Computations on Milnor $K_2$
of integer rings

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May 17, 2004
Milnor $K_2$ of integer rings:

$F$ number field, $S$ finite set of primes of $F$
$\mathcal{O}_{F,S}$ ring of $S$-integers of $F$

We define Milnor $K_2$ of $\mathcal{O}_{F,S}$ as

$$K_2^M(\mathcal{O}_{F,S}) = \frac{\mathcal{O}_{F,S}^\times \otimes \mathbb{Z} \mathcal{O}_{F,S}^\times}{\langle x \otimes (1 - x) \mid x, 1 - x \in \mathcal{O}_{F,S}^\times \rangle}.$$ 

For $x, y \in \mathcal{O}_{F,S}^\times$, let $\{x, y\}$ denote the image of $x \otimes y$ in $K_2^M(\mathcal{O}_{F,S})$. 


Let $k_v$ = the residue field of $F$ at a prime $v$.

We have a commutative diagram

$$K_2^M(\mathcal{O}_F,S) \to K_2^M(F)$$

$$0 \to K_2(\mathcal{O}_F,S) \to K_2(F) \to \bigoplus_{v \notin S} k_v^\times \to 0,$$

where the upper horizontal map is induced by the natural map on symbols and the right-hand lower horizontal map is given by tame symbols. The map $K_2^M(\mathcal{O}_F,S) \to K_2(\mathcal{O}_F,S)$ given by the natural map on symbols need not be injective/surjective.
Example: $F = \mathbb{Q}(\sqrt{-d})$, $d \geq 5$, $d$ square free
$\mathcal{O}_{F,\emptyset}^\times = \mathcal{O}_F^\times = \langle -1 \rangle$

$$K_2^M(\mathcal{O}_F) = \langle \{-1, -1\} \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

For $d = 5$, Tate showed that
$$K_2(\mathcal{O}_F) = 0.$$  

For $d = 39$, Browkin-Belabas-Gangl showed that
$$K_2(\mathcal{O}_F) \cong \mathbb{Z}/6\mathbb{Z}.$$
Our focus:
p an odd prime, \( \zeta \) a primitive \( p \)th root of 1
\( F = \mathbb{Q}(\mu_p), \ S = \{(1 - \zeta)\}, \ O_{F,S} = \mathbb{Z}[\frac{1}{p}, \zeta] \)
\( \mathcal{C} = \langle 1 - \zeta^i \mid 1 \leq i \leq p - 1 \rangle \) cyclotomic \( p \)-units
\[
\mathcal{C} \cong \mathbb{Z}^{(p-1)/2} \oplus \mathbb{Z}/2p\mathbb{Z}
\]
\([\mathcal{O}_{F,S}^\times : \mathcal{C}] = h^+ = \) class number of \( \mathbb{Q}(\zeta + \zeta^{-1}) \)

Vandiver’s Conjecture. \( p \nmid h^+ \)

Vandiver’s conjecture is known for \( p < 12,000,000 \) (BCEMS).

When \( p \nmid h^+ \), the group \( K_2^M(\mathcal{O}_{F,S}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \) is generated (over \( \mathbb{Z}_p \)) by symbols of the form \( \{x, y\} \) with \( x, y \in \mathcal{C} \).

Unless \( h^+ = 1 \), it is possible that there could exist \( x, y \in \mathcal{O}_{F,S}^\times \) with \( x + y = 1 \) and \( x \notin \mathcal{C} \).
**Specialization:**

Consider the set of polynomials

\[ P = \{X\} \cup \{\Phi_n(X) \mid n \geq 1, n \neq p\}. \]

where \( \Phi_n = n\)th cyclotomic polynomial. \( R = \mathbb{Z}[X][P^{-1}] \).

\( \text{Spec } R = \left((\mathbb{G}_m - \mu_{\neq p})_{\mathbb{Z}},\right)\) where \( \mu_{\neq p} = \) non-\(p\)th roots of 1. The surjection \( R \twoheadrightarrow \mathbb{Z}[\zeta, \frac{1}{p}], \) \( X \mapsto \zeta \) induces

\[ K_2^M(R) \rightarrow K_2^M(\mathcal{O}_{F,S}). \]

To put bounds on the size of \( K_2^M(\mathcal{O}_{F,S}) \), one can first try to find solutions of \( f + g = h \) with \( f, g, h \in R^\times \cap \mathbb{Z}[X] \).

**Question.** Given \( x, y \in \mathcal{C} \) with \( x+y = 1 \), do there exist \( f, g \in R^\times \) with \( f + g = 1 \)?
Some relations in Milnor $K_2$:

a. $\zeta^a + (1 - \zeta^a) = 1$

b. $\zeta^a(\zeta^b - 1) + (\zeta^a - 1) = \zeta^{a+b} - 1$

c. $\zeta^{b+f}(\zeta^{b+d} - \zeta^{c+e})(\zeta^a - \zeta^d) + \zeta^a(\zeta^{b+d} - \zeta^{e+f})(\zeta^{b+c} - \zeta^{d+f})$
   \[= \zeta^d(\zeta^a + e - \zeta^{b+c})(\zeta^{a+b} - \zeta^{e+f})\]

d. $(\zeta^{a+b} - 1)(\zeta^{2a} - 1) + \zeta^a(\zeta^{a-1})(\zeta^b - 1)(\zeta^{a+b} - 1) = (\zeta^a - 1)(\zeta^{2(a+b)-1})$

e. $\zeta^b(\zeta^{3a} - 1)(\zeta^{a+b} - 1) + (\zeta^{a-1})(\zeta^b - 1)(\zeta^{a+b} - 1)(\zeta^{2a+b} - 1) = (\zeta^{a} - 1)(\zeta^{3(a+b)} - 1)$

f. $\zeta^a(\zeta^{4b-1})(\zeta^{a+b} - 1)(\zeta^{a+2b} - 1) + (\zeta^{a-1})(\zeta^b - 1)(\zeta^{a+b} - 1)(\zeta^{a+2b} - 1)(\zeta^{a+3b} - 1)$
   \[= (\zeta^{2a+4b} - 1)(\zeta^{2a+2b} - 1)(\zeta^b - 1)\]

g. $(\zeta^{a-1})(\zeta^{2(a+b)} - 1)(\zeta^{a+c} - 1)(\zeta^c - 1) + \zeta^a(\zeta^{a-1})(\zeta^b - 1)(\zeta^{2c} - 1)(\zeta^{a+b} - 1) = (\zeta^{a+b+c} - 1)(\zeta^{2a} - 1)(\zeta^c - 1)(\zeta^{a+b} - 1)$

h. $\zeta^a(\zeta^{2b} - 1)^2(\zeta^{a+b} - 1)^2 + (\zeta^{a-1})(\zeta^b - 1)^2(\zeta^{a+b} - 1)^2(\zeta^{a+2b} - 1)$
   \[= (\zeta^b - 1)^2(\zeta^{2a+2b} - 1)^2\]
Eigenspaces:

\[ \Delta = \text{Gal}(F/Q) \cong (\mathbb{Z}/p\mathbb{Z})^\times \]

\[ \omega: \Delta \to \mathbb{Z}_p^\times, \text{ Teichmüller character} \]

For \( i \in \mathbb{Z} \), we have idempotents:

\[ \epsilon_i = \frac{1}{p-1} \sum_{\delta \in \Delta} \omega(\delta)^{-i} \delta \in \mathbb{Z}_p[\Delta]. \]

Given a \( \mathbb{Z}_p[\Delta] \)-module \( A \), we define the \( \omega^i \)-eigenspace of \( A \):

\[ A^{(i)} = \langle a \in A \mid \delta a = \omega(\delta)^i a, \delta \in \Delta \rangle, \]

We have an eigenspace decomposition

\[ A = \bigoplus_{i=0}^{p-2} A^{(i)}. \]
Cyclotomic $p$-units:

$$(C \otimes \mathbb{Z}_p)^{(i)} = \begin{cases} 
0 & i \text{ odd, } i \neq 1 \\
\mathbb{Z}/p\mathbb{Z} & i = 1 \\
\mathbb{Z}_p & i \text{ even}
\end{cases}$$

For $i$ odd, let $\eta_i \in C$ with

$$\eta_i \equiv (\zeta - 1)^{c_{1-i}} \mod C^p.$$ 

Then $\eta_i$ generates $(C/C^p)^{(1-i)}$.

For $c \in \mathbb{Z}$ with $p \nmid c$,

$$(\zeta^c - 1)^{c_{1-i}} \equiv \eta_i^{c_{1-i}} \mod C^p.$$
**Milnor $K_2$ modulo $p$:**

Vandiver at $p \Rightarrow (K_2^M(\mathcal{O}_{F,S})/p)^- = 0$.

Fix an even integer $k \geq 2$. Let $\alpha, \beta \in \mathcal{O}_{F,S}^\times$. Set $\{\alpha, \beta\}_k = \text{image of } \{\alpha, \beta\} \text{ in } (K_2^M(\mathcal{O}_{F,S})/p)^{(2-k)}$.

Eigenspace considerations yield

\[ \{\eta_i, \eta_j\}_k = 0 \text{ if } i + j \not\equiv k \mod p - 1 \]
\[ \{\zeta, \eta_j\}_k = 0 \text{ for all odd } j \]

Thus, Vandiver's conjecture at $p$ implies that the $\{\eta_i, \eta_{k-i}\}_k$ generate $(K_2^M(\mathcal{O}_{F,S})/p)^{(2-k)}$. 

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Identities:

One has

\[ \{ \alpha, \beta \}_k = \sum_{i=1}^{p-2} \{ \alpha^{e_1-i}, \beta^{e_1+k-i} \}_k. \]

In particular,

\[ \{ \zeta^a - 1, \zeta^b - 1 \}_k = \sum_{i=1}^{p-2} a^{1-i} b^{1-k+i} \{ \eta_i, \eta_{k-i} \}_k. \]
Applying the relations:

Consider, for example, the relations (d) with \( b = 1 - a \) and \( p \nmid a - 1, a \):

\[
\frac{(\zeta - 1)(\zeta^{2a} - 1)}{(\zeta^a - 1)(\zeta^2 - 1)} - \zeta \frac{(\zeta^{a-1} - 1)(\zeta - 1)}{\zeta^2 - 1} = 1
\]

These yield relations on the \( x_i = \{\eta_i, \eta_{k-i}\}_k \):

\[
\sum_{i=1}^{p-2} (1 + (a - 1)^{1-i} - 2^{1-i})(a^{1+k-i} - 1)(2^{1+k-i} - 1)x_i = 0 \quad (1)
\]

We can calculate the solution space in \( \mathbb{F}_p^{(p-1)/2} \) of the equations resulting from (a)–(h) and \( x_i = -x_{k-i} \) to bound the \( p \)-rank of \( (K_2^M(\mathcal{O}_F,S)/p)^{(2-k)} \).
Table of solutions \((x_1 \ x_3 \ \ldots \ x_{p-1})\):

\(p = 37, \ k = 32\)
\((1 \ 26 \ 0 \ 36 \ 1 \ 35 \ 31 \ 34 \ 3 \ 6 \ 2 \ 36 \ 1 \ 0 \ 11 \ 36 \ 11 \ 26)\)

\(p = 59, \ k = 44\)
\((1 \ 45 \ 21 \ 30 \ 14 \ 35 \ 5 \ 0 \ 48 \ 57 \ 7 \ 52 \ 2 \ 11 \ 0 \ 54 \ 24 \ 45 \ 29 \ 38 \ 14 \ 58 \ 27 \ 32 \ 15 \ 0 \ 44 \ 27 \ 32)\)

\(p = 67, \ k = 58\)
\((1 \ 45 \ 38 \ 56 \ 0 \ 47 \ 62 \ 9 \ 29 \ 15 \ 65 \ 26 \ 45 \ 57 \ 0 \ 10 \ 22 \ 41 \ 2 \ 52 \ 38 \ 58 \ 5 \ 20 \ 0 \ 11 \ 29 \ 22 \ 66 \ 2 \ 24 \ 43 \ 65)\)

\(p = 73, \ k = 38\)
\((1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 72 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)\)

\(p = 97, \ k = 50\)
\((1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)\)

\(p = 101, \ k = 68\)
\((1 \ 56 \ 40 \ 96 \ 26 \ 63 \ 0 \ 61 \ 81 \ 71 \ 35 \ 92 \ 73 \ 64 \ 6 \ 88 \ 0 \ 0 \ 13 \ 95 \ 37 \ 28 \ 9 \ 66 \ 30 \ 20 \ 40 \ 0 \ 38 \ 75 \ 5 \ 61 \ 45 \ 100 \ 17 \ 17 \ 12 \ 66 \ 72 \ 53 \ 86 \ 31 \ 70 \ 15 \ 48 \ 29 \ 35 \ 89 \ 84 \ 84)\)
Except for (73, 38) and (97, 50), the pairs \((p, k)\) on the previous slide are all irregular pairs, i.e., \(2 \leq k \leq p - 3\) even, \(p \mid B_k\), where \(B_k = k\)th Bernoulli number.

**Exceptional pairs:**
If 2 and 3 are squares mod \(p\), there is a solution of the form listed above for \((p, \frac{p+3}{2})\).
(These are not the only exceptional solutions.)

**Conjecture.** Let \(R_1 = \mathbb{Z}[X][P_1^{-1}]\) and \(R_2 = \mathbb{Z}[X][P_2^{-1}]\) with
\[
P_1 = \{X\} \cup \{\Phi_n(X) \mid n \geq 1\}
\]
\[
P_2 = \{X\} \cup \{\Phi_n(X) \mid n \geq 1, (n, 6) = 1\}.
\]

There do not exist \(f \in R_1^\times\) and \(g \in R_2^\times\) with \(f + g = 1\), \(f(1) = 0\), and \(g \notin \langle X \rangle\).

**Question.** For \(p = 73\), is \(K_2^M(\mathcal{O}_{F,S})^{(36)} \cong \mathbb{Z}/p\mathbb{Z}\)?
As for the irregular pairs, we used the relations (1) (for $a$ odd) and the antisymmetry relations to obtain:

**Theorem** (McCallum-S.). *For all irregular pairs* $(p,k)$ *with* $p < 10,000$, *one has*

$$|(K_2^M(O_F,S)/p)^{(2-k)}| \leq p.$$  

In fact, we computed:

**Proposition.** *For all irregular pairs* $(p,k)$ *with* $p < 4,000$, *one has*

$$|(K_2^M(O_F,S) \otimes \mathbb{Z}_p)^{(2-k)}| \leq p.$$
Relationship with cohomology:

For now, let $F$ be any number field. Assume $S \supset \{v | n\infty\}$ for some $n \geq 1$. $G_{F,S}$ = Galois group of the maximal extension of $F$ unramified outside $S$.

**Theorem** (Tate, Soulé). *There is a canonical isomorphism*

$$K_2(O_{F,S})/n \sim H^2(G_{F,S}, \mu_n \otimes^2).$$

Assume that $\mu_n \subset F$. Then

$$H^2(G_{F,S}, \mu_n \otimes^2) \cong H^2(G_{F,S}, \mu_n) \otimes \mu_n.$$
Cohomology groups:

\[ \text{Cl}_{F,S} = S\text{-class group of } F \text{ (class group } \text{Cl}_F \text{ modulo classes of finite primes in } S) \]

\[ \text{Br}_S(F') = \ker(\bigoplus_{v \in S} \text{Br}(F_v) \to \mathbb{Q}/\mathbb{Z}) \]

Kummer theory yields an exact sequence

\[ 0 \to \text{Cl}_{F,S}/n \to H^2(G_{F,S}, \mu_n) \to \text{Br}_S(F')[n] \to 0. \]

Cup product:

\[ H^1(G_{F,S}, \mu_n) \otimes H^1(G_{F,S}, \mu_n) \xrightarrow{\cup} H^2(G_{F,S}, \mu_n \otimes_n 2) \]

Kummer theory yields a natural injection

\[ \mathcal{O}_{F,S}^\times/\mathcal{O}_{F,S}^\times n \hookrightarrow H^1(G_{F,S}, \mu_n) \]
We define a pairing:

$$(\ , \ )_{S} = (\ , \ )_{n,F,S} : \mathcal{O}_{F,S}^\times \times \mathcal{O}_{F,S}^\times \cup H^2(G_{F,S}, \mu_n \otimes 2)$$

**Theorem** (Soulé). The diagram

$$\mathcal{O}_{F,S}^\times \otimes \mathcal{O}_{F,S}^\times \xrightarrow{(\ , \ )_{S}} H^2(G_{F,S}, \mu_n \otimes 2)$$

$$\downarrow \quad \quad \downarrow $$

$$K^M_2(\mathcal{O}_{F,S})/n \quad \quad \quad K_2(\mathcal{O}_{F,S})/n$$

of canonical maps is commutative.

The projection of $(\ , \ )_{S}$ to $\text{Br}_S(F)[n] \otimes \mu_n$ is the sum of norm residue symbols at $v \in S$. 

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Now take $F = \mathbb{Q}(\mu_p)$ again.

Since $\text{Cl}_{F,S} = \text{Cl}_F$ and $\text{Br}_S(F) = 0$, we have

$$H^2(G_{F,S}, \mu_p^{\otimes 2}) \cong \text{Cl}_F \otimes \mu_p.$$ 

Vandiver’s conjecture implies the following:

**Cyclicity conjecture.** $(\text{Cl}_F \otimes \mathbb{Z}_p)^{(i)}$ is cyclic for all $i$.

The cyclicity conjecture was a question of Iwasawa’s.

$(\text{Cl}_F \otimes \mathbb{Z}_p)^{(1-k)} \neq 0$ if and only if $p \mid \frac{B_k}{k}$. 

Cyclicity conjecture $\Rightarrow$ for $(p, k)$ irregular,

$$(K_2(O_{F,S})/p)^{(2-k)} \cong \mathbb{Z}/p\mathbb{Z}.$$
A surjectivity conjecture:

**Conjecture** (McCallum-S.). *The map*

\[
(K_2^M(O_{F,S} \otimes \mathbb{Z}_p))^+ \to (K_2(O_{F,S} \otimes \mathbb{Z}_p))^+
\]

*is surjective.*

**Remarks:**
1. If \( p \) violates Vandiver’s conjecture and one of the even eigenspaces of the \( p \)-part of the class group has \( p \)-rank \( \geq 2 \), then the map cannot be surjective on \((-1)\)-eigenspaces.
2. Injectivity: unlikely?
3. One always has \( \{\eta_{1-k}, \eta_{2k-1}\}_k = 0 \) for \((p, k)\) irregular.
An approach to surjectivity:
Let $H = F(\eta_{p-k}^{1/p})$, an unramified cyclic degree $p$ extension of $F$.

Let $\text{Gal}(H/F) = \langle \sigma \rangle$ and $N = \sum_{i=1}^{p-1} \sigma^i$, $D = \sum_{i=1}^{p-1} i \sigma^i \in \mathbb{Z}[\text{Gal}(H/F)]$.

$p_0$ prime of $H$ above $p$, $\Delta_0 = $ subgroup of $\text{Gal}(H/F)$ fixing $p_0$.

**Proposition.** Let $(p, k)$ be an irregular pair with $p \nmid B_{p+1-k}$. Assume $p = \beta^N$, with $\beta \in \mathcal{O}_{H,S}^\times$ such that its image in $\mathcal{O}_{H,S}^\times / \mathcal{O}_{H,S}^{\times p}$ is fixed by $\Delta_0$. Then

$$(\eta_{k-1}, p)_S \neq 0 \iff (\eta_{k-1}, \beta^D)_{p,H_{p_0}} \neq 1 \iff \beta^D \notin H_{p_0}^\times p.$$  

For $p = 37$, this is computationally verifiable.

**Theorem** (McCallum-S.). The surjectivity conjecture holds for $p = 37$.  

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Sketch of proof.  
With the help of W. Stein and C. Fieker, we determined the \( p \)-unit group of the fixed field \( E \) of \( \Delta_0 \) as follows.

\[
\begin{array}{c}
\Delta_0 \\
E \\
Q
\end{array}
\quad H \quad
\begin{array}{c}
\Delta \\
F
\end{array}
\]

The field \( E \) is generated by the trace \( x \) of a 37th root of \( \eta_5 \), and we found a minimal polynomial for this element using CRT. Various Magma routines then took 5 days on a 2GHz processor to produce an optimal representation of the integer ring of \( E \), from which its \( p \)-unit group was computable. One of the \( p \)-units \( \beta \) had norm \( p \). By computing the various embeddings of \( x \) in the \( H_p \), we were able to verify the condition of the proposition for \( \beta^D \).
The minimal polynomial of $x$:
\[x^{37} - 6483584x^{34} - 118234637824x^{33} - 123335506765824x^{32} - 7894900273815552x^{31} - 25584896141781024768x^{30} - 1961278666613992009728x^{29} - 2221784070205669762924544x^{28} - 33628014249666292632903483392x^{27} - 48057116976091024421471201472x^{26} - 2249002615426863992005848511545344x^{25} - 13099755496539209311468832290825568256x^{24} - 3171787436319383501703813676940597919744x^{23} + 476259323830076662111107898811789814530048x^{22} - 1396232608839552259966984463923520026947092480x^{21} - 331493134727514939719441018060252656606965137408x^{20} - 802686380624350745599184759300711777564488630272x^{19} - 872057565672136492561824204817812097995282872168087552x^{18} + 1772659418875854490177280483057352783210247369401565184x^{17} + 37244222236334875481641252538596552828631758622687299108864x^{16} - 20651404785477501467881895153357983415526349942938256921329664x^{15} + 311835441256087157637746419559980783743744537079124122814696652x^{14} + 28547054494846244416795330612386811215415869973011706932441160613888x^{13} - 18557314583560485308211477301528775481854373440798991639264756844462080x^{12} + 3087405021478910646130093242279350919332930043815268747163999299543498752x^{11} - 844861169134880185162881813189113039529594781451540816736236726469132320768x^{10} - 181305918878969636874702964809165511398336363048146870204995155207780939936x^9 + 484965911395764970871665544609840479278207020589886844109688505883361242049937408x^8 + 551140497767821995852622334540957461483624433957263207123073326516074293876028866560x^7 - 106850589825632789894896612887329721094911179135082410100519870323032421196184294522880x^6
A better polynomial for $E$:
\[
y^{37} + 4y^{36} + 12y^{35} + 36y^{34} - 336y^{33} - 268y^{32} - 3912y^{31} - 7555y^{30} + 60363y^{29} - 254771y^{28} + 1584299y^{27} - 4912687y^{26} + 17776688y^{25} - 51189497y^{24} + 135760742y^{23} - 339845565y^{22} + 729194231y^{21} - 1823351247y^{20} + 2954679204y^{19} - 7136330744y^{18} + 14870105096y^{17} - 19798475744y^{16} + 63485328194y^{15} - 69489469832y^{14} + 240906930339y^{13} - 130150428853y^{12} + 883058481925y^{11} - 525666202335y^{10} + 1336924708802y^9 - 2790390347185y^8 + 2312809893723y^7 - 300537388911y^6 + 6491297663291y^5 - 2826510585529y^4 + 4902736951337y^3 - 6453741855514y^2 + 3673618997547y - 1546779831802
\]

The embedding of $\beta^D$:
\[-445 + 13 \cdot 37t - 3t^{31} - 9t^{32} + 18t^{33} + 14t^{34} + 2t^{35} + O(t^{38}), \; t = \zeta - 1\]
A few consequences for $p = 37$:

The group $K_2^M(\mathcal{O}_{F,S}) \otimes \mathbb{Z}_{37}$ has order 37.

The product maps

$$K_{2i-1}(\mathbb{Z}) \otimes K_{63+72k-2i}(\mathbb{Z}) \rightarrow K_{62+72k}(\mathbb{Z}) \otimes \mathbb{Z}_{37}$$

are nontrivial for a given odd $i$ and any $k$ if $i \not\equiv 5, 27 \mod 36$ and nonsurjective otherwise.

(Under the Quillen-Lichtenbaum conjecture, we have surjectivity and triviality in the two respective cases.)

Let $M/F$ be a cyclic extension of degree 37 that is unramified outside 37. Then $|\text{Cl}_{M,S} \otimes \mathbb{Z}_{37}| = 37$ if and only if

$$M \not\subset \mathbb{Q}(\zeta_{37^2}, \eta_5^{1/37}, \eta_2^{1/37}).$$

Furthermore, only $M = F(\eta_5^{1/37})$ has trivial 37-class number.
Another approach to surjectivity: \( \chi_j : G_F \to \mathbb{Z}/p\mathbb{Z} \) Kummer character for a \( p \)th root of \( \eta_j \).

Fix \( i \) odd and let \( L \) be the field defined by \( \chi_i \).

Let \( H \) be defined by \( \chi_{p-k} \).

Then there exists a representation of \( G_{\mathbb{Q},S} \) in \( \text{Gl}_3(\mathbb{F}_p) \) such that for \( \sigma \in G_{F,S} \), we have

\[
\rho(\sigma) = \begin{pmatrix}
1 & \chi_i(\sigma) & \kappa(\sigma) \\
0 & 1 & \chi_{p-k}(\sigma) \\
0 & 0 & 1
\end{pmatrix}
\]

Let \( M \) denote the fixed field of the kernel of \( \rho \).

**Proposition.** Assume that \( (\text{Cl}_F \otimes \mathbb{Z}_p)^{1-k} \) is cyclic and that \( p \nmid B_{p-i} \). Then \( (\eta_i, \eta_{k-i})_S = 0 \) if and only if \( M/HL \) is completely split at all (any) primes above \( p \).
Hecke algebras:

Let \((p,k)\) be an irregular pair.

\(\mathbf{T}\) = the ordinary cuspidal Hecke algebra (over \(\mathbb{Z}_p\)) of weight 2, level \(p\), and character \(\omega^{k-2}\).
(By ordinary, we mean the maximal subfactor in which \(U_p\) is unit.)

\(\mathbf{T}\) is generated by Hecke operators \(T_l\) with \(l \neq p\) prime and \(U_p\)

Eisenstein ideal:

\[
\mathcal{I} = (U_p - 1) + (T_l - 1 - l\omega(l)^{k-2} | l \neq p).
\]

Idea: study \(Y = \text{eigenspace of } H^{1}_{\text{ét}}(X_1(p),\mathbb{Z}_p)^{\text{ord}}\) on which \(\langle i \rangle\) acts as \(i^{k-2}\) for \(i \in \mu_{p-1}(\mathbb{Z}_p)\) “modulo \(\mathcal{I}^2\).
The modular representation (following Ohta):
The action of $G_Q$ on $Y$ provides

$$\rho: G_Q \rightarrow \text{Aut}_T(Y).$$

Fix a decomposition group $D_p$ in $G_Q$ at $p$.
Let $I_p$ be the inertia subgroup of $D_p$.
Fix $\Delta_p \leq I_p$ with $\omega: \Delta_p \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^\times$.
Under $\Delta_p$, we have $Y = Y^+ \oplus Y^-$.

For $\sigma \in G_Q$, we write

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix},$$

$a(\sigma) \in \text{End}_T(Y^-), b(\sigma) \in \text{Hom}_T(Y^+, Y^-), \ldots$
For $\sigma \in I_p$, we have

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & 0 \\ c(\sigma) & 1 \end{pmatrix},$$

Note: $\text{End}_T(Y^\pm) \hookrightarrow T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

For $\sigma, \tau \in G_Q$, we have

$$d(\sigma) \equiv 1 \mod I,$$

$$b(\sigma)c(\tau) \equiv 0 \mod I.$$

In fact, let $B$ (resp., $C$) denote the $T$-module generated by all $b(\sigma)$ (resp., $c(\sigma)$) with $\sigma \in G_Q$.

One has $BC = I$ (Ohta, following Kurihara, Harder-Pink).
Let $m = \mathcal{I} + pT$.

Consider the maps

\[
\bar{a}: G_Q \to (T/m)^\times \quad \text{and} \quad \bar{b}: G_Q \to B/mB
\]
\[
\bar{c}: G_Q \to C/mC \quad \text{and} \quad \bar{d}: G_Q \to (T/\mathcal{I}m)^\times
\]

We remark that $\bar{a} = \omega^{1-k}$.

$H = \text{fixed field of the kernel of } \bar{b} \text{ on } G_F$.
$H$ is an unramified abelian $p$-extension of $F$ with an $\omega^{1-k}$-action of Galois.

$L = \text{fixed field of the kernel of } \bar{c} \text{ on } G_F$.
If $p \nmid B_{p+1-k}$, then $L = F(\eta_{k-1})$. 
Consider the homomorphism (note: $BC = I$)

$$
\phi_D(\sigma) = \begin{pmatrix}
\bar{a}(\sigma) & \bar{b}(\sigma) \\
\bar{c}(\sigma) & \bar{d}(\sigma)
\end{pmatrix}.
$$

We also have a homomorphism

$$
\psi(\sigma) = \begin{pmatrix}
1 & \bar{c}(\sigma) & \bar{d}(\sigma) - 1 \\
0 & \omega^{1-k}(\sigma) & \bar{b}(\sigma) \\
0 & 0 & 1
\end{pmatrix}
$$

defining the same extension $M/HL$ as $\phi_D$.

$\bar{d}(\varphi_p^{-1}) = U_p$ for a Frobenius $\varphi_p \in D_p$

$M/HL$ is non-split at primes above $p$ if and only if

$$
I = Im + (U_p - 1)
$$

By the earlier proposition, we have:
Theorem (S.). Assume that $p \nmid B_{p+1-k}$ and $(\Cl_F \otimes \mathbb{Z}_p)^{(1-k)}$ is cyclic. Then $U_p - 1$ generates the group $\mathcal{I}/\mathcal{I}^2$ if and only if $(p, \eta_{k-1})_S \neq 0$.

Theorem (S.). For any irregular pair $(p, k)$ with $p < 1000$, we have $\mathcal{I} = (U_p - 1) + \mathcal{I}^2$.

Corollary. For $p < 1000$, the map

$$K_2^M(\mathcal{O}_F, S) \otimes \mathbb{Z}_p \to K_2(\mathcal{O}_F, S) \otimes \mathbb{Z}_p$$

is surjective.
Sketch of proof.
Let $\mathcal{H}$ denote the full cuspidal Hecke algebra for $\Gamma_1(p)$ and $\omega^{k-2}$. We used built-in Magma routines for modular symbols (W. Stein) to compute the $U_p$ and $T_i$ in $\mathcal{H}$ for all $i$ with $1 \leq i < p/6$, viewed in $M_d(\mathbb{Z}_p)$ for some $d$. These $T_i$ generate $\mathcal{H}$ as a $\mathbb{Z}$-module. Let $\bar{T}_i$ and $\bar{U}_p$ be the images in $M_d(\mathbb{Z}/p^2\mathbb{Z})$. Let $M$ be the submodule of $M_d(\mathbb{Z}/p^2\mathbb{Z})$ spanned by the $\bar{T}_i$.

We computed $N$ minimal such that the $\bar{T}_i$ with $1 \leq i \leq N$ generate $M$. Then $p$ and the $T_i - \sum_{0 < e | i} \omega(e)^{k-2} e$ with $1 \leq i \leq N$ generate $\mathcal{I}$ over $\mathbb{Z}_p$, provided that

$$\mathcal{H}/\mathcal{I} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{T}/\mathcal{I} \cong \mathbb{Z}_p/L_p(-1, \omega^k)$$

is $p$-torsion, which we checked. We computed the images $I$ and $J$ of $\mathcal{I}$ and $\mathcal{I}^2$ in $M$ using these elements and their products. We verified that $I = J + (\bar{U}_p - 1)$. 33