Computations on Milnor K_2 of integer rings

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Milnor K_2 of integer rings:

F number field, S finite set of primes of F
$\mathcal{O}_{F,S}$ ring of S-integers of F

We define Milnor K_2 of $\mathcal{O}_{F,S}$ as

$$K_2^M(\mathcal{O}_{F,S}) = \frac{\mathcal{O}_{F,S}^\times \otimes_{\mathbf{Z}} \mathcal{O}_{F,S}^\times}{\langle x \otimes (1-x) \mid x, 1-x \in \mathcal{O}_{F,S}^\times \rangle}.$$

For $x, y \in \mathcal{O}_{F,S}^\times$, let $\{x, y\}$ denote the image of $x \otimes y$ in $K_2^M(\mathcal{O}_{F,S})$.

Let $k_v =$ the residue field of F at a prime v.

We have a commutative diagram

$$\begin{array}{ccccccccc} & & K_2^M(\mathcal{O}_{F,S}) & \longrightarrow & K_2^M(F) & & & & \\ & & \downarrow & & \| & & & & \\ 0 & \longrightarrow & K_2(\mathcal{O}_{F,S}) & \longrightarrow & K_2(F) & \longrightarrow & \oplus_{v \notin S} k_v^\times & \longrightarrow & 0, \end{array}$$

where the upper horizontal map is induced by the natural map on symbols and the right-hand lower horizontal map is given by tame symbols. The map $K_2^M(\mathcal{O}_{F,S}) \to K_2(\mathcal{O}_{F,S})$ given by the natural map on symbols need not be injective/surjective.

Example: $F = \mathbf{Q}(\sqrt{-d})$, $d \geq 5$, d square free
$\mathcal{O}_{F,\varnothing}^{\times} = \mathcal{O}_F^{\times} = \langle -1 \rangle$

$$K_2^M(\mathcal{O}_F) = \langle \{-1, -1\} \rangle \cong \mathbf{Z}/2\mathbf{Z}$$

For $d = 5$, Tate showed that

$$K_2(\mathcal{O}_F) = 0.$$

For $d = 39$, Browkin-Belabas-Gangl showed that

$$K_2(\mathcal{O}_F) \cong \mathbf{Z}/6\mathbf{Z}.$$

Our focus:
p an odd prime, ζ a primitive pth root of 1
$F = \mathbf{Q}(\mu_p)$, $S = \{(1-\zeta)\}$, $\mathcal{O}_{F,S} = \mathbf{Z}[\frac{1}{p}, \zeta]$
$\mathcal{C} = \langle 1 - \zeta^i \mid 1 \leq i \leq p-1 \rangle$ cyclotomic p-units

$$\mathcal{C} \cong \mathbf{Z}^{\oplus (p-1)/2} \oplus \mathbf{Z}/2p\mathbf{Z}$$

$[\mathcal{O}_{F,S}^{\times} : \mathcal{C}] = h^+ =$ class number of $\mathbf{Q}(\zeta + \zeta^{-1})$

Vandiver's Conjecture. $p \nmid h^+$

Vandiver's conjecture is known for $p < 12{,}000{,}000$ (BCEMS).

When $p \nmid h^+$, the group $K_2^M(\mathcal{O}_{F,S}) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is generated (over $\mathbf{Z}_p$) by symbols of the form $\{x, y\}$ with $x, y \in \mathcal{C}$.

Unless $h^+ = 1$, it is possible that there could exist $x, y \in \mathcal{O}_{F,S}^{\times}$ with $x + y = 1$ and $x \notin \mathcal{C}$.

Specialization:
Consider the set of polynomials

$$P = \{X\} \cup \{\Phi_n(X) \mid n \geq 1, n \neq p\}.$$

where $\Phi_n = n$th cyclotomic polynomial. $R = \mathbf{Z}[X][P^{-1}]$.
Spec R = "$(\mathbf{G}_m - \mu_{\neq p})_{\mathbf{Z}}$," where $\mu_{\neq p}$ = non-pth roots of 1.
The surjection $R \twoheadrightarrow \mathbf{Z}[\zeta, \frac{1}{p}]$, $X \mapsto \zeta$ induces

$$K_2^M(R) \to K_2^M(\mathcal{O}_{F,S}).$$

To put bounds on the size of $K_2^M(\mathcal{O}_{F,S})$, one can first try to find solutions of $f + g = h$ with $f, g, h \in R^\times \cap \mathbf{Z}[X]$.

Question. Given $x, y \in \mathcal{C}$ with $x+y = 1$, do there exist $f, g \in R^\times$ with $f + g = 1$?

Some relations in Milnor K_2:

a. $\zeta^a + (1 - \zeta^a) = 1$

b. $\zeta^a(\zeta^b - 1) + (\zeta^a - 1) = \zeta^{a+b} - 1$

c. $\zeta^{b+f}(\zeta^{b+d} - \zeta^{c+e})(\zeta^a - \zeta^d) + \zeta^a(\zeta^{b+d} - \zeta^{e+f})(\zeta^{b+c} - \zeta^{d+f})$
$= \zeta^d(\zeta^{a+e} - \zeta^{b+c})(\zeta^{a+b} - \zeta^{e+f})$

d. $(\zeta^{a+b} - 1)(\zeta^{2a} - 1) + \zeta^a(\zeta^a - 1)(\zeta^b - 1)(\zeta^{a+b} - 1) = (\zeta^a - 1)(\zeta^{2(a+b)-1})$

e. $\zeta^b(\zeta^{3a} - 1)(\zeta^{a+b} - 1) + (\zeta^a - 1)(\zeta^b - 1)(\zeta^{a+b} - 1)(\zeta^{2a+b} - 1)$
$= (\zeta^a - 1)(\zeta^{3(a+b)} - 1)$

f. $\zeta^a(\zeta^{4b}-1)(\zeta^{a+b}-1)(\zeta^{a+2b}-1)+(\zeta^a-1)(\zeta^b-1)(\zeta^{a+b}-1)(\zeta^{a+2b}-1)(\zeta^{a+3b}-1)$
$= (\zeta^{2a+4b} - 1)(\zeta^{2a+2b} - 1)(\zeta^b - 1)$

g. $(\zeta^a - 1)(\zeta^{2(a+b)} - 1)(\zeta^{a+c} - 1)(\zeta^c - 1) + \zeta^a(\zeta^a - 1)(\zeta^b - 1)(\zeta^{2c} - 1)(\zeta^{a+b} - 1)$
$= (\zeta^{a+b+c} - 1)(\zeta^{2a} - 1)(\zeta^c - 1)(\zeta^{a+b} - 1)$

h. $\zeta^a(\zeta^{2b} - 1)^2(\zeta^{a+b} - 1)^2 + (\zeta^a - 1)(\zeta^b - 1)^2(\zeta^{a+b} - 1)^2(\zeta^{a+2b} - 1)$
$= (\zeta^b - 1)^2(\zeta^{2a+2b} - 1)^2$

Eigenspaces:

$\Delta = \mathrm{Gal}(F/\mathbf{Q}) \cong (\mathbf{Z}/p\mathbf{Z})^\times$
$\omega\colon \Delta \to \mathbf{Z}_p^\times$, Teichmüller character

For $i \in \mathbf{Z}$, we have idempotents:

$$\epsilon_i = \frac{1}{p-1} \sum_{\delta \in \Delta} \omega(\delta)^{-i} \delta \in \mathbf{Z}_p[\Delta].$$

Given a $\mathbf{Z}_p[\Delta]$-module A, we define the ω^i-eigenspace of A:

$$A^{(i)} = \langle a \in A \mid \delta a = \omega(\delta)^i a,\, \delta \in \Delta \},$$

We have an eigenspace decomposition

$$A = \bigoplus_{i=0}^{p-2} A^{(i)}.$$

Cyclotomic p-units:

$$(\mathcal{C} \otimes \mathbf{Z}_p)^{(i)} = \begin{cases} 0 & i \text{ odd}, i \neq 1 \\ \mathbf{Z}/p\mathbf{Z} & i = 1 \\ \mathbf{Z}_p & i \text{ even} \end{cases}$$

For i odd, let $\eta_i \in \mathcal{C}$ with

$$\eta_i \equiv (\zeta - 1)^{\epsilon_{1-i}} \bmod \mathcal{C}^p.$$

Then η_i generates $(\mathcal{C}/\mathcal{C}^p)^{(1-i)}$.

For $c \in \mathbf{Z}$ with $p \nmid c$,

$$(\zeta^c - 1)^{\epsilon_{1-i}} \equiv \eta_i^{c^{1-i}} \bmod \mathcal{C}^p.$$

Milnor K_2 modulo p:

Vandiver at $p \Rightarrow (K_2^M(\mathcal{O}_{F,S})/p)^- = 0$.

Fix an even integer $k \geq 2$. Let $\alpha, \beta \in \mathcal{O}_{F,S}^\times$.
Set $\{\alpha, \beta\}_k$ = image of $\{\alpha, \beta\}$ in $(K_2^M(\mathcal{O}_{F,S})/p)^{(2-k)}$.

Eigenspace considerations yield

$$\begin{aligned} \{\eta_i, \eta_j\}_k &= 0 \text{ if } i + j \not\equiv k \bmod p - 1 \\ \{\zeta, \eta_j\}_k &= 0 \text{ for all odd } j \end{aligned}$$

Thus, Vandiver's conjecture at p implies that the $\{\eta_i, \eta_{k-i}\}_k$ generate $(K_2^M(\mathcal{O}_{F,S})/p)^{(2-k)}$.

Identities:

One has

$$\{\alpha, \beta\}_k = \sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} \{\alpha^{\epsilon_{1-i}}, \beta^{\epsilon_{1+k-i}}\}_k.$$

In particular,

$$\{\zeta^a - 1, \zeta^b - 1\}_k = \sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} a^{1-i} b^{1-k+i} \{\eta_i, \eta_{k-i}\}_k.$$

Applying the relations:

Consider, for example, the relations (d) with $b = 1 - a$ and $p \nmid a-1, a$:

$$\frac{(\zeta-1)(\zeta^{2a}-1)}{(\zeta^a-1)(\zeta^2-1)} - \zeta\frac{(\zeta^{a-1}-1)(\zeta-1)}{\zeta^2-1} = 1$$

These yield relations on the $x_i = \{\eta_i, \eta_{k-i}\}_k$:

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{p-2} (1 + (a-1)^{1-i} - 2^{1-i})(a^{1+k-i} - 1)(2^{1+k-i} - 1)x_i = 0 \tag{1}$$

We can calculate the solution space in $\mathbf{F}_p^{(p-1)/2}$ of the equations resulting from (a)–(h) and $x_i = -x_{k-i}$ to bound the p-rank of $(K_2^M(\mathcal{O}_{F,S})/p)^{(2-k)}$.

Table of solutions $(x_1\ x_3\ \ldots\ x_{p-1})$**:**
p = 37, k = 32
(1 26 0 36 1 35 31 34 3 6 2 36 1 0 11 36 11 26)

p = 59, k = 44
(1 45 21 30 14 35 5 0 48 57 7 52 2 11 0 54 24 45 29 38 14 58 27 32 15 0 44 27 32)

p = 67, k = 58
(1 45 38 56 0 47 62 9 29 15 65 26 45 57 0 10 22 41 2 52 38 58 5 20 0 11 29 22 66 2 24 43 65)

p = 73, k = 38
(1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 72 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0)

p = 97, k = 50
(1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 96 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0)

p = 101, k = 68
(1 56 40 96 26 63 0 61 81 71 35 92 73 64 6 88 0 0 13 95 37 28 9 66 30 20 40 0 38 75 5 61 45 100 17 17 12 66 72 53 86 31 70 15 48 29 35 89 84 84)

Except for $(73, 38)$ and $(97, 50)$, the pairs (p, k) on the previous slide are all irregular pairs, i.e., $2 \leq k \leq p-3$ even, $p \mid B_k$, where $B_k = k$th Bernoulli number.

Exceptional pairs:
If 2 and 3 are squares mod p, there is a solution of the form listed above for $(p, \frac{p+3}{2})$.
(These are not the only exceptional solutions.)

Conjecture. *Let* $R_1 = \mathbf{Z}[X][P_1^{-1}]$ *and* $R_2 = \mathbf{Z}[X][P_2^{-1}]$ *with*

$$\begin{aligned} P_1 &= \{X\} \cup \{\Phi_n(X) \mid n \geq 1\} \\ P_2 &= \{X\} \cup \{\Phi_n(X) \mid n \geq 1, (n, 6) = 1\}. \end{aligned}$$

There do not exist $f \in R_1^\times$ *and* $g \in R_2^\times$ *with* $f + g = 1$, $f(1) = 0$, *and* $g \notin \langle X \rangle$.

Question. For $p = 73$, is $K_2^M(\mathcal{O}_{F,S})^{(36)} \cong \mathbf{Z}/p\mathbf{Z}$?

As for the irregular pairs, we used the relations (1) (for a odd) and the antisymmetry relations to obtain:

Theorem (McCallum-S.). *For all irregular pairs* (p,k) *with* $p <$ 10,000, *one has*

$$|(K_2^M(\mathcal{O}_{F,S})/p)^{(2-k)}| \leq p.$$

In fact, we computed:

Proposition. *For all irregular pairs* (p,k) *with* $p < 4{,}000$, *one has*

$$|(K_2^M(\mathcal{O}_{F,S}) \otimes \mathbf{Z}_p)^{(2-k)}| \leq p.$$

Relationship with cohomology:

For now, let F be any number field.
Assume $S \supset \{v \mid n\infty\}$ for some $n \geq 1$.
$G_{F,S}$ = Galois group of the maximal extension of F unramified outside S.

Theorem (Tate, Soulé)**.** *There is a canonical isomorphism*

$$K_2(\mathcal{O}_{F,S})/n \xrightarrow{\sim} H^2(G_{F,S}, \mu_n^{\otimes 2}).$$

Assume that $\mu_n \subset F$. Then

$$H^2(G_{F,S}, \mu_n^{\otimes 2}) \cong H^2(G_{F,S}, \mu_n) \otimes \mu_n.$$

Cohomology groups:

$\mathsf{Cl}_{F,S} = S$-class group of F (class group Cl_F modulo classes of finite primes in S)
$\mathsf{Br}_S(F) = \ker(\oplus_{v\in S}\mathsf{Br}(F_v) \to \mathbf{Q}/\mathbf{Z})$
Kummer theory yields an exact sequence

$$0 \to \mathsf{Cl}_{F,S}/n \to H^2(G_{F,S}, \mu_n) \to \mathsf{Br}_S(F)[n] \to 0.$$

Cup product:

$$H^1(G_{F,S}, \mu_n) \otimes H^1(G_{F,S}, \mu_n) \xrightarrow{\cup} H^2(G_{F,S}, \mu_n^{\otimes 2})$$

Kummer theory yields a natural injection

$$\mathcal{O}_{F,S}^\times/\mathcal{O}_{F,S}^{\times n} \hookrightarrow H^1(G_{F,S}, \mu_n)$$

We define a pairing:

$$(\; , \;)_S = (\; , \;)_{n,F,S} \colon \mathcal{O}_{F,S}^{\times} \times \mathcal{O}_{F,S}^{\times} \xrightarrow{\cup} H^2(G_{F,S}, \mu_n^{\otimes 2})$$

Theorem (Soulé)**.** *The diagram*

$$\begin{array}{ccc} \mathcal{O}_{F,S}^{\times} \otimes \mathcal{O}_{F,S}^{\times} & \xrightarrow{(\; , \;)_S} & H^2(G_{F,S}, \mu_n^{\otimes 2}) \\ \downarrow & & \uparrow \wr \\ K_2^M(\mathcal{O}_{F,S})/n & \longrightarrow & K_2(\mathcal{O}_{F,S})/n \end{array}$$

of canonical maps is commutative.

The projection of $(\; , \;)_S$ to $\mathrm{Br}_S(F)[n] \otimes \mu_n$ is the sum of norm residue symbols at $v \in S$.

Now take $F = \mathbf{Q}(\mu_p)$ again.

Since $\mathrm{Cl}_{F,S} = \mathrm{Cl}_F$ and $\mathrm{Br}_S(F) = 0$, we have

$$H^2(G_{F,S}, \mu_p^{\otimes 2}) \cong \mathrm{Cl}_F \otimes \mu_p.$$

Vandiver's conjecture implies the following:

Cyclicity conjecture. $(\mathrm{Cl}_F \otimes \mathbf{Z}_p)^{(i)}$ *is cyclic for all i.*

The cyclicity conjecture was a question of Iwasawa's.

$(\mathrm{Cl}_F \otimes \mathbf{Z}_p)^{(1-k)} \neq 0$ if and only if $p \mid \frac{B_k}{k}$.

Cyclicity conjecture $\Rightarrow$ for (p, k) irregular,

$$(K_2(\mathcal{O}_{F,S})/p)^{(2-k)} \cong \mathbf{Z}/p\mathbf{Z}.$$

A surjectivity conjecture:

Conjecture (McCallum-S.). *The map*

$$(K_2^M(\mathcal{O}_{F,S}) \otimes \mathbf{Z}_p)^+ \to (K_2(\mathcal{O}_{F,S}) \otimes \mathbf{Z}_p)^+$$

is surjective.

Remarks:

1. If p violates Vandiver's conjecture and one of the even eigenspaces of the p-part of the class group has p-rank ≥ 2, then the map cannot be surjective on (-1)-eigenspaces.
2. Injectivity: unlikely?
3. One always has $\{\eta_{1-k}, \eta_{2k-1}\}_k = 0$ for (p, k) irregular.

An approach to surjectivity:
Let $H = F(\eta_{p-k}^{1/p})$, an unramified cyclic degree p extension of F.

Let $\mathrm{Gal}(H/F) = \langle\sigma\rangle$ and $N = \sum_{i=1}^{p-1} \sigma^i$, $D = \sum_{i=1}^{p-1} i\sigma^i \in \mathbf{Z}[\mathrm{Gal}(H/F)]$.

$\mathfrak{p}_0$ prime of H above p, $\Delta_0 =$ subgroup of $\mathrm{Gal}(H/F)$ fixing $\mathfrak{p}_0$.

Proposition. *Let (p, k) be an irregular pair with $p \nmid B_{p+1-k}$. Assume $p = \beta^N$, with $\beta \in \mathcal{O}_{H,S}^\times$ such that its image in $\mathcal{O}_{H,S}^\times/\mathcal{O}_{H,S}^{\times p}$ is fixed by Δ_0. Then*

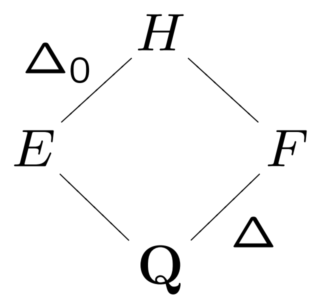
$$(\eta_{k-1}, p)_S \neq 0 \Longleftrightarrow (\eta_{k-1}, \beta^D)_{p,H_{\mathfrak{p}_0}} \neq 1 \Longleftrightarrow \beta^D \notin H_{\mathfrak{p}_0}^{\times p}.$$

For $p = 37$, this is computationally verifiable.

Theorem (McCallum-S.)**.** *The surjectivity conjecture holds for $p = 37$.*

Sketch of proof.

With the help of W. Stein and C. Fieker, we determined the p-unit group of the fixed field E of Δ_0 as follows.



The field E is generated by the trace x of a 37th root of η_5, and we found a minimal polynomial for this element using CRT. Various Magma routines then took 5 days on a 2GHz processor to produce an optimal representation of the integer ring of E, from which its p-unit group was computable. One of the p-units β had norm p. By computing the various embeddings of x in the $H_{\mathfrak{p}}$, we were able to verify the condition of the proposition for β^D.

The minimal polynomial of x:

$x^{37} - 6483584x^{34} - 118234637824x^{33} - 123335506765824x^{32} - 7894900273815552x^{31}$
$-25584896141781024768x^{30} - 19612786666813992009728x^{29} - 2221784070205669762924544x^{28}$
$- 336280142496662926329034833392x^{27} - 4805711697609190244214712041472x^{26}$
$-22490026154268639920058485115453344x^{25} - 1309975549653920931146883229082556825 6x^{24}$
$- 3171787436319383501703813676940597919744x^{23}$
$+ 476259323830076662111107898811789814530048x^{22}$
$- 13962326088395522599669844639235200269470924 80x^{21}$
$- 3314931347275149397194410180602526566069651374 08x^{20}$
$- 80268638062435074559599184759300711777564488630272x^{19}$
$- 872057565672136492561824204817812097995282872168087552x^{18}$
$+ 17726594188758544901772804830573527832102473694015651 84x^{17}$
$+ 37244222236334875481641252538596552828631758622687299108864x^{16}$
$- 206514047854775014678818951533579834155263499429382569213296 64x^{15}$
$+ 311835441256087157637746419559980783743744453707912412281469665 28x^{14}$
$+ 28547054494846244167953306123868112154158699730117069324411606138 88x^{13}$
$-1855731458356048530821147730152877548185437344079899163926475684446 2080x^{12}$
$+30874050214789106461300932422793509193329300438152687471639992995434987 52x^{11}$
$-8448611691348801851628818131891130395295947814515408167362637264691323207 68x^{10}$
$-1816305918878969636874702964809165551139833636304814687020499515520778093$
$19936x^{9}$
$+484965911395764970871665544609840479278207020589886844109688505883361 2420$
$49937408x^{8}$
$+551140497767821995826223345409574614836244339572632071230733265160742 9387$
$6028866560x^{7}$
$-106850589825632789894896612887329721094911179135082410100519870323032 4211$
$96184294522880x^{6}$

$+1225313897986066030513017243242734500242304084259199969989081481911199828$
$17601008959488x^5$
$+1982469259694895457314935195126430029297012127795195501396552566165464092$
$269185291272585216x^4$
$-6606779528917029897795831254093541405143192723743455088686470004371281822$
$96246506433818918912x^3$
$-2649881947577745980599970091092292918947828671888834337650679348784388269$
$01051317961493560950784x^2$
$+6728762060805419616052269030627338051927895313522501375177099188651104750$
$53588494379165834704584704x$
$-26229302920145682793735730674797865320906253597999312333151589937525338452$
$7718903616471614294706880512$

A better polynomial for E:

$y^{37}+4y^{36}+12y^{35}+36y^{34}-336y^{33}-268y^{32}-3912y^{31}-7555y^{30}+60363y^{29}-254771y^{28}+1584299y^{27}-4912687y^{26}+17776688y^{25}-51189497y^{24}+135760742y^{23}-339845565y^{22}+729194231y^{21}-1823351247y^{20}+2954679204y^{19}-7136330744y^{18}+14870105096y^{17}-19798475744y^{16}+63485328194y^{15}-69489469832y^{14}+240906930339y^{13}-130150428853y^{12}+883058481925y^{11}-525666202335y^{10}+1336924708802y^{9}-2790390347185y^{8}+2312809893723y^{7}-3005373888911y^{6}+6491297663291y^{5}-2826510585529y^{4}+4902736951337y^{3}-6453741855514y^{2}+3673618997547y-1546779831802$

The embedding of β^D: $-445+13\cdot 37t-3t^{31}-9t^{32}+18t^{33}+14t^{34}+2t^{35}+O(t^{38}),\ t=\zeta-1$

A few consequences for $p = 37$:

The group $K_2^M(\mathcal{O}_{F,S}) \otimes \mathbf{Z}_{37}$ has order 37.

The product maps

$$K_{2i-1}(\mathbf{Z}) \otimes K_{63+72k-2i}(\mathbf{Z}) \to K_{62+72k}(\mathbf{Z}) \otimes \mathbf{Z}_{37}$$

are nontrivial for a given odd i and any k if $i \not\equiv 5, 27 \bmod 36$ and nonsurjective otherwise.
(Under the Quillen-Lichtenbaum conjecture, we have surjectivity and triviality in the two respective cases.)

Let M/F be a cyclic extension of degree 37 that is unramified outside 37. Then $|\mathrm{Cl}_{M,S} \otimes \mathbf{Z}_{37}| = 37$ if and only if

$$M \not\subset \mathbf{Q}(\zeta_{37^2}, \eta_5^{1/37}, \eta_{27}^{1/37}).$$

Furthermore, only $M = F(\eta_5^{1/37})$ has trivial 37-class number.

Another approach to surjectivity: $\chi_j \colon G_F \to \mathbf{Z}/p\mathbf{Z}$ Kummer character for a pth root of η_j.
Fix i odd and let L be the field defined by χ_i.
Let H be defined by χ_{p-k}.
Then there exists a representation of $G_{\mathbf{Q},S}$ in $\mathrm{Gl}_3(\mathbf{F}_p)$ such that for $\sigma \in G_{F,S}$, we have

$$\rho(\sigma) = \begin{pmatrix} 1 & \chi_i(\sigma) & \kappa(\sigma) \\ 0 & 1 & \chi_{p-k}(\sigma) \\ 0 & 0 & 1 \end{pmatrix}$$

Let M denote the fixed field of the kernel of ρ.

Proposition. *Assume that* $(\mathrm{Cl}_F \otimes \mathbf{Z}_p)^{(1-k)}$ *is cyclic and that* $p \nmid B_{p-i}$. *Then* $(\eta_i, \eta_{k-i})_S = 0$ *if and only if* M/HL *is completely split at all (any) primes above* p.

Hecke algebras:

Let (p, k) be an irregular pair.

$\mathbf{T}=$ the ordinary cuspidal Hecke algebra (over $\mathbf{Z}_p$) of weight 2, level p, and character ω^{k-2}.
(By ordinary, we mean the maximal subfactor in which U_p is unit.)

$\mathbf{T}$ is generated by Hecke operators T_l with $l \neq p$ prime and U_p
Eisenstein ideal:

$$\mathcal{I}=(U_p-1)+(T_l-1-l\omega(l)^{k-2} \mid l \neq p).$$

Idea: study $Y=$ eigenspace of $H^1_{\text{ét}}(X_1(p), \mathbf{Z}_p)^{\text{ord}}$ on which $\langle i\rangle$ acts as i^{k-2} for $i \in \mu_{p-1}(\mathbf{Z}_p)$ "modulo $\mathcal{I}^2$".

The modular representation (following Ohta):
The action of $G_{\mathbf{Q}}$ on Y provides

$$\rho\colon G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathbf{T}}(Y).$$

Fix a decomposition group D_p in $G_{\mathbf{Q}}$ at p.
Let I_p be the inertia subgroup of D_p.
Fix $\Delta_p \le I_p$ with $\omega\colon \Delta_p \xrightarrow{\sim} (\mathbf{Z}/p\mathbf{Z})^{\times}$.
Under Δ_p, we have $Y = Y^{+} \oplus Y^{-}$.

For $\sigma \in G_{\mathbf{Q}}$, we write

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix},$$

$a(\sigma) \in \mathrm{End}_{\mathbf{T}}(Y^{-})$, $b(\sigma) \in \mathrm{Hom}_{\mathbf{T}}(Y^{+}, Y^{-})$, ...

For $\sigma \in I_p$, we have

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & 0 \\ c(\sigma) & 1 \end{pmatrix},$$

Note: $\mathrm{End}_{\mathbf{T}}(Y^{\pm}) \hookrightarrow \mathbf{T} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

For $\sigma, \tau \in G_{\mathbf{Q}}$, we have

$$\begin{aligned} d(\sigma) &\equiv 1 \bmod \mathcal{I}, \\ b(\sigma)c(\tau) &\equiv 0 \bmod \mathcal{I}. \end{aligned}$$

In fact, let B (resp., C) denote the $\mathbf{T}$-module generated by all $b(\sigma)$ (resp., $c(\sigma)$) with $\sigma \in G_{\mathbf{Q}}$.

One has $BC = \mathcal{I}$ (Ohta, following Kurihara, Harder-Pink).

Let $\mathfrak{m} = \mathcal{I} + p\mathbf{T}$.

Consider the maps

$$\begin{aligned} \bar{a}\colon G_{\mathbf{Q}} \to (\mathbf{T}/\mathfrak{m})^{\times} \quad &\text{and} \quad \bar{b}\colon G_{\mathbf{Q}} \to B/\mathfrak{m}B \\ \bar{c}\colon G_{\mathbf{Q}} \to C/\mathfrak{m}C \quad &\text{and} \quad \bar{d}\colon G_{\mathbf{Q}} \to (\mathbf{T}/\mathcal{I}\mathfrak{m})^{\times} \end{aligned}$$

We remark that $\bar{a} = \omega^{1-k}$.

$H =$ fixed field of the kernel of $\bar{b}$ on G_F.
H is an unramified abelian p-extension of F with an ω^{1-k}-action of Galois.

$L =$ fixed field of the kernel of $\bar{c}$ on G_F.
If $p \nmid B_{p+1-k}$, then $L = F(\eta_{k-1})$.

Consider the homomorphism (note: $BC = \mathcal{I}$)

$$\phi_D(\sigma) = \begin{pmatrix} \bar{a}(\sigma) & \bar{b}(\sigma) \\ \bar{c}(\sigma) & \bar{d}(\sigma) \end{pmatrix}.$$

We also have a homomorphism

$$\psi(\sigma) = \begin{pmatrix} 1 & \bar{c}(\sigma) & \bar{d}(\sigma) - 1 \\ 0 & \omega^{1-k}(\sigma) & \bar{b}(\sigma) \\ 0 & 0 & 1 \end{pmatrix}$$

defining the same extension M/HL as ϕ_D.

$\bar{d}(\varphi_p^{-1}) = U_p$ for a Frobenius $\varphi_p \in D_p$

M/HL is non-split at primes above p if and only if

$$\mathcal{I} = \mathcal{I}\mathfrak{m} + (U_p - 1)$$

By the earlier proposition, we have:

Theorem (S.). *Assume that* $p \nmid B_{p+1-k}$ *and* $(\mathrm{Cl}_F \otimes \mathbf{Z}_p)^{(1-k)}$ *is cyclic. Then* $U_p - 1$ *generates the group* $\mathcal{I}/\mathcal{I}^2$ *if and only if* $(p, \eta_{k-1})_S \neq 0$.

Theorem (S.). *For any irregular pair* (p, k) *with* $p < 1000$, *we have* $\mathcal{I} = (U_p - 1) + \mathcal{I}^2$.

Corollary. *For* $p < 1000$, *the map*

$$K_2^M(\mathcal{O}_{F,S}) \otimes \mathbf{Z}_p \to K_2(\mathcal{O}_{F,S}) \otimes \mathbf{Z}_p$$

is surjective.

Sketch of proof.
Let $\mathcal{H}$ denote the full cuspidal Hecke algebra for $\Gamma_1(p)$ and ω^{k-2}. We used built-in Magma routines for modular symbols (W. Stein) to compute the U_p and T_i in $\mathcal{H}$ for all i with $1 \leq i < p/6$, viewed in $M_d(\mathbf{Z}_p)$ for some d. These T_i generate $\mathcal{H}$ as a $\mathbf{Z}$-module. Let $\bar{T}_i$ and $\bar{U}_p$ be the images in $M_d(\mathbf{Z}/p^2\mathbf{Z})$. Let M be the submodule of $M_d(\mathbf{Z}/p^2\mathbf{Z})$ spanned by the $\bar{T}_i$.

We computed N minimal such that the $\bar{T}_i$ with $1 \leq i \leq N$ generate M. Then p and the $T_i - \sum_{0<e|i} \omega(e)^{k-2} e$ with $1 \leq i \leq N$ generate $\mathcal{I}$ over $\mathbf{Z}_p$, provided that

$$\mathcal{H}/\mathcal{I} \otimes_{\mathbf{Z}} \mathbf{Z}_p \cong \mathbf{T}/\mathcal{I} \cong \mathbf{Z}_p/L_p(-1, \omega^k)$$

is p-torsion, which we checked. We computed the images I and J of $\mathcal{I}$ and $\mathcal{I}^2$ in M using these elements and their products. We verified that $I = J + (\bar{U}_p - 1)$.