}} F(V) \rightrightarrows \prod_{V_1,V_2\in\mathscr{V}} F(V_1 \cap V_2),$$

where $\mathscr{V}$ is an open covering of U by open subsets via the two maps on products induced by the application of F to inclusion maps $V_1 \cap V_2 \to V_i$ for each pair (V_1, V_2) of open sets in $\mathscr{V}$. The category of sheaves is the full subcategory of the category of presheaves with objects the sheaves.

EXAMPLE 2.3.6. Let X be a topological space and $\mathscr{C}$ an abelian category. For any $A \in \operatorname{Obj}(\mathscr{C})$, we can view A as a discrete space and consider the constant sheaf

$$F(U) = \{f\colon U \to A \mid f \text{ continuous}\}$$

for all $U \in \mathscr{U}_X$. If U is connected, then $F(U) = A$.

EXAMPLE 2.3.7. Let X be a topological space and for $U \in \mathscr{U}_X$. Then $\mathscr{O}_X$ is the sheaf with values in **Ab** of continuous functions to $\mathbb{C}$ that for $U \in \mathscr{U}_X$ satisfies

$$\mathscr{O}_X(U) = \{\text{continuous maps } U \to \mathbb{C}\}.$$

EXAMPLE 2.3.8. Let $X = \mathbb{R}$, and let $F\colon \mathscr{U}_{\mathbb{R}} \to$ **Ab** be the presheaf such that $F(U)$ is the abelian group of bounded continuous functions on U. Then F is not a sheaf, as we may consider the open covering of $\mathbb{R}$ by intervals $I_n = (-n, n)$ for $n \in \mathbb{Z}$, and then the function $f(x) = x$ is contained in $F(I_n)$ for each n, so in the equalizer, but it is not in $F(\mathbb{R})$.

TERMINOLOGY 2.3.9. In an abelian category $\mathbb{C}$, we will typically refer to a coproduct (when it exists) as a direct sum, and we write $\bigoplus_{i \in I} A_i$ in place of $\amalg_{i \in I} A_i$.

2.4. Exact sequences

Suppose that $f: A \to B$ and $g: B \to C$ are morphisms in an abelian category $\mathscr{C}$ with $g \circ f = 0$. Note that $f: \kappa \circ \lambda$, where $\lambda: A \to \operatorname{im} f$ is an epimorphism and $\kappa: \operatorname{im} f \to B$ is a monomorphism. We have $g \circ \kappa \circ \lambda = 0$, so the fact that λ is an epimorphism tells us that $g \circ \kappa = 0$. The universal property of the kernel then provides a morphism $\alpha: \operatorname{im} f \to \ker g$ such that when composed with the canonical monomorphism $\beta: \ker g \to B$ is $\beta \circ \alpha = \kappa$. Since β and κ are both monomorphisms, which is to say have trivial kernel, we have that α is a monomorphism as well.

DEFINITION 2.4.1. We say that a diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in an abelian category $\mathscr{C}$ is *exact* if $g \circ f = 0$ and the induced monomorphism $\operatorname{im} f \to \ker g$ is an isomorphism. We call such a diagram a *three term exact sequence*.

We generalize this notion. First, we define a sequence.

DEFINITION 2.4.2. A diagram of the form

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}^A} A_n \xrightarrow{d_n^A} A_{n-1} \to \cdots$$

in a category $\mathscr{C}$ is a *sequence*, with the understanding that the diagram may terminate (i.e., be of finite length) on either, both, or neither side. We denote the data of a sequence by $A_\cdot = (A_i, d_i^A)$, where i runs for A_i over the interval of integers on which A_i is defined and for d_i^A over every such i but the left endpoint, if it exists.

DEFINITION 2.4.3. Let $\mathscr{C}$ be an abelian category.

a. A *chain complex* in $\mathscr{C}$ is a sequence $A_\cdot = (A_i, d_i^A)_{i \in \mathbb{Z}}$ in $\mathscr{C}$ such that $d_i^A \circ d_{i+1}^A = 0$ for all $i \in \mathbb{Z}$.

b. For a chain complex $A_\cdot$ and $i \in \mathbb{Z}$, the morphism $d_i^A: A_i \to A_{i-1}$ is called the ith *differential* in the complex $A_\cdot$.

NOTATION 2.4.4. Unless otherwise specified, the ith object in a chain complex $A_\cdot$ will be denoted A_i and the ith differential by $d_i: A_i \to A_{i-1}$. If we have multiple complexes, we will use d_i^A to specify the differential on $A_\cdot$.

NOTATION 2.4.5. In an abelian category, if A is a subobject of an object B, we will often write $A \subseteq B$ to denote this and B/A for the cokernel of the inclusion morphism $A \to B$. For $f: B \to C$, we will let $f(A)$ denote the image of the composite of the inclusion with f.

DEFINITION 2.4.6. A sequence

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_{n-1}} A_{n-1} \to \cdots$$

in an abelian category $\mathscr{C}$ is *exact* if the subdiagram $A_{m+1} \to A_m \to A_{m-1}$ is exact for each m such that there are terms A_{m+1} and A_{m-1} in the diagram.

DEFINITION 2.4.7. A *long exact sequence* in an abelian category $\mathscr{C}$ is an exact sequence $A_\cdot = (A_i, d_i^A)$, where i runs over all integers.

REMARK 2.4.8. A complex is often said to be *acyclic* if it is an exact sequence.

DEFINITION 2.4.9. Let $\mathscr{C}$ be an abelian category.

a. A *short exact sequence* in $\mathscr{C}$ is an exact sequence in $\mathscr{C}$ of the form

$$0 \to A \to B \to C \to 0.$$

b. A *left short exact sequence* in $\mathscr{C}$ is an exact sequence in $\mathscr{C}$ of the form

$$0 \to A \to B \to C.$$

c. A *right short exact sequence* in $\mathscr{C}$ is an exact sequence in $\mathscr{C}$ of the form

$$A \to B \to C \to 0.$$

DEFINITION 2.4.10. Let $\mathscr{C}$ be an abelian category.

a. We say that an epimorphism $g\colon B \to C$ is split if there exists a morphism $t\colon C \to B$ in $\mathscr{C}$ with $g \circ t = \mathrm{id}_C$. In this case, we say that t is a splitting of g.

b. We say that a monomorphism $f\colon A \to B$ is split if there exists a morphism $s\colon B \to A$ in $\mathscr{C}$ with $s \circ f = \mathrm{id}_A$. In this case, we say that s is a splitting of f.

c. We say that a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{2.4.1}$$

in an abelian category $\mathscr{C}$ *splits* if there exists an isomorphism $A \oplus C \xrightarrow{w} B$ in $\mathscr{C}$ with $w \circ \iota_A = f$ and $w \circ \iota_C \circ g = \mathrm{id}_B$, where $\iota_A\colon A \to A \oplus C$ and $\iota_C\colon C \to A \oplus C$ are the inclusion morphisms.

EXAMPLE 2.4.11. Any exact sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is split in **Ab**, but

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is not.

LEMMA 2.4.12. *The following conditions on a short exact sequence* (2.4.1) *are equivalent:*

i. The sequence (2.4.1) *splits.*

ii. The monomorphism $f\colon A \to B$ splits.

iii. The epimorphism $g\colon B \to C$ splits.

PROOF.

i $\Rightarrow$ ii Suppose we have a splitting map $t: C \to B$. Then define $s: B \to A$ by $s(b) = a$ where $f(a) = b - t(g(b))$. This is well-defined as f is injective, and such an a exists since

$$g(b - t(g(b))) = g(b) - g(t(g(b))) = g(b) - g(b) = 0.$$

It splits f as

$$s(f(a)) = s(b) - s(t(g(b))) = s(b),$$

the latter step using the fact that $s \circ t = 0$, which follows in turn from

$$f(s(t(c))) = t(c) - t(g(t(c))) = t(c) - t(c) = 0.$$

ii $\Rightarrow$ i Suppose that we have a splitting map $s: B \to A$. Then define $t: C \to B$ by $t(c) = b - f(s(b))$ where $g(b) = c$. To see this is well defined, note that $f(a) - f(s(f(a))) = 0$ for any $a \in A$.

i+ii $\Rightarrow$ iii Define $w(a, c) = f(a) + t(c)$. Its inverse is $w'(b) = (s(b), g(b))$. We check

$$w' \circ w(a, c) = (s(f(a) + t(c)), g(f(a) + t(c))) = (s(f(a)), g(t(c))) = (a, c)$$

and

$$w \circ w'(b) = w(s(b), g(b)) = f(s(b)) + t(g(b)).$$

It remains to see that $f \circ s + t \circ g = \mathrm{id}_B$. It is easy to see this on elements of the form $f(a)$ or $t(c)$ and that an arbitrary element of b is a sum of these.

iii $\Rightarrow$ i Set $t(b) = w(0, b)$.

$\square$

2.5. Exact functors

DEFINITION 2.5.1. Let $F: \mathscr{C} \to \mathscr{D}$ be an additive functor between abelian categories.

a. We say that F is *left exact* if it preserves exact sequences of the form

$$0 \to A \to B \to C,$$

which is to say that the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact.

b. We say that F is *right exact* if it preserves exactness of exact sequences of the form

$$A \to B \to C \to 0.$$

c. We say that F is *exact* if it is both left and right exact.

The reader will verify the following.

LEMMA 2.5.2. *An additive functor $F: \mathscr{C} \to \mathscr{D}$ between abelian categories is exact (resp., left exact, resp., right exact) if and only if it takes short exact sequences in $\mathscr{C}$ to short exact (resp., left short exact, resp. right short exact) sequences in $\mathscr{D}$. Moreover, F is exact if and only if it preserves three term exact sequences.*

REMARK 2.5.3. We may extend the definition of left and right exact functors to contravariant functors. The requirement that a contravariant functor $F\colon \mathscr{C} \to \mathscr{D}$ be left exact is that it preserves exactness of sequences of the form $0 \to A \to B \to C$, which is to say that $F(C) \to F(B) \to F(A) \to 0$ is exact.

EXAMPLES 2.5.4.

a. If $\mathscr{C}$ is abelian, then the functor $F\colon \mathscr{C} \to \mathscr{C}$ by $F(A) = A \oplus A$ with $F(f) = f \oplus f$ is exact.

b. In R-**mod** for a commutative ring R, the functor t_A is right exact for any R-module A.

c. Let G be a group. For a $\mathbb{Z}[G]$-module A, consider the abelian group

$$A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}$$

of G-invariants in A, with $\mathbb{Z}[G]$-module homomorphisms $A \to B$ restricting to homomorphisms $A^G \to B^G$. The resulting functor $\mathbb{Z}[G]$-**mod** $\to$ **Ab** is left exact but not in general right exact.

Recall that for an additive category $\mathscr{C}$, the functors h^X (and h_X) may be viewed as taking values in **Ab**, and clearly such functors are additive. In fact, they are also left exact.

LEMMA 2.5.5. *Let $\mathscr{C}$ be an abelian category, and let X be an object of $\mathscr{C}$.*

*a. The functor $h_X\colon \mathscr{C} \to$ **Ab** is left exact.*

*b. The functor $h^X\colon \mathscr{C}^{\mathrm{op}} \to$ **Ab** is left exact.*

PROOF. Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be an exact sequence in $\mathscr{C}$. Applying h_X, we obtain homomorphisms

$$0 \to \mathrm{Hom}_{\mathscr{C}}(X, A) \xrightarrow{h_X(f)} \mathrm{Hom}_{\mathscr{C}}(X, B) \xrightarrow{h_X(g)} \mathrm{Hom}_{\mathscr{C}}(X, C)$$

of abelian groups, and we claim this sequence is exact. If $h_X(f)(\alpha) = 0$, then $f \circ \alpha = 0$, but f is a monomorphism, so $\alpha = 0$. Since h_X is a functor, we have $h_X(g) \circ h_X(f) = 0$, and if $\beta \in \ker h_X(g)$, then $g \circ \beta = 0$. Naturality of the kernel implies that β factors through a morphism $X \to \ker g$. But we have canonical isomorphisms

$$A \xrightarrow{\sim} \mathrm{coim}\, f \xrightarrow{\sim} \mathrm{im}\, f \xrightarrow{\sim} \ker g,$$

the first as f is a monomorphism, and the composite of the composite of these with the canonical morphism $\ker g \to B$ is g. Therefore, we obtain a morphism $\alpha\colon X \to A$ satisfying $f \circ \alpha = g$. This proves part a, and part b is just part a with $\mathscr{C}$ replaced by $\mathscr{C}^{\mathrm{op}}$. □

LEMMA 2.5.6. *Let $\mathscr{C}$ be an abelian category. A sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if every sequence

$$\mathrm{Hom}_{\mathscr{C}}(X, A) \xrightarrow{h_X(f)} \mathrm{Hom}_{\mathscr{C}}(X, B) \xrightarrow{h_X(g)} \mathrm{Hom}_{\mathscr{C}}(X, C)$$

is exact.

PROOF. For $X = A$, we get

$$g \circ f = h_X(g) \circ h_X(f)(\mathrm{id}_A) = 0,$$

so we have a monomorphism $s\colon \operatorname{im} f \to \ker g$. For $X = \ker g$ and $\beta\colon \ker g \to B$ the natural monomorphism defined by the kernel, we have $h^X(g)(\beta) = g \circ \iota = 0$, so there exists $\alpha\colon \ker g \to A$ with $f \circ \alpha = \beta$. We then have that β factors a morphism $t\colon \ker g \to \operatorname{im} f$ inverse to s. $\square$

PROPOSITION 2.5.7. *Suppose that $G\colon \mathscr{C} \to \mathscr{D}$ is an additive functor between abelian categories that admits a left (resp., right) adjoint. Then F is left (resp., right) exact.*

PROOF. We treat the case of left exactness, the other case simply being the corresponding statement in opposite categories. Suppose that

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C$$

is a left exact sequence in $\mathscr{C}$. Let F be a left adjoint to G. Then for any $D \in \mathrm{Obj}(\mathscr{D})$, the sequence

$$0 \to h_{F(D)}(A) \xrightarrow{h_{F(D)}(f)} h_{F(D)}(B) \xrightarrow{h_{F(D)}(g)} h_{F(D)}(C)$$

is left exact. Since F is left adjoint to G, this sequence is isomorphic to

$$0 \to h_D(G(A)) \xrightarrow{h_D(G(f))} h_D(G(B)) \xrightarrow{h_D(G(g))} h_D(G(C))$$

as a sequence of abelian groups. Since this holds for all D, the sequence

$$0 \to G(A) \xrightarrow{G(f)} G(B) \xrightarrow{G(g)} G(C)$$

is exact. $\square$

EXAMPLE 2.5.8. The inclusion functor from the category of sheaves on a space X to that of presheaves on X has a left adjoint called sheafification. The inclusion functor is therefore left exact, and the sheafification functor is in fact exact.

The following embedding theorem allows us to do most of the homological algebra that can be done in the category of R-modules for any R in an arbitrary abelian category.

THEOREM 2.5.9 (Freyd-Mitchell). *If $\mathscr{C}$ is a small abelian category, then there exists a ring R and an exact, fully faithful functor $\mathscr{C} \to R$-**mod**.*

In other words, $\mathscr{C}$ is equivalent to a full, abelian subcategory of R-**mod** for some ring R. We can use this as follows: suppose there is a result we can prove about exact diagrams in R-modules for all R, like the snake lemma. We then have the result in all abelian categories, since we can take a small full, abelian subcategory containing the objects in which we are interested and embed it into some category of left R-modules. If the result holds in that category, then by exactness of the embedding, the result will hold in the original category.

While we do not prove the Freyd-Mitchell embedding theorem, we will give a few remarks on its details (which hopefully are not so far off from the truth).

REMARK 2.5.10. To prove Theorem 2.5.9, the first step is to embed $\mathscr{C}$ in the (opposite category of the) full subcategory of $\mathbf{Func}(\mathscr{C},\mathbf{Ab})$ consisting of left exact, additive functors, which is an abelian category using $h_{\mathscr{C}}$. While $h_{\mathscr{C}}$ is not exact as an embedding into the category of additive functors in $\mathbf{Func}(\mathscr{C},\mathbf{Ab})$, it is exact as an embedding into the latter category. (Another method is to embed $\mathscr{C}$ in the category of $\mathbf{Pro}(\mathscr{C})$ of cofiltered limits of the objects of $\mathscr{C}$, viewed as a subcategory of $\mathbf{Func}(\mathscr{C},\mathbf{Ab})^{\mathrm{op}}$.) In either case, the category $\mathscr{E}$ in question has what is called a projective generator G which we can choose so that every object in $\mathscr{C}$ is a quotient object of G, and the point is that $R_G = \mathrm{Hom}_{\mathscr{E}}(G,G)$ is a ring under its usual addition and composition. Then h_G induces a functor $\mathscr{C} \to R_G$-**mod**, which is the exact, fully faithful functor in question.

2.6. Standard lemmas

We begin with the following extremely useful result.

THEOREM 2.6.1 (Snake lemma). *Suppose that we have a commutative diagram*

$$\begin{tikzcd} D \arrow[r, "e"] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \end{tikzcd} \tag{2.6.1}$$

with exact rows. Then there is a homomorphism $\psi\colon \ker\gamma \to \operatorname{coker}\alpha$ fitting into a larger commutative diagram

$$\begin{tikzcd} & \ker\alpha \arrow[r, dashed] \arrow[d] & \ker\beta \arrow[r, dashed] \arrow[d] & \ker\gamma \arrow[d] \arrow[dddll, dashed] & \\ D \arrow[r] \arrow[ur, dashed] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & D' \\ & \operatorname{coker}\alpha \arrow[r, dashed] & \operatorname{coker}\beta \arrow[r, dashed] & \operatorname{coker}\gamma \arrow[ur, dashed] & \end{tikzcd}$$

and the resulting eight-term sequence

$$D \to \ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to D'$$

is exact.

PROOF. We define ψ as follows. For $c \in \ker\gamma$, find $b \in B$ with $g(b) = c$. Then $g' \circ \beta(b) = \gamma(c) = 0$, so $\beta(b) = f'(a')$ for some $a' \in A'$. Let $\psi(c)$ denote the image $\bar{a}'$ of a' in $\operatorname{coker}\alpha$. To see that this is well-defined, note that if $b_2 \in B$ also satisfies $g(b_2) = c$, then $g(b - b_2) = 0$, so $b_2 - b = f(a)$ for some $a \in A$. We then have

$$\beta(b_2) = \beta(b) + \beta \circ f(a) = \beta(b) + f' \circ \alpha(a),$$

so $b_2 = f'(a' + \alpha(a))$. But $a' + \alpha(a)$ has image $\bar{a}'$ in $\operatorname{coker}\alpha$, so ψ is well-defined.

We now check that the other maps are well-defined. Since

$$f' \circ \alpha \circ e = \beta \circ f \circ e = 0$$

and f' is injective, we have $\alpha \circ e = 0$, so $\operatorname{im} e \subseteq \ker\alpha$. Since

$$\beta \circ f(\ker\alpha) = f' \circ \alpha(\ker\alpha) = 0,$$

we have $f(\ker\alpha) \subseteq \ker\beta$. Similarly, $g(\ker\beta) \subset \ker\gamma$. Also, if $\bar{a}' \in \operatorname{coker}\alpha$, then we may lift it to $a' \in A'$, map to $b' \in B'$, and then project to $\bar{b}' \in \operatorname{coker}\beta$. This is well-defined as any other choice of a' differs by some $a \in A$, which causes $\bar{b}'$ to change by the image of $\beta(f(a))$, which is zero. Thus f' induces a well-defined map

$$\bar{f}' \colon \operatorname{coker}\alpha \to \operatorname{coker}\beta.$$

Similarly, we have a well-defined

$$\bar{g}' \colon \operatorname{coker}\beta \to \operatorname{coker}\gamma.$$

Finally, $\operatorname{im} g = C$, so $\operatorname{im}\gamma \subseteq \operatorname{im} g'$, and therefore $h'(\operatorname{im}\gamma) = 0$. Thus h' induces a map

$$\bar{h}' \colon \operatorname{coker}\gamma \to D'.$$

It is clear from the above definitions that the 8-term sequence is a complex at all the terms but $\ker\gamma$ and $\operatorname{coker}\alpha$. Let $b \in \ker\beta$. Then $\psi(g(b))$ is given by considering $\beta(b) = 0$, lifting it to some $a' \in A'$, which we may take to be 0, and projecting to $\operatorname{coker}\alpha$. Hence $\psi(g(\ker\beta)) = 0$. On the other hand, if $c \in \ker\gamma$, then $\bar{f}'(\psi(c))$ is given by definition by projecting $\beta(b)$ to $\operatorname{coker}\beta$, where $g(b) = c$, hence is zero.

We now check exactness at each term. We have

$$\ker\alpha \cap \ker f = \ker\alpha \cap \operatorname{im} e = \operatorname{im} e,$$

so the sequence is exact at $\ker\alpha$. Next, if $b \in \ker\beta \cap \ker g$, then there exists $a \in A$ with $f(a) = b$.

Since $f' \circ \alpha(a) = \beta \circ f(a) = 0$ and f' is injective, we have $\alpha(a) = 0$, or $a \in \ker\alpha$. Hence

$$f(\ker\alpha) = \ker(\ker\beta \to \ker\gamma),$$

and we have exactness at $\ker\beta$.

If $c \in \ker\psi$, then whenever $g(b) = c$ and $f'(a') = \beta(b)$, we have $\bar{a}' = 0$, letting $\bar{a}'$ denote the image of $a' \in \operatorname{coker}\alpha$. We then have $a' = f(a)$ for some $a \in A$, so $b_2 = b - f(a)$ still satisfies $f(b_2) = c$, but $\beta(b_2) = 0$. So $b_2 \in \ker\beta$, and we have exactness at $\ker\gamma$.

If $\bar{a}' \in \operatorname{coker}\alpha$ is the image of $a' \in A'$ and $\bar{f}'(\bar{a}') = 0$, then there exists $b \in B$ with $\beta(b) = f'(a')$. Now

$$\gamma(g(b)) = g'(\beta(b)) = g'(f'(a')) = 0,$$

so $g(b) \in \ker\gamma$, and $\psi(g(b)) = \bar{a}'$. Hence, we have exactness at $\operatorname{coker}\alpha$.

If $\bar{b}' \in \operatorname{coker}\beta$ is the image of $b' \in B'$ and $\bar{g}'(\bar{b}') = 0$, then there exists $c \in C$ with $g(b') = \gamma(c)$. Now $c = g(b)$ for some $b \in B$. And $b_2' = b' - b$ has image $\bar{b}'$ in $\operatorname{coker}\beta$. On the other hand, $f'(b_2') = 0$, so $b_2' = f'(a')$ for some $a' \in A'$. If $\bar{a}' \in \operatorname{coker}\alpha$ is the image of a', then $\bar{f}'(\bar{a}') = \bar{b}'$ as the image of b_2' in $\operatorname{coker}\beta$. Thus, we have exactness at $\operatorname{coker}\beta$.

Finally, let $\bar{c}' \in \operatorname{coker}\gamma$ be the image of $c' \in C'$. Note that $\bar{h}'(\bar{c}') = h(c')$. If this is zero, then $c' = g'(b')$ for some $b' \in B'$, and $\bar{c}'$ is the image of the projection $\bar{b}'$ of b' to $\operatorname{coker}\beta$ under $\bar{g}'$. □

LEMMA 2.6.2. *Suppose we have a commutative diagram as in* (2.6.1) *and that both g and f' are split. Then the snake map $\psi\colon \ker\gamma \to \operatorname{coker}\alpha$ is zero.*

PROOF. Let $c \in \ker\gamma$. Let t split g and s' split f'. By Lemma **??**, we also have a splitting map
$$t'\colon \ker h' \to A'.$$
It is easy to check that $t' \circ \gamma = \beta \circ t$. It follows that $t(\ker\gamma) \subseteq \ker\beta$. Thus $\psi(c)$, which is the image of $s'(\beta(t(c)))$ in $\operatorname{coker}\alpha$, is zero. □

2.7. Complexes

Let $\mathscr{C}$ be an abelian category. We have already defined the notion of a chain complex in $\mathscr{C}$ above. In addition to chain complexes, we will also deal with cochain complexes, which are likewise defined, but with increasing superscripts replacing decreasing subscripts. This is primarily a notational convenience, but it is a very useful one. When the choice of chain or cochain complexes matters little, we will typically choose to work with cochain complexes.

DEFINITION 2.7.1. A *cochain complex* $A^\cdot = (A^i, d_A^i)$ is a diagram
$$\cdots \to A^{i-1} \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \to \cdots,$$
such that $d_A^i \circ d_A^{i-1} = 0$ for all $i \in \mathbb{Z}$. Again, the d_A^i are referred to as differentials.

REMARK 2.7.2. We refer to both chain complexes and cochain complexes as *complexes*, though we will usually mean the latter if we do not specify. We will often state facts simply for cochain complexes that have a direct translation to the setting of chain complexes.

DEFINITION 2.7.3. Let $A^\cdot = (A^i, d_A^i)$ and $B_\cdot = (B^i, d_B^i)$ be complexes. A *morphism of complexes $f^\cdot\colon A^\cdot \to B^\cdot$* is a sequence of morphisms $f^i\colon A^i \to B^i$ commuting with the differentials of the complexes in the sense that
$$d_B^i \circ f^i = f^{i-1} \circ d_A^i$$
for all $i \in \mathbb{Z}$.

REMARK 2.7.4. We may consider the category $\mathbf{Ch}(\mathscr{C})$ of cochain complexes in $\mathscr{C}$, where morphisms of chain complexes $A^\cdot = (A^i, d^i)$ and $B^\cdot = (B^i, d'^i)$ are morphisms $A^i \to B^i$ for each i commuting with the differentials.

The reader can easily check the following.

PROPOSITION 2.7.5. *Let $\mathscr{C}$ be an abelian category. Then the category* $\mathbf{Ch}(\mathscr{C})$ *is an abelian category as well.*

DEFINITION 2.7.6. Let $A_\cdot$ (resp., $A^\cdot$) be a chain complex (resp., cochain complex).

a. We say that $A_\cdot$ (resp., $A^\cdot$) is *bounded below* if $A_i = 0$ (resp., $A^i = 0$) for all $i < N$ for some $N \in \mathbb{Z}$.

b. We say that $A_\cdot$ (resp., $A^\cdot$) is *bounded above* if $A_i = 0$ (resp., $A^i = 0$) for all $i > N$ for some $N \in \mathbb{Z}$.

c. We say that $A_\cdot$ (resp., $A^\cdot$) is *bounded* if it is both bounded below and bounded above.

In most examples that we shall explore, our chain complexes (resp., cochain complexes) will be bounded below (resp., bounded above), usually with $N = 0$.

DEFINITION 2.7.7.

a. We define ith *homology* (object) of a chain complex $A_\cdot$ by

$$H_i(A_\cdot) = \ker d_i^A / \operatorname{im} d_{i+1}^A$$

for each $i \in \mathbb{Z}$.

b. We define the ith *cohomology* (object) of a cochain complex $A^\cdot$ by

$$H^i(A^\cdot) = \ker d_A^i / \operatorname{im} d_A^{i-1}$$

for $i \in \mathbb{Z}$.

REMARK 2.7.8. For a complex of abelian groups, we speak of homology and cohomology groups, as opposed to objects.

EXAMPLE 2.7.9. Consider the chain complex $A^\cdot = (\mathbb{Z} \oplus \mathbb{Z}, d_A^i)$ with $d_A^i = 3\,\mathrm{id}_{\mathbb{Z}} \oplus 0$ and $d_A^{i+1} = 0 \oplus 2\,\mathrm{id}_{\mathbb{Z}}$ for i even. Then $H_i(A_\cdot) = \mathbb{Z}/2\mathbb{Z}$ and $H_{i+1}(A_\cdot) = \mathbb{Z}/3\mathbb{Z}$ for i even.

LEMMA 2.7.10. *Let $f^\cdot : A^\cdot \to B^\cdot$ be a morphism of complexes. Then we have natural homomorphisms*

$$f_*^i : H^i(A^\cdot) \to H^i(B^\cdot)$$

for each $i \in \mathbb{Z}$.

PROOF. Let $A^\cdot = (A^i, d_A^i)$ and $B^\cdot = (B^i, d_B^i)$. We have $f^i \circ d_A^{i+1} = d_B^{i+1} \circ f^{i+1}$ for all i, so

$$f^i(\operatorname{im} d_A^{i+1}) \subseteq \operatorname{im} d_B^{i+1},$$

and

$$d_B^i(f^i(\ker d_A^i)) = f^{i-1}(d_A^i(\ker d_A^i)) = 0,$$

so

$$f^i(\ker d_A^i) \subseteq \ker d_B^i.$$

Hence, we have an induced morphism

$$f_*^i : \frac{\ker d_A^i}{\operatorname{im} d_A^{i+1}} \to \frac{\ker d_B^i}{\operatorname{im} d_B^{i+1}},$$

as desired. □

REMARK 2.7.11. Of course, the analogous result holds in cohomology as a consequence.

REMARK 2.7.12. We note that a sequence

$$0 \to A^\cdot \xrightarrow{f^\cdot} B^\cdot \xrightarrow{g^\cdot} C^\cdot \to 0$$

in $\mathbf{Ch}(\mathscr{C})$ is exact if and only if each sequence

$$0 \to A^i \xrightarrow{f^i} B^i \xrightarrow{g^i} C^i \to 0$$

is exact in $\mathscr{C}$.

THEOREM 2.7.13. *Let*

$$0 \to A^\cdot \xrightarrow{f^\cdot} B^\cdot \xrightarrow{g^\cdot} C^\cdot \to 0$$

be an exact sequence of cochain complexes of R-modules. Then we have a long exact sequence in cohomology

$$\cdots \to H^i(A^\cdot) \xrightarrow{f_*^i} H^i(B^\cdot) \xrightarrow{g_*^i} H^i(C^\cdot) \xrightarrow{\partial^i} H^{i+1}(A^\cdot) \to \cdots,$$

where ∂_i is the map given by the snake lemma applied to the diagram (2.7.1) *below.*

PROOF. Consider the following diagram

$$\begin{tikzcd} & \operatorname{coker} d_A^{i-1} \arrow[r] \arrow[d, "d_A^i"] & \operatorname{coker} d_B^{i-1} \arrow[r] \arrow[d, "d_B^i"] & \operatorname{coker} d_C^{i-1} \arrow[r] \arrow[d, "d_C^i"] & 0 \\ 0 \arrow[r] & \ker d_A^{i+1} \arrow[r] & \ker d_B^{i+1} \arrow[r] & \ker d_C^{i+1}. & \end{tikzcd} \tag{2.7.1}$$

By the snake lemma, the two rows are exact, and the diagram commutes since we have $d_B^i \circ f^i = f^{i+1} \circ d_A^i$ and $d_C^i \circ g^i = g^{i+1} \circ d_B^i$ as morphisms $A^i \to B^{i+1}$ and $B^i \to C^{i+1}$, respectively, by naturality of the cokernel and kernel. Since we have natural isomorphisms

$$H^i(A^\cdot) \cong \ker(A^i/\operatorname{im} d_A^{i-1} \to A^{i+1}) \quad \text{and} \quad H^{i+1}(A^\cdot) \cong \operatorname{coker}(A^i \to \ker d_A^{i+1}),$$

we apply the snake lemma to obtain an exact sequence

$$H^i(A^\cdot) \xrightarrow{f_*^i} H^i(B^\cdot) \xrightarrow{g_*^i} H^i(C^\cdot) \xrightarrow{\partial^i} H^{i+1}(A^\cdot) \xrightarrow{f_*^{i+1}} H^{i+1}(B^\cdot) \xrightarrow{g_*^{i+1}} H^{i+1}(C^\cdot),$$

and the result follows by splicing together these sequences. □

DEFINITION 2.7.14. Let $A^\cdot = (A^i, d_A^i)$ and $B^\cdot = (B^i, d_B^i)$ be chain complexes. Let $f^\cdot, g^\cdot : A^\cdot \to B^\cdot$ be morphisms of chain complexes.

a. A *chain homotopy* from $f^\cdot$ to $g^\cdot$ is a sequence $s^\cdot = (s^i)_{i\in\mathbb{Z}}$ of morphisms $s^i : A^i \to B^{i-1}$ satisfying

$$f^i - g^i = d_B^{i-1} \circ s^i + s^{i+1} \circ d_A^i$$

for all $i \in \mathbb{Z}$.

b. We say that $f^\cdot$ and $g^\cdot$ are *chain homotopic*, and write $f^\cdot \sim g^\cdot$, if there exists a homotopy from $f^\cdot$ to $g^\cdot$.

c. If $f^\cdot$ is (chain) homotopic to 0, then $f^\cdot$ is said to be *null-homotopic*.

The maps $s_\cdot$ defining a null-homotopy fit into a (not usually) diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"'] \arrow[dl, "s^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"'] \arrow[dl, "s^{i+1}"'] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \cdots.
\end{tikzcd}$$

PROPOSITION 2.7.15. *Assume that $f^\cdot$ and $g^\cdot$ are homotopic as maps $A^\cdot \to B^\cdot$. Then the maps f_*^i and g_*^i on homology are equal for all $i \in \mathbb{Z}$.*

PROOF. It suffices to assume that $g = 0$, since the ith homology functor from $\mathbf{Ch}(\mathscr{C})$ to $\mathscr{C}$ is additive. So, we must show that $f_*^i = 0$ for all i, which is to say that $f^i(\ker d_A^i) \subseteq \operatorname{im} d_A^{i-1}$. Since $f^i = d_B^{i-1} \circ s^i + s^{i+1} \circ d_A^i$, we have

$$f^i(\ker d_A^i) = d_B^{i-1}(s^i(\ker d_A^i)) \subseteq \operatorname{im} d_B^{i-1},$$

so $f_*^i = 0$. $\square$

DEFINITION 2.7.16. A morphism of complexes $f^\cdot : A^\cdot \to B^\cdot$ is a *homotopy equivalence* if there exists a morphism $g^\cdot : B^\cdot \to A^\cdot$ such that $g^\cdot \circ f^\cdot \sim \mathrm{id}_{A^\cdot}$ and $f^\cdot \circ g^\cdot \sim \mathrm{id}_{B^\cdot}$.

DEFINITION 2.7.17. Let $A^\cdot$ be a complex, and let $j \in \mathbb{Z}$. The *shift* by j of the complex A is the complex $A[j]^\cdot$ with $A[j]^i = A^{i+j}$ and $d_{A[j]}^i = (-1)^j d_A^i$ for all $i \in \mathbb{Z}$.

NOTATION 2.7.18. Let $n \geq 1$, let $A_i, B_i \in \mathscr{C}$ be objects in an abelian category $\mathscr{C}$ for $1 \leq i \leq n$, and let $f_{i,j} : A_i \to B_j$ be morphisms in $\mathscr{C}$ for $1 \leq i, j \leq n$. Let $A = \bigoplus_{i=1}^n A_i$ and $B = \bigoplus_{i=1}^n B_i$, and let $\iota_i : B_i \to B$ and $\pi_i : A \to A_i$ denote the canonical inclusions and projections. Then we use the matrix $(f_{i,j})$ to represent the morphism $f : A \to B$ in $\mathscr{C}$ that is the sum

$$f = \sum_{i=1}^n \sum_{j=1}^n \iota_j \circ f_{i,j} \circ \pi_j.$$

DEFINITION 2.7.19. Let $f^\cdot : A^\cdot \to B^\cdot$ be a morphism of complexes. The *cone* of $f^\cdot$ is the complex $\operatorname{Cone}(f)^\cdot$ with

$$\operatorname{Cone}(f)^i = A^{i+1} \oplus B^i$$

and

$$d_{\operatorname{Cone}(f)}^i = \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} : A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1}.$$

PROPOSITION 2.7.20. *The cone of $f^\cdot : A^\cdot \to B^\cdot$ fits in a short exact sequence*

$$0 \to B^\cdot \to \operatorname{Cone}(f)^\cdot \to A[1]^\cdot \to 0,$$

and the resulting long exact sequence has ith connecting homomorphism

$$H^i(A^\cdot[1]) = H^{i+1}(A^\cdot) \to H^{i+1}(B^\cdot)$$

equal to f_*^{i+1}.

PROOF. The differential on $\mathrm{Cone}(f)^{\cdot}$ preserves $B^{\cdot}$ and agrees with the differential on $B^{\cdot}$, so $B^{\cdot}$ is a subcomplex. The quotient complex has ith term A^{i+1} and differential induced by the negative of the differential on A, so is canonically isomorphic to $A[1]^{\cdot}$. Since $\mathrm{Cone}(f)^i = A^{i+1} \oplus B^i$, we have a natural splitting of the surjection $\mathrm{Cone}(f)^i \to A^{i+1}$ given by the inclusion. The connecting homomorphism is then induced by the composition

$$\ker d_A^{i+1} \to A^{i+1} \to \mathrm{Cone}(f)^i \xrightarrow{d^i_{\mathrm{Cone}(f)}} \mathrm{Cone}(f)^{i+1},$$

which is given by the sum of $-d_A^{i+1}$ and f^i as maps to $A^{i+2} \oplus B^{i+1}$, but then has image in B^{i+1} (since d_A^{i+1} is zero on $-d_A^{i+1}$) and therefore agrees with f^{i+1} as a morphism to B^{i+1}. In other words, the connecting homomorphism is canonically identified with f_*^{i+1}. □

2.8. Total complexes

DEFINITION 2.8.1. A double (cochain) complex in an abelian category $\mathscr{C}$ is a complex in $\mathbf{Ch}(\mathscr{C})$.

The data of a double complex $A^{\cdot\cdot}$ consists of complexes $(A^{i,\cdot}, d^{i,\cdot})$ of objects of $\mathrm{Ch}(\mathscr{C})$, which is to say that each $A^{i,\cdot}$ is a complex in $\mathscr{C}$ with differentials of its own, and the $d^{i,\cdot}$ are morphisms between $A^{i,\cdot} \to A^{i+1,\cdot}$, so commute with the differentials on these complexes. We rephrase this as follows.

REMARK 2.8.2. A double complex $(A^{\cdot\cdot}, d^{\cdot\cdot})$ in $\mathscr{C}$ is a diagram of the form

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & A^{i-1,j+1} \arrow[r, "d_h^{i-1,j+1}"] \arrow[u] & A^{i,j+1} \arrow[r, "d_h^{i,j+1}"] \arrow[u] & A^{i+1,j+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & A^{i-1,j} \arrow[r, "d_h^{i-1,j}"] \arrow[u, "d_v^{i-1,j}"] & A^{i,j} \arrow[r, "d_h^{i,j}"] \arrow[u, "d_v^{i,j}"] & A^{i+1,j} \arrow[r] \arrow[u, "d_v^{i+1,j}"] & \cdots \\
\cdots \arrow[r] & A^{i-1,j-1} \arrow[r, "d_h^{i-1,j-1}"] \arrow[u, "d_v^{i-1,j-1}"] & A^{i,j-1} \arrow[r, "d_h^{i,j-1}"] \arrow[u, "d_v^{i,j-1}"] & A^{i+1,j-1} \arrow[r] \arrow[u, "d_v^{i+1,j-1}"] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

such that (for all $i, j \in \mathbb{Z}$)

i. each row is a complex: i.e., $d_h^{i+1,j} \circ d_h^{i,j} = 0$,

ii. each column is a complex: i.e., $d_v^{i,j+1} \circ d_v^{i,j} = 0$, and

iii. the squares commute: i.e., $d_h^{i,j+1} \circ d_v^{i,j} = d_v^{i+1,j} \circ d_h^{i,j}$.

DEFINITION 2.8.3. The degree of the term $A^{i,j}$ of a double complex $A^{\cdot\cdot}$ is $i + j$.

DEFINITION 2.8.4. Let $A^{\cdot\cdot}$ be a double complex in an abelian category $\mathscr{C}$.

a. Suppose that $\mathscr{C}$ admits coproducts. The total sum complex $\mathrm{Tot}^{\oplus}(A)^{\cdot}$ of $A^{\cdot\cdot}$ is the complex

$$\mathrm{Tot}^{\oplus}(A)^k = \bigoplus_{i\in\mathbb{Z}} A^{i,k-i}$$

with the differential

$$d^k = \sum_{i\in\mathbb{Z}} (d_h^{i,k-i} + (-1)^i d_v^{i,k-i})\colon \bigoplus_{i\in\mathbb{Z}} A^{i,k-i} \to \bigoplus_{i\in\mathbb{Z}} A^{i,k+i-1}$$

where the differentials $d_h^{i,k-i}$ and $d_v^{i,k-i}$ are taken to be zero on the $A^{j,k-j}$ with $j \neq i$.

b. Suppose that $\mathscr{C}$ admits products. The total product complex $\mathrm{Tot}^{\Pi}(A)^{\cdot}$ of $A^{\cdot\cdot}$ is the complex

$$\mathrm{Tot}^{\Pi}(A)^k = \prod_{i\in\mathbb{Z}} A^{i,k-i}$$

with the differential

$$d^k = (d_h^{i,k-i} + (-1)^i d_v^{i,k-i})_{i\in\mathbb{Z}}\colon \prod_{i\in\mathbb{Z}} A^{i,k-i} \to \prod_{i\in\mathbb{Z}} A^{i,k+1-i},$$

where the differentials $d_h^{i,k-i}$ and $d_v^{i,k-i}$ are taken to be zero on the $A^{j,k-j}$ with $j \neq i$.

REMARK 2.8.5. We remark that $\mathrm{Tot}^{\oplus}(A)^{\cdot}$ is in fact a complex (when it exists). That is,

$$\begin{aligned}
d^{k+1} \circ d^k &= \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} (d_h^{i,k+1-i} + (-1)^i d_v^{i,k+1-i}) \circ (d_h^{j,k-j} + (-1)^j d_v^{j,k-j}) \\
&= \sum_{i\in\mathbb{Z}} ((-1)^i d_h^{i,k+1-i} \circ d_v^{i,k-i} + (-1)^i d_v^{i,k+1-i} \circ d_h^{i-1,k+1-i}) \\
&= \sum_{i\in\mathbb{Z}} ((-1)^i d_h^{i,k+1-i} \circ d_v^{i,k-i} + (-1)^{i+1} d_v^{i+1,k-i} \circ d_h^{i,k-i}) = 0.
\end{aligned}$$

Similarly, $\mathrm{Tor}^{\Pi}(A)^{\cdot}$ is a complex as well.

DEFINITION 2.8.6. An nth quadrant double complex, for $1 \le n \le 4$, is a double complex $A^{\cdot\cdot}$ such that $A^{i,j} = 0$ for (i,j) not in the closed nth quadrant of $\mathbb{R}^2$. An upper, lower, left, or right half-plane double complex is one that is zero outside of said closed half-plane.

REMARK 2.8.7. Recall that the closed 1st, 2nd, 3rd, and 4th quadrants of $\mathbb{R}^2$ are $\mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$, $\mathbb{R}_{\le 0} \times \mathbb{R}_{\ge 0}$, $\mathbb{R}_{\le 0} \times \mathbb{R}_{\le 0}$, and $\mathbb{R}_{\ge 0} \times \mathbb{R}_{\le 0}$, respectively. The closed upper, lower, left, and right half-planes are $\mathbb{R} \times \mathbb{R}_{\ge 0}$, $\mathbb{R} \times \mathbb{R}_{\le 0}$, $\mathbb{R}_{\le 0} \times \mathbb{R}$, and $\mathbb{R}_{\ge 0} \times \mathbb{R}$, respectively.

REMARK 2.8.8. If $A^{\cdot\cdot}$ is either a first or a fourth quadrant double complex, then

$$\mathrm{Tot}^{\oplus}(A)^{\cdot} = \mathrm{Tot}^{\Pi}(A)^{\cdot},$$

and it exists for any abelian category $\mathscr{C}$, since there are only finitely many nonzero terms in the complex of a given degree. In this case, we simply write $\mathrm{Tot}(A)^{\cdot}$ for either total complex.

DEFINITION 2.8.9. Let $A^{\cdot\cdot}$ be a double complex. We define the truncated quotient complexes $({}^{I}\tau^{\geq n}A)^{\cdot\cdot}$ and $({}^{II}\tau^{\geq n}A)^{\cdot\cdot}$ of $A^{\cdot\cdot}$ by

$$({}^{I}\tau^{\geq n}A)^{i,j} = \begin{cases} A^{i,j} & \text{if } i > n \\ \operatorname{coker} d_h^{n-1,j} & \text{if } i = n \\ 0 & \text{if } i < n \end{cases} \quad \text{and} \quad ({}^{II}\tau^{\geq n}A)^{i,j} = \begin{cases} A^{i,j} & \text{if } j > n \\ \operatorname{coker} d_v^{j,n-1} & \text{if } j = n \\ 0 & \text{if } j < n, \end{cases}$$

as well as subcomplexes $({}^{I}\tau^{\leq n}A)^{\cdot\cdot}$ and $({}^{II}\tau^{\leq n}A)^{\cdot\cdot}$ by

$$({}^{I}\tau^{\leq n}A)^{i,j} = \begin{cases} A^{i,j} & \text{if } i < n \\ \ker d_h^{n,j} & \text{if } i = n \\ 0 & \text{if } i > n \end{cases} \quad \text{and} \quad ({}^{II}\tau^{\leq n}A)^{i,j} = \begin{cases} A^{i,j} & \text{if } j < n \\ \ker d_v^{j,n} & \text{if } j = n \\ 0 & \text{if } j > n. \end{cases}$$

PROPOSITION 2.8.10. *Suppose that $\mathscr{C}$ is an abelian category that admits direct products (resp., direct sums). Let $A^{\cdot\cdot}$ be an upper (or left) double half-plane complex in $\mathscr{C}$ with exact columns (resp., exact rows) or a right (or lower) double half-plane complex with exact rows (resp., exact columns). Then* $\mathrm{Tot}^{\oplus}(A)^{\cdot}$ *(resp.,* $\mathrm{Tot}^{\Pi}(A)^{\cdot}$*) is exact.*

PROOF. By interchanging rows and columns, we may suppose that $A^{\cdot\cdot}$ is an upper or lower half-plane complex with either exact rows or exact columns. By working in the opposite category (and changing the signs of the degrees), we may focus on the case of a lower half-plane complex. Suppose first that $A^{\cdot\cdot}$ has exact rows. For $n \in \mathbb{Z}$, consider the truncated complex $({}^{I}\tau^{\leq n}A)^{\cdot\cdot}$, which then has exact rows as well. Then a shift by n in the horizontal direction is a third quadrant double complex $B_n^{\cdot\cdot}$. If we can show $\mathrm{Tot}(B_n)^{\cdot}$ to be exact (for all n), then $\mathrm{Tot}^{\oplus}(A)^{\cdot}$ will be exact as well, as the reader may check that

$$\mathrm{Tot}^{\oplus}(A)^{\cdot} \cong \varinjlim_{n} \mathrm{Tot}(B_n)^{\cdot},$$

the limit taken with respect to the morphisms induced by the inclusion morphisms $B_n^{\cdot} \to B_{n+1}^{\cdot}$. Thus, it suffices to consider the case of an exact left half-plane complex with exact rows, which is equivalent to the case of an exact lower half-plane complex $A^{\cdot\cdot}$ with exact columns, for which we must show that $\mathrm{Tot}^{\Pi}(A)^{\cdot}$ is exact. Since we can always shift $A^{\cdot\cdot}$ horizontally, it suffices to check that $H^0(\mathrm{Tot}^{\Pi}(A)^{\cdot}) = 0$.

We now work in the category of R-modules for a ring R without loss of generality. Let

$$a = (a_i) \in \mathrm{Tot}^{\Pi}(A)^0 = \prod_{i=0}^{\infty} A^{i,-i}.$$

with $d^0(a) = 0$. Choose $b_0 \in A^{0,-1}$ with $d_v^{0,-1}(b_0) = a_0$ by exactness of the 0th column, and set $b_0' = 0 \in A^{-1,0}$. Suppose we have defined $b_j \in A^{i,-i-1}$ for $0 \leq j \leq i$ with

$$(-1)^i d_v^{i,-i-1}(b_i) + d_h^{i-1,-i}(b_{i-1}) = a_i.$$

Then

$$\begin{aligned}(-1)^{i+1} d_v^{i+1,-i-1}(a_{i+1} - d_h^{i,-i-1}(b_i)) &= (-1)^{i+1} d_v^{i+1,-i-1}(a_{i+1}) + (-1)^i d_h^{i,-i} \circ d_v^{i,-i-1}(b_i) \\ &= (-1)^{i+1} d_v^{i+1,-i-1}(a_{i+1}) + d_h^{i,-i}(a_i),\end{aligned}$$

which is the $(i+1,-i)$-coordinate of $d^{i+1}(a)$, hence 0. By exactness of the $(i+1)$th column, we may then choose $b_{i+1} \in A^{i+1,-i}$ with

$$(-1)^{i+1} d_v^{i+1,-i}(b_{i+1}) = a_{i+1} - d_h^{i,-i-1}(b_i),$$

completing the induction. We then set $b = (b_i) \in \mathrm{Tot}^{\Pi}(A)^{-1}$ and note that $d^{-1}(b) = a$, finishing the proof. $\square$

CHAPTER 3

Derived Functors

3.1. δ-functors

Let us return briefly to the general setting. Suppose that $\mathscr{C}$ and $\mathscr{D}$ are abelian categories.

DEFINITION 3.1.1. A homological δ-functor is a sequence of additive functors $F_i\colon \mathscr{C} \to \mathscr{D}$ for $i \in \mathbb{Z}$, together with, for every exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $\mathscr{C}$, morphisms $\delta_i\colon F_i(C) \to F_{i-1}(A)$ fitting in a long exact sequence

$$\cdots \to F_i(A) \xrightarrow{F_i(f)} F_i(B) \xrightarrow{F_i(g)} F_i(C) \xrightarrow{\delta_i} F_{i-1}(A) \to \cdots$$

which are natural in the sense that if we have a morphism of short exact sequences in $\mathscr{C}$,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0, \end{tikzcd}$$

then we obtain a morphism of long exact sequences in $\mathscr{D}$,

$$\begin{tikzcd} \cdots \arrow[r] & F_i(A) \arrow[r] \arrow[d] & F_i(B) \arrow[r] \arrow[d] & F_i(C) \arrow[r] \arrow[d] & F_{i-1}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & F_i(A') \arrow[r] & F_i(B') \arrow[r] & F_i(C') \arrow[r] & F_{i-1}(A') \arrow[r] & \cdots. \end{tikzcd}$$

EXAMPLE 3.1.2. Define functors F_0, $F_1\colon \mathbf{Ab} \to \mathbf{Ab}$ by $F_0(A) = A/pA$ and

$$F_1(A) = A[p] = \{a \in A \mid pa = 0\}$$

for any abelian group A, and set $F_i = 0$ otherwise. Given an exact sequence

$$0 \to A \to B \to C \to 0$$

in **Ab**, we obtain a long exact sequence

$$0 \to A[p] \to B[p] \to C[p] \xrightarrow{\delta_1} A/pA \to B/pB \to C/pC \to 0$$

from the snake lemma applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\cdot p"] & B \arrow[r] \arrow[d, "\cdot p"] & C \arrow[r] \arrow[d, "\cdot p"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

This defines a δ-functor.

DEFINITION 3.1.3. A (homological) *universal δ-functor* is a δ-functor $F_. = (F_i, \delta_i)$ with $F_i : \mathscr{C} \to \mathscr{D}$ such that if $G_. = (G_i, \delta'_i)$ is any other δ-functor with $G_i : \mathscr{C} \to \mathscr{D}$ for which there exists a natural transformation $\eta_0 : G_0 \rightsquigarrow F_0$, then η_0 extends to a *morphism of δ-functors*, i.e., a sequence of natural transformations $\eta_i : G_i \rightsquigarrow F_i$ such that

$$\begin{tikzcd} G_i(C) \arrow[r, "\delta'_i"] \arrow[d, "(\eta_i)_C"] & G_{i-1}(A) \arrow[d, "(\eta_{i-1})_A"] \\ F_i(C) \arrow[r, "\delta_i"] & F_{i-1}(A) \end{tikzcd}$$

commutes for any short exact sequence in $\mathscr{C}$:

$$0 \to A \to B \to C \to 0.$$

(That is, we get a morphism of the associated long exact sequences.)

We have analogous notions in cohomology.

DEFINITION 3.1.4. A cohomological δ-functor is a sequence of additive functors $F^i : \mathscr{C} \to \mathscr{D}$ for $i \in \mathbb{Z}$, together with, for every exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

in $\mathscr{C}$, morphisms $\delta_i : F^i(C) \to F^{i+1}(A)$ fitting in a long exact sequence

$$\cdots \to F^i(A) \xrightarrow{F^i(f)} F_i(B) \xrightarrow{F^i(g)} F^i(C) \xrightarrow{\delta_i} F^{i+1}(A) \to \cdots$$

which are natural in the sense that if we have a morphism of short exact sequences in $\mathscr{C}$,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0, \end{tikzcd}$$

then we obtain a morphism of long exact sequences in $\mathscr{D}$,

$$\begin{tikzcd} \cdots \arrow[r] & F^i(A) \arrow[r] \arrow[d] & F^i(B) \arrow[r] \arrow[d] & F^i(C) \arrow[r] \arrow[d] & F^{i+1}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & F^i(A') \arrow[r] & F^i(B') \arrow[r] & F^i(C') \arrow[r] & F^{i+1}(A') \arrow[r] & \cdots. \end{tikzcd}$$

REMARK 3.1.5. A cohomological δ-functor (F^i, δ^i) is *universal* if there exists a unique extension of any natural transformation $F^0 \rightsquigarrow G^0$, where $(G^i, (\delta')^i)$ is another δ-functor, to a morphism of δ-functors.

Our situation will be as follows. Suppose that we have a right exact functor $F\colon \mathscr{C} \to \mathscr{D}$ of abelian categories (which have certain hypothesis on them). Our goal will be to construct a universal δ-functor $F_\cdot$ with $F_i\colon \mathscr{C} \to \mathscr{D}$ with $F_0 = F$ and $F_i = 0$ for $i < 0$. That is, given a short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathscr{C}$, we will have a long exact sequence

$$\cdots \to F_2(C) \to F_1(A) \to F_1(B) \to F_1(C) \to F(A) \to F(B) \to F(C) \to 0$$

in $\mathscr{D}$. Suppose that $G\colon \mathscr{C} \to \mathscr{D}$ is another right exact functor which has a natural transformation $\eta\colon G \rightsquigarrow F$ to F. Let $G_\cdot$ be the associated universal δ-functor we assume exists. Then universality of $F_\cdot$ then produces for us a morphism of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & G_1(B) \arrow[r] \arrow[d] & G_1(C) \arrow[r] \arrow[d] & G(A) \arrow[r] \arrow[d, "\eta_A"] & G(B) \arrow[r] \arrow[d, "\eta_B"] & G(C) \arrow[r] \arrow[d, "\eta_C"] & 0 \\
\cdots \arrow[r] & F_1(B) \arrow[r] & F_1(C) \arrow[r] & F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[r] & 0,
\end{tikzcd}$$

depending only on F, G, η, and the short exact sequence.

3.2. Projective objects

DEFINITION 3.2.1. An object P in an abelian category $\mathscr{C}$ is said to be *projective* if, given any epimorphism $g\colon A \to B$ in $\mathscr{C}$ and morphism $\beta\colon P \to B$, there exists a morphism $\alpha\colon P \to A$ with $\beta = g \circ \alpha$.

We draw the corresponding diagram:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\alpha"'] \arrow[d, "\beta"] & \\
A \arrow[r, "g"] & B \arrow[r] & 0.
\end{tikzcd}$$

LEMMA 3.2.2. *An object P in an abelian category $\mathscr{C}$ is projective if and only if every exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{p} P \to 0$$

in $\mathscr{C}$ splits.

PROOF. If P is projective, then this is the special case of Definition 3.2.1 in which $\beta = \mathrm{id}_B$ and $g = p$. Conversely, suppose we have an epimorphism g and a morphism β as in Definition 3.2.1. We then consider the pullback diagram

$$\begin{tikzcd}
P \times_B A \arrow[r, "p_1"] \arrow[d, "p_2"'] & P \arrow[d, "\beta"] \\
A \arrow[r, "g"] & B.
\end{tikzcd}$$

Now g is an epimorphism, and it follows that p_1 is an epimorphism. By assumption, p_1 has a splitting map u. We set $\alpha = p_2 \circ u$. Then

$$g \circ \alpha = g \circ p_2 \circ u = \beta \circ p_1 \circ u = \beta.$$

□

We say that a left R-module F is free if it is isomorphic to an arbitrary direct sum of copies of R as a left R-modules. Any free R-module F has a basis B in bijection with its indexing set, and therefore a map $F \to A$ for some left R-module A is prescribed uniquely by its (arbitrary) values on B. Any free module F is projective, since given $\beta\colon F \to B$ and an epimorphism $f\colon A \to B$, we merely have to lift the values $\beta(x)$ for x in a basis of F to A to define a map $\alpha\colon F \to A$.

EXAMPLE 3.2.3. Not every projective module need be free. For example, consider $R = \mathbb{Z}/6\mathbb{Z}$. We claim that $P = \mathbb{Z}/3\mathbb{Z}$ is a projective R-module. To see this, suppose that B is a $\mathbb{Z}/6\mathbb{Z}$-module and $g\colon B \to \mathbb{Z}/3\mathbb{Z}$ is surjective. Take any $b \in B$ with $g(b) = 1$. Then the $\mathbb{Z}/6\mathbb{Z}$ submodule generated by b is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, and hence $1 \mapsto b$ defines a splitting of g.

Note that P is not projective as a $\mathbb{Z}$-module (abelian group) since the quotient map $\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ does not split. In fact, every projective $\mathbb{Z}$-module is free.

We also have the following equivalent condition for a module to be projective, specific to the category R-**mod**.

LEMMA 3.2.4. *An R-module P is projective if and only if it is the direct summand of a free R-module.*

PROOF. Suppose P is projective. Find a generating set of P, and let F be the free left R-module on this set. Then we have an epimorphism $F \to P$ defined on this basis. Since this is split, we obtain a direct sum decomposition.

On the other hand, suppose we can write some P as a direct summand of a free module F, i.e., $F \cong P \oplus M$ for some M. Using Lemma 3.2.2, suppose we have an exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$$

Consider the commutative diagram

$$\begin{tikzcd} & & & F \arrow[dl, "q"'] \arrow[d, shift left, "p"] & \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & P \arrow[r] \arrow[u, shift left, "\iota"] & 0, \end{tikzcd}$$

where the maps p, ι arise from the direct sum decomposition and q exists by the projectivity of F. We take $\beta = q \circ \iota$. □

In the case that R is a principal ideal domain, we have the following.

COROLLARY 3.2.5. *If R is a principal ideal domain, then every projective R-module is free.*

PROOF. By the classification of finitely generated modules over a principal ideal domain, it suffices for finitely generated R-modules to show that any R-module of the form

$$A = R/(a_1) \oplus R/(a_2) \oplus \cdots R/(a_n)$$

for nonzero and nonunit $a_1, a_2, \dots, a_n \in R$ is not projective. Consider the obvious quotient map $R^n \to A$. That it splits means that each $R \to R/(a_i)$ splits. Then $R \cong (a_i) \oplus (x)$ for some $x \in R$, which means that R is free of rank 2 over itself, which is impossible (e.g., by the classification theorem).

The general case is left as an exercise. $\square$

LEMMA 3.2.6. *An object P in an abelian category $\mathscr{C}$ is projective if and only if* $\mathrm{Hom}_R(P,\cdot)\colon \mathscr{C} \to$ **Ab** *is an exact functor.*

PROOF. Suppose that the functor is exact. Then for any epimorphism $g\colon B \to P$ we have an epimorphism

$$\mathrm{Hom}_R(P,B) \to \mathrm{Hom}_R(P,P),$$

and any inverse image t of id_P is the desired splitting map of g.

On the other hand, suppose that P is projective. Consider an exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

Then we have a diagram

$$\mathrm{Hom}_R(P,A) \xrightarrow{h_P(f)} \mathrm{Hom}_R(P,B) \xrightarrow{h_P(g)} \mathrm{Hom}_R(P,C) \to 0.$$

That this is a complex is immediate. Surjectivity of $h_P(g)$ follows immediately from the definition of a projective module. Finally, let $h \in \ker h_P(g)$, so $h\colon P \to \ker g$. Then $A \to \ker g$ is an epimorphism, and we have by projectivity of P a map $j\colon P \to A$ with with $f \circ j = h$, i.e., $h_P(f)(j) = h$. $\square$

REMARK 3.2.7. If R is a commutative ring, then $\mathrm{Hom}_R(A,B)$ for R-modules A and B may be viewed as an R-module under $(r \cdot f)(a) = r \cdot f(a)$. It follows easily that $\mathrm{Hom}_R(A,\cdot)$ is an additive functor from the category of R-modules to itself which is exact if A is projective.

We now proceed to introduce projective resolutions in abelian categories.

DEFINITION 3.2.8. An abelian category $\mathscr{C}$ is said to have *sufficiently many* (or *enough*) projectives if for every $A \in \mathrm{Obj}(\mathscr{C})$, there exists a projective object $P \in \mathrm{Obj}(\mathscr{C})$ and an epimorphism $P \to A$.

Clearly, R-**mod** has enough projectives.

DEFINITION 3.2.9. Let $\mathscr{C}$ be an abelian category, and let A be an object in $\mathscr{C}$.

a. A *resolution* of A is a complex $C_\cdot$ of objects in $\mathscr{C}$ together with an *augmentation morphism* $\epsilon^C\colon C_0 \to A$ such that the *augmented complex*

$$\cdots \to C_1 \xrightarrow{d_1^C} C_0 \xrightarrow{\epsilon^C} A \to 0$$

is exact.

b. A *projective resolution* of A is a resolution of A by projective objects.

REMARK 3.2.10. If $\mathscr{C}$ has enough projectives, then every object in $\mathscr{C}$ has a projective resolution. (We leave the proof as an exercise.)

EXAMPLES 3.2.11. We have the following examples of projective resolutions, all of which are in fact resolutions by free modules:

a. In **Ab**, the abelian group $\mathbb{Z}/n\mathbb{Z}$ has a projective resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

b. Consider $R = \mathbb{Z}[X]$, and let $A = \mathbb{Z}[X]/(n, X^2+1)$. Then we have a projective resolution

$$0 \to \mathbb{Z}[X] \xrightarrow{(\cdot(X^2+1), \cdot n)} \mathbb{Z}[X] \oplus \mathbb{Z}[X] \xrightarrow{(\cdot n) - (\cdot(X^2+1))} \mathbb{Z}[X] \to \mathbb{Z}[X]/(n, X^2+1) \to 0.$$

c. Consider the ring $R = \mathbb{Z}[X]/(X^n - 1)$. (This is isomorphic to the *group ring* $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$.) We have a projective resolution of $\mathbb{Z}$:

$$\cdots \to R \xrightarrow{N} R \xrightarrow{X-1} R \xrightarrow{N} R \xrightarrow{X-1} R \to \mathbb{Z} \to 0,$$

where $N = \sum_{i=0}^{n-1} X^i$.

PROPOSITION 3.2.12. *Let $P. \to A$ and $Q. \to B$ be projective resolutions in R-***mod***, and suppose that $g\colon A \to B$ is an R-module homomorphism. Then g extends to a map $f.\colon P. \to Q.$ of chain complexes such that*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

commutes. Furthermore, any other lift of g is chain homotopic to $f.$.

PROOF. Let $P = (P_i, d_i)$ and $Q = (Q_i, d_i')$, and let ϵ and ϵ' denote the respective augmentation maps. Then $g \circ \epsilon\colon P_0 \to B$. Since ϵ' is an epimorphism, we have a map $f_0\colon P_0 \to Q_0$ lifting $g \circ \epsilon$. Now f_0 induces a map

$$\bar{f}_0\colon \ker \epsilon \to \ker \epsilon',$$

and since $\operatorname{im} d_1 = \ker \epsilon$ and $\ker d_1' = \ker \epsilon'$, we have an epimorphism $Q_1 \to \ker d_1'$, and we again use projectivity, this time of Q_1, to lift $\bar{f}_0 \circ d_1$ to a map f_1 as in the diagram. We continue in this manner to obtain $f.$.

Now, for uniqueness up to chain homotopy, it suffices to show that if $g = 0$, then $f.$ is chain homotopic to zero. Well, $d_0' \circ f_0 = g \circ d_0 = 0$, so $f_0(P_0) \subseteq \operatorname{im} d_1'$. By projectivity of P_0, we have $s_0\colon P_0 \to Q_1$ with

$$f_0 = d_1' \circ s_0 + s_{-1} \circ d_0 = d_1' \circ s_0,$$

where we have set $s_i = 0$ for $i < 0$ (and $d_i = 0$ for $i \le 0$). Now $h_1 = f_1 - s_0 \circ d_1$ satisfies

$$d_1' \circ h_1 = d_1' \circ f_1 - d_1' \circ s_0 \circ d_1 = f_0 \circ d_1 - f_0 \circ d_1 = 0,$$

so $\operatorname{im} h_1 \subseteq \operatorname{im} d_2'$. Thus, we have $s_1\colon P_1 \to Q_1$ lifting h_1, i.e., so that

$$d_2' \circ s_1 = h_1 = f_1 - s_0 \circ d_1,$$

as desired. We continue in this fashion to obtain all s_i. □

PROPOSITION 3.2.13 (Horseshoe lemma). *Suppose that*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence of R-modules and that $(P_\cdot^A, \epsilon^A)$ *and* $(P_\cdot^C, \epsilon^C)$ *are projective resolutions of A and C respectively. Then there exists a projective resolution* $(P_\cdot^B, \epsilon^B)$ *of B with* $P_i^B = P_i^A \oplus P_i^C$ *for each i and such that the diagram*

(3.2.1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_\cdot^A & \xrightarrow{\iota_\cdot} & P_\cdot^B & \xrightarrow{p_\cdot} & P_\cdot^C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \epsilon^A} & & \downarrow{\scriptstyle \epsilon^B} & & \downarrow{\scriptstyle \epsilon^C} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

commutes, where $\iota_\cdot$ *and* $p_\cdot$ *are the natural maps on each term.*

PROOF. Choose a lift t_0 of ϵ^C to $P_0^C \to B$, and let

$$\epsilon^B = f \circ \epsilon^A + t_0 \circ p_0.$$

Then ϵ^B is clearly surjective, and we have the desired commutativity of the "first two" squares. Next, letting $d_\cdot^X$ denote the boundary maps with $X = A$, C, we may define the boundary map d_1^B for B similarly. That is, consider a lift of $d_1^C \colon P_1^C \to \ker \epsilon^C$ to a map $t_1 \colon P_1^B \to \ker \epsilon^B$, and define

$$d_1^B = \iota_0 \circ d_1^A + t_1 \circ p_1.$$

Then d_1^B maps onto $\ker d_1^B$ and makes the next two squares commute. We then continue in this fashion. □

LEMMA 3.2.14. *Let* $\mathscr{C}$ *be an abelian category and* $P_\cdot$ *a split long exact sequence of projectives. Then* $P_\cdot$ *is a projective object in* $\mathbf{Ch}(\mathscr{C})$.

PROOF. Let $P_\cdot$ be a split exact sequence of projectives in $\mathscr{C}$. In other words, we may write each $P_0 = Q_0$ and $P_i = Q_i \oplus Q_{i-1}$ for $i \geq 1$, where Q_i is a projective object in $\mathscr{C}$, and the morphism $P_i \to P_{i-1}$ is simply the composition of the projection $P_i \to Q_{i-1}$ with the inclusion $Q_{i-1} \to P_i$. Suppose that $\pi_\cdot \colon A_\cdot \to P_\cdot$ is a epimorphism of complexes. Since Q_i is projective, there exists a splitting $s_i \colon Q_i \to A_i$ of the composition of π_i with projection to Q_i. Then

$$t_i = s_i \oplus s_{i-1} \colon P_i = Q_i \oplus Q_{i-1} \to A_i$$

is a splitting of π_i. Since $t_\cdot$ is a morphism of complexes, it is a splitting of $\pi_\cdot$. □

REMARK 3.2.15. Every split exact complex is the cone of a complex with zero differentials.

REMARK 3.2.16. Though we shall not prove it, every projective object in the category of chain complexes over an abelian category is a split exact sequence of projectives. Also, the projective objects in the category of bounded below chain complexes (or those in nonnegative degrees) are the bounded below exact sequences of projectives (which automatically split).

DEFINITION 3.2.17. We say that a functor $F\colon \mathscr{C} \to \mathscr{D}$ between categories preserves projectives if it takes projective objects in $\mathscr{C}$ to projective objects in $\mathscr{D}$.

PROPOSITION 3.2.18. *Let $\mathscr{C}$ and $\mathscr{D}$ be an abelian category. Let $F\colon \mathscr{C} \to \mathscr{D}$ be an additive functor that is left adjoint to an exact functor $G\colon \mathscr{D} \to \mathscr{C}$. Then F preserves projectives.*

PROOF. Let P be a projective object in $\mathscr{C}$. Let $f\colon A \to B$ be a epimorphism in $\mathscr{D}$. We must show that $h_{F(P)}(f)\colon F(A) \to F(B)$ is an epimorphism. Note that we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{C}}(P, G(A)) & \xrightarrow{h_P(G(f))} & \mathrm{Hom}_{\mathscr{C}}(P, G(B)) \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{Hom}_{\mathscr{D}}(F(P), A) & \xrightarrow{h_{F(P)}(f)} & \mathrm{Hom}_{\mathscr{D}}(F(P), B),
\end{array}$$

Exactness of G tells us that $G(f)$ is an epimorphism, and the upper horizontal map is then an epimorphism by the projectivity of P, hence the result. $\square$

3.3. Left derived functors

Suppose that $F\colon \mathscr{C} \to \mathscr{D}$ is a right exact functor between abelian categories $\mathscr{C}$ and $\mathscr{D}$ and that $\mathscr{C}$ has enough projectives. Then we could try to define the ith left derived functor of F (for $i \geq 0$) on an object A of $\mathscr{C}$ by $H_i(F(P_\cdot))$, where $P_\cdot \to A$ is a projective resolution of A. Of course, we must check that this definition is independent of the projective resolution chosen, and that we obtain induced maps on morphisms so that our map becomes a functor.

In the following, one may suppose that $\mathscr{C}$ is the category of R-**mod**, but it is often the case that $\mathscr{D}$ is some other category, like **Ab** or S-**mod** for some ring S.

LEMMA 3.3.1. *Suppose that $P_\cdot \to A$ and $Q_\cdot \to A$ are projective resolutions of A. Then there is a canonical isomorphism between $H_i(F(P_\cdot))$ and $H_i(F(Q_\cdot))$ for each $i \geq 0$.*

PROOF. By Proposition 3.2.12, the identity morphism on A extends to a map $P_\cdot \to Q_\cdot$ of chain complexes. Note also that we have a map $Q_\cdot \to P_\cdot$, and again by Proposition 3.2.12, both compositions are chain-homotopic to the identity. By Lemma 2.7.15, we therefore have that the induced maps $H_i(F(P_\cdot)) \to H_i(F(Q_\cdot))$ are inverse to each other. $\square$

We set $L_i F(A) = H_i(F(P_\cdot))$ for any projective resolution $P_\cdot \to A$ and $i \geq 0$. This is well-defined up to canonical isomorphism.

We have the following obvious corollaries of Lemma 3.3.1.

COROLLARY 3.3.2. *We have canonical isomorphisms $L_0 F(A) \cong F(A)$ for all R-modules A.*

PROOF. Since F is right exact, the sequence

$$F(P_1) \to F(P_0) \to F(A) \to 0$$

is exact. Hence, we have

$$L_0 F(A) = H_0(F(P_\cdot)) \cong F(A).$$

$\square$

COROLLARY 3.3.3. *If P is a projective object, then $L_i F(P) = 0$ for $i \geq 1$.*

PROOF. Consider the projective resolution that is P in degree zero and 0 elsewhere, where the augmentation map $P \to P$ is the identity. This has the desired homology. □

Next, we prove that the $L_i F$ are functors.

PROPOSITION 3.3.4. *To each morphism $f: A \to B$ in $\mathscr{C}$, we can associate canonical morphisms*

$$L_i F(f): L_i F(A) \to L_i F(B)$$

for all $i \geq 0$ in such a way that $L_i F: \mathscr{C} \to \mathscr{D}$ becomes a functor and $L_0 F(f) = F(f)$. Furthermore, each $L_i F$ is additive.

PROOF. The unique morphism $L_i F(f)$ is induced on homology by the morphism of chain complexes given in Proposition 3.2.12. Functoriality follows by canonicality of the map of homology.

To see additivity, note that $L_i F(0_A)$ is induced by the zero morphism of chain complexes and hence is is zero map on $L_i F(A)$. Similarly $L_i F(f+g)$, for $f, g: A \to B$, can be given by the sum of the induced maps on chain complexes, hence is given by the sum of the maps on homology. □

We refer to $L_i F$ as the ith *left derived functor* of F. We see that $L_0 F$ and F are canonically naturally isomorphic functors.

THEOREM 3.3.5. *For every short exact sequence*

$$0 \to A \to B \to C \to 0$$

in $\mathscr{C}$, there exist morphisms $\delta_i: L_i F(C) \to L_{i-1} F(A)$ such that the functors $L_\cdot F$ together with the maps $\delta_\cdot$ form a homological δ-functor.

PROOF. By the Horseshoe lemma, we have a projective resolution $P_\cdot^X \to X$ for $X = A, B, C$ fitting in a diagram (3.2.1). Now, applying F to the resolutions, we have split exact sequences

$$0 \to F(P_i^A) \to F(P_i^B) \to F(P_i^C) \to 0$$

for each i. The resulting exact sequence of complexes (which need not be split) yields a long exact sequence in homology

$$\cdots L_1 F(B) \to L_1 F(C) \xrightarrow{\delta_1} F(A) \to F(B) \to F(C) \to 0,$$

as desired.

It remains to check naturality. Consider a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "q^A"] & B \arrow[r, "g"] \arrow[d, "q^B"] & C \arrow[r] \arrow[d, "q^C"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0.
\end{tikzcd}$$

By Proposition 3.2.12, can extend q^A and q^C to maps of complexes $q^A_\cdot\colon P^A_\cdot \to P^{A'}_\cdot$ and $q^C_\cdot\colon P^C_\cdot \to P^{C'}_\cdot$. Suppose we have constructed $P^B_\cdot$ and $P^{B'}_\cdot$ via the Horseshoe lemma. We fit this all into a commutative diagram

(3.3.1)
$$\begin{tikzcd}
0 \arrow[r] & P^A_\cdot \arrow[rr, "\iota_\cdot"] \arrow[dr, "\epsilon^A"] \arrow[dd, "q^A_\cdot"'] & & P^B_\cdot \arrow[rr, "p_\cdot"] \arrow[dr] & & P^C_\cdot \arrow[rr] \arrow[dr] \arrow[dd, "q^C_\cdot"'] & & 0 & \\
& 0 \arrow[r] & A \arrow[rr, "f"] \arrow[dd] & & B \arrow[rr, "g"] \arrow[dd] & & C \arrow[rr] \arrow[dd, "q^C"] & & 0 \\
0 \arrow[r] & P^{A'}_\cdot \arrow[rr, "\iota'_\cdot"] \arrow[dr] & & P^{B'}_\cdot \arrow[rr, "p'_\cdot"] \arrow[dr] & & P^{C'}_\cdot \arrow[rr] \arrow[dr] & & 0 & \\
& 0 \arrow[r] & A' \arrow[rr, "f'"] & & B' \arrow[rr, "g'"] & & C' \arrow[rr] & & 0.
\end{tikzcd}$$

We also have splitting maps $j_i\colon P^C_i \to P^B_i$ and $k_i\colon P^B_i \to P^A_i$ for each i (and, similarly, maps j'_i and k'_i). For each X, let us denote the augmentation map by ϵ^X.

We must define a map $q^B_\cdot\colon P^B_\cdot \to P^{B'}_\cdot$ making the entire diagram (3.3.1) commute. We first note that

$$\begin{aligned}
g' \circ (q^B \circ \epsilon^B - \epsilon^{B'} \circ j'_0 \circ q^C_0 \circ p_0) &= q^C \circ g \circ \epsilon^B - \epsilon^{C'} \circ p'_0 \circ j_0 \circ q^C_0 \circ p_0 \\
&= q^C \circ \epsilon^C \circ p_0 - \epsilon^{C'} \circ q^C_0 \circ p_0 = (q^C \circ \epsilon^C - \epsilon^{C'} \circ q^C_0) \circ p_0 = 0.
\end{aligned}$$

Hence, there exists a map $\beta_0\colon P^B_0 \to A'$ with
$$f' \circ \beta_0 = q^B \circ \epsilon^B - \epsilon^{B'} \circ j'_0 \circ q^C_0 \circ p_0.$$
Since $\epsilon^{A'}$ is an epimorphism, we may choose $\alpha_0\colon P^B_0 \to P^{A'}_0$ with $\epsilon^{A'} \circ \alpha_0 = \beta_0$. Now set
$$q^B_0 = \iota'_0 \circ q^A_0 \circ k_0 + \iota'_0 \circ \alpha_0 \circ p_0 + j'_0 \circ q^C_0 \circ p_0.$$
The trickiest check of commutativity is that $\epsilon^{B'} \circ q^B_0 = q^B \circ \epsilon^B$. We write this mess out:
$$\begin{aligned}
\epsilon^{B'} \circ q^B_0 &= \epsilon^{B'} \circ \iota'_0 \circ q^A_0 \circ k_0 + \epsilon^{B'} \circ \iota'_0 \circ \alpha_0 \circ p_0 + \epsilon^{B'} \circ j'_0 \circ q^C_0 \circ p_0 \\
&= f' \circ \epsilon^{A'} \circ q^A_0 \circ k_0 + f' \circ \epsilon^{A'} \circ \alpha_0 \circ p_0 + \epsilon^{B'} \circ j'_0 \circ q^C_0 \circ p_0 \\
&= f' \circ q^A \circ \epsilon^A \circ k_0 + f' \circ \beta_0 \circ p_0 + \epsilon^{B'} \circ j'_0 \circ q^C_0 \circ p_0 \\
&= f' \circ q^A \circ \epsilon^A \circ k_0 + (j'_0 \circ q^C \circ \epsilon^C - \epsilon^{B'} \circ j'_0 \circ q^C_0) \circ p_0 + \epsilon^{B'} \circ j'_0 \circ q^C_0 \circ p_0 \\
&= f' \circ q^A \circ \epsilon^A \circ k_0 + j'_0 \circ q^C \circ \epsilon^C \circ p_0 \\
&= q^B \circ \epsilon^B.
\end{aligned}$$

The other q^B_i are defined similarly. For instance, one can see there exists a map $\beta_1\colon P^C_1 \to P^{A'}_0$ such that
$$\iota_0 \circ \beta_1 = \iota_0 \circ \beta_0 \circ d^C_0 + j'_1 \circ q^C_0 \circ d^C_0 - d^{B'}_0 \circ j'_1 \circ q^C_1 \circ p_1,$$
and we set
$$q^B_1 = \iota'_1 \circ q^A_1 \circ k_1 + \iota'_1 \circ \alpha_1 \circ p_1 + j'_1 \circ q^C_1.$$

□

Next, we see that the δ-functor of left derived functors of F is universal.

THEOREM 3.3.6. *The δ-functor $(L_i F, \delta_i)$ is universal.*

DEFINITION 3.3.7. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a left exact functor between abelian categories. We say that an object Q in $\mathscr{C}$ is F-acyclic if $L_i F(Q) = 0$ for all $i \geq 1$.

Note that the $L_i F(A)$ for any $A \in \mathrm{Obj}(\mathscr{C})$ may be computed using resolutions by F-acyclic objects, as opposed to just projectives.

PROPOSITION 3.3.8. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a left exact functor between abelian categories, and let A be an object of $\mathscr{C}$. Suppose that $C. \to A$ is a resolution of A by F-acyclic objects. Then $L_i F(A) \cong H_i(F(C.))$ for each $i \geq 0$.*

PROOF. Note that we have an exact sequence

$$F(C_1) \xrightarrow{F(d_1^C)} F(C_0) \xrightarrow{F(\epsilon^C)} F(A) \to 0,$$

so $F(A) \cong H_0(F(C.))$. Set $K_0 = \ker \epsilon^C$. We then have an exact sequence

$$0 \to L_1 F(A) \to F(K_0) \to F(C_0) \to F(A) \to 0,$$

which yields

$$L_1 F(A) \cong \ker(\mathrm{coker}(F(C_2) \to F(C_1)) \to F(C_0)) \cong \frac{\ker F(d_1^C)}{\mathrm{im}\, F(d_2^C)} \cong H_1(F(C.)).$$

We also have isomorphisms $L_i F(A) \cong L_{i-1} F(K_0)$ for each $i \geq 2$.

For $i \geq 1$, set $K_i = \ker d_i^C \cong \mathrm{im}\, d_{i+1}^C$. The exact sequences

$$0 \to K_i \to C_i \to K_{i-1} \to 0, \tag{3.3.2}$$

then yield isomorphisms used in the following for $i \geq 2$:

$$L_i F(A) \cong L_{i-1} F(K_0) \cong \cdots \cong L_1 F(K_{i-2}) \cong \ker(F(K_{i-1}) \to F(C_{i-1})) \cong \frac{\ker F(d_i^C)}{\mathrm{im}\, F(d_{i+1}^C)} \cong H_i(F(C.)).$$

□

More generally, we have the following characterization of universal δ-functors.

THEOREM 3.3.9. *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories such that $\mathscr{C}$ has enough projectives. Suppose that (F_i, δ_i) form a δ-functor $F_i\colon \mathscr{C} \to \mathscr{D}$ and $F_i(P) = 0$ for every projective $P \in \mathrm{Obj}(\mathscr{C})$ and $i \geq 1$. Then (F_i, δ_i) is universal.*

PROOF. Suppose that (G_i, δ'_i) is another δ-functor and that we have a natural transformation $G_0 \rightsquigarrow F_0$. Let $A \in \mathrm{Obj}(\mathscr{C})$ and let $\pi\colon P \to A$ be an epimorphism with P projective. Let $K = \ker \pi$. Let $i \geq 1$, and suppose that we have constructed a natural transformation $G_{i-1} \rightsquigarrow F_{i-1}$. Since $F_i(P) = 0$ is projective, we have a commutative diagram

$$\begin{tikzcd}
 & G_i(A) \arrow[r] \arrow[d, dashed] & G_{i-1}(K) \arrow[r] \arrow[d] & G_{i-1}(P) \arrow[d] \\
0 \arrow[r] & F_i(A) \arrow[r] & F_{i-1}(K) \arrow[r] & F_{i-1}(P).
\end{tikzcd}$$

The morphism $G_i(A) \to F_i(A)$ is the unique map which makes the diagram commute.

Now let $f: A \to B$ be a morphism in $\mathscr{C}$. We create a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K' \arrow[r] & P' \arrow[r] & B \arrow[r] & 0,
\end{tikzcd}$$

by taking P' to be projective, P to be any projective with an epimorphism to the pullback of the diagram $P' \to B \leftarrow A$, and K and K' to be the relevant kernels. We then have a diagram

$$\begin{tikzcd}
G_i(A) \arrow[rr] \arrow[dd] \arrow[dr] & & G_{i-1}(K) \arrow[dd] \arrow[dr] & \\
& F_i(A) \arrow[rr] \arrow[dd] & & F_{i-1}(K) \arrow[dd] \\
G_i(B) \arrow[rr] \arrow[dr] & & G_{i-1}(K') \arrow[dr] & \\
& F_i(B) \arrow[rr] & & F_{i-1}(K').
\end{tikzcd}$$

We need only see that the leftmost square commutes, but this follows easily from a diagram chase and the fact that the two horizontal maps on the frontmost square are monomorphisms.

Hence, we have constructed a sequence of natural transformations $G_i \rightsquigarrow F_i$. It remains only to see that these form a morphism of δ-functors. This being an inductive argument of the above sort, we leave it to the reader. $\square$

As a corollary, we have a natural isomorphism of δ-functors between the left derived functors $L_i F_0$ of a right exact functor F_0 and any δ-functor (F_i, δ_i) with $F_i(P) = 0$ for P projective and $i \geq 1$.

3.4. Injectives and right derived functors

We next wish to study right derived functors of left exact functors. For this, we need the analogues of projective objects, which are called injective objects.

DEFINITION 3.4.1. An object I in an abelian category $\mathscr{C}$ is called *injective* if, for every monomorphism $f: A \to B$ and morphism $\alpha: A \to I$, there exists a morphism $\beta: B \to I$ with $\alpha = \beta \circ f$.

In this case, the appropriate diagram is

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[dl, dashed, "\beta"] \\
& I &
\end{tikzcd}$$

One sees easily that I is injective if and only if it is projective as an object of $\mathscr{C}^{\mathrm{op}}$. By Lemma 3.2.6 (which holds in an arbitrary abelian category), it follows that I is injective if and only if h^I is exact. Similarly, by Lemma 3.2.2, we have that I is injective if and only if every exact sequence

$$0 \to I \to B \to C \to 0$$

in $\mathscr{C}$ splits.

We also have the analogue of Proposition 3.2.18.

PROPOSITION 3.4.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be an abelian category. Let $G \colon \mathscr{D} \to \mathscr{C}$ be an additive functor that is right adjoint to an exact functor $F \colon \mathscr{C} \to \mathscr{D}$. Then G preserves injectives.*

We now look at the category R-**mod**. One can prove the following using Zorn's Lemma. (Zorn's Lemma, which is equivalent to the Axiom of Choice, states that if $(I, \leq)$ is a partially ordered set such that every chain $i_1 \leq i_2 \leq i_3 \leq \ldots$ in I has an upper bound in I, then I has a maximal element, i.e., an element $j \in I$ such that $j \not\leq i$ for all $i \in I$.) We omit the proof.

LEMMA 3.4.3 (Baer's criterion). *A left R-module I is injective if and only if every homomorphism $J \to I$ with J a left ideal of R may be extended to a map $R \to I$.*

EXAMPLE 3.4.4.

a. $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

b. $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$-module for any $n \geq 1$.

c. $\mathbb{Z}/3\mathbb{Z}$ is an injective $\mathbb{Z}/6\mathbb{Z}$-module, but not an injective $\mathbb{Z}/9\mathbb{Z}$-module.

We have a very nice description of injective objects in **Ab**.

DEFINITION 3.4.5. An abelian group D is called *divisible* if multiplication by n is surjective on D for every natural number n.

PROPOSITION 3.4.6. *An abelian group is injective if and only if it is divisible.*

PROOF. Let D be injective, and take $d \in D$. Then there exists a group homomorphism $\phi \colon \mathbb{Z} \to D$ with $1 \mapsto d$. We also have the multiplication-by-n map on $\mathbb{Z}$, which is injective. By injectivity of D, we have a map $\theta \colon \mathbb{Z} \to D$ with $\phi = n\theta$. Then $d = n\theta(1)$, so D is divisible.

Conversely, let D be divisible. By Lemma 3.4.3, it suffices to show that every homomorphism $\phi \colon n\mathbb{Z} \to D$ with $n \geq 1$ extends to a homomorphism $\theta \colon \mathbb{Z} \to D$. Such a ϕ is determined by $d = \phi(n)$. Let $d' \in D$ be such that $nd' = d$. Set $\theta(1) = d'$. □

DEFINITION 3.4.7. We say that an abelian category $\mathscr{C}$ has *enough* (or *sufficiently many*) injectives if for every $A \in \mathrm{Obj}(\mathscr{C})$, there exists an injective object $I \in \mathrm{Obj}(\mathscr{C})$ and a monomorphism $A \to I$.

LEMMA 3.4.8. *The category R-**mod** has enough injectives.*

PROOF. First take the case that $R = \mathbb{Z}$. Let A be an abelian group, and write it as a quotient of a free abelian group

$$A = \Big(\bigoplus_{j \in J} \mathbb{Z}\Big)/T$$

for some indexing set J and submodule T of $\bigoplus_{j\in J}\mathbb{Z}$. Then we may embed A in

$$I = (\bigoplus_{j\in J}\mathbb{Q})/T,$$

which is divisible as a quotient of a divisible group.

Next, let A be a left R-module. We have an injection of left R-modules,

$$\phi\colon A \to \mathrm{Hom}_{\mathbb{Z}}(R,A),$$

by $\phi(a)(r) = ra$. Now, embed A in a divisible group D, so that the resulting map

$$\mathrm{Hom}_{\mathbb{Z}}(R,A) \to \mathrm{Hom}_{\mathbb{Z}}(R,D)$$

is an injection. The proof that $\mathrm{Hom}_{\mathbb{Z}}(R,D)$ is an injective R-module is left to the reader. □

DEFINITION 3.4.9. An *injective resolution* of an object A of an abelian category $\mathscr{C}$ is a cochain complex $I^{\cdot}$ of injective objects with $I^i = 0$ for $i < 0$ and a morphism $A \to I^0$ such that the resulting diagram

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

is exact.

Again, any object in an abelian category with enough injectives has an injective resolution. We also have the analogues of Propositions 3.2.12 and 3.2.13 for injective resolutions.

Suppose now that $F\colon \mathscr{C} \to \mathscr{D}$ is a left exact functor between abelian categories and that $\mathscr{C}$ has enough injectives. For each $i \geq 0$, we define additive functors $R^iF\colon \mathscr{C} \to \mathscr{D}$ by

$$R^iF(A) = H^i(F(I^{\cdot})),$$

where $A \to I^{\cdot}$ is any injective resolution of $A \in \mathrm{Obj}(\mathscr{C})$ and, for $f\colon A \to B$ in $\mathscr{C}$, by

$$R^iF(f)\colon R^iF(A) \to R^iF(B)$$

to be the map on homology induced by any morphism of chain complexes $I^{\cdot} \to J^{\cdot}$ extending f, where $A \to I^{\cdot}$ and $B \to J^{\cdot}$ are injective resolutions. We have $R^0F = F$. The functors $R^{\cdot}F$ are called the *right-derived functors* of F.

THEOREM 3.4.10. *The functors $R^{\cdot}F$ form a cohomological universal δ-functor.*

The proof is dual to that of Theorems 3.3.5 and 3.3.6.

We also have the following.

THEOREM 3.4.11. *Let $\mathscr{C}$ be an abelian category that admits kernels and cokernels and has enough injectives. Then the cohomology functors $H^i\colon \mathbf{Ch}^{\geq 0}(\mathscr{C}) \to \mathscr{C}$ for $i \geq 0$ on complexes in nonnegative degrees together with the connecting homomorphisms δ^i attached to a short exact sequence of complexes form a universal δ-functor.*

Restricting to chain complexes concentrated in two terms, we have the following.

COROLLARY 3.4.12. *Let $\mathscr{C}$ be an abelian category that admits kernels and cokernels and has enough injectives. Then the functors $F^0 = \ker$ and $F^1 = \mathrm{coker}\colon \mathrm{Mor}(\mathscr{C}) \to \mathscr{C}$ (along with $F^i = 0$ for $i \geq 2$) and the morphisms $\delta^0 = \delta$ given by the snake lemma (and $\delta^i = 0$ for $i \geq 1$) are canonically naturally isomorphic to the right derived functors of* $\ker$.

3.5. Tor and Ext

DEFINITION 3.5.1. Let Ω and Λ be rings.

a. An Ω-Λ-bimodule is a module for the ring $\Omega \otimes_{\mathbb{Z}} \Lambda^\circ$, where Λ° is the opposite ring to Λ.

b. The category Ω-Λ-**mod** of Ω-Λ-bimodules is the category of $\Omega \otimes_{\mathbb{Z}} \Lambda^\circ$-modules.

DEFINITION 3.5.2. Let Ω and Λ be rings, and let A be an Ω-Λ-bimodule. For $i \geq 0$, the Tor-functors

$$\mathrm{Tor}_i^\Lambda(A,\cdot)\colon \Lambda\text{-}\mathbf{mod} \to \Omega\text{-}\mathbf{mod}$$

are the left derived functors of t_A.

REMARK 3.5.3. If R is a commutative ring, then an R-module A provides functors

$$\mathrm{Tor}_i^R(A,\cdot)\colon R\text{-}\mathbf{mod} \to R\text{-}\mathbf{mod}$$

since R-modules are automatically R-R-bimodules.

REMARK 3.5.4. As $\mathrm{Tor}_i^\Lambda(A,B) = H_i(A \otimes_\Lambda Q_\cdot)$ for any projective resolution $Q_\cdot$ of B by Λ-modules, the composition of the functor

$$\mathrm{Tor}_i^\Lambda(A,\cdot)\colon \Lambda\text{-}\mathbf{mod} \to \Omega\text{-}\mathbf{mod}$$

with the forgetful functor $F\colon \Omega\text{-}\mathbf{mod} \to \mathbf{Ab}$ agrees with the functor

$$\mathrm{Tor}_i^\Lambda(F(A),\cdot)\colon \Lambda\text{-}\mathbf{mod} \to \mathbf{Ab},$$

hence the omission of the notation for Ω in the definition of $\mathrm{Tor}_i^\Lambda(A,\cdot)$.

EXAMPLE 3.5.5. In **Ab**, consider the projective resolution

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

of B. Computing the homology of $0 \to A \xrightarrow{n} A \to 0$, we obtain

$$\mathrm{Tor}_i^{\mathbb{Z}}(A,\mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} A/nA & \text{if } i=0 \\ A[n] = \{a \in A \mid na = 0\} & \text{if } i=1 \\ 0 & \text{if } i \geq 2. \end{cases}$$

DEFINITION 3.5.6. We say that a right Λ-module A is *flat* as a right Λ-module if $t_A\colon \Lambda\text{-}\mathbf{mod} \to \mathbf{Ab}$ is an exact functor.

LEMMA 3.5.7. *Let Λ be a ring. The following conditions on a right Λ-module A are equivalent:*

i. A is flat,

ii. $\mathrm{Tor}_1^\Lambda(A,\cdot) = 0$,

iii. $\mathrm{Tor}_i^\Lambda(A,\cdot) = 0$ *for all* $i \geq 1$.

PROOF. Clearly, (iii) implies (ii). If

$$0 \to B_1 \to B_2 \to B_3 \to 0$$

is an exact sequence of right Λ-modules, then we have a long exact sequence for any Λ-module that ends with

$$\mathrm{Tor}_1^\Lambda(A, B_3) \to A \otimes_\Lambda B_1 \to A \otimes_\Lambda B_2 \to A \otimes_\Lambda B_3 \to 0,$$

from which it is clear that (ii) implies (i).

Finally, if (i) holds and $Q_\cdot$ is a projective resolution of B in Λ-**mod**, then the complex

$$\cdots \to A \otimes_\Lambda Q_1 \to A \otimes_\Lambda Q_0 \to A \otimes_\Lambda B \to 0$$

is exact by the flatness of A. It follows that

$$\mathrm{Tor}_i^\Lambda(A, B) = H_i(A \otimes_\Lambda Q_\cdot) = 0$$

for all $i \geq 1$. □

LEMMA 3.5.8. *Any projective right Λ-module is flat.*

PROOF. Let P be a projective right Λ-module. Then P is a direct summand of a free right Λ-module F, let us say isomorphic as a right Λ-module to $\bigoplus_{i \in I} \Lambda$ for an indexing set I. If

$$0 \to B_1 \to B_2 \to B_3 \to 0$$

is exact, then

$$0 \to F \otimes_\Lambda B_1 \to F \otimes_\Lambda B_2 \to F \otimes_\Lambda B_3 \to 0$$

is exact, being isomorphic to the direct sum of one copy of the original sequence for each element of I. Moreover, the sequence

$$0 \to P \otimes_\Lambda B_1 \to P \otimes_\Lambda B_2 \to P \otimes_\Lambda B_3 \to 0$$

is a direct summand of the latter sequence, hence is exact. □

PROPOSITION 3.5.9. *Let A be a right Λ-module and B a left Λ-module. Let $P_\cdot \to A$ be a resolution of A by projective right Λ-modules. Then*

$$\mathrm{Tor}_i^\Lambda(A, B) \cong H_i(P_\cdot \otimes_\Lambda B)$$

for all $i \geq 0$. In particular, the functors $\mathrm{Tor}_i^\Lambda(\cdot, B)$ are the left derived functors of Λ-tensor product with B.

We sketch a proof.

PROOF. Form projective resolutions $P_\cdot \to A$ and $Q_\cdot \to B$. We then have a double complex $P_\cdot \otimes_\Lambda Q_\cdot$, and we can consider homology of the total complex

$$\mathrm{Tot}(P_\cdot \otimes_\Lambda Q_\cdot)_k = \bigoplus_{i+j=k} P_i \otimes_\Lambda Q_j,$$

where the boundary maps from each term $P_i \otimes_\Lambda Q_j$ are given by the sums $d_i^A + (-1)^i d_j^B$. We claim that the homology of this complex is isomorphic to the homology of the complexes $P_\cdot \otimes_\Lambda B$ and $A \otimes_\Lambda Q_\cdot$, from which the lemma follows.

We have maps of complexes

$$\mathrm{Tot}(P_\cdot \otimes_\Lambda Q_\cdot) \to P_\cdot \otimes_\Lambda B \tag{3.5.1}$$

and

$$\mathrm{Tot}(P_\cdot \otimes_\Lambda Q_\cdot) \to A \otimes_\Lambda Q_\cdot \tag{3.5.2}$$

induced by augmentation morphisms (up to sign, and zero maps otherwise). The double complex $P_\cdot \otimes_\Lambda Q_\cdot \to P_\cdot \otimes_\Lambda B$ (i.e., with $P_i \otimes_\Lambda B$ in the $(i,-1)$-position) has exact columns, since each projective module is acyclic. One can show that this implies that the total complex of this complex is exact. This says precisely that the map in (3.5.1) induces an isomorphism on homology. Similarly, so does the map in (3.5.2). □

We have the following almost immediate corollary, since left and right tensor product with a module over a commutative ring are naturally isomorphic functors.

COROLLARY 3.5.10. *Let R be commutative. We have* $\mathrm{Tor}_i^R(A,B) \cong \mathrm{Tor}_i^R(B,A)$ *for all R-modules A, B and $i \geq 0$.*

We now give an alternate proof of Proposition 3.5.9.

PROOF. Let $Q_\cdot \to B$ be a projective resolution of A by right Λ-modules. Suppose that

$$0 \to A_1 \to A_2 \to A_3 \to 0$$

is an exact sequence. Then

$$0 \to A_1 \otimes_\Lambda Q_\cdot \to A_2 \otimes_\Lambda Q_\cdot \to A_3 \otimes_\Lambda Q_\cdot \to 0$$

is exact. This yields a long exact sequence in homology of the form

$$\cdots \to \mathrm{Tor}_i^\Lambda(A_1,B) \to \mathrm{Tor}_i^\Lambda(A_2,B) \to \mathrm{Tor}_i^\Lambda(A_3,B) \to \mathrm{Tor}_{i-1}^\Lambda(A_1,B) \to \cdots,$$

so the functors $\mathrm{Tor}_i^\Lambda(\cdot\, B)$ do in fact form a δ-functor. Futhermore, since any projective right Λ-module P is flat, we have that $\mathrm{Tor}_i^\Lambda(P,B) = 0$ for all $i \geq 1$. By Theorem 3.3.9, it follows that the $\mathrm{Tor}_i^\Lambda(\cdot,B)$ are a universal δ-functor extending t_B. The proposition therefore follows by Theorem 3.3.6. □

REMARK 3.5.11. It follows from Proposition 3.5.9 and Proposition 3.3.8 that the $\mathrm{Tor}_i^\Lambda(A,B)$ can be computed via a flat resolution of either A or B.

The following explains something more of the name "Tor".

LEMMA 3.5.12. *The functor* $\mathrm{Tor}_1^{\mathbb{Z}}(A,\cdot) = 0$ *if and only if A is torsion-free.*

PROOF. We prove this for finitely generated abelian groups. (The general result then follows from the fact that left derived functors commute with colimits.) By Proposition 3.5.10, we may compute $\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$ by finding a projective resolution of A. Say

$$A \cong \mathbb{Z}^m \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \mathbb{Z}/n_r\mathbb{Z}$$

with $r \geq 0$ and the $n_i \geq 2$. Then we have a projective resolution of the form

$$0 \to \mathbb{Z}^{m+r} \xrightarrow{(1,\cdots,1,n_1,\cdots,n_r)} \mathbb{Z}^{m+r} \to A \to 0.$$

Tensoring with B and computing H_1, we obtain $B[n_1]\oplus\cdots\oplus B[n_r]$. This will always be trivial if and only if $r=0$. □

By Lemma 3.5.12, a $\mathbb{Z}$-module is flat if and only if it is torsion-free. This is seen to hold in the same manner with $\mathbb{Z}$ replaced by any PID. Note that this does not hold for all commutative rings.

EXAMPLE 3.5.13. Consider $R=\mathbb{Q}[x,y]$. Then the exact sequence

$$0\to R\xrightarrow{(y,-x)}R^2\xrightarrow{(a,b)\mapsto ax+by}R\to\mathbb{Q}\to 0$$

is a free resolution of $\mathbb{Q}$. Let J be the ideal (x,y) of R, so $\mathbb{Q}\cong R/J$. Then we have isomorphisms

$$\operatorname{Tor}_1^R(J,\mathbb{Q})\cong\operatorname{Tor}_2^R(\mathbb{Q},\mathbb{Q})\cong\ker(\mathbb{Q}\xrightarrow{0}\mathbb{Q}^2)=\mathbb{Q}.$$

Thus J is not flat as an R-module, even though it is torsion-free.

Here is another class of examples.

LEMMA 3.5.14. *Let S be a subset of R that is multiplicatively closed. Then the localization $S^{-1}R$ is a flat R-module.*

PROOF. We claim that $S^{-1}A\cong S^{-1}R\otimes_R A$ for any R-module A. The map inducing this isomorphism is $s^{-1}a\mapsto s^{-1}\otimes a$. This is well-defined as if $s^{-1}a=t^{-1}b$ for some $t\in S$ and $b\in B$, then $ta=sb$, and then

$$t^{-1}\otimes b=(st)^{-1}\otimes sb=(st)^{-1}\otimes ta=s^{-1}\otimes a.$$

On the other hand, the inverse map is induced by the R-bilinear map

$$S^{-1}R\times A\to S^{-1}A$$

by $(s^{-1}r,a)\mapsto s^{-1}ra$.

Suppose that $f\colon A\to B$ is an injection of R-modules. Then we obtain an induced R-module homomorphism $\tilde{f}\colon S^{-1}A\to S^{-1}B$, which we must show is an injection. Suppose $\tilde{f}(s^{-1}a)=0$. Then

$$0=s\tilde{f}(s^{-1}a)=\tilde{f}(a)=f(a),$$

so $a=0$. □

Here are some interesting results on flat modules:

PROPOSITION 3.5.15. *The following are equivalent for a module A over a commutative ring R.*

i. A is flat,

ii. for every ideal J of R, the sequence

$$0\to A\otimes_R J\to A\to A/JA\to 0$$

is exact,

iii. the R-module $A^*=\operatorname{Hom}_{\mathbf{Ab}}(A,\mathbb{Q}/\mathbb{Z})$ is injective, where R acts on A^* by $(rf)(a)=f(ra)$ for all $f\in A^*$, $r\in R$, and $a\in A$.

PROPOSITION 3.5.16. *Let R be a local ring. Then any finitely generated flat module over R is free.*

DEFINITION 3.5.17. Let Ω and Λ be rings, and let A be an Λ-Ω-bimodule. The Ext-functors

$$\mathrm{Ext}^i_\Lambda(A,\cdot)\colon \Lambda\text{-}\mathbf{mod} \to \Omega\text{-}\mathbf{mod}$$

are the right derived functors of h_A.

EXAMPLE 3.5.18. For $R=\mathbb{Z}$, we may consider the injective resolution

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{n} \mathbb{Q}/\mathbb{Z} \to 0$$

of $\mathbb{Z}/n\mathbb{Z}$. For any abelian group B, we write $B^* = \mathrm{Hom}(B,\mathbb{Q}/\mathbb{Z})$. We must compute the cohomology of $A^* \xrightarrow{n} A^*$. This yields

$$\mathrm{Ext}^i_{\mathbb{Z}}(A,\mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} A^*[n] & \text{if } i=0 \\ A^*/nA^* & \text{if } i=1 \\ 0 & \text{if } i=2. \end{cases}$$

One has that $\mathrm{Ext}^1_R(P,B)=0$ for all B if P is a projective module, as follows from the exactness of $\mathrm{Hom}_R(P,\cdot)$. We have the analogous result to Proposition 3.5.10 for Ext-groups, which says that such groups may be computed using projective resolutions.

PROPOSITION 3.5.19. *We have* $\mathrm{Ext}^i_R(A,B) \cong H^i(\mathrm{Hom}_R(P_\cdot,B))$, *where* $P_\cdot \to A$ *is any projective resolution of* A.

We end with a characterization of Ext^1_R in terms of extensions.

DEFINITION 3.5.20. An *extension* of an R-module A by an R-module B is an exact sequence $0 \to B \to E \to A \to 0$, where E is an R-module. Two extensions of A by B are called equivalent if there is an isomorphism of exact sequences between them that is the identity on A and B.

Note that all split extensions (i.e., those with split exact sequences) are split.

EXAMPLE 3.5.21. There are p equivalence classes of extensions of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}$-modules:

$$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{pi} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\bmod p} \mathbb{Z}/p\mathbb{Z} \to 0$$

with $1 \le i \le p-1$, and

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

THEOREM 3.5.22. *There is a one-to-one correspondence between equivalence classes of extensions of A by B and* $\mathrm{Ext}^1_R(A,B)$.

PROOF. Suppose that $\mathscr{E}$ is an equivalence class of extensions of A by B, representative by an exact sequence

$$0 \to B \to E \to A \to 0. \tag{3.5.3}$$

We then have an exact sequence

$$\mathrm{Hom}_R(E,B) \to \mathrm{Hom}_R(B,B) \xrightarrow{\partial_{\mathscr{E}}} \mathrm{Ext}^1_R(A,B),$$

and we set $\Phi(\mathscr{E}) = \partial_{\mathscr{E}}(\mathrm{id}_B)$. This is clearly independent of the choice of representative.

Conversely, suppose $u \in \mathrm{Ext}^1_R(A, B)$. Fix an exact sequence

$$0 \to K \xrightarrow{\iota} P \to A \to 0$$

with P projective. We then have an exact sequence

$$\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(K, B) \xrightarrow{\partial} \mathrm{Ext}^1_R(A, B) \to 0.$$

Let $t \in \mathrm{Hom}_R(K, B)$ with $\partial(t) = u$. Let E be the pushout

$$E = P \amalg_K B = P \oplus B / \{(\iota(k), t(k)) \mid k \in K\}.$$

We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "t"] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A \arrow[r] & 0. \end{tikzcd} \tag{3.5.4}$$

Here, the map $E \to A$ is defined by universality of the pushout (via the map $P \to A$ and the zero map $B \to A$). We define $\Psi(u)$ to be the equivalence class $\mathscr{E}'$ of the extension given by the lower row. Though it is not immediately clear that this is independent of the choice of t with $\partial(t) = u$, this follows if we can show that Ψ and Φ as constructed are mutually inverse.

To see that $\Phi(\Psi(u)) = u$, set $\mathscr{E} = \Psi(u)$, again choosing any t with $\partial(t) = u$. The diagram

$$\begin{tikzcd} \mathrm{Hom}_R(B, B) \arrow[r, "\partial_{\mathscr{E}}"] \arrow[d, "h^B(t)"] & \mathrm{Ext}^1_R(A, B) \arrow[d, equal] \\ \mathrm{Hom}_R(K, B) \arrow[r, "\partial"] & \mathrm{Ext}^1_R(A, B), \end{tikzcd} \tag{3.5.5}$$

commutes. Hence, we have

$$\Phi(\Psi(u)) = \Phi(\mathscr{E}) = \partial_{\mathscr{E}}(\mathrm{id}_B) = \partial(t) = u,$$

as desired.

On the other hand, suppose given $\mathscr{E}$ with exact sequence (3.5.3). By projectivity of P, the map $P \to A$ lifts to a map $P \to E$. Hence, we have a diagram as in (3.5.4). Furthermore, the map t in the diagram (3.5.4) satisfies $\partial(t) = \partial_{\mathscr{E}}(\mathrm{id}_B)$ by the commutativity of (3.5.5). Now, there exists a map $P \amalg_K B \to E$ by universality of the pushout, and it is the identity on A and B, hence an isomorphism by the 5-lemma. It follows by construction that

$$\Psi(\Phi(\mathscr{E})) = \Psi(\partial_{\mathscr{E}}(\mathrm{id}_B)) = \mathscr{E}.$$

$\square$

3.6. Group homology and cohomology

Let Λ be a ring and G a group.

DEFINITION 3.6.1. The Λ-*group ring* $\Lambda[G]$ of G consists of the set of finite formal sums of group elements with coefficients in R

$$\{\sum_{g\in G} a_g g \mid a_g \in \Lambda \text{ for all } g\in G, \text{ almost all } a_g = 0\}.$$

with addition given by addition of coefficients and multiplication induced by the group law on G and Λ-linearity. (Here, "almost all" means all but finitely many.)

In other words, the operations are

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G}(a_g + b_g)g$$

and

$$(\sum_{g\in G} a_g g)(\sum_{g\in G} b_g g) = \sum_{g\in G}(\sum_{k\in G} a_k b_{k^{-1}g})g.$$

We shall work here only with the case that $\Lambda = \mathbb{Z}$.

DEFINITION 3.6.2. If G is a finite group, then we have the *norm element*

$$N_G = \sum_{g\in G} g \in \mathbb{Z}[G]^G.$$

DEFINITION 3.6.3. Let A be a left $\mathbb{Z}[G]$-module. We define the G-*invariants* A^G of A to be the set of elements of A fixed under G:

$$A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}.$$

EXAMPLE 3.6.4. If G is a finite group, then $\mathbb{Z}[G]^G = (N_G)$. Otherwise, $\mathbb{Z}[G]^G = 0$.

If $f: A \to B$ is an $\mathbb{Z}[G]$-module homomorphism, it clearly induces a group homomorphism on subsets $A^G \to B^G$. Hence, $A \mapsto A^G$ defines a functor $\mathbb{Z}[G]$-**mod** $\to$ **Ab**. We leave it to the reader to check that this functor is in fact left exact.

DEFINITION 3.6.5. The ith G-*cohomology group* $H^i(G,A)$ with coefficients in A is the ith left derived functor of the functor of G-invariants applied to A.

In particular, $H^0(G,A) = A^G$. Group cohomology may be computed by taking the *standard resolution* of a module A. That is, we have a projective resolution by $\mathbb{Z}[G]$-modules of $\mathbb{Z}$ viewed as a trivial G-module:

$$\cdots \mathbb{Z}[G\times G\times G] \xrightarrow{d_2} \mathbb{Z}[G\times G] \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where

$$d_i(g_1,\cdots,g_{i+1}) = \sum_{j=0}^{i+1}(-1)^j(g_1,\cdots,g_{j-1},g_{j+1},\cdots,g_{i+1})$$

and the augmentation map ϵ is defined by $\epsilon(g) = 1$. Apply $\mathrm{Hom}_{\mathbb{Z}}(\cdot, A)$ to the sequence. Then we obtain an injective resoulution of A, and taking invariants of the injective resolution yields our result. (Note: this is the same as applying $\mathrm{Hom}_{\mathbb{Z}[G]}(\cdot, A)$ to the standard resolution.)

We omit the proof of the following:

EXAMPLE 3.6.6. Suppose that G acts trivially on A (i.e., $A^G = A$). Then $H^1(G, A) = \mathrm{Hom}(G, A)$.

We also have a notion of group homology $H_i(G, A)$, arising as the left derived functors of the coinvariant functor $A \to A_G$. Here

$$A_G = A/\langle ga - a \mid g \in G \rangle.$$

This may also be computed using the standard resolution, this time by tensoring it with A over $\mathbb{Z}[G]$. Here is another example.

EXAMPLE 3.6.7. We have $H_1(G, \mathbb{Z}) \cong G^{\mathrm{ab}}$, where G^{ab} is the *abelianization*, or maximal abelian quotient, of G. To see this, note that we have an exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}[G] \to \mathbb{Z}$ is the augmentation homomorphism and $I_G = (g - 1 \mid g \in G)$ is the *augmentation ideal* of G. Since $\mathbb{Z}[G]$ is projective, the long exact sequence in homology becomes

$$0 \to H_1(G, \mathbb{Z}) \to (I_G)_G \to \mathbb{Z}[G]_G \to \mathbb{Z} \to 0.$$

We have $\mathbb{Z}[G]_G \cong \mathbb{Z}$ via the augmentation map, so

$$H_1(G, \mathbb{Z}) \cong (I_G)_G \cong I_G/I_G^2 \cong G^{\mathrm{ab}},$$

where the latter isomorphism takes the image of $g - 1$ to the image of g. (Check this!)

REMARK 3.6.8. It follows by Proposition 3.5.19 that

$$H^i(G, A) \cong \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$

and by Proposition 3.5.19 that

$$H_i(G, A) \cong \mathrm{Tor}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

3.7. Derived functors of limits

In this section, we will focus on the abelian category Λ-**mod** of left modules over a ring Λ. This category is both complete and cocomplete: a functor $F\colon I \to \Lambda$-**mod** from a small category I has limit

$$\lim F = \left\{ (x_i) \in \prod_{i \in I} F(i) \mid F(\kappa)(a_i) = a_j \text{ for all } \kappa\colon i \to j \text{ in } I \right\}$$

and colimit

$$\operatorname{colim} F = \left(\bigoplus_{i \in I} F(i) \right) / M,$$

where

$$M = \sum_{\kappa\colon i \to j} \{ F(\kappa)(a) - a \mid a \in F(i) \}.$$

Thus, we have a limit (resp., colimit) functor

$$\varinjlim : \mathbf{Func}(I, \Lambda\text{-}\mathbf{mod}) \to \Lambda\text{-}\mathbf{mod} \qquad (\text{resp., } \varprojlim : \mathbf{Func}(I^{\mathrm{op}}, \Lambda\text{-}\mathbf{mod}) \to \Lambda\text{-}\mathbf{mod}),$$

for any small filtered category I, and it is clearly additive.

LEMMA 3.7.1. *Let I be a small filtered category, and let $F: I \to \Lambda$-**mod** be a functor.*

a. For each $x \in \varinjlim F$, there exist $i \in I$ and $x_i \in F(i)$ such that x is the image of x_i under the natural map $F(i) \to \varinjlim F$.

b. Let $x_k \in F(k)$ for some $k \in I$, and suppose that x_k has zero image in $\varinjlim F$. Then there exists $\kappa: k \to j$ in I such that $F(\kappa)(x_k) = 0$.

PROOF. Let $x \in \varinjlim F$. By construction of $\operatorname{colim} F$, there exist $n \geq 1$, distinct $i_t \in I$, and $x_{i_t} \in F(i_t)$ for $1 \leq t \leq n$ such that x is the image of

$$\sum_{t=1}^{n} x_{i_t} \in \bigoplus_{i \in I} F(i)$$

in the quotient $\operatorname{colim} F$. But there exists $j \in I$ such that there are morphisms $\kappa_t : i_t \to j$ for each $1 \leq t \leq n$. But then x is the image of the sum of the $F(\kappa_t)(x_{i_t})$ in $F(j)$, proving part a.

Next, let $x_k \in F(k)$ with zero image in $\varinjlim F$. Then, there exist $m \geq 1$, $\kappa_s : i_s \to j_s$ and $a_s \in F(i_s)$ for each $1 \leq s \leq m$ such that

$$x_k = \sum_{s=1}^{m} (F(\kappa_s)(a_s) - a_s)$$

in $\bigoplus_{i \in I} F(i)$. Find l and $\lambda_s : j_s \to l$ for each s. Then

$$x_k = -\sum_{s=1}^{m} (F(\lambda_s \circ \kappa_s)(a_s) - F(\kappa_s)(a_s)) + \sum_{s=1}^{m} (F(\lambda_s \circ \kappa_s)(a_s) - a_s),$$

which is to say that we may assume all of the elements j_s are equal to l and then by combining terms that all of the i_s are distinct. If $\ell = k$, then by the equality in the direct sum, we must have that each $a_s = 0$, so $x_k = 0$. If $\ell \neq k$, then we have that $x_k = -a_u$ for some u and all other a_s are zero. It then follows that $F(\kappa_u)(a_u) = -F(\kappa_u)(x_k) = 0$ as well, proving part b. □

PROPOSITION 3.7.2. *For any small filtered category I, the colimit functor*

$$\varinjlim : \mathbf{Func}(I, \Lambda\text{-}\mathbf{mod}) \to \Lambda\text{-}\mathbf{mod}$$

is exact.

PROOF. By Proposition 1.5.6, the functor $\varinjlim$ has a left adjoint, so is right exact by Proposition 2.5.7.

Let F and G be functors $I \to \Lambda$-**mod**. Suppose that $\eta: F \rightsquigarrow G$ is a monomorphism. We claim that each η_i with $i \in I$ is injective. Choose $i \in I$ and $y \in F(i)$. We take $E: I \to \Lambda$-**mod** on

$j \in I$ to be

$$E(j) = \sum_{\kappa\colon i\to j} \Lambda \cdot F(\kappa)(y)$$

(so $E(j) = 0$ if $j \notin I_{\geq i}$) and $E(\lambda)\colon E(j) \to E(k)$ to be the restriction of $F(\lambda)$ for $\lambda\colon j \to k$ in I. We define $\xi\colon E \rightsquigarrow F$ by taking ξ_i to be the inclusion of $E(i)$ in $F(i)$. If $\eta_i(y) = 0$, then $\eta \circ \xi = 0$, but then it follows that $\xi = 0$ as η is a monomorphism, and hence $y = 0$. Thus, each η_i is injective.

Let $x \in \varinjlim F$ with $(\varinjlim \eta)(x) = 0$. By Lemma 3.7.1a, we can find $i \in I$ such that x is t he image of some $x_i \in F(i)$. By definition, $\eta_i(x_i)$ has 0 image in $\varinjlim G$, and so by Lemma 3.7.1b, we can find $\kappa\colon i \to j$ in I such that

$$\eta_j(F(\kappa)(x_i)) = G(\kappa)(\eta_i(x_i)) = 0.$$

The injectivity of η_i then tells us that $F(\kappa)(x_i) = 0$, so we have that $x = 0$. □

As in the proof of Proposition 3.7.2, we have that limits of functors to Λ-**mod** are left exact. However, they are not always right exact, as we shall see. We may, of course, consider the derived functors

$$R^n \varprojlim \colon \mathbf{Func}(I, \Lambda\text{-}\mathbf{mod}) \to \Lambda\text{-}\mathbf{mod}$$

for a cofiltered category I. Let us focus the special case that I is the category given by the natural numbers ordered by $\geq$. That is, we consider sequential limits in Λ-**mod**. To give a functor $F\colon I \to \Lambda$-**mod** is equivalent to giving a sequence of Λ-modules $A_i = F(i)$ and morphisms $\pi_{i+1}\colon A_{i+1} \to A_i$ for $i \geq 1$.

Definition 3.7.3. Let I be the category of natural numbers ordered by $\geq$.

a. We define a functor $\Phi\colon \mathbf{Func}(I, \Lambda\text{-}\mathbf{mod}) \to \mathrm{Mor}(\Lambda)$ by taking $\Phi(F)$ for $F\colon I \to \Lambda$-**mod** to be

$$\Phi(F)\colon \prod_{i=1}^{\infty} F(i) \to \prod_{i=1}^{\infty} F(i), \qquad \Phi(F)((a_i)_{i\geq 1}) = (a_i - \pi_{i+1}(a_{i+1}))_{i\geq 0}$$

for $a_i \in F(i)$ for $i \geq 1$ and $\Phi(\eta)$, for $\eta\colon F \rightsquigarrow G$, to be the pair $(\prod \eta_i, \prod \eta_i)$.

b. For $n \geq 0$, let $\varprojlim^n \colon \mathbf{Func}(I, \Lambda\text{-}\mathbf{mod}) \to \Lambda\text{-}\mathbf{mod}$ be the functor

$$\varprojlim^n = \begin{cases} \ker \circ \Phi & \text{if } n = 0 \\ \operatorname{coker} \circ \Phi & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

Remark 3.7.4. Since Φ is exact, Corollary 3.4.12 tells us that the functors $\varprojlim^n$ are the right derived functors of

$$\varprojlim^0 = \ker \circ \Phi = \varprojlim .$$

Notation 3.7.5. If (A_i, π_i) is a collection of Λ-modules A_i and morphisms $\pi_{i+1}\colon A_{i+1} \to A_i$ for $i \geq 1$, then we write $\Delta_A = \Phi(F)$ for the functor F resulting from the collection, and we write $\varprojlim^1 A_i$ for $\varprojlim^1 F$. For $j \geq k \geq 1$, we set $\pi_{j,k} = \pi_k \circ \pi_{k-1} \circ \cdots \circ \pi_{j+1}$.

LEMMA 3.7.6. *Let (A_i, π_i) be a collection of nonzero Λ-modules A_i and surjective morphisms $\pi_{i+1}\colon A_{i+1} \to A_i$ for $i \geq 1$. Then $\varprojlim A_i \neq 0$ and $\varprojlim^1 A_i = 0$.*

PROOF. Note that if $a_1 \in A_1$ is nonzero, then we can inductively find $a_{i+1} \in A_{i+1}$ with $\pi_{i+1}(a_{i+1}) = a_i$, so $\varprojlim A_i \neq 0$.

Let $b_i \in A_i$ for each $i \geq 1$, and let $a_1 \in A_1$. Inductively choose $a_{i+1} \in A_{i+1}$ such that $\pi_{i+1}(a_{i+1}) = a_i - b_i$. We then have that

$$\Delta_A((a_i)_i) = (a_i - \pi_{i+1}(a_{i+1}))_i = (b_i)_i,$$

It follows that $\varprojlim^1 A_i = \operatorname{coker} \Delta_A = 0$. $\square$

EXAMPLE 3.7.7. Let $A_i = p^i\mathbb{Z}$ for each $i \geq 1$ and $\pi_i\colon p^i\mathbb{Z} \to p^{i-1}\mathbb{Z}$ be the natural injection. Consider the commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & p^{i+1}\mathbb{Z} \arrow[r] \arrow[d, hook] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}/p^{i+1}\mathbb{Z} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & p^i\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}/p^i\mathbb{Z} \arrow[r] & 0,
\end{tikzcd}$$

which gives rise to a long exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}_p \to \varprojlim^1(p^i\mathbb{Z}) \to 0,$$

so $\varprojlim^1(p^i\mathbb{Z}) \neq 0$.

DEFINITION 3.7.8. Let (A_i, π_i) be a collection of Λ-modules A_i and morphisms $\pi_{i+1}\colon A_{i+1} \to A_i$ for $i \geq 1$. We say that (A_i, π_i) satisfies the *Mittag-Leffler criterion* (ML) if for each $k \geq 1$, there exists some $j \geq k$ such that $\operatorname{im}\pi_{j,k} = \operatorname{im}\pi_{i,k}$ for all $i \geq j$.

REMARK 3.7.9. In other words, (A_i, π_i) satisfies the Mittag-Leffler criterion if for each $k \geq 0$, the images of the morphisms $\pi_{j,k}$ in A_k stabilize for sufficiently large j. (Note that $\operatorname{im}\pi_{j+1,k} \subseteq \operatorname{im}\pi_{j,k}$ for any $j \geq k$.)

EXAMPLES 3.7.10.

a. Any sequence of finite Λ-modules satisfies ML.

b. Any sequence of subspaces of a finite dimensional vector space over a field satisfies ML.

THEOREM 3.7.11. *If (A_i, π_i) satisfies ML, then $\varprojlim^1 A_i = 0$.*

PROOF. First suppose that for each i there exists $k \geq i$ with $\pi_{k,i} = 0$. Given $b_i \in A_i$ for each $i \geq 1$, we set

$$a_i = \sum_{j=1}^{\infty} \pi_{j,i}(b_j),$$

which is a finite sum by assumption and hence gives a well-defined element of A_i. Moreover, the sequence $(a_i)_i$ satisfies $\Delta_A((a_i)_i) = b_i$, so $\operatorname{coker}\Delta_A = 0$.

In the general case, set

$$B_i = \bigcap_{j \geq i} \pi_{j,i}(A_j) \subseteq A_i$$

for each $i \geq 1$. The restriction of π_i to B_i is surjective for each $i \geq 2$, and so the resulting system satisfies $\varprojlim^1 B_i = 0$. On the other hand, since the image of $\pi_{j,i}$ is contained in B_i for sufficiently large j, the system of quotients A_i/B_i has $\varprojlim^1 A_i/B_i = 0$ by the case already proven. Since we have a an exact sequence $0 \to B_i \to A_i \to A_i/B_i \to 0$ for each i, together with compatible maps, the fact that $\varprojlim^1 A_i$ is forced by the resulting long exact sequence. $\square$

CHAPTER 4

Spectral Sequences

4.1. Spectral sequences

DEFINITION 4.1.1. A (cohomological) spectral sequence $(E_r^{p,q}, d_r^{p,q})_{r\geq r_0}$ starting at $r = r_0$ in an abelian category $\mathscr{C}$ consists of a nonnegative integer a, and for each $r \geq a$ and $p, q \in \mathbb{Z}$,

i. an object $E_r^{p,q}$ of $\mathscr{C}$

ii. a morphism $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$, and

iii. an isomorphism

$$\frac{\ker(d_r^{p,q})}{\operatorname{im}(d_r^{p-r,q+r-1})} \xrightarrow{\sim} E_{r+1}^{p,q}$$

such that we have $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all r, p, and q.

DEFINITION 4.1.2. Let $\mathbb{E} = (E_r^{p,q}, d_r^{p,q})_{r\geq r_0}$ be a spectral sequence starting at $r = r_0$. Fix $(p,q) \in \mathbb{Z}^2$ and $r \geq r_0$.

a. The object $E_r^{p,q}$ is called the (p,q)-*term* in the rth layer of $\mathbb{E}$.

b. The morphism $d_r^{p,q}$ is called the rth *differential* at (p,q).

c. The *degree* of the (p,q)-term $E_r^{p,q}$ is $p+q$.

As one might expect, there exists a category of spectral sequences in $\mathscr{C}$. We define a morphism between spectral sequences as follows.

DEFINITION 4.1.3. A *morphism of spectral sequences* $\psi \colon \mathbb{E} \to \mathbb{D}$ with $\mathbb{E} = (E_r^{p,q}, d_r^{p,q})_{r\geq r_0}$ and $\mathbb{D} = (D_r^{p,q}, \delta_r^{p,q})_{r\geq s_0}$ spectral sequences in $\mathscr{C}$ is a family of morphisms

$$\psi_r^{p,q} \colon E_r^{p,q} \to D_r^{p,q}$$

in $\mathscr{C}$ for sufficiently large r such that

$$\psi_r^{p+r,q-r+1} \circ d_r^{p,q} = \delta_r^{p,q} \circ \psi_r^{p,q}$$

and $\psi_{r+1}^{p,q}$ is the map induced by $\psi_r^{p,q}$ on subquotients for all r, p, and q.

The reader will check the following.

LEMMA 4.1.4. *Let $\psi \colon \mathbb{E} \to \mathbb{D}$ be a morphism of spectral sequences such that there exists an integer r_1 for which the morphisms $\psi_{r_1}^{p,q}$ are all isomorphisms. Then the morphisms $\psi_r^{p,q}$ are isomorphisms for all $r \geq r_1$.*

Throughout the remainder of this section $\mathbb{E} = (E_r^{p,q}, d_r^{p,q})_{r\geq r_0}$ will denote a spectral sequence starting at $r = r_0$ in an abelian category $\mathscr{C}$.

REMARK 4.1.5. Given $p,q \in \mathbb{Z}$, the object $E_r^{p,q}$ is for any $r \geq r_0$ a subquotient of $E_{r_0}^{p,q}$, equal to $Z_r^{p,q}/B_r^{p,q}$ for subobjects $B_r^{p,q} \subseteq Z_r^{p,q}$ of $E_{r_0}^{p,q}$ that satisfy

$$B_r^{p,q} \subseteq B_{r+1}^{p,q} \quad \text{and} \quad Z_{r+1}^{p,q} \subseteq Z_r^{p,q}$$

for each $r \geq r_0$.

NOTATION 4.1.6. We refer to $B_r^{p,q}$ and $Z_r^{p,q}$ as the *r-coboundaries* and *r-cocycles* in the (p,q)-term of $\mathbb{E}$.

DEFINITION 4.1.7. For a spectral sequence $\mathbb{E}$, we set

$$B_\infty^{p,q} = \bigcup_{r \geq r_0} B_r^{p,q} \quad \text{and} \quad Z_\infty^{p,q} = \bigcap_{r \geq r_0} Z_r^{p,q}$$

if these colimits and limits exist, in which case we set $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$.

TERMINOLOGY 4.1.8. When they exist, the objects $B_\infty^{p,q}$, $Z_\infty^{p,q}$, and $E_\infty^{p,q}$ are respectively called the limit coboundaries, cocycles, and term at (p,q).

REMARK 4.1.9. If $\mathscr{C}$ is a complete category, then $B_\infty^{p,q}$ exists. If $\mathscr{C}$ is a cocomplete category, then $Z_\infty^{p,q}$ exists.

Often, we have that for each pair (p,q), there exists $r \geq r_0$ such that $B_r^{p,q} = B_\infty^{p,q}$ and $Z_r^{p,q} = Z_\infty^{p,q}$. As an example, we have the first quadrant spectral sequences.

DEFINITION 4.1.10. We say that $\mathbb{E}$ is a *nth quadrant spectral sequence*, for $1 \leq n \leq 4$, if $E_{r_0}^{p,q} = 0$ for all $(p,q) \in \mathbb{Z}^2$ lying outside of the closed nth quadrant of the plane $\mathbb{R}^2$.

LEMMA 4.1.11. *Let $\mathbb{E}$ be a first or third quadrant spectral sequence. Then for each $(p,q) \in \mathbb{Z}^2$, there exists an $r_1 \geq r_0$ such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for all $r \geq r_1$.*

PROOF. One need only take $r_1 = \max\{|p|+1, |q|+2\}$. $\square$

REMARK 4.1.12. That $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ says exactly that

$$\ker(d_r^{p,q}) = E_r^{p,q} \quad \text{and} \quad \operatorname{im}(d_r^{p-r,q+r-1}) = 0,$$

which is to say that the morphism given by property (iii) of a spectral sequence is an isomorphism $E_r^{p,q} \xrightarrow{\sim} E_{r+1}^{p,q}$. In other words, if this holds for all $r \geq r_1$, then $E_{r_1}^{p,q}$ is $E_\infty^{p,q}$.

Often, something even stronger occurs.

DEFINITION 4.1.13. We say that a spectral sequence $\mathbb{E}$ *degenerates* at $r \geq r_0$ if $d_r^{p,q} = 0$ for all $p,q \in \mathbb{Z}$.

We now define the notion of convergence of a spectral sequence.

DEFINITION 4.1.14. Let $(E^n)_{n\in\mathbb{Z}}$ be a collection of objects in $\mathscr{C}$ such that each E^n is endowed with a nonincreasing filtration $(F^p E^n)_{p\in\mathbb{Z}}$ by subobjects that satisfy $F^p E^n = 0$ for sufficiently small p and $F^p E^n = E^n$ for sufficiently large p. For each pair $(p,q) \in \mathbb{Z}^2$, let us set

$$\operatorname{gr}_p E^{p+q} = F^p E^{p+q}/F^{p+1} E^{p+q}.$$

We say that the spectral sequence $\mathbb{E}$ *converges to* (E^n) and write

$$E_{r_0}^{p,q} \Rightarrow E^{p+q}$$

if the $E_\infty^{p,q}$ exist and there are isomorphisms

$$\alpha^{p,q} \colon E_\infty^{p,q} \xrightarrow{\sim} \operatorname{gr}_p E^{p+q}$$

for each pair $(p,q) \in \mathbb{Z}^2$.

REMARK 4.1.15. There exist definitions of convergence that weaken the conditions that the graded terms in the filtration vanish in sufficiently low and high degrees. We do not treat those here.

REMARK 4.1.16. In a first quadrant spectral sequence $\mathbb{E}$, each term $E_{r+1}^{p,0}$ for $r \geq r_0$ is a quotient of the term $E_r^{p,0}$ and each term $E_{r+1}^{0,q}$ is a subobject of $E_r^{0,q}$. So, if $E_{r_0}^{p,q} \Rightarrow E^{p+q}$, we have epimorphisms $E_r^{p,0} \to E^p$ and monomorphisms $E^q \to E_r^{0,q}$ induced by $\alpha^{p,0}$ and $(\alpha^{q,0})^{-1}$.

TERMINOLOGY 4.1.17. Let $\mathbb{E}$ be a first quadrant spectral sequence.

a. The terms $E_r^{p,0}$ and $E_r^{0,q}$ are known as edge terms.

b. If $E_{r_0}^{p,q} \Rightarrow E^{p+q}$, then the morphisms

$$E_r^{p,0} \to E^p \quad \text{and} \quad E^q \to E_r^{0,q}$$

are known as edge maps (or morphisms).

LEMMA 4.1.18. *Let $\mathbb{E}$ be a first quadrant spectral sequence starting at $r_0 \leq 2$ and converging to $(E^n)_{n\in\mathbb{Z}}$. Then there is an exact sequence*

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \to E^2,$$

in which the maps not labelled are edge maps.

PROOF. We must have $F^0 E^n = E^n$ and $F^{n+1} E^n = 0$ for each $n \geq 0$ and $E^n = 0$ for $n < 0$ since $\mathbb{E}$ is a first quadrant spectral sequence. Note that $E_2^{1,0} = E_\infty^{1,0}$, and $E_3^{p,q} = E_\infty^{p,q}$ for $(p,q) = (0,1)$ and $(p,q) = (2,0)$. The result is then almost immediate from the definition of the edge maps, since we have an exact sequence

$$0 \to F^1 E^1 \to E^1 \to \operatorname{gr}^0 E^1 \to 0$$

with $E_2^{1,0} \xrightarrow{\sim} F^1 E^1$ and $\operatorname{gr}^0 E^1 \xrightarrow{\sim} \ker d_2^{0,1}$ the isomorphisms induced by edge maps. □

TERMINOLOGY 4.1.19. The exact sequence of Lemma 4.1.18 is known as the *exact sequence of edge terms* of $\mathbb{E}$.

4.2. Filtrations on complexes

In this section, we let $(A^\cdot, d^\cdot)$ be a cochain complex in an abelian category $\mathscr{C}$ equipped with a nonincreasing filtration of subcomplexes

$$\cdots \subseteq F^{p+1}A^\cdot \subseteq F^p A^\cdot \subseteq F^{p-1}A^\cdot \subseteq \cdots$$

such that the differentials on each $F^p A^\cdot$ are the restrictions of the differentials on $A^\cdot$. Note that F^p induces a filtration on the homology of $A^\cdot$.

Definition 4.2.1.

a. A filtered cochain complex is a pair, denoted $(A^\cdot, F^\cdot)$, consisting of a cochain complex $(A^\cdot, d^\cdot)$ and a nonincreasing filtration $F_p A^\cdot$ of $A^\cdot$ by subcomplexes $(F^p A^\cdot, F^p d^\cdot)$ for $p \in \mathbb{Z}$.

b. We say that a filtered cochain complex is bounded if for each $i \in \mathbb{Z}$, there exist $t < s \in \mathbb{Z}$ such that $F^s A^i = 0$ and $F^t A^i = A^i$.

c. The pth graded piece in the filtration is the complex $(\mathrm{gr}_p A^\cdot, \mathrm{gr}_p d^\cdot)$, where the differentials are induced by the $d^\cdot$ via the projection morphism of complexes $\pi^{p,\cdot}\colon F^p A^\cdot \to \mathrm{gr}_p A^\cdot$.

Notation 4.2.2. Let $(A^\cdot, F^\cdot)$ be a filtered cochain complex. Let $(p,q) \in \mathbb{Z}^2$ and $r \geq 0$.

a. We set $\tilde{Z}_0^{p,q} = F^p A^{p+q}$ and $\tilde{B}_0^{p,q} = 0$, and if $r \geq 1$, we set

$$\tilde{Z}_r^{p,q} = (d^{p+q})^{-1}(F^{p+r}A^{p+q+1}) \cap F^p A^{p+q} \quad \text{and} \quad \tilde{B}_r^{p,q} = d^{p+q-1}(\tilde{Z}_{r-1}^{p-r+1,q+r-2}) \subseteq \tilde{Z}_r^{p,q}.$$

b. We set

$$Z_r^{p,q} = \pi^{p,q}(\tilde{Z}_r^{p,q}) \subseteq \mathrm{gr}_p A^{p+q} \quad \text{and} \quad B_r^{p,q} = \pi^{p,q}(\tilde{B}_r^{p,q}) \subseteq Z_r^{p,q}.$$

c. We set $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$.

Theorem 4.2.3. *The collection $\mathbb{E}(A^\cdot) = (E_r^{p,q}, d_r^{p,q})$ attacted to a filtered complex $(A^\cdot, F^\cdot)$, where*

$$d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

is induced by $\mathrm{gr}_p d^{p+q}$, is a spectral sequence. If the filtration is bounded, then $\mathbb{E}(A^\cdot)$ converges to $H^{p+q}(A^\cdot)$.

Proof. Note that

$$d^{p+q}(F^p A^{p+q}) \subseteq F^p A^{p+q+1},$$

so the differential $d_0^{p,q}\colon E_0^{p,q} \to E_0^{p,q+1}$ is well defined. Moreover, we have that

$$d^{p+q}(\tilde{Z}_r^{p,q}) \subseteq F_p A^{p+r,q-r+1} \cap \ker d^{p+q+1} \subseteq \tilde{Z}_r^{p+r,q-r+1}$$

and $d^{p+q}(\tilde{B}_r^{p,q}) = 0$, so $d_r^{p,q}$ is well-defined for all $r \geq 0$.

Since

$$\tilde{Z}_{r-1}^{p+1,q-1} = \tilde{Z}_r^{p,q} \cap F^{p+1}A^{p+q},$$

we have

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}} \cong \frac{\tilde{Z}_r^{p,q}}{\tilde{B}_r^{p,q} + \tilde{Z}_{r-1}^{p+1,q-1}}.$$

From this, the reader may check that

$$\ker d_r^{p,q} \cong \frac{(d^{p+q})^{-1}(\tilde{B}_r^{p+r,q-r+1} + \tilde{Z}_{r-1}^{p+r+1,q-r}) \cap \tilde{Z}_r^{p,q}}{\tilde{B}_r^{p,q} + \tilde{Z}_{r-1}^{p+1,q-1}} \cong \frac{\tilde{Z}_{r+1}^{p,q} + \tilde{Z}_{r-1}^{p+1,q-1}}{\tilde{B}_r^{p,q} + \tilde{Z}_{r-1}^{p+1,q-1}} \cong \frac{Z_{r+1}^{p,q}}{B_r^{p,q}}.$$

and

$$\operatorname{im} d_r^{p-r,q+r-1} \cong \frac{\tilde{B}_{r+1}^{p,q} + \tilde{Z}_{r-1}^{p+1,q-1}}{\tilde{B}_r^{p,q} + \tilde{Z}_{r-1}^{p+1,q-1}} \cong \frac{B_{r+1}^{p,q}}{B_r^{p,q}}.$$

Therefore, we have

$$E_{r+1}^{p,q} = \frac{Z_{r+1}^{p,q}}{B_{r+1}^{p,q}} \cong \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}.$$

Therefore, taken together with these isomorphisms, $\mathbb{E}(A^\cdot)$ forms a spectral sequence.

Now suppose that the filtration is bounded. Then the $E_r^{p,q}$ are successive subquotients which eventually stabilize for sufficiently large r, with the $\tilde{Z}_r^{p,q}$ stabilizing to

$$\tilde{Z}_\infty^{p,q} = \ker d^{p+q} \cap F^p A^{p+q}$$

and the

$$\tilde{B}_r^{p,q} = d^{p+q-1}((d^{p+q-1})^{-1}(F^p A^{p+q}) \cap F^{p-r+1} A^{p+q-1})$$

stabilizing to

$$\tilde{B}_\infty^{p,q} = \operatorname{im} d^{p+q-1} \cap F^p A^{p+q},$$

with $Z_\infty^{p,q} = \pi^{p,q}(\tilde{Z}_\infty^{p,q})$ and $B_\infty^{p,q} = \pi^{p,q}(\tilde{B}_\infty^{p,q})$. Thus, we have

$$\operatorname{gr}^p H^{p+q}(A^\cdot) \cong \frac{F^p H^{p+q}(A^\cdot)}{F^{p+1} H^{p+q}(A^\cdot)} \cong \frac{\ker d^{p+q} \cap F^p A^{p+q}}{(\ker d^{p+q} \cap F^{p+1} A^{p+q}) + (\operatorname{im} d^{p+q-1} \cap F^p A^{p+q})} \cong \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} = E_\infty^{p,q}.$$

□

We next consider a setting in which filtrations on complexes naturally arise.

Definition 4.2.4. Let $C^{\cdot\cdot}$ be a double (cochain) complex in $\mathscr{C}$, and let $n \in \mathbb{Z}$.

a. Let ${}^I\sigma^{\geq n} C$ be the double subcomplex of C with

$${}^I\sigma^{\geq n} C^{i,j} = \begin{cases} C^{i,j} & \text{if } i \geq n \\ 0 & \text{if } i < n. \end{cases}$$

b. Let ${}^{II}\sigma^{\geq n} C$ be the double subcomplex of C with

$${}^{II}\sigma^{\geq n} C^{i,j} = \begin{cases} C^{i,j} & \text{if } j \geq n \\ 0 & \text{if } j < n. \end{cases}$$

Notation 4.2.5. Let $C^{\cdot\cdot}$ be a double (cochain) complex, let $A^\cdot = \operatorname{Tot}^\oplus(C)$, and let $p \in \mathbb{Z}$.

a. Let ${}^I F^p A^\cdot$ be the subcomplex of $A^\cdot$ that is the total (sum) complex of ${}^I\sigma^{\geq p} C^{i,j}$.

b. Let ${}^{II} F^p A^\cdot$ be the subcomplex of $A^\cdot$ that is the total (sum) complex of ${}^{II}\sigma^{\geq p} C^{i,j}$.

We may make the analogous definitions with the total product complex $A^\cdot = \operatorname{Tot}^\Pi(C)$.

REMARK 4.2.6. If $C^{\cdot\cdot}$ is a first quadrant cochain complex, then the filtrations ${}^{I}F^{p}$ and ${}^{II}F^{p}$ on $A = \mathrm{Tot}(C)$ are bounded. That is, for each n, we have that $F^{n+1}A^{n} = 0$, while we always have that $F^{0}A^{n} = A^{n}$.

NOTATION 4.2.7. Let $C^{\cdot\cdot}$ be a first quadrant double (cochain) complex. We let ${}^{I}\mathbb{E}(C)$ (resp., ${}^{II}\mathbb{E}(C)$, with terms ${}^{I}E_{r}^{p,q}$ and ${}^{II}E_{r}^{p,q}$, respectively) denote the spectral sequence attached to the filtration ${}^{I}F^{p}$ (resp., ${}^{II}F^{p}$) on $\mathrm{Tot}^{\oplus}(C)$ by Theorem 4.2.3.

REMARK 4.2.8. The spectral sequences ${}^{I}\mathbb{E}(C)$ and ${}^{II}\mathbb{E}(C)$ both converge to $H_{p+q}(\mathrm{Tot}(C))$. We have ${}^{I}E_{0}^{p,q} = C^{p,q}$ and ${}^{I}d_{0}^{p,q} = d_{v}^{p,q}$, so

$$ {}^{I}E_{1}^{p,q} = H^{q}(C^{p,\cdot}), $$

and ${}^{I}d_{1}^{p,q}$ is induced by $d_{h}^{p,q}$, so we have

$$ {}^{I}E_{2}^{p,q} \cong H_{h}^{p}(H_{v}^{q}(C)) \Rightarrow H^{p+q}(\mathrm{Tot}(C)). $$

On the other hand, we have ${}^{II}E_{0}^{p,q} = C^{q,p}$ and ${}^{II}d_{0}^{p,q} = d_{h}^{q,p}$, and the maps ${}^{II}d_{1}^{p,q}$ are induced by the $d_{v}^{q,p}$. We therefore have

$$ {}^{II}E_{2}^{p,q} \cong H_{v}^{p}(H_{h}^{q}(C)) \Rightarrow H^{p+q}(\mathrm{Tot}(C)). $$

We can often play these spectral seqeunces off of each other to obtain interesting results. For instance, we may shed new light on our proof of Proposition 3.5.9.

REMARK 4.2.9. Let Λ be a ring, A be a right Λ-module, and B be a left Λ-module. Let $P_{\cdot} \to A$ be a projective resolution of A by right Λ-modules, and let $Q_{\cdot} \to B$ be a projective resolution of B by left Λ-modules. We form the double complex $P_{\cdot} \otimes_{\Lambda} Q_{\cdot}$ and consider the resulting (homological) first quadrant spectral sequence with E^{1}-terms

$$ {}^{I}E_{p,q}^{1} \cong H_{q}(P_{p} \otimes_{\Lambda} Q_{\cdot}) \cong P_{p} \otimes_{\Lambda} H_{q}(Q_{\cdot}) \cong \begin{cases} P_{p} \otimes_{\Lambda} B & \text{if } q = 0 \\ 0 & \text{if } q \geq 1. \end{cases} $$

We then have that the spectral sequences degenerates at $r = 2$, with

$$ {}^{I}E_{p,q}^{\infty} = {}^{I}E_{p,q}^{2} \cong \begin{cases} H_{p}(P_{\cdot} \otimes_{\Lambda} B) & \text{if } q = 0 \\ 0 & \text{if } q \geq 1. \end{cases} $$

On the other hand, if we consider the spectral sequence ${}^{II}E_{p,q}^{r}$, we similarly obtain

$$ {}^{II}E_{p,q}^{\infty} = {}^{II}E_{p,q}^{2} \cong \begin{cases} H_{p}(A \otimes_{\Lambda} Q_{\cdot}) & \text{if } q = 0 \\ 0 & \text{if } q \geq 1, \end{cases} $$

and both ${}^{I}E_{p,0}^{\infty}$ and ${}^{II}E_{p,0}^{\infty}$ are then isomorphic $H_{p}(\mathrm{Tot}(P_{\cdot} \otimes_{\Lambda} Q_{\cdot}))$. Now, note that

$$ \mathrm{Tor}_{i}^{\Lambda}(A, B) \cong H_{i}(A \otimes_{\Lambda} Q_{\cdot}) $$

by definition. So, we have a reinterpretation of the proof of the assertion

$$ \mathrm{Tor}_{i}^{\Lambda}(A, B) \cong H_{i}(P_{\cdot} \otimes_{\Lambda} B) $$

of Proposition 3.5.9.

We can generalize this as follows.

THEOREM 4.2.10 (Künneth spectral sequence). *Let $M.$ be a bounded below chain complex of flat right Λ-modules, and let B be a Λ-module. Then there is a convergent spectral sequence*

$$E^2_{p,q} = \operatorname{Tor}^{\Lambda}_p(H_q(M.), B) \Rightarrow H_{p+q}(M. \otimes_\Lambda B).$$

PROOF. We take a projective resolution $Q. \to B$ of B by left Λ-modules and form the double complex $M. \otimes_\Lambda Q..$ Since the terms of $M.$ are flat and $H_q(Q.) = 0$ for $q \geq 1$, we obtain

$$^{I}E^{\infty}_{p,q} = {}^{I}E^2_{p,q} \cong \begin{cases} H_p(M. \otimes_\Lambda B) & \text{if } q = 0 \\ 0 & \text{if } q \geq 1, \end{cases}$$

which then implies that

$$H_p(M. \otimes_\Lambda B) \cong H_{p+q}(\operatorname{Tot}(M. \otimes_\Lambda Q.)).$$

Since the terms of $Q.$ are flat as well, we have

$$^{II}E^2_{p,q} = H_p(H_q(M.) \otimes_\Lambda B) \cong \operatorname{Tor}^{\Lambda}_p(H_q(M.), B),$$

which finishes the proof. □

REMARK 4.2.11. If $M.$ is a projective resolution of a right Λ-module A, the spectral sequence degenerates at $r = 2$, and the resulting isomorphisms are again simply those of Proposition 3.5.9.

4.3. Grothendieck spectral sequences

Let $\mathscr{C}$ be an abelian category. Recall that an injective object in $\mathbf{Ch}(\mathscr{C})$ is a split exact sequence of injective objects in $\mathscr{C}$. However, unless a complex $A^\cdot$ of objects in $\mathscr{C}$ is bounded below, it is not clear that there will exists an injective object $I^\cdot$ of $\mathbf{Ch}(\mathscr{C})$ and a monomorphism $A^\cdot \to I^\cdot$. Therefore, we not be able to find injective resolutions of unbounded complexes. However, we do always have the following substitute.

DEFINITION 4.3.1. Let $\mathscr{C}$ be an abelian category. A Cartan-Eilenberg resolution of a cochain complex $A^\cdot$ is a resolution $(I^{\cdot\cdot}, \epsilon^\cdot)$ of $A^\cdot$ in $\mathbf{Ch}(\mathscr{C})$, where $\epsilon^i \colon A^i \to I^{i,0}$ is a morphism in $\mathscr{C}$, such that

i. each $I^{i,j}$ for $i, j \in \mathbb{Z}$ is injective,

ii. $I^{i,\cdot} = 0$ for each $i \in \mathbb{Z}$ with $A^i = 0$,

iii. each of the objects

$$B^{i,j}_h(I) = \operatorname{im} d^{i-1,j}_h \quad \text{and} \quad H^{i,j}_h(I) = \ker d^{i,j}_h / \operatorname{im} d^{i-1,j}_h$$

is injective, and

iv. each of the augmented complexes

$$\operatorname{im} d^{i-1}_A \to B^{i,\cdot}_h(I)$$

and

$$H^i(A) \to H^{i,\cdot}_h(I),$$

with the morphisms induced by the $d_v^{i,j}$ and the augmentation maps by ϵ^i, is exact.

REMARK 4.3.2. If $(I^{\cdot\cdot}, \epsilon^{\cdot})$ is a Cartan-Eilenberg resolution of a complex $A^{\cdot}$, then the augmented complex $\ker d^i \to Z_h^{i,\cdot}(I)$, with $Z_h^{i,j}(I) = \ker d_h^{i,j}$, is an injective resolution.

REMARK 4.3.3. If $A^{\cdot}$ is concentrated in degree 0, then a Cartan-Eilenberg resolution of $A^{\cdot}$ consists of a double complex concentrated in the 0th column, an injective resolution of A^0.

The following is a clever application of the Horseshoe lemma.

PROPOSITION 4.3.4. *Let $\mathscr{C}$ be an abelian categories, and suppose that $\mathscr{C}$ has enough injectives. Then every cochain complex in $\mathscr{C}$ has a Cartan-Eilenberg resolution.*

PROOF. Set $Z^i = \ker d_A^i$, $B^i = \operatorname{im} d_A^{i+1}$, and $H^i = H^i(A)$. We consider $Z^{\cdot}$, $B^{\cdot}$, and $H^{\cdot}$ as complexes with zero differentials, since the d_A^i induce trivial maps on their ith terms. Choose injective resolutions $\iota_B^i \colon B^i \to I_B^{i,\cdot}$ and $\iota_H^i \colon H^i \to I_H^{i,\cdot}$ for each i, taking $I_B^i = I_H^i = 0$ if $B^i = 0$ or $I^i = 0$. These form double complexes $I_B^{\cdot\cdot}$ and $I_H^{\cdot\cdot}$ with zero horizontal differentials. Set $I_Z^{\cdot,j} = I_B^{\cdot,j} \oplus I_H^{\cdot,j}$ for each $j \geq 0$, and then apply the Horseshoe lemma to obtain the vertical differentials that give the augmented complex $Z^{\cdot} \to I_Z^{\cdot\cdot}$ (again with zero horizontal differentials). Finally, apply the Horseshoe lemma to the exact sequence

$$0 \to Z^{\cdot} \to A^{\cdot} \to B^{\cdot} \to 0$$

and the augmented complexes $Z^{\cdot} \to I_Z^{\cdot\cdot}$ and $B^{\cdot} \to I_B^{\cdot\cdot}$ to obtain the Cartan-Eilenberg resolution. □

The use of Cartan-Eilenberg resolutions is seen in the following lemmas.

LEMMA 4.3.5. *Let $A^{\cdot}$ be a complex and $(I^{\cdot\cdot}, \epsilon^{\cdot})$ a Cartan-Eilenberg resolution of $A^{\cdot}$. Suppose that either $A^{\cdot}$ is bounded below or $\mathscr{C}$ admits direct products. Then the induced augmentation morphism $A^{\cdot} \to \mathrm{Tot}^{\Pi}(I^{\cdot\cdot})$ is a quasi-isomorphism. In particular, every bounded below complex admits a quasi-isomorphism to a bounded below complex of injective objects.*

PROOF. Consider the augmented complex $B^{\cdot\cdot}$ attached to $A^{\cdot} \to I^{\cdot\cdot}$ (with $A^{\cdot}$ in the row of degree -1). Then $\mathrm{Tot}^{\Pi}(B^{\cdot\cdot})$ is exact by Proposition 2.8.10. Since the latter total complex is isomorphic to the cone of the morphism $A^{\cdot} \to \mathrm{Tot}^{\Pi}(I^{\cdot\cdot})$, we have the result. □

LEMMA 4.3.6. *Let $f^{\cdot} \colon A^{\cdot} \to B^{\cdot}$ be a morphism of cochain complexes in $\mathscr{C}$, and let $A^{\cdot} \to I^{\cdot\cdot}$ and $B^{\cdot} \to J^{\cdot\cdot}$ be Cartan-Eilenberg resolutions.*

a. There is a morphism $\alpha^{\cdot\cdot} \colon I^{\cdot\cdot} \to J^{\cdot\cdot}$ such that the pair $(f^{\cdot}, \alpha)$ is a morphism of the augmented complexes, and any two such morphisms are chain homotopic (as morphisms of double complexes).

b. If $f^{\cdot} = \mathrm{id}_{A^{\cdot}}$, then the morphism $\alpha^{\cdot\cdot}$ is a homotopy equivalence, as is the induced morphism $\mathrm{Tot}^{\Pi}(I^{\cdot\cdot}) \to \mathrm{Tot}^{\Pi}(J^{\cdot\cdot})$.

Let $\mathscr{C}$ be an abelian category. We will discuss cochain complexes, resolutions by injectives, and cohomological spectral sequences, though everything can be done for chain complexes, resolutions by projectives and homological spectral sequences.

DEFINITION 4.3.7. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a left exact functor of abelian categories, and suppose that $\mathscr{C}$ has enough injectives. For any cochain complex $A^\cdot$ in $\mathscr{C}$ nonnegative degrees, we define the ith right hyper-derived functor

$$\mathbb{R}^i F\colon \mathbf{Ch}(\mathscr{C}) \to \mathscr{D}$$

by

$$\mathbb{R}^i F(A^\cdot) \cong H^i(F(C^\cdot)),$$

where $C^\cdot$ is the total complex of a Cartan-Eilenberg resolution of $A^\cdot$ in $\mathbf{Ch}(\mathscr{C})$.

PROPOSITION 4.3.8. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a left exact functor of abelian categories, and suppose that $\mathscr{C}$ has enough injectives. Then there is a natural isomorphism of δ-functors between the restrictions $\mathbb{R}^i F\colon \mathbf{Ch}^{\geq 0}(\mathscr{C}) \to \mathscr{D}$ of the right derived hyper-functors of F and the right derived functors of the left exact functors $H^0 \circ F$, where $H^0\colon \mathbf{Ch}^{\geq 0}(\mathscr{D}) \to \mathscr{D}$ is the 0th cohomology functor.*

PROPOSITION 4.3.9. *Let $F\colon \mathscr{C} \to \mathscr{D}$ be a left exact functor of abelian categories, and suppose that $\mathscr{C}$ has enough injectives. Let $A^\cdot$ be an object of $\mathbf{Ch}^{\geq 0}(\mathscr{C})$. Then we have first quadrant convergent spectral sequences*

$${}^{I}E_2^{p,q}(A^\cdot) = H^p(R^q F(A^\cdot)) \Rightarrow \mathbb{R}^{p+q} F(A^\cdot).$$

and

$${}^{II}E_2^{p,q}(A^\cdot) = (R^p F)(H^q(A^\cdot)) \Rightarrow \mathbb{R}^{p+q} F(A^\cdot).$$

PROOF. These are simply the spectral sequences ${}^{I}\mathbb{E}(F(I^{\cdot\cdot}))$ and ${}^{II}\mathbb{E}(F(I^{\cdot\cdot}))$, where $A^\cdot \to I^{\cdot\cdot}$ is a Cartan-Eilenberg resolution. As explained in Remark 4.2.8, these spectral sequences both converge to $H^n(\mathrm{Tot}(F(I^{\cdot\cdot})))$, and we have that they satisfy

$${}^{I}E_2^{p,q}(A^\cdot) \cong H_h^p(H_v^q(F(I^{\cdot\cdot}))) \cong H^p(R^q F(A^\cdot))$$

and

$${}^{II}E_2^{p,q}(A^\cdot) \cong H_v^p(H_h^q(F(I^{\cdot\cdot}))) \cong H_v^p(F(H_h^q(I^{\cdot\cdot}))) \cong R^p F(H^q(A^\cdot)).$$

□

We are now able to construct Grothendieck spectral sequences.

THEOREM 4.3.10 (Grothendieck). *Let $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$ be abelian categories such that both $\mathscr{B}$ and $\mathscr{C}$ have enough injectives. Let $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be left exact functors of abelian categories, and suppose that G sends injective objects in $\mathscr{B}$ to F-acyclic objects in $\mathscr{C}$. Let B be an object of $\mathscr{B}$. Then there exists a first quadrant convergent cohomological spectral sequence*

$$E_2^{p,q} = (R^p F)(R^q G)(B) \Rightarrow R^{p+q}(F \circ G)(B),$$

and this construction is natural in B.

PROOF. Let $B \to I^\cdot$ be an injective resolution. Then $G(I^\cdot)$ is an object of $\mathbf{Ch}^{\geq 0}(\mathscr{C})$. Consider the hyper-derived functors $\mathbb{R}^i F$ applied to $G(I^\cdot)$. We have convergent spectral sequences as in Proposition 4.3.9 with E_2-terms:

$${}^{I}E_2^{p,q} = H^p((R^q F)(G(I^\cdot)))$$

and

$$^{II}E_2^{p,q} = R^p F(H^q(G(I^\cdot))) \cong (R^p F)(R^q G)(B).$$

Since $G(I^\cdot)$ is F-acyclic, the spectral sequence has $^{I}E_2^{p,q} = 0$ for $q > 0$ and

$$^{I}E_2^{p,0} = H^p(F(G(I^\cdot))) \cong R^p(F \circ G)(B)$$

so degenerates at $r = 2$ and therefore converges to the sequence of $R^n(F \circ G)(B)$. It follows immediately that the second spectral sequence converges to the sequence of $R^n(F \circ G)(B)$ as well, which is what we required. □

REMARK 4.3.11. In view of Proposition 4.3.8, the spectral sequence $^{II}E_2^{p,q}(A^\cdot)$ of Proposition 4.3.9 is the Grothendieck spectral sequence for the functors H^0 and F. Note that F takes injective objects in $\mathbf{Ch}^{\geq 0}(\mathscr{C})$, which are exact complexes of injectives in nonnegative degrees, to complexes in $\mathbf{Ch}^{\geq 0}(\mathscr{D})$ that are exact in degree 0 by left exactness of F. Of course, the complexes that are exact in degree 0 are exactly the H^0-acyclic complexes.

We give an extremely useful example.

THEOREM 4.3.12 (Hochschild-Serre). *Let G be a group, N a normal subgroup, and A a $\mathbb{Z}[G]$-module. Then there is a first quadrant convergent cohomological spectral sequence*

$$E_2^{p,q} = H^p(G/N, H^q(N, A)) \Rightarrow H^{p+q}(G, A).$$

PROOF. We consider the functors $\mathbb{Z}[N]$-**mod** $\to$ **Ab** and $\mathbb{Z}[G]$-**mod** $\to \mathbb{Z}[G/N]$-**mod** given by taking (G/N)-invariants and N-invariants, respectively. We claim that the N-invariant functor preserves injectives. The Hochschild-Serre spectral sequence is then simply the Grothendieck spectral sequence for these two functors.

So, let B be a $\mathbb{Z}[G]$-module and A a $\mathbb{Z}[G/N]$-module, and note that we have isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}[G/N]}(A, B^N) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(A, B)$$

natural in A and B. In other words, the (additive) N-invariant functor is right adjoint to the forgetful functor, which is exact. Therefore, Proposition 3.4.2 yields the claim. □