

# Determination of Conductors from Galois Module Structure

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## Abstract

Let  $E$  denote an unramified extension of  $\mathbf{Q}_p$ , and set  $F = E(\zeta_{p^n})$  for an odd prime  $p$  and  $n \geq 1$ . We determine the conductors of the Kummer extensions  $F(\sqrt[n]{a})$  of  $F$  by those elements  $a \in F^\times$  such that  $F(\sqrt[n]{a})/E$  is Galois. This follows from a comparison of the Galois module structure of  $F^\times$  with the unit filtration of  $F$ .

## 1 Introduction

We consider ramification in those Galois extensions  $K$  of an unramified extension  $E$  of  $\mathbf{Q}_p$  which are also Kummer extensions of  $F = E(\zeta_{p^n})$  by the  $p^n$ th root of an element of  $F^\times$ , where  $p$  is an odd prime and  $\zeta_{p^n}$  is a primitive  $p^n$ th root of unity. We determine the conductors of the abelian subextensions of  $F$  in  $K$ . From these, the ramification groups of the entire extension  $K/E$  are computable (see [S1], for example).

The two step solvable extensions we consider arise as fixed fields of the kernels of certain upper triangular representations  $\rho: G_E \rightarrow GL_2(\mathbf{Z}/p^n\mathbf{Z})$  of the absolute Galois group  $G_E$  of  $E$ . More specifically, for  $\sigma \in G_E$  we should have

$$\rho(\sigma) = \begin{pmatrix} \chi^{r+t}(\sigma) & \kappa(\sigma) \\ 0 & \chi^t(\sigma) \end{pmatrix},$$

where  $\chi$  denotes the cyclotomic character with  $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi(\sigma)}$  and  $\kappa$  is a map which makes  $\rho$  a homomorphism (in other words,  $\kappa\chi^{-t}$  should be a 1-cocycle of  $G_E$  with values in  $\mathbf{Z}/p^n\mathbf{Z}(r)$  for some  $r$ ). Such extensions can often be found as localizations of interesting global extensions ramified only at primes above  $p$  (see [Ih], for example).

We fix the fields  $E$  and  $F$  as above throughout the remainder of this article. Let  $U_i$  denote the  $i$ th unit group of  $F$  for  $i \geq 0$ . We also let  $(\cdot, \cdot)_{n,F}$  denote the  $p^n$ th norm residue symbol of  $F$  (and similarly for  $\mathbf{Q}_p(\zeta_{p^n})$ ). For  $a \in F^\times$ , we denote by  $f_n(a)$  the

conductor of  $F(\sqrt[p^n]{a})/F$  (considered as an integer) [Se, XV.2]. Then  $f = f_n(a)$  is the smallest nonnegative integer for which  $(a, u)_{n,F} = 1$  for all  $u \in U_f$ .

For any integer  $r$ , we define a group  $F^r$  by

$$F^r = \{x \in F^\times \mid \sigma(x)x^{-\chi(\sigma)^r} \in F^{\times p^n} \text{ for all } \sigma \in G_E\}.$$

We shall compute the conductors  $f_n(x)$  for all  $x \in F^r$ . In particular, in Section 4, we shall prove the following result which gives the answer for those  $r$  with  $r \not\equiv 0, 1 \pmod{p-1}$ .

**Theorem.** *Let  $r \not\equiv 0, 1 \pmod{p-1}$ , and let  $t$  denote the smallest positive integer with  $t \equiv 2 - r \pmod{p-1}$ . Let  $x \in F^r$  with  $x \notin F^{\times p}$ . Then we have that  $f_m(x) = p^{m-1}t$  for any positive integer  $m$  with  $m \leq n$ .*

The cases  $r \equiv 0, 1 \pmod{p-1}$  have more complicated statements and are worked out in Sections 5 and 6. The case  $r = 0$ , in which one considers extensions of  $F$  by roots of elements of  $E^\times$ , was described using completely different methods in [CM] for  $E = \mathbf{Q}_p$  and [S3] in general.

The determination of the conductors follows from the determination of generators of certain unit subgroups as Galois submodules of  $F^\times$ . More specifically, one can determine a simple formula for the norm residue symbol  $(x, y)_{n,F}$  with  $x \in F^r$  and  $y \in U_1$  in terms of certain basic symbols  $(x, z)_{n,F}$  with  $z$  running over a set of generators of  $U_1$  as a Galois module. Using this formula, one can evaluate  $(x, y)_{n,F}$ , with  $y$  running over the generators of the particular unit subgroups  $U_{f-1}$  and  $U_f$ , in order to determine that  $f_n(x) = f$ .

The determination of the generators can be done for all of the unit subgroups, but we do not follow this course here, as it is not needed for the determination of the conductors.

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## 2 Eigenspaces of the multiplicative group

We maintain the notation of the introduction. We may decompose  $G = G_{F/\mathbf{Q}_p}$  with  $p$  odd into a direct product of cyclic subgroups

$$G = \Delta \times \Gamma \times \Phi,$$

where  $\Sigma = G_{F/E} = \Delta \times \Gamma$ , the group  $\Delta$  has order  $p - 1$ ,  $\Gamma$  has order  $p^{n-1}$  and  $\Phi = G_{F/\mathbf{Q}_p(\zeta_p^n)}$ .

Let  $D$  denote the pro- $p$  completion of  $F^\times$ . As the action of  $\Delta$  on  $D$  is semisimple,  $D$  decomposes into a direct sum of eigenspaces for the powers of the cyclotomic character on  $\Delta$ . For  $r \in \mathbf{Z}$ , we let  $D_r$  denote the eigenspace consisting of  $x \in D$  such that

$$\delta(x) = x^{\omega(\delta)^r},$$

for all  $\delta \in \Delta$ , where  $\omega: \Delta \rightarrow \mathbf{Z}_p^*$  is the character with  $\delta(\zeta_p) = \zeta_p^{\omega(\delta)}$ . In other words,  $D_r = D^{\epsilon_r}$  where

$$\epsilon_r = \sum_{\delta \in \Delta} \omega(\delta)^{-r} \delta.$$

It is the goal of this section to gain an understanding of the  $A = \mathbf{Z}_p[\Gamma \times \Phi]$ -module structure of  $D_r$ , viewed inside  $D$ . The result which follows shortly is clearly very important in this regard and results from an examination of the arguments of Greither in [Gr]. (See [Ja] for similar results of a less explicit nature.)

Let  $\varphi \in \Phi$  denote the Frobenius element. Let  $\gamma$  denote a generator of  $\Gamma$  with  $\chi(\gamma) \equiv 1 + p \pmod{p^n}$  (considering  $\chi$  as a character of  $G_{F/E}$ ), and let  $\sigma$  be a generator of  $\Sigma$  such that  $\sigma^{p-1} = \gamma^{p-1}$ . We let  $\chi_\sigma$  denote a lift of  $\chi(\sigma)$  to  $\mathbf{Z}_p^*$ . Let  $q$  denote the order of the residue field of  $E$ . Let  $\xi \in \mu_{q-1}(F)$  be such that  $\text{Tr}_\Phi \xi = 1$  and the conjugates of  $\xi$  form a normal basis of  $F$  over  $\mathbf{Q}_p(\zeta_p^n)$ .

**Theorem 2.1.** *The  $\mathbf{Z}_p[G]$ -module  $D$  has a presentation*

$$D = \langle u, v, w, \pi \mid \sigma v = v, \varphi \pi = \pi, N_\Sigma u = v^{1-\varphi}, N_\Phi u = \pi^{\sigma-1}, w^{\chi_\sigma^{-1}\sigma-1} = u^{\varphi-1} \rangle, \quad (1)$$

where  $N_\Sigma \pi = p$ ,  $w \in U_1 - U_2$ ,  $u \in U_2 - U_3$  and  $v \equiv 1 + p\xi \pmod{p^2}$ .

*Proof.* In the notation of [Gr],  $u$  is the element  $u_{n-1}^{(1)}$ ,  $w$  is  $y_{n-1}^{r-1}$ ,  $v$  is  $v_{n-1}^{(1)}$ , and  $\pi$  is  $\pi_{n-1}$ . The properties of the elements listed at the end of the theorem follow from their definitions and the expansion of  $\log u_{n-1}^{(1)}$  found in [Gr, p. 12].

In Theorem 3.3 of [Gr], it is shown that there is an exact sequence

$$1 \rightarrow Q \rightarrow D \rightarrow \text{Ind}_\Sigma^G \mu_{p^n} \rightarrow 1,$$

where  $Q$  is the submodule of  $D$  generated by  $u$ ,  $v$  and  $\pi$  subject to all but the last relation in (1). By their definitions,  $u$  and  $w$  generate the kernel of the absolute norm on  $F^\times$  as a  $\mathbf{Z}_p[G]$ -module, and together with  $v$ , generate  $U_1$ . Hence, the given elements generate  $D$ . Furthermore, Lemma 2.6 of [Gr] states that the final relation in (1) holds. Therefore,  $D$  is a quotient of the group  $P$  given by the presentation in (1). It suffices to show that this quotient becomes an isomorphism modulo  $Q$ . This follows from the following isomorphisms of  $\mathbf{Z}_p[G]$ -modules:

$$P/Q \cong \langle \bar{w} \mid \bar{w}^{\chi_\sigma^{-1}\sigma-1} = 1 \rangle \cong \text{Ind}_\Sigma^G \mu_{p^n} \cong D/Q,$$

where the middle step is Lemma 2.4 of [Gr].  $\square$

Note that, as a corollary,  $N_{\Phi}w = \zeta_{p^n}$  for a proper choice of  $\zeta_{p^n}$ , which we make. Theorem 2.1 also gives us the structure of the groups  $D_r$ .

**Corollary 2.2.** *We maintain the notation of Theorem 2.1 and set  $u_r = u^{\epsilon_r}$ ,  $\pi_0 = \pi^{\epsilon_0}$  and  $w_1 = w^{\epsilon_1}$ . If  $r \not\equiv 0, 1 \pmod{p-1}$ , then  $D_r$  is a free  $A$ -module on  $u_r$ . Furthermore, we have the  $A$ -module presentations*

$$D_0 = \langle \pi_0, u_0, v \mid \gamma v = v, \varphi \pi_0 = \pi_0, N_{\Gamma}u_0 = v^{1-\varphi}, N_{\Phi}u_0 = \pi_0^{\gamma-1} \rangle \quad (2)$$

and

$$D_1 = \langle u_1, w_1 \mid w_1^{(1+p)^{-1}\gamma-1} = u_1^{\varphi-1} \rangle. \quad (3)$$

*Proof.* We leave this primarily to the reader to verify. We do note, however, that  $\epsilon_0(\chi_{\sigma}^{-1}\sigma - 1)$  and  $\epsilon_1(\sigma - 1)$  are units of  $A$ , and so by (1) we need not include  $w^{\epsilon_0}$  and  $\pi^{\epsilon_1}$  in the presentations of  $D_0$  and  $D_1$ , respectively.  $\square$

For  $t \geq 1$ , set  $V_{t,r} = U_t \cap D_r$  and  $V'_{t,r} = V_{t,r} - V_{t+1,r}$ . We now briefly study the module structures of the  $V_{t,r}$ .

**Lemma 2.3.** *We have  $V_{t,r}/V_{t+p-1,r} \cong \mathbf{F}_q$  for every  $t \geq 1$ , and  $V'_{t,r} \neq 0$  if and only if  $t \equiv r \pmod{p-1}$ .*

*Proof.* We sketch the proof. Set  $\lambda_n = 1 - \zeta_{p^n}$ , and let  $\alpha = 1 + x\lambda_n^t$  for some  $x \in U$ . For  $\delta \in \Delta$ , we have

$$\delta(\alpha) \equiv 1 + \omega(\delta)^t x \lambda_n^t \pmod{\lambda_n^{t+1}}.$$

Then

$$\alpha^{\epsilon_r} \equiv 1 + \sum_{\delta \in \Delta} \omega(\delta)^{t-r} x \lambda_n^t \pmod{\lambda_n^{t+1}}, \quad (4)$$

and this is 1 modulo  $\lambda_n^{t+1}$  unless  $t \equiv r \pmod{p-1}$ . On the other hand, if  $t \equiv r \pmod{p-1}$ , then equation (4) implies that

$$\alpha^{\epsilon_r} \equiv \alpha^{-1} \pmod{\lambda_n^{t+1}}.$$

The assertions of the lemma follow immediately.  $\square$

From now on, we set  $V_t = V_{t,r}$  and  $V'_t = V'_{t,r}$  if  $t \equiv r \pmod{p-1}$ .

**Lemma 2.4.** *Let  $z \in V'_t$ . If  $p \nmid t$ , then  $z^{\gamma-1} \in V'_{t+p-1}$ . Otherwise,  $z^{\gamma-1} \in V_{t+2(p-1)}$ .*

*Proof.* From the ramification groups of  $F/E$  in the lower numbering [Se, IV.4], we see that  $v(\gamma(\pi)/\pi - 1) = p - 1$ , where  $v$  is the valuation on  $F$  and  $\pi$  is any prime element of  $F$ . Hence, for any  $x \in U_t$  with  $p \nmid v(x - 1)$ , we have  $v(\gamma(x) - x) = t + p - 1$ . On the other hand, if  $p \mid v(x - 1)$  then  $v(\gamma(x) - x) > t + p - 1$ . Since  $z^{\gamma-1} \in D_t$ , the result follows from Lemma 2.3.  $\square$

**Lemma 2.5.** *Let  $z \in V'_t$ . Let  $i$  denote the smallest nonnegative integer with  $i \equiv t \pmod{p}$ . For  $j \geq 1$ , we have that*

$$z^{(\gamma-1)^j} \in V_{t+(j+\lceil \frac{j-i}{p-1} \rceil)(p-1)},$$

where  $\lceil \cdot \rceil$  denotes the ceiling function.

*Proof.* Lemma 2.4 shows that  $V_a^{\gamma-1} \subseteq V_{a+p-1}$  for all  $a \equiv t \pmod{p-1}$ . Hence

$$z^{(\gamma-1)^j} \in V_{t+j(p-1)}. \tag{5}$$

Furthermore, Lemma 2.4 implies that

$$z^{(\gamma-1)^i} \in V'_{t+(p-1)i}$$

and

$$y^{(\gamma-1)^{p-1}} \in V_{a+p(p-1)}$$

for  $y \in V'_a$  with  $a \equiv t \pmod{p-1}$ . Together these show that for every  $p-1$  applications of  $\gamma-1$  after the  $i$ th and starting with the  $(i+1)$ st, we can add an additional  $p-1$  to the subscript in (5), which yields the lemma.  $\square$

The following easy lemma will also be useful. The use of additive notation for a given  $V_t$  indicates that we are viewing it as an  $A$ -module.

**Lemma 2.6.** *Let  $a, b$  and  $j$  be nonnegative integers with  $1 \leq j \leq n-1$  and  $a < p^{n-j}$ . Then*

$$p^j V_{a+b} \subseteq V_{p^j a+b}.$$

### 3 The norm residue symbol on the eigenspaces

We consider the norm residue pairing on the eigenspaces. As  $F^r \subseteq D_r F^{\times p^n}$ , it will prove useful to consider the restriction of the norm residue symbol to elements lying in eigenspaces. Let  $(\cdot, \cdot)$  denote the pairing induced by  $(\cdot, \cdot)_{n,F}$  on  $D \times D$ .

**Lemma 3.1.** *The pairing*

$$D_r \times D_s \rightarrow \mu_{p^n}$$

*induced by the Hilbert norm residue symbol is trivial unless  $s \equiv 1-r \pmod{p-1}$  and nondegenerate otherwise.*

*Proof.* Let us evaluate  $(a_r, a_s)$  with  $a_r \in D_r$  and  $a_s \in D_s$ . Let  $i = \omega(\delta)$  for a generator  $\delta$  of  $\Delta$ . Then we have

$$(a_r, a_s) = \delta(a_r, a_s)^{i^{-1}}.$$

By Galois equivariance of the norm residue symbol, the last term equals

$$(\delta(a_r), \delta(a_s))^{i^{-1}} = (a_r^{i^r}, a_s^{i^s})^{i^{-1}} = (a_r, a_s)^{i^{r+s-1}}.$$

Hence either  $i^{r+s-1} \equiv 1 \pmod{p}$  or  $(a_r, a_s) = 1$ . However, as  $i$  has order dividing  $p-1$ , the former condition is exactly that  $i^{r+s-1} \equiv 1 \pmod{p^n}$ . This proves the first statement, and the second follows by nondegeneracy of the norm residue symbol.  $\square$

From now on, we fix  $r$  and let  $s = 1 - r$ . We have the following easy corollary of Lemmas 2.3 and 3.1.

**Corollary 3.2.** *Let  $x \in D_r \cap F^\times$ . Then either*

$$f_n(x) \equiv s + 1 \pmod{p - 1}$$

or  $f_n(x) = 0$ .

Define the symbol  $[\cdot, \cdot]$  with values in  $\mathbf{Z}/p^n\mathbf{Z}$  by

$$(\alpha, \beta) = \zeta_{p^n}^{[\alpha, \beta]}$$

for  $\alpha, \beta \in D$ .

**Lemma 3.3.** *Let  $x \in F^r$  and  $y \in D_s$ . Set*

$$\alpha = y^{f(\gamma)}$$

with  $f \in \mathbf{Z}_p[X]$ . Then

$$[x, \alpha] \equiv f((1+p)^s)[x, y] \pmod{p^n}.$$

*Proof.* We need only show that  $[x, \gamma(y)] = (1+p)^s[x, y]$ . We have

$$(x, \gamma(y)) = \gamma(\gamma^{-1}(x), y) = (x^{(1+p)^{-r}}, y)^{1+p} = (x, y)^{(1+p)^s}.$$

$\square$

Let  $T = \gamma - 1 \in \mathbf{Z}_p[\Gamma]$ .

**Corollary 3.4.** *Let  $x \in F^r$ , and let  $m$  be a positive integer. Choose  $y \in D_s$ , and let  $i$  be such that the order of  $[x, y]$  is  $p^{n-i}$ . Set  $j = \min\{m + i, n\}$ . Then*

$$[x, y^{p^k T^{m-k}}] \equiv 0 \pmod{p^j}$$

for  $0 \leq k \leq m$ .

*Proof.* We calculate:

$$[x, y^{p^k T^{m-k}}] = p^k ((1+p)^s - 1)^{m-k} [x, y] \equiv 0 \pmod{p^j}.$$

□

For  $\alpha, \beta \in D$ , let

$$\Psi(\alpha, \beta) = \min\{k \mid p^k [\alpha, \varphi^i \beta] = 0 \text{ for all } i\}.$$

**Lemma 3.5.** *Let  $x \in F^r$  with  $x \notin F^{\times p}$ . If  $S$  is a set of generators for  $D_s$  as an  $A$ -module, then  $\Psi(x, y) = n$  for some  $y \in S$ .*

*Proof.* Lemma 3.1 implies that  $(x, z)$  is a primitive  $p^n$ th root of unity for some  $z \in D_s$ . We may write  $z$  as a product of powers of elements of the form  $\varphi^i \gamma^j y$  with  $y \in S$ ,  $i, j \in \mathbf{Z}$ . Hence  $(x, \varphi^i \gamma^j y)$  is a primitive  $p^n$ th root of unity for some such  $i$  and  $j$ . By Lemma 3.3, this cannot happen unless  $(x, \varphi^i y)$  is a primitive  $p^n$ th root of unity for some  $i$  as well. □

## 4 Conductors for $r \not\equiv 0, 1 \pmod{p-1}$

We assume in this section that  $r \not\equiv 0, 1 \pmod{p-1}$ . The goal of this section is the determination of the conductors of the extensions  $F(\sqrt[p^n]{x})/F$  for  $x \in F^r$ . By Corollary 2.2,  $D_s$  is a free  $A$ -module of rank 1 generated by an element  $u_s$ . We identify  $D_s$  with  $A$  via the isomorphism

$$A \xrightarrow{\sim} D_s, \quad \alpha \mapsto u_s^\alpha$$

of  $A$ -modules. Each submodule  $V_i$  with  $i \equiv s \pmod{p-1}$  is thereby canonically identified with an ideal of  $A$  and by abuse of notation we treat them as equal. In addition, setting  $\bar{F}^r = F^r / F^{\times p^n}$ , we see that

$$\bar{F}^r \cong \mathbf{Z}/p^n \mathbf{Z}[\Phi],$$

as  $D_r$  is also free of rank 1.

Note that

$$\mathbf{Z}_p[\Gamma] \cong \mathbf{Z}_p[Y] / ((Y+1)^{p^{n-1}} - 1).$$

We shall require the following easy lemma.

**Lemma 4.1.** *In  $\mathbf{Z}_p[Y]$ , the polynomial  $(Y+1)^{p^{n-1}} - 1$  is contained in the ideal*

$$(p^{n-1}Y, p^{n-2}Y^p, \dots, Y^{p^{n-1}}).$$

Let  $t$  denote the smallest positive integer congruent to  $s+1$  modulo  $p-1$ . To determine the conductor  $f_n(x)$  for  $x \in F^r$ , it suffices to look at two ideals of  $A$ .

**Proposition 4.2.** *Let  $m$  be a nonnegative integer with  $m \leq n - 1$ . The ideal  $V_{p^{m_t-1}}$  of  $A$  contains an element  $a$  of the form*

$$a = p^m + \sum_{k=1}^m d_k p^{m-k} T^{p^{k-1}t-1}$$

with  $d_k \in \mathbf{Z}_p[\Phi]$  for  $1 \leq k \leq m$ . Furthermore, we have

$$V_{p^{m_t+p-2}} = (p^{m+1}, p^m T, p^{m-k} T^{p^{k-1}t} \mid 1 \leq k \leq m).$$

*Proof.* We begin with the second statement. By Lemma 4.1, the ideal

$$I = (p^{m+1}, p^m Y, p^{m-k} Y^{p^{k-1}t} \mid 1 \leq k \leq m)$$

of  $\mathbf{Z}_p[Y]$  contains  $(Y + 1)^{p^{n-1}} - 1$ . Hence, the image  $\bar{I}$  of  $I$  in  $\mathbf{Z}_p[\Gamma]$  has

$$[\mathbf{Z}_p[\Gamma] : \bar{I}] = [\mathbf{Z}_p[Y] : I] = p^h,$$

where

$$h = m + 1 + (t - 1)m + (p - 1)t(m - 1) + \dots + p^{m-2}(p - 1)t = (p^{m-1} + \dots + p + 1)t + 1.$$

Let  $J = \bar{I}[\Phi]$  as an ideal of  $A$ . Then Lemma 2.3 implies that

$$[V_{t-1} : V_{p^{m_t+p-2}}] = q^h = [A : J].$$

To prove the second statement, we are left only to verify the claim that  $J \subseteq V_{p^{m_t+p-2}}$ . Note that  $1 \in V_{t-1}$ , so clearly  $p^{m+1} \in V_{p^{m_t+p-2}}$ . By Lemma 2.4, we see that

$$p^m T \in p^m V_{t+p-2}.$$

Similarly, Lemma 2.5 yields

$$p^{m-k} T^{p^{k-1}t} \in p^{m-k} V_{p^{k_t+p-2}}$$

for  $1 \leq k \leq m$ . By Lemma 2.6, we have

$$p^{m-k} V_{p^{k_t+p-2}} \subseteq V_{p^{m_t+p-2}},$$

for  $0 \leq k \leq m$ , proving the claim.

For the first statement, we proceed by induction, the case of  $m = 0$  being obvious. Assume that we have proven the proposition for  $m \leq n - 2$ . Let  $i = p^m t - 1$ . Then we have found  $b \in V'_i$  of the desired form. We claim that

$$c = dT^i \in V'_{p^i} \tag{6}$$

for any  $d \in \mathbf{Z}_p[\Phi]$  which is not a multiple of  $p$ . By the second statement,  $c \notin V_{pi+2(p-1)}$ , and by Lemma 2.5 we have  $c \in V_{pi}$ . If it were that  $c \in V'_{pi+p-1}$ , then by Lemmas 2.3 and 2.4 we would have  $dT^{i-1} \in V'_{pj}$  with  $j \leq i - p + 1$ , but this would contradict Lemma 2.5. Having demonstrated the claim, we see that  $a = pb + c \in V'_{pi+p-1}$  for an appropriate choice of  $d$  in (6) by Lemma 2.3 and the freeness of  $D_r$  as a  $\mathbf{Z}_p[\Phi]$ -module. It is clear that  $a$  is the desired element for the case  $m + 1$  of the first statement.  $\square$

We now determine the desired conductors.

**Theorem 4.3.** *Let  $r \not\equiv 0, 1 \pmod{p-1}$ , and let  $t$  denote the smallest positive integer with  $t \equiv 2 - r \pmod{p-1}$ . Let  $x \in F^r$  with  $x \notin F^{\times p}$ . Then we have that  $f_i(x) = p^{i-1}t$  for any positive integer  $i$  with  $i \leq n$ .*

*Proof.* By Corollary 3.4, we have  $[x, \varphi^j b] \equiv 0 \pmod{p^i}$  for every  $j$  and each generator  $b$  of  $V_{p^{i-1}t+p-2}$  listed in Proposition 4.2 (with  $m = i - 1$ ). On the other hand, by Lemma 3.5, we may choose  $j$  such that for  $u = \varphi^j(u_s)$ , the symbol  $(x, u)$  is a primitive  $p^n$ th root of unity. In this case, we have

$$[x, u^a] = [x, u^{p^{i-1}}] = p^{i-1}[x, u] \not\equiv 0 \pmod{p^i},$$

where  $a \in \mathbf{Z}_p[\Gamma]$  is the element of Proposition 4.2. Noting Corollary 3.2, we see that

$$f_i(x) = f_n(x^{p^{n-i}}) = p^{i-1}t.$$

$\square$

## 5 Conductors for $r \equiv 0 \pmod{p-1}$

Over the final two sections, we determine the conductors for the remaining values of  $r$ . For an element  $x \in F^r$ , we will first determine  $f_n(x)$  in terms of the values  $\Psi(x, y)$ , where  $y$  runs over a set of  $A$ -module generators of  $D_s$ . Then, we shall explicitly determine the values  $\Psi(x, y)$ . The situation is complicated by the existence of multiple generators of  $\bar{F}^r$  and  $D_s$ . We shall twice appeal to the case of  $r = 0$  from [S3].

In this section, we let  $r \equiv 0 \pmod{p-1}$ . Recall the notation of Corollary 2.2.

**Lemma 5.1.** *For  $n \geq 2$ , there exists an element  $c \in \mathbf{Z}_p[\Phi]$  such that  $y_1 = u_1 w_1^{pc}$  lies in  $V'_{2p-1}$  and has image generating  $V_{2p-1}/V_{3p-2}$  as a  $\mathbf{Z}_p[\Phi]$ -module. In fact, we have  $c \equiv \varphi^{-1} \pmod{(N_\Phi)}$ .*

*Proof.* Recall from Theorem 2.1 that  $w_1 \in V'_1$  and  $u_1 \in V_p$ . From this, Lemma 2.4, Corollary 2.2 and the relation

$$w_1^{\gamma-1} = w_1^p u_1^{(\varphi-1)(1+p)}$$

found in (3), we see that  $V_p/V_{3p-2}$  is a  $\mathbf{Z}/p\mathbf{Z}[\Phi]$ -module generated by (the images of)  $w_1^p$  and  $u_1$ . Now, the submodule  $M$  generated by  $w_1^{\gamma^{-1}}$  and  $w_1^p$  does not contain the submodule  $N = V_{2p-1}/V_{3p-2}$ , since the fact that  $N_{\Phi}w_1 = \zeta_{p^n}$  implies  $N \cap N_{\Phi}M = 0$ . The existence of  $c \in \mathbf{Z}_p[\Phi]$  follows.

Note that

$$w_1 \equiv 1 + \eta\lambda_n \pmod{\lambda_n^2}$$

with  $\eta \in \mu_{q-1}$  and  $b = \text{Tr}_{\Phi}\eta \not\equiv 0 \pmod{p}$ . It follows that from the equation

$$\gamma(\lambda_n) \equiv \lambda_n + \lambda_n^p \pmod{\lambda_n^{p+1}}$$

that

$$w_1^{(1+p)^{-1}\gamma^{-1}} \equiv 1 + (1 - \varphi)\eta\lambda_n^p \pmod{\lambda_n^{p+1}}.$$

By (3), we must have

$$u_1 \equiv 1 + \eta'\lambda_n^p \pmod{\lambda_n^{p+1}} \quad (7)$$

with

$$(\varphi - 1)(\eta' + \eta) \equiv 0 \pmod{p}.$$

Thus,  $\eta' \equiv -\eta + a \pmod{p}$  for some  $a \in \mathbf{Z}_p$ . Take  $c = \varphi^{-1} - ab^{-1}N_{\Phi}$ , so that

$$w_1^{pc} \equiv 1 + c\varphi(\eta)\lambda_n^p \equiv 1 - \eta'\lambda_n^p \pmod{\lambda_n^{p+1}}. \quad (8)$$

From (7), (8) and the uniqueness of  $c$ , we see that  $y_1 = u_1w_1^{pc}$  satisfies  $y_1 \in V'_{2p-1}$ .  $\square$

Clearly,  $w_1$  and the element  $y_1$  of Lemma 5.1 generate  $D_1$  as a  $A$ -module for  $n \geq 2$ . Let us determine the generators of the relevant submodules of  $D_1$ .

**Proposition 5.2.** *Let  $m$  be a positive integer with  $m \leq n - 1$ .*

*a. The submodule  $V_{p^m+p^{m-1}-1}$  of  $D_1$  contains an element of the form*

$$a_1 = p^m w_1 + \sum_{k=2}^m d_{k,1} p^{m-k} T^{p^{k-1}+p^{k-2}-2} y_1$$

*with  $d_{k,1} \in \mathbf{Z}_p[\Phi]$ , and*

$$V_{p^m+p^{m-1}+p-2} = (p^{m+1}w_1, p^{m-1}y_1, p^{m-k}T^{p^{k-1}+p^{k-2}-1}y_1 \mid 2 \leq k \leq m). \quad (9)$$

*b. The submodule  $V_{2p^m-1}$  of  $D_1$  contains an element of the form*

$$a_2 = p^{m-1}y_1 + \sum_{k=2}^m d_{k,2} p^{m-k} T^{2p^{k-1}-2} y_1,$$

*with  $d_{k,2} \in \mathbf{Z}_p[\Phi]$ , and*

$$V_{2p^m+p-2} = (p^{m+1}w_1, p^m y_1, p^{m-k} T^{2p^{k-1}-1} y_1 \mid 1 \leq k \leq m).$$

*Proof.* We proceed as in the proof of Proposition 4.2. Let us focus on part a. Let  $J$  be the submodule of  $D_1$  given by the right hand side of (9). This has index in  $D_1$  equal to  $q^h$ , where

$$\begin{aligned} h &= m + 1 + p(m - 1) + \sum_{i=2}^{m-1} p^{i-2}(p^2 - 1)(m - i) \\ &= 3 + 2p + 2p^2 + \dots + 2p^{m-2} + p^{m-1} = \frac{p^m + p^{m-1} + p - 3}{p - 1}. \end{aligned}$$

On the other hand, Lemma 2.3 implies that  $[V_1 : V_{i+p-1}] = q^h$  for  $i = p^m + p^{m-1} - 1$ . Next, we remark that as  $y_1 \in V'_{2p-1}$ , we have

$$p^{m-k} T^{p^{k-1} + p^{k-2} - 1} y_1 \in p^{m-k} V_{p^k + p^{k-1} + p - 2},$$

for  $2 \leq k \leq m$  by Lemma 2.5. Applying Lemma 2.6, we have that  $J \subseteq V_{i+p-1}$  and hence equality by equality of the indexes. This proves the second statement of part a.

For the first statement, the base case of  $m = 1$  is obvious. Assume that we have proven the proposition for  $m \leq n - 2$ . Then we have found  $b \in V'_i$  of the desired form. Using Lemmas 2.4, 2.5 and the second statement of the proposition, one can check (as in Proposition 4.2) that  $c = dT^{i-1}y_1 \in V'_{pi}$  with  $d \in \mathbf{Z}_p[\Phi] - p\mathbf{Z}_p[\Phi]$ . We conclude that  $a_1 = pb + c \in V'_{pi+p-1}$  is the desired element for some choice of  $d = d_{m+1,1}$ .

Part b follows from the same argument with the obvious modifications.  $\square$

If  $n = 1$ , we set  $y_1 = u_1$ .

**Proposition 5.3.** *Let  $x \in F^r - F^{\times p^n}$ . Let  $i = \Psi(x, y_1)$  and  $j = \Psi(x, w_1)$ . Then*

$$f_n(x) = \begin{cases} p^n + p^{n-1} & \text{if } i = n, \\ 2p^i & \text{if } 1 \leq i \leq n - 1 \text{ and } j \leq i + 1, \\ p^{j-1} + p^{j-2} & \text{if } 2 \leq j \leq n \text{ and } i + 2 \leq j. \end{cases}$$

*Proof.* For  $n \geq 2$ , with the exception of  $j = 1$  and  $i = 0$ , this follows directly from Proposition 5.2 after noting that

$$V_{p^n + p^{n-1} - 1} = pV_{2p^{n-1} - 1}$$

and similarly for  $V_{p^n + p^{n-1} + p - 2}$ . If  $j = 1$  and  $i = 0$ , it follows from the fact that  $w_1 \in V'_1$ . For  $n = 1$  and  $i = 1$ , it follows from  $y_1 \in V'_p$  by Lemma 2.3.  $\square$

Let  $N_r : D_r \rightarrow \bar{F}^r$  be defined by

$$N_r(x) = \prod_{i=1}^{p^n-1} \gamma^i(\bar{x})^{(1+p)^{-ir}}$$

for  $x \in D_r$  and  $\bar{x}$  its image in  $F^\times/F^{\times p^n}$ . By abuse of notation, we will use the same letter to denote an element of  $\bar{F}^r$  and a chosen lift of it to  $F^r$  (or conversely, an element of  $F^r$  and its image in  $\bar{F}^r$ ).

Let  $k \leq n$  be maximal such that  $r \equiv 0 \pmod{p^{k-1}}$ . In the notation of Theorem 2.1, we set  $t_r = N_r \pi_0$ ,  $x_r = N_r u_0$  and  $v_r = v^{p^{n-k}}$ . Then  $\bar{F}^r$  has presentation as a (multiplicative)  $(\mathbf{Z}/p^n \mathbf{Z})[\Phi]$ -module given as

$$\bar{F}^r = \langle t_r, x_r, v_r \mid t_r^{\varphi-1} = 1, N_\Phi x_r = t_r^{(1+p)^{r-1}}, x_r^{p^{n-k}} = v_r^{1-\varphi}, v_r^{p^k} = 1 \rangle. \quad (10)$$

Note that  $k$  is maximal among  $k \leq n$  such that  $p^k$  divides  $(1+p)^r - 1$ .

For  $\alpha \in \mathbf{Z}_p[\Phi]$ , let  $\nu(\alpha)$  denote the largest integer (possibly infinite) such that  $\alpha \in p^{\nu(\alpha)} \mathbf{Z}_p[\Phi]$ . Note that  $\nu$  is exactly the  $p$ -adic valuation on elements of  $\mathbf{Z}_p$ .

**Lemma 5.4.** *Let  $\alpha \in \mathbf{Z}_p[\Phi]$  with  $\nu(\alpha) = 0$ . Let  $\delta \in \mathbf{Z}_p[\Phi]$  with  $\nu(\delta) = 0$  and  $(\varphi-1)\delta \neq 0$ . The following statements hold:*

$$\begin{array}{ll} a. \Psi(t_r, y_1) = n & b. \Psi(t_r, w_1) = 0 \\ c. \Psi(v^\alpha, y_1) = n - 1 & d. \Psi(v^\alpha, w_1) = n \\ e. \Psi(x_r^\delta, y_1) = n - 1 & f. \Psi(x_r^\delta, w_1) = n \end{array}$$

*Proof.* We proceed case by case.

a. This follows easily from Lemma 3.5 and part b (to be proven).

b. Set  $\lambda = \sigma^i(\pi)$  for any  $i$ . Since  $N_\Sigma \lambda = p$  from Theorem 2.1, we have  $\lambda = (1 - \zeta_{p^n})\eta$  for some  $\eta \in U_1$  with  $N_\Sigma \eta = 1$ . Then

$$(\lambda, w_1)_{n, F} = (\lambda, \zeta_{p^n})_{n, \mathbf{Q}_p(\zeta_{p^n})} = (\eta, \zeta_{p^n})_{n, \mathbf{Q}_p(\zeta_{p^n})} = 1,$$

the first step following from  $\lambda^{\varphi-1} = 1$  and  $N_\Phi w_1 = \zeta_{p^n}$  and the last since  $\zeta_{p^n}$  pairs trivially with any element in the kernel of the norm. The result now follows, as  $t_r = N_r N_\Delta \pi$ .

c. In [S3, Theorem 8], it was shown that  $f_n(v^\alpha) = 2p^{n-1}$ , and hence Proposition 5.3 forces that  $\Psi(v^\alpha, y_1) = n - 1$ .

d. This follows from part c and Lemma 3.5.

e. By (10) and part c, we see that

$$p^{n-k}[x_r^\delta, \varphi^i y_1] = p^{n-k}[v^{\delta(1-\varphi)}, \varphi^i y_1] \quad (11)$$

is a primitive  $p^{k-1}$ st root of unity for some  $i$ . Hence, we are done if  $k \geq 2$ .

For  $k = 1$ , equation (11) still implies  $\Psi(x_r^\delta, y_1) \leq n - 1$ . Write  $\delta$  as  $\delta = a + \beta(\varphi^{-1} - 1)$ , with  $a \in \mathbf{Z}_p$  and  $\beta \in \mathbf{Z}_p[\Phi]$ . Note that

$$[x_r^\delta, N_\Phi y_1] = a[N_\Phi x_r, y_1] = a((1+p)^r - 1)[t_r, y_1].$$

Since  $\nu(r) = 0$ , we are done if  $\nu(a) = 0$ , by part a. If  $\nu(a) \geq 1$ , one checks using (3) and Lemma 5.1 that

$$[x_r^{\delta-a}, \varphi^i y_1] = [x_r^\beta, y_1^{\varphi^i(\varphi-1)}] = [x_r^\beta, w_1^{\varphi^i((1+p)^{-1}\gamma-1)} w_1^{p\varphi^i(1-\varphi^{-1})}],$$

so, using Lemma 3.3, we have

$$[x_r^\delta, \varphi^i y_1] \equiv p[x_r^{\beta(1-r-\varphi)}, \varphi^i w_1] \pmod{p^2}.$$

Since  $\nu(\beta) = 0$ , the result follows from part f, proven below.

f. Since  $x_r^\delta v^{(\varphi-1)\delta} \in F^{\times p^k}$  by (10), this follows from part d.

□

The following describes the conductors of all elements of  $F^r$  for  $r \equiv 0 \pmod{p-1}$ .

**Theorem 5.5.** *Let  $x = t_r^\alpha v^\beta x_r^\delta$  with  $\alpha \in \mathbf{Z}_p$ ,  $\beta \in p^{n-k}\mathbf{Z}_p[\Phi]$  and  $\delta \in \mathbf{Z}_p[\Phi]$  satisfying either  $\delta = 0$ , or  $\nu(\delta) \leq n - k - 1$  and  $(\varphi - 1)\delta \neq 0$ . Let*

$$i = n - \min\{\nu(\alpha), \nu(\beta) + 1, \nu(\delta) + 1\}$$

and

$$j = n - \max\{\nu(p\beta - N_\Phi \alpha), 0\}.$$

Then

$$f_n(x) = \begin{cases} p^{i-1}(p+1) & \text{if } i = n, \text{ or } \delta = 0 \text{ and } j < i, \\ 2p^i & \text{if } i \leq n-1 \text{ and either } \delta \neq 0 \text{ or } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Applying Proposition 5.3, part a of Lemma 5.4 yields that

$$f_n(t_r^\alpha) = \begin{cases} p^{n-1}(p+1) & \text{if } \nu(\alpha) = 0, \\ 2p^{n-\nu(\alpha)} & \text{if } \nu(\alpha) \leq n-1, \end{cases}$$

for  $\alpha \in \mathbf{Z}_p$ . Parts c and d yield

$$f_n(v^\beta) = 2p^{n-1-\nu(\beta)}$$

if  $\beta \in \mathbf{Z}_p[\Phi]$  satisfies  $\nu(\beta) \leq n - 1$ . Parts e and f yield

$$f_n(x_r^\delta) = 2p^{n-1-\nu(\delta)}$$

with  $\delta$  satisfying  $(\varphi - 1)\delta \neq 0$  and  $\nu(\delta) \leq n - 1$ .

By the conditions on  $\delta$  and  $\beta$  that  $\nu(\delta) \leq n - k - 1$ , unless  $\delta = 0$ , and  $\nu(\beta) \geq n - k$ , we have

$$f_n(v^\beta x_r^\delta) = \max\{f_n(v^\beta), f_n(x_r^\delta)\}.$$

While  $\Psi(t_r, y_1^{\varphi-1}) = 0$ , on the other hand, we have

$$\Psi(v^\beta, y_1^{\varphi-1}) = n - 1 - \nu(\beta)$$

by part c, unless  $(\varphi - 1)\beta = 0$ , and

$$\Psi(x_r^\delta, y_1^{\varphi-1}) = n - 1 - \nu(\delta)$$

by part e. Therefore, by Proposition 5.3, we have

$$f_n(t_r^\alpha v^\beta x_r^\delta) = \max\{f_n(t_r^\alpha), f_n(v^\beta), f_n(x_r^\delta)\}$$

unless  $\delta = 0$ ,  $(\varphi - 1)\beta = 0$  and  $\nu(\alpha) = \nu(\beta) + 1 \leq n - 1$ .

In this “exceptional” case, there exists  $d \in \mathbf{Z}_p$ , unique modulo  $p$ , such that

$$\Psi(t_r^{d\alpha} v^\beta, y_1) \leq n - 2 - \nu(\beta).$$

In [S3, Theorem 8] (or [CM, Theorem 6.1]), it is shown that

$$f_n(p^p(1 + p)) = p^{n-2}(p + 1) < f_n(1 + p).$$

Proposition 5.3 immediately implies that

$$\Psi(p^p(1 + p), y_1) = n - 2.$$

By Theorem 2.1, we have  $N_\Phi v \equiv 1 + p \pmod{p^2}$  and  $t_r p^{-1} \in F^{\times p^k}$ . Hence, we must have  $\Psi(t_r^p N_\Phi v, y_1) \leq n - 2$ , with equality for  $k \geq 3$ . The case statement of the theorem now follows.  $\square$

The use of [S3, Theorem 8] in Lemma 5.4 (aside from the case  $(\varphi - 1)\alpha = 0$  in part c) could have been avoided, but this would not have alleviated the need for its use in Theorem 5.5. We remark that the methods used in [S3] are quite elementary, relying upon basic properties of the norm residue symbol and the Artin-Hasse law for the symbol  $(\zeta_{p^n}, \cdot)_{n,F}$ .

## 6 Conductors for $r \equiv 1 \pmod{p-1}$

Note that  $u_0 \in V'_{p-1}$  and  $v \in V'_{p^{n-1}(p-1)}$ .

**Proposition 6.1.** *Let  $m$  denote a positive integer with  $m \leq n$ .*

a. *The submodule  $V_{p^m-1}$  of  $D_0$  contains an element of the form*

$$a_1 = p^{m-1}u_0 + \sum_{k=2}^m d_{k,1}p^{m-k}T^{p^{k-1}-1}u_0$$

with  $d_{k,1} \in \mathbf{Z}_p[\Phi]$ , and if  $m \leq n-1$ , then

$$V_{p^m+p-2} = (v, p^m u_0, p^{m-k} T^{p^{k-1}-1} u_0 \mid 1 \leq k \leq m).$$

b. *The submodule  $V_{p^n-1}$  of  $D_0$  contains an element of the form*

$$a_2 = v + \sum_{k=2}^n d_{k,2}p^{n-k}T^{p^{k-1}-1}u_0$$

with  $d_{k,2} \in \mathbf{Z}_p[\Phi]$ , and

$$V_{p^n+p-2} = (pv, p^n u_0, p^{n-k} T^{p^{k-1}-1} u_0 \mid 1 \leq k \leq n-1). \quad (12)$$

*Proof.* The proof of part a is virtually identical to those of Propositions 4.2 and 5.2. Note that the key to proving the first statement is the fact that

$$dT^{p^{k-1}-1}u_0 \in V'_{p^k-p} \quad (13)$$

for  $2 \leq k \leq n$  and any  $d \in \mathbf{Z}_p[\Phi]$  with  $\nu(d) = 0$ .

We focus on part b. Its second statement (12) follows the fact that  $pV_{p^{n-1}+p-2} = V_{p^n+p-2}$ . As for the first statement, we begin by proving the claim that  $N_\Phi a_1 \in V_{p^n+p-2}$ , where  $a_1$  is the element of part a for  $m = n$ . First, we remark that

$$N_\Gamma = \frac{(1+T)^{p^{n-1}} - 1}{T} = \sum_{j=1}^{p^{n-1}} \binom{p^{n-1}}{j} T^{j-1}.$$

Using (12) (and some simple congruences for binomial symbols), we see that

$$N_\Gamma u_0 \equiv \sum_{k=1}^n p^{n-k} T^{p^{k-1}-1} u_0 \pmod{V_{p^n+p-2}}.$$

By (2), we have  $N_\Gamma N_\Phi u_0 = 0$ , and therefore,

$$N_\Phi a_1 \equiv \sum_{k=2}^{n-1} N_\Phi(d_{k,1} - 1) p^{n-k} T^{p^{k-1}-1} u_0 \pmod{V_{p^n+p-2}}.$$

Assume that the claim does not hold, so that

$$N_\Phi(d_{k,1} - 1) \not\equiv 0 \pmod{p}$$

for some minimal  $k$ . Noting (13), we see that  $N_\Phi a_1 \in V'_{p^n-p^{n-k+1}}$ , contradicting  $a_1 \in V'_{p^{n-1}}$ .

Consider the submodule

$$W = Au_0 \cap V_{p^n-1} = Aa_1 + V_{p^n+p-2}$$

of  $D_0$ . From the claim, we see that  $[W : V_{p^n+p-2}] = q/p$ , whereas  $[V_{p^n-1} : V_{p^n+p-2}] = q$  by Lemma 2.3. Hence, by (2) and (12), there exists an element  $b \in V'_{p^n-1}$  of the form

$$b = v + \sum_{k=1}^n c_{k,2} p^{n-k} T^{p^{k-1}-1} u_0$$

with  $c_{k,2} \in \mathbf{Z}_p[\Phi]$ . We take  $a_2 = b - c_{1,2} a_1$ .  $\square$

Now let  $r \equiv 1 \pmod{p-1}$ . The following is a direct consequence of Proposition 6.1.

**Proposition 6.2.** *Let  $x \in F^r$ . Let  $i = \Psi(x, u_0)$  and  $j = \Psi(x, v)$ . Then*

$$f_n(x) = \begin{cases} p^{n-1}(j(p-1) + 1) & \text{if } j \geq 1, \\ p^i & \text{if } j = 0, i \geq 1. \end{cases}$$

Let  $k \leq n$  be maximal such that  $r \equiv 1 \pmod{p^{k-1}}$ . Let  $x_r = N_r u_1$ ,  $z_r = N_r w_1$  and  $\kappa_r = \zeta_{p^n}^{p^{n-k}}$ , and recall that  $N_\Phi w = \zeta_{p^n}$ . From (3), we obtain a presentation

$$\bar{F}^r = \langle x_r, z_r, \kappa_r \mid x_r^{\varphi-1} = z_r^{(1+p)^{r-1}-1}, N_\Phi z_r = \kappa_r^{p^{k-1}}, \kappa_r^{p^k} = 1, \kappa_r^{\varphi-1} = 1 \rangle. \quad (14)$$

We make the following useful remark.

**Lemma 6.3.** *If  $x \in F^r$  and  $x \notin F^{\times p}$ , then the larger of  $\Psi(x, u_0)$  and  $\Psi(x, v)$  is at least  $n - k$ .*

*Proof.* We recall that for an element  $x \in F^\times$  and  $m \leq n$ , the extension  $F(\sqrt[m]{x})/F$  is unramified if and only if  $(x, u)_{m,F} = 1$  for all  $u \in U_1$ . Furthermore,  $p^k$  is the degree of the maximal unramified extension of  $F$  by the  $p^n$ th root of an element of  $F^r$ , since such an extension must be abelian over  $\mathbf{Q}_p$ . The result now follows.  $\square$

**Lemma 6.4.** *Let  $\alpha \in \mathbf{Z}_p[\Phi]$  with  $\nu(\alpha) = 0$ . Let  $\delta \in \mathbf{Z}_p[\Phi]$  with  $\nu(\delta) = 0$  and  $(\varphi - 1)\delta \neq 0$ . The following statements hold:*

$$\begin{array}{ll} a. \Psi(x_r^\alpha, v) = 0 & b. \Psi(x_r, u_0) = n - k \\ c. \Psi(z_r^\alpha, v) = 1 & d. \Psi(z_r^\delta, u_0) = n \\ e. \Psi(\zeta_{p^n}, v) = n & f. \Psi(\zeta_{p^n}, u_0) = 0 \end{array}$$

*Proof.* Again, we proceed case by case.

a. It suffices to consider  $\alpha = 1$ . We have (abusing notation)

$$[x_r, \varphi^i v] = [N_r u_1, \varphi^i v] = [N_r y_1, \varphi^i v] - p[N_r w_1^c, \varphi^i v], \quad (15)$$

with  $c \in \mathbf{Z}_p[\Phi]$ , by Lemma 5.1. Now apply Lemma 3.3 and the definition of  $N_r$  to see that the right hand side of (15) reduces to  $p^{n-1}[y_1, \varphi^i v]$ . The conclusion now follows from Lemma 5.4c.

b. Note that  $x_0 = v^{1-\varphi}$  by (2). Hence, for any  $i$ , we have

$$(x_0, u_1^{\varphi^i}) = (v, u_1^{\varphi^{i-1}(\varphi-1)}) = (v, w_1^{\varphi^{i-1}(1-(1+p)^{-1}\gamma)}) = 1$$

by (3) and Lemma 3.3. Since  $x_{1-r}x_0^{-1} \in F^{\times p^k}$ , we have  $\Psi(x_{1-r}, u_1) \leq n - k$ . We conclude by remarking that

$$(x_r, u_0^{\varphi^i}) = (u_1^{\varphi^{-i}}, x_{1-r})$$

and applying Lemma 6.3.

c. This follows from Lemma 5.4d in the same way that part a of this lemma followed from Lemma 5.4c.

d. This follows from Lemma 5.4f since

$$(z_r^\delta, u_0) = (w_1, x_{1-r}^{\iota(\delta)}),$$

where  $\iota$  is the involution of  $\mathbf{Z}_p[\Phi]$  defined by  $\varphi \mapsto \varphi^{-1}$ .

e. Recall that

$$[\zeta_{p^n}, x] = \frac{N_G x - 1}{p^n} \quad (16)$$

for any  $x \in U_1$ . Since  $N_G v \equiv 1 + p^n \pmod{p^{n+1}}$ , we have the result.

f. This follows from (16) as well, since  $N_G u_0 = 1$ .

□

The conductors of the elements of  $F^r$  for  $r \equiv 1 \pmod{p-1}$  are now described by the following.

**Theorem 6.5.** *Let  $x \in F^r$  with image  $x_r^\alpha z_r^\beta \kappa_r^\delta \in \bar{F}^r$ , where  $\alpha, \delta \in \mathbf{Z}_p$  and  $\beta \in \mathbf{Z}_p[\Phi]$  satisfies  $(\varphi - 1)\beta \neq 0$  if  $\beta \neq 0$ . Let  $i = k - \nu(\delta)$  and  $j = n - \min\{\nu(\beta), \nu(\alpha) + k\}$ . Then*

$$f_n(x) = \begin{cases} p^{n-1}(i(p-1) + 1) & \text{if } i \geq 1, \\ p^j & \text{if } i \leq 0 \text{ and } j \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Applying Proposition 6.2, parts a and b of Lemma 6.4 yield

$$f_n(x_r^\alpha) = p^{n-k-\nu(\alpha)}$$

if  $\alpha \in \mathbf{Z}_p$  with  $\nu(\alpha) \leq n - k - 1$  (and 0 otherwise). Parts c and d yield

$$f_n(z_r^\beta) = p^{n-\nu(\beta)}$$

if  $\nu(\beta) \leq n - 1$  and  $(\varphi - 1)\beta \neq 0$ . Part e yields

$$f_n(\kappa_r^\delta) = p^{n-1}((k - \nu(\delta))(p - 1) + 1)$$

if  $\delta \in \mathbf{Z}_p$  with  $\nu(\delta) \leq k - 1$ .

Furthermore, we see from the values in Lemma 6.4 along with Proposition 6.2 that

$$f_n(x) = \max\{f_n(x_r^\alpha), f_n(z_r^\beta), f_n(\kappa_r^\delta)\}$$

with  $\alpha, \beta$  and  $\delta$  as in the theorem, unless perhaps if  $\nu(\beta) = \nu(\alpha) + k \leq n - 1$  and  $\nu(\delta) \geq k$ . In this case, writing  $\beta = ((1 + p)^{r-1} - 1)\beta'$ , we have

$$x_r^\alpha z_r^\beta = x_r^{\alpha + (\varphi - 1)\beta'}.$$

Denote this element by  $y$ . By Lemma 6.4, we have that  $\Psi(y, v) = 0$  and  $\Psi(y, u_0) \leq n - k - \nu(\alpha)$ . Lemma 6.3 forces  $\Psi(y, u_0) = n - k - \nu(\alpha)$ , and so  $f_n(y) = p^j$  by Proposition 6.2. The case statement now follows easily. □

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