Compactifications of S-arithmetic quotients for the projective general linear group

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Dedicated to Professor John Coates on the occasion of his 70th birthday

Abstract

Let F be a global field, let S be a nonempty finite set of places of F which contains the archimedean places of F, let $d \geq 1$, and let $X = \prod_{v \in S} X_v$ where X_v is the symmetric space (resp., Bruhat-Tits building) associated to $\mathrm{PGL}_d(F_v)$ if v is archimedean (resp., non-archimedean). In this paper, we construct compactifications $\Gamma \backslash \bar{X}$ of the quotient spaces $\Gamma \backslash X$ for S-arithmetic subgroups Γ of $\mathrm{PGL}_d(F)$. The constructions make delicate use of the maximal Satake compactification of X_v (resp., the polyhedral compactification of X_v of Gérardin and Landvogt) for v archimedean (resp., non-archimedean). We also consider a variant of $\bar{X}$ in which we use the standard Satake compactification of X_v (resp., the compactification of X_v due to Werner).

1 Introduction

1.1. Let $d \geq 1$, and let $X = \mathrm{PGL}_d(\mathbb{R})/\mathrm{PO}_d(\mathbb{R}) \cong \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$. The Borel-Serre space (resp., reductive Borel-Serre space) $\bar{X}$ contains X as a dense open subspace [3] (resp., [26]). If Γ is a subgroup of $\mathrm{PGL}_d(\mathbb{Z})$ of finite index, this gives rise to a compactification $\Gamma \backslash \bar{X}$ of $\Gamma \backslash X$.

1.2. Let F be a global field, which is to say either a number field or a function field in one variable over a finite field. For a place v of F, let F_v be the local field of F at v. Fix $d \geq 1$.

In this paper, we will consider the space X_v of all homothety classes of norms on F_v^d and a certain space $\bar{X}_{F,v}$ which contains X_v as a dense open subset. For $F = \mathbb{Q}$ and v the real place, X_v is identified with $\mathrm{PGL}_d(\mathbb{R})/\mathrm{PO}_d(\mathbb{R})$, and $\bar{X}_{F,v}$ is identified with the reductive Borel-Serre space associated to $\mathrm{PGL}_d(F_v)$. We have the following analogue of 1.1.

Theorem 1.3. *Let F be a function field in one variable over a finite field, let v be a place of F, and let O be the subring of F consisting of all elements which are integral outside v. Then for any subgroup Γ of $\mathrm{PGL}_d(O)$ of finite index, the quotient $\Gamma\backslash\bar{X}_{F,v}$ is a compact Hausdorff space which contains $\Gamma\backslash X_v$ as a dense open subset.*

1.4. Our space $\bar{X}_{F,v}$ is not a very new object. In the case that v is non-archimedean, X_v is identified as a topological space with the Bruhat-Tits building of $\mathrm{PGL}_d(F_v)$. In this case, $\bar{X}_{F,v}$ is similar to the polyhedral compactification of X_v of Gérardin [7] and Landvogt [19], which we denote by $\bar{X}_v$. To each element of $\bar{X}_v$ is associated a parabolic subgroup of PGL_{d,F_v}. We define $\bar{X}_{F,v}$ as the subset of $\bar{X}_v$ consisting of all elements for which the associated parabolic subgroup is F-rational. We endow $\bar{X}_{F,v}$ with a topology which is different from its topology as a subspace of $\bar{X}_v$.

In the case $d=2$, the boundary $\bar{X}_v \setminus X_v$ of $\bar{X}_v$ is $\mathbb{P}^1(F_v)$, whereas the boundary $\bar{X}_{F,v} \setminus X_v$ of $\bar{X}_{F,v}$ is $\mathbb{P}^1(F)$. Unlike $\bar{X}_v$, the space $\bar{X}_{F,v}$ is not compact, but the arithmetic quotient as in 1.1 and 1.3 is compact (see 1.6).

1.5. In §4, we derive the following generalization of 1.1 and 1.3.

Let F be a global field. For a nonempty finite set S of places of F, let $\bar{X}_{F,S}$ be the subspace of $\prod_{v\in S}\bar{X}_{F,v}$ consisting of all elements $(x_v)_{v\in S}$ such that the F-parabolic subgroup associated to x_v is independent of v. Let X_S denote the subspace $\prod_{v\in S} X_v$ of $\bar{X}_{F,S}$.

Let S_1 be a nonempty finite set of places of F containing all archimedean places of F, let S_2 be a finite set of places of F which is disjoint from S_1, and let $S = S_1 \cup S_2$. Let O_S be the subring of F consisting of all elements which are integral outside S.

Our main result is the following theorem (see Theorem 4.1.4).

Theorem 1.6. *Let Γ be a subgroup of $\mathrm{PGL}_d(O_S)$ of finite index. Then the quotient $\Gamma\backslash(\bar{X}_{F,S_1}\times X_{S_2})$ is a compact Hausdorff space which contains $\Gamma\backslash X_S$ as a dense open subset.*

1.7. If F is a number field and S_1 coincides with the set of archimedean places of F, then the space $\bar{X}_{F,S_1}$ is the maximal Satake space of the Weil restriction of $\mathrm{PGL}_{d,F}$ from F to $\mathbb{Q}$. In this case, the theorem is known for $S=S_1$ through the work of Satake [23] and in general through the work of Ji, Murty, Saper, and Scherk [14, 4.4].

1.8. We also consider a variant $\bar{X}^\flat_{F,v}$ of $\bar{X}_{F,v}$ and a variant $\bar{X}^\flat_{F,S}$ of $\bar{X}_{F,S}$ with continuous surjections

$$\bar{X}_{F,v} \to \bar{X}^\flat_{F,v}, \quad \bar{X}_{F,S} \to \bar{X}^\flat_{F,S}.$$

In the case v is non-archimedean (resp., archimedean), $\bar{X}^\flat_{F,v}$ is the part with "F-rational boundary" in Werner's compactification (resp., the standard Satake compactification) $\bar{X}^\flat_v$ of

X_v [24, 25] (resp., [22]), endowed with a new topology. We will obtain an analogue of 1.6 for this variant.

To grasp the relationship with the Borel-Serre compactification [3], we also consider a variant $\bar{X}^{\sharp}_{F,v}$ of $\bar{X}_{F,v}$ which has a continuous surjection $\bar{X}^{\sharp}_{F,v} \to \bar{X}_{F,v}$, and we show that in the case that $F = \mathbb{Q}$ and v is the real place, $\bar{X}^{\sharp}_{\mathbb{Q},v}$ coincides with the Borel-Serre space associated to $\mathrm{PGL}_{d,\mathbb{Q}}$ (3.7.4). If v is non-archimedean, the space $\bar{X}^{\sharp}_{F,v}$ is not Hausdorff (3.7.6) and does not seem useful.

1.9. What we do in this paper is closely related to what Satake did in [22] and [23]. In [22], he defined a compactification of a symmetric Riemannian space. In [23], he took the part of this compactification with "rational boundary" and endowed it with the Satake topology. Then he showed that the quotient of this part by an arithmetic group is compact. We take the part $\bar{X}_{F,v}$ of $\bar{X}_v$ with "F-rational boundary" to have a compact quotient by an arithmetic group. So, the main results and their proofs in this paper might be evident to the experts in the theory of Bruhat-Tits buildings, but we have not found them in the literature.

1.10. We intend to apply the compactification 1.3 to the construction of toroidal compactifications of the moduli space of Drinfeld modules of rank d in a forthcoming paper. In Section 4.7, we give a short explanation of this plan, along with two other potential applications, to asymptotic behavior of heights of motives and to modular symbols over function fields.

1.11. We plan to generalize the results of this paper from PGL_d to general reductive groups in another forthcoming paper. The reason why we separate the PGL_d-case from the general case is as follows. For PGL_d, we can describe the space $\bar{X}_{F,v}$ via norms on finite-dimensional vector spaces over F_v (this method is not used for general reductive groups), and these norms play an important role in the analytic theory of toroidal compactifications.

1.12. In §2, we review the compactifications of Bruhat-Tits buildings in the non-archimedean setting and symmetric spaces in the archimedean setting. In §3 and §4, we discuss our compactifications.

1.13. We plan to apply the results of this paper to the study of Iwasawa theory over a function field F. We dedicate this paper to John Coates, who has played a leading role in the development of Iwasawa theory.

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2 Spaces associated to local fields

In this section, we briefly review the compactification of the symmetric space (resp., of the Bruhat-Tits building) associated to PGL_d of an archimedean (resp., non-archimedean) local field. See the papers of Satake [22] and Borel-Serre [3] (resp., Gérardin [7], Landvogt [19], and Werner [24, 25]) for details.

Let E be a local field. This means that E is a locally compact topological field with a non-discrete topology. That is, E is isomorphic to $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$ for some prime number p, or $\mathbb{F}_q(\!(T)\!)$ for a finite field $\mathbb{F}_q$.

Let $|\ |\colon E \to \mathbb{R}_{\geq 0}$ be the normalized absolute value. If $E \cong \mathbb{R}$, this is the usual absolute value. If $E \cong \mathbb{C}$, this is the square of the usual absolute value. If E is non-archimedean, this is the unique multiplicative map $E \to \mathbb{R}_{\geq 0}$ such that $|a| = \sharp(O_E/aO_E)^{-1}$ if a is a nonzero element of the valuation ring O_E of E.

Fix a positive integer d and a d-dimensional E-vector space V.

2.1 Norms

2.1.1. We recall the definitions of norms and semi-norms on V.

A norm (resp., semi-norm) on V is a map $\mu\colon V \to \mathbb{R}_{\geq 0}$ for which there exist an E-basis $(e_i)_{1\leq i\leq d}$ of V and an element $(r_i)_{1\leq i\leq d}$ of $\mathbb{R}^d_{>0}$ (resp., $\mathbb{R}^d_{\geq 0}$) such that

$$\mu(a_1e_1+\cdots+a_de_d) = \begin{cases} (r_1^2|a_1|^2\cdots+r_d^2|a_d|^2)^{1/2} & \text{if } E\cong\mathbb{R},\\ r_1|a_1|+\cdots+r_d|a_d| & \text{if } E\cong\mathbb{C},\\ \max(r_1|a_1|,\ldots,r_d|a_d|) & \text{otherwise.}\end{cases}$$

for all $a_1,\ldots,a_d \in E$.

2.1.2. We will call the norm (resp., semi-norm) μ in the above, the norm (resp., semi-norm) given by the basis $(e_i)_i$ and by $(r_i)_i$.

2.1.3. We have the following characterizations of norms and semi-norms.

(1) If $E \cong \mathbb{R}$ (resp., $E \cong \mathbb{C}$), then there is a one-to-one correspondence between semi-norms on V and symmetric bilinear (resp., Hermitian) forms $(\ ,\ )$ on V such that $(x,x)\geq 0$ for all $x \in V$. The semi-norm μ corresponding to $(\ ,\ )$ is given by $\mu(x)=(x,x)^{1/2}$ (resp., $\mu(x)=(x,x)$). This restricts to a correspondence between norms and forms that are positive definite.

(2) If E is non-archimedean, then (as in [9]) a map $\mu\colon V \to \mathbb{R}_{\geq 0}$ is a norm (resp., semi-norm) if and only if μ satisfies the following (i)–(iii) (resp., (i) and (ii)):

(i) $\mu(ax) = |a|\mu(x)$ for all $a \in E$ and $x \in V$,

(ii) $\mu(x+y) \leq \max(\mu(x), \mu(y))$ for all $x, y \in V$, and

(iii) $\mu(x) > 0$ if $x \in V \setminus \{0\}$.

These well-known facts imply that if μ is a norm (resp., semi-norm) on V and V' is an E-subspace of V, then the restriction of μ to V' is a norm (resp., semi-norm) on V'.

2.1.4. We say that two norms (resp., semi-norms) μ and μ' on V are equivalent if $\mu' = c\mu$ for some $c \in \mathbb{R}_{>0}$.

2.1.5. The group $\mathrm{GL}_V(E)$ acts on the set of all norms (resp., semi-norms) on V: for $g \in \mathrm{GL}_V(E)$ and a norm (resp., semi-norm) μ on V, $g\mu$ is defined as $\mu \circ g^{-1}$. This action preserves the equivalence in 2.1.4.

2.1.6. Let V^* be the dual space of V. Then there is a bijection between the set of norms on V and the set of norms on V^*. That is, for a norm μ on V, the corresponding norm μ^* on V^* is given by

$$\mu^*(\varphi) = \sup\left(\frac{|\varphi(x)|}{\mu(x)} \mid x \in V \setminus \{0\}\right) \qquad \text{for } \varphi \in V^*.$$

For a norm μ on V associated to a basis $(e_i)_i$ of V and $(r_i)_i \in \mathbb{R}^d_{>0}$, the norm μ^* on V^* is associated to the dual basis $(e_i^*)_i$ of V^* and $(r_i^{-1})_i$. This proves the bijectivity.

2.1.7. For a norm μ on V and for $g \in \mathrm{GL}_V(E)$, we have

$$(\mu \circ g)^* = \mu^* \circ (g^*)^{-1},$$

which is to say $(g\mu)^* = (g^*)^{-1}\mu^*$, where $g^* \in \mathrm{GL}_{V^*}(E)$ is the transpose of g.

2.2 Definitions of the spaces

2.2.1. Let X_V denote the set of all equivalence classes of norms on V (as in 2.1.4). We endow X_V with the quotient topology of the subspace topology on the set of all norms on V inside $\mathbb{R}^V$.

2.2.2. In the case that E is archimedean, we have

$$X_V \cong \begin{cases} \mathrm{PGL}_d(\mathbb{R})/\mathrm{PO}_d(\mathbb{R}) \cong \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R}) & \text{if } E \cong \mathbb{R} \\ \mathrm{PGL}_d(\mathbb{C})/\mathrm{PU}(d) \cong \mathrm{SL}_d(\mathbb{C})/\mathrm{SU}(d) & \text{if } E \cong \mathbb{C}. \end{cases}$$

In the case E is non-archimedean, X_V is identified with (a geometric realization of) the Bruhat-Tits building associated to PGL_V [4] (see also [5, Section 2]).

2.2.3. Recall that for a finite-dimensional vector space $H \neq 0$ over a field I, the following four objects are in one-to-one correspondence:

(i) a parabolic subgroup of the algebraic group GL_H over I,

(ii) a parabolic subgroup of the algebraic group PGL_H over I,

(iii) a parabolic subgroup of the algebraic group SL_H over I, and

(iv) a flag of I-subspaces of H (i.e., a set of subspaces containing $\{0\}$ and H and totally ordered under inclusion).

The bijections (ii) $\mapsto$ (i) and (i) $\mapsto$ (iii) are the taking of inverse images. The bijection (i) $\mapsto$ (iv) sends a parabolic subgroup P to the set of all P-stable I-subspaces of H, and the converse map takes a flag to its isotropy subgroup in GL_H.

2.2.4. Let $\bar{X}_V$ be the set of all pairs (P,μ), where P is a parabolic subgroup of the algebraic group PGL_V over E and, if

$$0 = V_{-1} \subsetneq V_0 \subsetneq \cdots \subsetneq V_m = V$$

denotes the flag corresponding to P (2.2.3), then μ is a family $(\mu_i)_{0\leq i\leq m}$, where μ_i is an equivalence class of norms on V_i/V_{i-1}.

We have an embedding $X_V \hookrightarrow \bar{X}_V$ which sends μ to (PGL_V, μ).

2.2.5. Let $\bar{X}_V^\flat$ be the set of all equivalence classes of nonzero semi-norms on the dual space V^* of V (2.1.4). We have an embedding $X_V \hookrightarrow \bar{X}_V^\flat$ which sends μ to μ^* (2.1.6).

This set $\bar{X}_V^\flat$ is also identified with the set of pairs (W,μ) with W a nonzero E-subspace of V and μ an equivalence class of a norm on W. In fact, μ corresponds to an equivalence class μ^* of a norm on the dual space W^* of W (2.1.6), and μ^* is identified via the projection $V^* \to W^*$ with an equivalence class of semi-norms on V^*.

We call the understanding of $\bar{X}_V^\flat$ as the set of such pairs (W,μ) the definition of $\bar{X}_V^\flat$ in the second style. In this interpretation of $\bar{X}_V^\flat$, the above embedding $X_V \to \bar{X}_V^\flat$ is written as $\mu \mapsto (V,\mu)$.

2.2.6. In the case that E is non-archimedean, $\bar{X}_V$ is the polyhedral compactification of the Bruhat-Tits building X_V by Gérardin [7] and Landvogt [19] (see also [11, Proposition 19]), and $\bar{X}_V^\flat$ is the compactification of X_V by Werner [24, 25]. In the case that E is archimedean, $\bar{X}_V$ is the maximal Satake compactification, and $\bar{X}_V^\flat$ is the minimal Satake compactification for the standard projective representation of $\mathrm{PGL}_V(E)$, as constructed by Satake in [22] (see also [2, 1.4]). The topologies of $\bar{X}_V$ and $\bar{X}_V^\flat$ are reviewed in Section 2.3 below.

2.2.7. We have a canonical surjection $\bar{X}_V \to \bar{X}_V^\flat$ which sends (P,μ) to (V_0,μ_0), where V_0 is as in 2.2.4, and where we use the definition of $\bar{X}_V^\flat$ of the second style in 2.2.5. This surjection is compatible with the inclusion maps from X_V to these spaces.

2.2.8. We have the natural actions of $\mathrm{PGL}_V(E)$ on X_V, $\bar{X}_V$ and $\bar{X}_V^\flat$ by 2.1.5. These actions are compatible with the canonical maps between these spaces.

2.3 Topologies

2.3.1. We define a topology on $\bar{X}_V$.

Take a basis $(e_i)_i$ of V. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{PGL}_V(E)\times\mathbb{R}_{>0}^{d-1} & \longrightarrow & X_V \\ \downarrow & & \downarrow \\ \mathrm{PGL}_V(E)\times\mathbb{R}_{\geq 0}^{d-1} & \longrightarrow & \bar{X}_V. \end{array}$$

Here the upper arrow is $(g,t)\mapsto g\mu$, where μ is the class of the norm on V associated to $((e_i)_i,(r_i)_i)$ with $r_i=\prod_{1\leq j<i} t_j^{-1}$, and where $g\mu$ is defined by the action of $\mathrm{PGL}_V(E)$ on X_V (2.2.8). The lower arrow is $(g,t)\mapsto g(P,\mu)$, where $(P,\mu)\in\bar{X}_V$ is defined as follows, and $g(P,\mu)$ is then defined by the action of $\mathrm{PGL}_V(E)$ on $\bar{X}_V$ (2.2.8). Let

$$I=\{j \mid t_j=0\}\subset\{1,\dots,d-1\},$$

and write

$$I=\{c(i)\mid 0\leq i\leq m-1\},$$

where $m=\sharp I$ and $1\leq c(0)<\cdots<c(m-1)\leq d-1$. If we also let $c(-1)=0$ and $c(m)=d$, then the set of

$$V_i=\sum_{j=1}^{c(i)} Fe_j$$

with $-1\leq i\leq m$ forms a flag in V, and P is defined to be the corresponding parabolic subgroup of PGL_V (2.2.3). For $0\leq i\leq m$, we take μ_i to be the equivalence class of the norm on V_i/V_{i-1} given by the basis $(e_j)_{c(i-1)<j\leq c(i)}$ and the sequence $(r_j)_{c(i-1)<j\leq c(i)}$ with $r_j=\prod_{c(i-1)<k<j} t_k^{-1}$.

Both the upper and the lower horizontal arrows in the diagram are surjective, and the topology on X_V coincides with the topology as a quotient space of $\mathrm{PGL}_V(E)\times\mathbb{R}_{>0}^{d-1}$ via the upper horizontal arrow. The topology on $\bar{X}_V$ is defined as the quotient topology of the topology on $\mathrm{PGL}_V(E) \times \mathbb{R}_{\geq 0}^{d-1}$ via the lower horizontal arrow. It is easily seen that this topology is independent of the choice of the basis $(e_i)_i$.

2.3.2. The space $\bar{X}_V^{\flat}$ has the following topology: the space of all nonzero semi-norms on V^* has a topology as a subspace of the product $\mathbb{R}^{V^*}$, and $\bar{X}_V^{\flat}$ has a topology as a quotient of it.

2.3.3. Both $\bar{X}_V$ and $\bar{X}_V^{\flat}$ are compact Hausdorff spaces containing X_V as a dense open subset. This is proved in [7, 19, 24, 25] in the case that E is non-archimedean and in [22, 2] in the archimedean case.

2.3.4. The topology on $\bar{X}_V^{\flat}$ coincides with the image of the topology on $\bar{X}_V$. In fact, it is easily seen that the canonical map $\bar{X}_V \to \bar{X}_V^{\flat}$ is continuous (using, for instance, [25, Theorem 5.1]). Since both spaces are compact Hausdorff and this continuous map is surjective, the topology on $\bar{X}_V^{\flat}$ is the image of that of $\bar{X}_V$.

3 Spaces associated to global fields

Let F be a global field, which is to say, either a number field or a function field in one variable over a finite field. We fix a finite-dimensional F-vector space V of dimension $d \geq 1$. For a place v of F, let $V_v = F_v \otimes_F V$. We set $X_v = X_{V_v}$ and $X_v^{\flat} = X_{V_v}^{\flat}$ for brevity. If v is non-archimedean, we let O_v, k_v, q_v, ϖ_v denote the valuation ring of F_v, the residue field of O_v, the order of k_v, and a fixed uniformizer in O_v, respectively.

In this section, we define sets $\bar{X}_{F,v}^{\star}$ containing X_v for $\star \in \{\sharp, \ , \flat\}$, which serve as our rational partial compactifications. Here, $\bar{X}_{F,v}$ (resp., $\bar{X}_{F,v}^{\flat}$) is defined as a subset of X_v (resp., $\bar{X}_v^{\flat}$), and $\bar{X}_{F,v}^{\sharp}$ has $\bar{X}_{F,v}$ as a quotient. In §3.2, by way of example, we describe these sets and various topologies on them in the case that $d = 2$, $F = \mathbb{Q}$, and v is the real place. For $\star \neq \sharp$, we construct more generally sets $\bar{X}_{F,S}^{\star}$ for a nonempty finite set S of places of F. In §3.1, we describe $\bar{X}_{F,S}^{\star}$ as a subset of $\prod_{v \in S} \bar{X}_{F,v}^{\star}$.

In §3.3 and §3.4, we define topologies on these sets. That is, in §3.3, we define the "Borel-Serre topology", while in §3.4, we define the "Satake topology" on $\bar{X}_{F,v}$ and, assuming S contains all archimedean places of F, on $\bar{X}_{F,S}^{\flat}$. In §3.5, we prove results on $\bar{X}_{F,v}$. In §3.6, we compare the following topologies on $\bar{X}_{F,v}$ (resp., $\bar{X}_{F,v}^{\flat}$): the Borel-Serre topology, the Satake topology, and the topology as a subspace of $\bar{X}_v$ (resp., $\bar{X}_v^{\flat}$). In §3.7, we describe the relationship between these spaces and Borel-Serre and reductive Borel-Serre spaces.

3.1 Definitions of the spaces

3.1.1. Let $\bar{X}_{F,v} = \bar{X}_{V,F,v}$ be the subset of $\bar{X}_v$ consisting of all elements (P,μ) such that P is F-rational. If P comes from a parabolic subgroup P' of PGL_V over F, we also denote (P,μ) by (P',μ).

3.1.2. Let $\bar{X}^{\flat}_{F,v}$ be the subset of $\bar{X}^{\flat}_v$ consisting of all elements (W,μ) such that W is F-rational (using the definition of $\bar{X}^{\flat}_v$ in the second style in 2.2.5). If W comes from an F-subspace W' of V, we also denote (W,μ) by (W',μ).

3.1.3. Let $\bar{X}^{\sharp}_{F,v}$ be the set of all triples (P,μ,s) such that $(P,\mu) \in \bar{X}_{F,v}$ and s is a splitting

$$s\colon \bigoplus_{i=0}^{m} (V_i/V_{i-1})_v \xrightarrow{\sim} V_v$$

over F_v of the filtration $(V_i)_{-1\le i\le m}$ of V corresponding to P.

We have an embedding $X_v \hookrightarrow \bar{X}^{\sharp}_{F,v}$ that sends μ to (PGL_V,μ,s), where s is the identity map of V_v.

3.1.4. We have a diagram with a commutative square

$$\begin{array}{ccccc} \bar{X}^{\sharp}_{F,v} & \twoheadrightarrow & \bar{X}_{F,v} & \twoheadrightarrow & \bar{X}^{\flat}_{F,v} \\ & & \downarrow & & \downarrow \\ & & \bar{X}_v & \twoheadrightarrow & \bar{X}^{\flat}_v. \end{array}$$

Here, the first arrow in the upper row forgets the splitting s, and the second arrow in the upper row is $(P,\mu) \mapsto (V_0,\mu_0)$, as is the lower arrow (2.2.7).

3.1.5. The group $\mathrm{PGL}_V(F)$ acts on the sets $\bar{X}_{F,v}$, $\bar{X}^{\flat}_{F,v}$ and $\bar{X}^{\sharp}_{F,v}$ in the canonical manner.

3.1.6. Now let S be a nonempty finite set of places of F.

- Let $\bar{X}_{F,S}$ be the subset of $\prod_{v\in S} \bar{X}_{F,v}$ consisting of all elements $(x_v)_{v\in S}$ such that the parabolic subgroup of $G = \mathrm{PGL}_V$ associated to x_v is independent of v.
- Let $\bar{X}^{\flat}_{F,S}$ be the subset of $\prod_{v\in S} \bar{X}^{\flat}_{F,v}$ consisting of all elements $(x_v)_{v\in S}$ such that the F-subspace of V associated to x_v is independent of v.

We will denote an element of $\bar{X}_{F,S}$ as (P,μ), where P is a parabolic subgroup of G and $\mu \in \prod_{v\in S, 0\le i\le m} X_{(V_i/V_{i-1})_v}$ with $(V_i)_i$ the flag corresponding to P. We will denote an element of $\bar{X}^{\flat}_{F,S}$ as (W,μ), where W is a nonzero F-subspace of V and $\mu \in \prod_{v\in S} X_{W_v}$. We have a canonical surjective map

$$\bar{X}_{F,S} \to \bar{X}^{\flat}_{F,S}$$

which commutes with the inclusion maps from X_S to these spaces.

3.2 Example: Upper half-plane

3.2.1. Suppose that $F = \mathbb{Q}$, v is the real place, and $d = 2$.

In this case, the sets X_v, $\bar{X}_v = \bar{X}_v^\flat$, $\bar{X}_{\mathbb{Q},v} = \bar{X}_{\mathbb{Q},v}^\flat$, and $\bar{X}_{\mathbb{Q},v}^\sharp$ are described by using the upper half-plane. In §2, we discussed topologies on the first two spaces. The remaining spaces also have natural topologies, as will be discussed in §3.3 and §3.4: the space $\bar{X}_{\mathbb{Q},v}^\sharp$ is endowed with the Borel-Serre topology, and $\bar{X}_{\mathbb{Q},v}$ has two topologies, the Borel-Serre topology and Satake topology, which are both different from its topology as a subspace of $\bar{X}_v$. In this section, as a prelude to §3.3 and §3.4, we describe what the Borel-Serre and Satake topologies look like in this special case.

3.2.2. Let $\mathfrak{H} = \{x + yi \mid x, y \in \mathbb{R}, y > 0\}$ be the upper half-plane. Fix a basis $(e_i)_{i=1,2}$ of V. For $z \in \mathfrak{H}$, let μ_z denote the class of the norm on V corresponding to the class of the norm on V^* given by $ae_1^* + be_2^* \mapsto |az + b|$ for $a, b \in \mathbb{R}$. Here $(e_i^*)_{1 \le i \le d}$ is the dual basis of $(e_i)_i$, and $|\ |$ denotes the usual absolute value (not the normalized absolute value) on $\mathbb{C}$. We have a homeomorphism

$$\mathfrak{H} \xrightarrow{\sim} X_v, \quad z \mapsto \mu_z$$

which is compatible with the actions of $\mathrm{SL}_2(\mathbb{R})$.

For the square root $i \in \mathfrak{H}$ of -1, the norm $ae_1 + be_2 \mapsto (a^2 + b^2)^{1/2}$ has class μ_i. For $z = x + yi$, we have

$$\mu_z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mu_i.$$

The action of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ on X_v corresponds to $x + yi \mapsto -x + yi$ on $\mathfrak{H}$.

3.2.3. The inclusions

$$X_v \subset \bar{X}_{\mathbb{Q},v} \subset \bar{X}_v$$

can be identified with

$$\mathfrak{H} \subset \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q}) \subset \mathfrak{H} \cup \mathbb{P}^1(\mathbb{R}).$$

Here $z \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ corresponds to the class in $\bar{X}_v^\flat = \bar{X}_v$ of the semi-norm $ae_1^* + be_2^* \mapsto |az + b|$ (resp., $ae_1^* + be_2^* \mapsto |a|$) on V^* if $z \in \mathbb{R}$ (resp., $z = \infty$). These identifications are compatible with the actions of $\mathrm{PGL}_V(\mathbb{Q})$.

The topology on $\bar{X}_v$ of 2.3.1 is the topology as a subspace of $\mathbb{P}^1(\mathbb{C})$.

3.2.4. Let B be the Borel subgroup of PGL_V consisting of all upper triangular matrices for the basis $(e_i)_i$, and let $0 = V_{-1} \subsetneq V_0 = \mathbb{Q}e_1 \subsetneq V_1 = V$ be the corresponding flag. Then $\infty \in \mathbb{P}^1(\mathbb{Q})$ is understood as the point (B, μ) of $\bar{X}_{\mathbb{Q},v}$, where μ is the unique element of $X_{(V_0)_v} \times X_{(V/V_0)_v}$.

Let $\bar{X}_{\mathbb{Q},v}(B) = \mathfrak{H} \cup \{\infty\} \subset \bar{X}_{\mathbb{Q},v}$ and let $\bar{X}^{\sharp}_{\mathbb{Q},v}(B)$ be the inverse image of $\bar{X}_{\mathbb{Q},v}(B)$ in $\bar{X}^{\sharp}_{\mathbb{Q},v}$. Then for the Borel-Serre topology defined in §3.3, we have a homeomorphism

$$\bar{X}^{\sharp}_{\mathbb{Q},v}(B) \cong \{x + yi \mid x \in \mathbb{R}, 0 < y \leq \infty\} \supset \mathfrak{H}.$$

Here $x + \infty i$ corresponds to (B, μ, s) where s is the splitting of the filtration $(V_{i,v})_i$ given by the embedding $(V/V_0)_v \to V_v$ that sends the class of e_2 to $xe_1 + e_2$.

The Borel-Serre topology on $\bar{X}^{\sharp}_{\mathbb{Q},v}$ is characterized by the properties that

(i) the action of the discrete group $\mathrm{GL}_V(\mathbb{Q})$ on $\bar{X}^{\sharp}_{\mathbb{Q},v}$ is continuous,

(ii) the subset $\bar{X}^{\sharp}_{\mathbb{Q},v}(B)$ is open, and

(iii) as a subspace, $\bar{X}^{\sharp}_{\mathbb{Q},v}(B)$ is homeomorphic to $\{x + yi \mid x \in \mathbb{R}, 0 < y \leq \infty\}$ as above.

3.2.5. The Borel-Serre and Satake topologies on $\bar{X}_{\mathbb{Q},v}$ (defined in §3.3 and §3.4) are characterized by the following properties:

(i) The subspace topology on $X_v \subset \bar{X}_{\mathbb{Q},v}$ coincides with the topology on $\mathfrak{H}$.

(ii) The action of the discrete group $\mathrm{GL}_V(\mathbb{Q})$ on $\bar{X}_{\mathbb{Q},v}$ is continuous.

(iii) The following sets (a) (resp., (b)) form a base of neighborhoods of ∞ for the Borel-Serre (resp., Satake) topology:

  (a) the sets $U_f = \{x + yi \in \mathfrak{H} \mid y \geq f(x)\} \cup \{\infty\}$ for continuous $f : \mathbb{R} \to \mathbb{R}$,
  (b) the sets $U_c = \{x + yi \in \mathfrak{H} \mid y \geq c\} \cup \{\infty\}$ with $c \in \mathbb{R}_{>0}$.

The Borel-Serre topology on $\bar{X}_{\mathbb{Q},v}$ is the image of the Borel-Serre topology on $\bar{X}^{\sharp}_{\mathbb{Q},v}$.

3.2.6. For example, the set $\{x + yi \in \mathfrak{H} \mid y > x\} \cup \{\infty\}$ is a neighborhood of ∞ for the Borel-Serre topology, but it is not a neighborhood of ∞ for the Satake topology.

3.2.7. For any subgroup Γ of $\mathrm{PGL}_2(\mathbb{Z})$ of finite index, the Borel-Serre and Satake topologies induce the same topology on the quotient space $X(\Gamma) = \Gamma \backslash \bar{X}_{\mathbb{Q},v}$. Under this quotient topology, $X(\Gamma)$ is compact Hausdorff. If Γ is the image of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then this is the usual topology on the modular curve $X(\Gamma)$.

3.3 Borel-Serre topology

3.3.1. For a parabolic subgroup P of PGL_V, let $\bar{X}_{F,v}(P)$ (resp., $\bar{X}^{\sharp}_{F,v}(P)$) be the subset of $\bar{X}_{F,v}$ (resp., $\bar{X}^{\sharp}_{F,v}$) consisting of all elements (Q,μ) (resp., (Q,μ,s)) such that $Q \supset P$.

The action of $\mathrm{PGL}_V(F_v)$ on $\bar{X}_v$ induces an action of $P(F_v)$ on $\bar{X}_{F,v}(P)$. We have also an action of $P(F_v)$ on $\bar{X}^{\sharp}_{F,v}(P)$ given by

$$g(\alpha,s) = (g\alpha, g\circ s\circ g^{-1})$$

for $g \in P(F_v)$, $\alpha \in \bar{X}_{F,v}(P)$, and s a splitting of the filtration.

3.3.2. Fix a basis $(e_i)_i$ of V. Let P be a parabolic subgroup of PGL_V such that

- if $0 = V_{-1} \subsetneq V_0 \subsetneq \cdots \subsetneq V_m = V$ denotes the flag of F-subspaces corresponding to P, then each V_i is generated by the e_j with $1 \le j \le c(i)$, where $c(i) = \dim(V_i)$.

This condition on P is equivalent to the condition that P contains the Borel subgroup B of PGL_V consisting of all upper triangular matrices with respect to $(e_i)_i$. Where useful, we will identify PGL_V over F with PGL_d over F via the basis $(e_i)_i$.

Let

$$\Delta(P) = \{\dim(V_j) \mid 0 \le j \le m-1\} \subset \{1,\dots,d-1\},$$

and let $\Delta'(P)$ be the complement of $\Delta(P)$ in $\{1,\dots,d-1\}$. Let $\mathbb{R}^{d-1}_{\ge 0}(P)$ be the open subset of $\mathbb{R}^{d-1}_{\ge 0}$ given by

$$\mathbb{R}^{d-1}_{\ge 0}(P) = \{(t_i)_{1\le i\le d-1} \in \mathbb{R}^{d-1}_{\ge 0} \mid t_i > 0 \text{ for all } i \in \Delta'(P)\}.$$

In particular, we have

$$\mathbb{R}^{d-1}_{\ge 0}(P) \cong \mathbb{R}^{\Delta'(P)}_{>0} \times \mathbb{R}^{\Delta(P)}_{\ge 0}.$$

3.3.3. With P as in 3.3.2, the map $\mathrm{PGL}_V(F_v) \times \mathbb{R}^{d-1}_{\ge 0} \to \bar{X}_v$ in 2.3.1 induces a map

$$\bar{\pi}_{P,v} \colon P(F_v) \times \mathbb{R}^{d-1}_{\ge 0}(P) \to \bar{X}_{F,v}(P),$$

which restricts to a map

$$\pi_{P,v} \colon P(F_v) \times \mathbb{R}^{d-1}_{>0} \to X_{F,v}.$$

The map $\bar{\pi}_{P,v}$ is induced by a map

$$\bar{\pi}^{\sharp}_{P,v} \colon P(F_v) \times \mathbb{R}^{d-1}_{\ge 0}(P) \to \bar{X}^{\sharp}_{F,v}(P)$$

defined as $(g,t) \mapsto g(P,\mu,s)$ where (P,μ) is as in 2.3.1 and s is the splitting of the filtration $(V_i)_{-1\le i\le m}$ defined by the basis $(e_i)_i$. For this splitting s, we set

$$V^{(i)} = s(V_i/V_{i-1}) = \sum_{c(i-1)<j\le c(i)} Fe_j$$

for $0 \le i \le m$ so that $V_i = V_{i-1} \oplus V^{(i)}$ and $V = \bigoplus_{i=0}^m V^{(i)}$. If $P = B$, then we will often omit the subscript B from our notation for these maps.

3.3.4. We review the Iwasawa decomposition. For v archimedean (resp., non-archimedean), let $A_v \le \mathrm{PGL}_d(F_v)$ be the subgroup of elements of that lift to diagonal matrices in $\mathrm{GL}_d(F_v)$ with positive real entries (resp., with entries that are powers of ϖ_v). Let K_v denote the standard maximal compact subgroup of $\mathrm{PGL}_d(F_v)$ given by

$$K_v = \begin{cases} \mathrm{PO}_d(\mathbb{R}) & \text{if } v \text{ is real,} \\ \mathrm{PU}_d & \text{if } v \text{ is complex,} \\ \mathrm{PGL}_d(O_v) & \text{otherwise.} \end{cases}$$

Let B_u denote the upper-triangular unipotent matrices in the standard Borel B. The Iwasawa decomposition is given by the equality

$$\mathrm{PGL}_d(F_v) = B_u(F_v) A_v K_v.$$

3.3.5. If v is archimedean, then the expression of a matrix in $\mathrm{PGL}_d(F_v)$ as a product in the Iwasawa decomposition is unique.

3.3.6. If v is non-archimedean, then the Bruhat decomposition is $\mathrm{PGL}_d(k_v) = B(k_v) S_d B(k_v)$, where the symmetric group S_d of degree d is viewed as a subgroup of PGL_d over any field via the permutation representation on the standard basis. This implies that $\mathrm{PGL}_d(O_v) = B(O_v) S_d \,\mathrm{Iw}(O_v)$, where $\mathrm{Iw}(O_v)$ is the Iwahori subgroup consisting of those matrices in with upper triangular image in $\mathrm{PGL}_d(k_v)$. Combining this with the Iwasawa decomposition (in the notation of 3.3.4), we have

$$\mathrm{PGL}_d(F_v) = B_u(F_v) A_v S_d \,\mathrm{Iw}(O_v).$$

This decomposition is not unique, since $B_u(F_v) \cap \mathrm{Iw}(O_v) = B_u(O_v)$.

3.3.7. If v is archimedean, then there is a bijection $\mathbb{R}_{>0}^{d-1} \xrightarrow{\sim} A_v$ given by

$$t = (t_k)_{1 \le k \le d-1} \mapsto a = \begin{cases} \mathrm{diag}(r_1, \dots, r_d)^{-1} & \text{if } v \text{ is real,} \\ \mathrm{diag}(r_1^{1/2}, \dots, r_d^{1/2})^{-1} & \text{if } v \text{ is complex,} \end{cases}$$

where $r_i = \prod_{k=1}^{i-1} t_k^{-1}$ as in 2.3.1.

Proposition 3.3.8.

(1) Let P be a parabolic subgroup of PGL_V *as in 3.3.2. Then the maps*

$$\bar{\pi}_{P,v}\colon P(F_v)\times\mathbb{R}^{d-1}_{\geq 0}(P)\to\bar{X}_{F,v}(P)\quad\textit{and}\quad\bar{\pi}^{\sharp}_{P,v}\colon P(F_v)\times\mathbb{R}^{d-1}_{\geq 0}(P)\to\bar{X}^{\sharp}_{F,v}(P)$$

of 3.3.3 are surjective.

(2) For the Borel subgroup B of 3.3.2, the maps

$$\pi_v\colon B_u(F_v)\times\mathbb{R}^{d-1}_{>0}\to X_v,\qquad \bar{\pi}_v\colon B_u(F_v)\times\mathbb{R}^{d-1}_{\geq 0}\to\bar{X}_{F,v}(B),$$
$$\textit{and}\quad\bar{\pi}^{\sharp}_v\colon B_u(F_v)\times\mathbb{R}^{d-1}_{\geq 0}\to\bar{X}^{\sharp}_{F,v}(B).$$

of 3.3.3 are all surjective.

(3) If v is archimedean, then π_v and $\bar{\pi}^{\sharp}_v$ are bijective.

(4) If v is non-archimedean, then $\bar{\pi}_v$ induces a bijection

$$(B_u(F_v)\times\mathbb{R}^{d-1}_{\geq 0})/\sim\;\to\bar{X}_{F,v}(B)$$

where $(g,(t_i)_i)\sim(g',(t'_i)_i)$ if and only if

(i) $t_i=t'_i$ for all i and

(ii) $|(g^{-1}g')_{ij}|\leq(\prod_{i\leq k<j}t_k)^{-1}$ for all $1\leq i<j\leq d$, considering any $c\in\mathbb{R}$ to be less than $0^{-1}=\infty$.

Proof. If $\bar{\pi}^{\sharp}_v$ is surjective, then for any parabolic P containing B, the restriction of $\bar{\pi}^{\sharp}_v$ to $B_u(F_v)\times\mathbb{R}^{d-1}_{\geq 0}(P)$ has image $\bar{X}^{\sharp}_{F,v}(P)$. Since $B_u(F_v)\subset P(F_v)$, this forces the surjectivity of $\bar{\pi}^{\sharp}_{P,v}$, hence of $\bar{\pi}_{P,v}$ as well. So, we turn our attention to (2)–(4). If $r\in\mathbb{R}^d_{>0}$, we let $\mu^{(r)}\in X_v$ denote the class of the norm attached to the basis $(e_i)_i$ and r.

Suppose first that v is archimedean. By the Iwasawa decomposition 3.3.4, and noting 3.3.5 and 2.2.2, we see that $B_u(F_v)A_v\to X_v$ given by $g\mapsto g\mu^{(1)}$ is bijective, where $\mu^{(1)}$ denotes the class of the norm attached to $(e_i)_i$ and $1=(1,\dots,1)\in\mathbb{R}^d_{>0}$. For $t\in\mathbb{R}^{d-1}_{>0}$, let $a\in A_v$ be its image under the bijection in 3.3.7. Since $pa\mu^{(1)}=p\mu^{(r)}$, for $p\in B_u(F_v)$ and r as in 2.3.1, we have the bijectivity of π_v.

Consider the third map in (2). For $t\in\mathbb{R}^{d-1}_{\geq 0}$, let P be the parabolic that contains B and is determined by the set $\Delta(P)$ of $k\in\{1,\dots,d-1\}$ for which $t_k=0$. Let $(V_i)_{-1\leq i\leq m}$ be the corresponding flag. Let M denote the Levi subgroup of P. (It is the quotient of $\prod_{i=0}^m\mathrm{GL}_{V^{(i)}}<\mathrm{GL}_V$ by scalars, where $V=\bigoplus_{i=0}^m V^{(i)}$ as in 3.3.3, and $M\cap B_u$ is isomorphic to the product of the upper-triangular unipotent matrices in each $\mathrm{PGL}_{V^{(i)}}$.) The product of the first maps in (2) for the blocks of M is a bijection

$$(M\cap B_u)(F_v)\times\mathbb{R}^{\Delta'(P)}_{>0}\xrightarrow{\sim}\prod_{i=0}^m X_{V_v^{(i)}}\subset\bar{X}_{F,v}(P),$$

such that (g,t') is sent to $(P, g\mu)$ in $\bar{X}_{F,v}$, where μ is the sequence of classes of norms determined by t' and the standard basis. The stabilizer of μ in B_u is the unipotent radical P_u of P, and this P_u acts simply transitively on the set of splittings for the graded quotients $(V_i/V_{i-1})_v$. Since $B_u = (M \cap B_u)P_u$, and this decomposition is unique, we have the desired bijectivity of $\bar{\pi}^{\sharp}_v$, proving (3).

Suppose next that v is non-archimedean. We prove the surjectivity of the first map in (2). Using the natural actions of A_v and the symmetric group S_d on $\mathbb{R}^d_{>0}$, we see that any norm on V_v can be written as $g\mu^{(r)}$, where $g \in \mathrm{PGL}_d(F_v)$ and $r = (r_i)_i \in \mathbb{R}^d_{>0}$, with r satisfying

$$r_1 \le r_2 \le \cdots \le r_d \le q_v r_1.$$

For such an r, the class $\mu^{(r)}$ is invariant under the action of $\mathrm{Iw}(O_v)$. Hence for such an r, any element of $S_d\,\mathrm{Iw}(O_v)\mu^{(r)} = S_d\mu^{(r)}$ is of the form $\mu^{(r')}$, where $r' = (r_{\sigma(i)})_i$ for some $\sigma \in S_d$. Hence, any element of $A_v S_d\,\mathrm{Iw}(O_v)\mu^{(r)}$ for such an r is of the form $\mu^{(r')}$ for some $r' = (r'_i)_i \in \mathbb{R}^d_{>0}$. This proves the surjecivity of the first map of (2). The surjectivity of the other maps in (2) is then shown using this, similarly to the archimedean case.

Finally, we prove (4). It is easy to see that the map $\bar{\pi}_v$ factors through the quotient by the equivalence relation. We can deduce the bijectivity in question from the bijectivity of $(B_u(F_v)\times\mathbb{R}^{d-1}_{>0})/{\sim} \to X_v$, replacing V by V_i/V_{i-1} as in the above arguments for the archimedean case. Suppose that $\pi_v(g,t) = \pi_v(1,t')$ for $g \in B_u(F_v)$ and $t, t' \in \mathbb{R}^{d-1}_{>0}$. We must show that $(g,t) \sim (1,t')$. Write $\pi_v(g,t) = g\mu^{(r)}$ and $\pi_v(1,t') = \mu^{(r')}$ with $r = (r_i)_i$ and $r' = (r'_i)_i \in \mathbb{R}^d_{>0}$ such that $r_1 = 1$ and $r_j/r_i = (\prod_{i\le k<j} t_k)^{-1}$ for all $1 \le i < j \le d$, and similarly for r' and t'. It then suffices to check that $r' = r$ and $r_i|g_{ij}| \le r_j$ for all $i < j$. Since $\mu^{(r)} = g^{-1}\mu^{(r')}$, there exists $c \in \mathbb{R}_{>0}$ such that

$$\max\{r_i|x_i| \mid 1 \le i \le d\} = c \max\{r'_i|(gx)_i| \mid 1 \le i \le d\}$$

for all $x = (x_i)_i \in F_v^d$. Taking $x = e_1$, we have $gx = e_1$ as well, so $c = 1$. Taking $x = e_i$, we obtain $r_i \ge r'_i$, and taking $x = g^{-1}e_i$, we obtain $r_i \le r'_i$. Thus $r = r'$, and taking $x = e_j$ yields $r_j = \max\{r_i|g_{ij}| \mid 1 \le i \le j\}$, which tells us that $r_j \ge r_i|g_{ij}|$ for $i < j$. □

Proposition 3.3.9. *There is a unique topology on $\bar{X}_{F,v}$ (resp., $\bar{X}^{\sharp}_{F,v}$) satisfying the following conditions (i) and (ii).*

(i) For every parabolic subgroup P of PGL_V, the set $\bar{X}_{F,v}(P)$ (resp., $\bar{X}^{\sharp}_{F,v}(P)$) is open in $\bar{X}_{F,v}$ (resp., $\bar{X}^{\sharp}_{F,v}$).

(ii) For every parabolic subgroup P of PGL_V and basis $(e_i)_i$ of V such that P contains the Borel subgroup with respect to $(e_i)_i$, the topology on $\bar{X}_{F,v}(P)$ (resp., $\bar{X}^{\sharp}_{F,v}(P)$) is the topology as a quotient of $P(F_v) \times \mathbb{R}^{d-1}_{\ge 0}(P)$ under the surjection of 3.3.8(1).

This topology is also characterized by (i) and the following (ii)'.

(ii)' If B is a Borel subgroup of PGL_V *consisting of upper triangular matrices with respect to a basis $(e_i)_i$ of V, then the topology on $\bar{X}_{F,v}(B)$ (resp., $\bar{X}^{\sharp}_{F,v}(B)$) is the topology as a quotient of $B_u(F_v)\times\mathbb{R}^{d-1}_{\geq 0}$ under the surjection of 3.3.8(2).*

Proof. The uniqueness is clear if we have existence of a topology satisfying (i) and (ii). Let $(e_i)_i$ be a basis of V, let B be the Borel subgroup of PGL_V with respect to this basis, and let P be a parabolic subgroup of PGL_V containing B. It suffices to prove that for the topology on $\bar{X}_{F,v}(B)$ (resp., $\bar{X}^{\sharp}_{F,v}(B)$) as a quotient of $B_u(F_v)\times\mathbb{R}^{d-1}_{\geq 0}(B)$, the subspace topology on $\bar{X}_{F,v}(P)$ (resp., $\bar{X}^{\sharp}_{F,v}(P)$) coincides with the quotient topology from $P(F_v)\times\mathbb{R}^{d-1}_{\geq 0}(P)$. For this, it is enough to show that the action of the topological group $P(F_v)$ on $\bar{X}_{F,v}(P)$ (resp., $\bar{X}^{\sharp}_{F,v}(P)$) is continuous with respect to the topology on $\bar{X}_{F,v}(P)$ (resp., $\bar{X}^{\sharp}_{F,v}(P)$) as a quotient of $B_u(F_v)\times\mathbb{R}^{d-1}_{\geq 0}(P)$. We must demonstrate this continuity.

Let $(V_i)_{-1\leq i\leq m}$ be the flag corresponding to P, and let $c(i)=\dim(V_i)$. For $0\leq i\leq m$, we regard $\mathrm{GL}_{V^{(i)}}$ as a subgroup of GL_V via the decomposition $V=\bigoplus_{i=0}^m V^{(i)}$ of 3.3.3.

Suppose first that v is archimedean. For $0\leq i\leq m$, let K_i be the compact subgroup of $\mathrm{GL}_{V^{(i)}}(F_v)$ that is the isotropy group of the norm on $V^{(i)}$ given by the basis $(e_j)_{c(i-1)<j\leq c(i)}$ and $(1,\dots,1)\in\prod_{c(i-1)<j\leq c(i)}\mathbb{R}_{>0}$. We identify $\mathbb{R}^{d-1}_{>0}$ with A_v as in 3.3.7. By the Iwasawa decomposition 3.3.4 and its uniqueness in 3.3.5, the product on $P(F_v)$ induces a homeomorphism

$$(a,b,c)\colon P(F_v)\xrightarrow{\sim} B_u(F_v)\times\mathbb{R}^{d-1}_{>0}\times\left(\prod_{i=0}^m K_i\right)/\{z\in F_v^\times \mid |z|=1\}.$$

We also have a product map $\phi\colon P(F_v)\times B_u(F_v)\times\mathbb{R}^{\Delta'(P)}_{>0}\to P(F_v)$, where we identify $t'\in\mathbb{R}^{\Delta'(P)}_{>0}$ with the diagonal matrix $\mathrm{diag}(r_1,\dots,r_d)^{-1}$ if v is real and $\mathrm{diag}(r_1^{1/2},\dots,r_d^{1/2})^{-1}$ if v is complex, with $r_j^{-1}=\prod_{c(i-1)<k<j}t'_k$ for $c(i-1)<j\leq c(i)$ as in 2.3.1. These maps fit in a commutative diagram

$$\begin{array}{ccccc} P(F_v)\times\mathbb{R}^{\Delta(P)}_{\geq 0} & \xleftarrow{(\phi,\mathrm{id})} & P(F_v)\times B_u(F_v)\times\mathbb{R}^{\Delta'(P)}_{>0}\times\mathbb{R}^{\Delta(P)}_{\geq 0} & \xrightarrow{(\mathrm{id},\bar{\pi}_{P,v})} & P(F_v)\times\bar{X}_{F,v}(P) \\ \downarrow & & & & \downarrow \\ B_u(F_v)\times\mathbb{R}^{d-1}_{\geq 0}(P) & & \xrightarrow{\bar{\pi}_{P,v}} & & \bar{X}_{F,v}(P) \end{array}$$

in which the right vertical arrow is the action of $P(F_v)$ on $\bar{X}_{F,v}(P)$, and the left vertical arrow is the continuous map

$$(u,t)\mapsto(a(u),b(u)\cdot(1,t)),\quad (u,t)\in P(F_v)\times\mathbb{R}^{\Delta(P)}_{\geq 0}$$

for $(1,t)$ the element of $\mathbb{R}^{d-1}_{\geq 0}(P)$ with $\mathbb{R}^{\Delta'(P)}_{>0}$-component 1 and $\mathbb{R}^{\Delta(P)}_{\geq 0}$-component t. (To see the commutativity, note that $c(u)$ commutes with the block-scalar matrix determined by

$(1, t)$.) We also have a commutative diagram of the same form for $\bar{X}^{\sharp}_{F,v}$. Since the surjective horizontal arrows are quotient maps, we have the continuity of the action of $P(F_v)$.

Next, we consider the case that v is non-archimedean. For $0 \le i \le m$, let $S^{(i)}$ be the group of permutations of the set

$$I_i = \{j \in \mathbb{Z} \mid c(i-1) < j \le c(i)\},$$

and regard it as a subgroup of $\mathrm{GL}_{V^{(i)}}(F)$. Let A_v be the subgroup of the diagonal torus of $\mathrm{PGL}_V(F_v)$ with respect to the basis $(e_i)_i$ with entries powers of a fixed uniformizer, as in 3.3.4.

Consider the action of $A_v \prod_{i=0}^m S^{(i)} \subset P(F_v)$ on $\mathbb{R}^{d-1}_{\ge 0}(P)$ that is compatible with the action of $P(F_v)$ on $\bar{X}_{F,v}(P)$ via the embedding $\mathbb{R}^{d-1}_{\ge 0}(P) \to \bar{X}_{F,v}(P)$. This action is described as follows. Any matrix $a = \mathrm{diag}(a_1, \dots, a_d) \in A_v$ sends $t \in \mathbb{R}^{d-1}_{\ge 0}(P)$ to $(t_j |a_{j+1}||a_j|^{-1})_j \in \mathbb{R}^{d-1}_{\ge 0}(P)$. The action of $\prod_{i=0}^m S^{(i)}$ on $\mathbb{R}^{d-1}_{\ge 0}(P)$ is the unique continuous action which is compatible with the evident action of $\prod_{i=0}^m S^{(i)}$ on $\mathbb{R}^d_{>0}$ via the map $\mathbb{R}^d_{>0} \to \mathbb{R}^{d-1}_{\ge 0}(P)$ that sends $(r_i)_i$ to $(t_j)_j$, where $t_j = r_j / r_{j+1}$. That is, for

$$\sigma = (\sigma_i)_{0 \le i \le m} \in \prod_{i=0}^m S^{(i)},$$

let $f \in S_d$ be the unique permutation with $f|_{I_i} = \sigma_i^{-1}$ for all i. Then σ sends $t \in \mathbb{R}^{d-1}_{\ge 0}(P)$ to the element $t' = (t'_j)_j$ given by

$$t'_j = \begin{cases} \prod_{f(j) \le k < f(j+1)} t_k & \text{if } f(j) < f(j+1), \\ \prod_{f(j+1) \le k < f(j)} t_k^{-1} & \text{if } f(j+1) < f(j). \end{cases}$$

Let C be the compact subset of $\mathbb{R}^{d-1}_{\ge 0}(P)$ given by

$$C = \left\{ t = (t_j)_j \in \mathbb{R}^{d-1}_{\ge 0}(P) \cap [0,1]^{d-1} \mid \prod_{c(i-1) < j < c(i)} t_j \ge q_v^{-1} \text{ for all } 0 \le i \le m \right\}.$$

We claim that for each $x \in \mathbb{R}^{d-1}_{\ge 0}(P)$, there is a finite family $(h_k)_k$ of elements of $A_v \prod_{i=0}^m S^{(i)}$ such that the union $\bigcup_k h_k C$ is a neighborhood of x. This is quickly reduced to the following claim.

Claim. Consider the natural action of $H = A_v S_d \subset \mathrm{PGL}_V$ on the quotient space $\mathbb{R}^d_{>0}/\mathbb{R}_{>0}$, with the class of $(a_j)_j$ in A_v acting as multiplication by $(|a_j|)_j$. Let C be the image of

$$\{r \in \mathbb{R}^d_{>0} \mid r_1 \le r_2 \le \cdots \le r_d \le q_v r_1\}$$

in $\mathbb{R}^d_{>0}/\mathbb{R}_{>0}$. Then for each $x \in \mathbb{R}^d_{>0}/\mathbb{R}_{>0}$, there is a finite family $(h_k)_k$ of elements of H such that $\bigcup_k h_k C$ is a neighborhood of x.

Proof of Claim. This is a well-known statement in the theory of Bruhat-Tits buildings: the quotient $\mathbb{R}^d_{>0}/\mathbb{R}_{>0}$ is called the apartment of the Bruhat-Tits building X_v of PGL_V, and the set C is a $(d-1)$-simplex in this apartment. Any $(d-1)$-simplex in this apartment has the form hC for some $h \in H$, for any $x \in \mathbb{R}^d_{>0}/\mathbb{R}_{>0}$ there are only finitely many $(d-1)$-simplices in this apartment which contain x, and the union of these is a neighborhood of x in $\mathbb{R}^d_{>0}/\mathbb{R}_{>0}$.

By compactness of C, the topology on the neighborhood $\bigcup_k h_k C$ of x is the quotient topology from $\coprod_k h_k C$. Thus, it is enough to show that for each $h \in A_v \prod_{i=0}^m S^{(i)}$, the composition

$$P(F_v) \times B_u(F_v) \times hC \xrightarrow{(\mathrm{id}, \pi_{P,v})} P(F_v) \times \bar{X}_{F,v}(P) \to \bar{X}_{F,v}(P)$$

(where the second map is the action) and its analogue for $\bar{X}^\sharp_{F,v}$ are continuous.

For $0 \le i \le m$, let Iw_i be the Iwahori subgroup of $\mathrm{GL}_{V^{(i)}}(F_v)$ for the basis $(e_j)_{c(i-1)<j\le c(i)}$. By the the Iwasawa and Bruhat decompositions as in 3.3.6, the product on $P(F_v)$ induces a continuous surjection

$$B_u(F_v) \times A_v \prod_{i=0}^m S^{(i)} \times \prod_{i=0}^m \mathrm{Iw}_i \to P(F_v),$$

and it admits continuous sections locally on $P(F_v)$. (Here, the middle group $A_v \prod_{i=0}^m S^{(i)}$ has the discrete topology.) Therefore, there exist an open covering $(U_\lambda)_\lambda$ of $P(F_v)$ and, for each λ, a subset $\mathscr{U}_\lambda$ of the above product mapping homeomorphically to U_λ, together with a continuous map

$$(a_\lambda, b_\lambda, c_\lambda)\colon \mathscr{U}_\lambda \to B_u(F_v) \times A_v \prod_{i=0}^m S^{(i)} \times \prod_{i=0}^m \mathrm{Iw}_i$$

such that its composition with the above product map is the map $\mathscr{U}_\lambda \xrightarrow{\sim} U_\lambda$. Let U'_λ denote the inverse image of U_λ under

$$P(F_v) \times B_u(F_v) \to P(F_v), \quad (g, g') \mapsto gg'h,$$

so that $(U'_\lambda)_\lambda$ is an open covering of $P(F_v) \times B_u(F_v)$. For any γ in the indexing set of the cover, let $\mathscr{U}'_{\lambda,\gamma}$ be the inverse image of U'_λ in $\mathscr{U}_\gamma \times B_u(F_v)$. Then the images of the $\mathscr{U}'_{\lambda,\gamma}$ form an open cover of $P(F_v) \times B_u(F_v)$ as well. Let $(a'_{\lambda,\gamma}, b'_{\lambda,\gamma})$ be the composition

$$\mathscr{U}'_{\lambda,\gamma} \to \mathscr{U}_\lambda \xrightarrow{(a_\lambda, b_\lambda)} B_u(F_v) \times A_v \prod_{i=0}^m S^{(i)}.$$

As $\prod_{i=0}^m \mathrm{Iw}_i$ fixes every element of C under its embedding in $\bar{X}_{F,v}(P)$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{U}'_{\lambda,\gamma} \times hC & \hookrightarrow & P(F_v)\times B_u(F_v)\times hC \\ \downarrow & & \downarrow \\ B_u(F_v)\times \mathbb{R}^{d-1}_{\geq 0}(P) & \twoheadrightarrow & \bar{X}_{F,v}(P) \end{array}$$

in which the left vertical arrow is

$$(u, hx) \mapsto (a'_{\lambda,\gamma}(u), b'_{\lambda,\gamma}(u)x)$$

for $x \in C$. We also have a commutative diagram of the same form for $\bar{X}^{\sharp}_{F,v}$. This proves the continuity of the action of $P(F_v)$. □

3.3.10. We call the topology on $\bar{X}_{F,v}$ (resp., $\bar{X}^{\sharp}_{F,v}$) in 3.3.9 the Borel-Serre topology. The Borel-Serre topology on $\bar{X}_{F,v}$ coincides with the quotient topology of the Borel-Serre topology on $\bar{X}^{\sharp}_{F,v}$. This topology on $\bar{X}_{F,v}$ is finer than the subspace topology from $\bar{X}_v$.

We define the Borel-Serre topology on $\bar{X}^{\flat}_{F,v}$ as the quotient topology of the Borel-Serre topology of $\bar{X}_{F,v}$. This topology on $\bar{X}^{\flat}_{F,v}$ is finer than the subspace topology from $\bar{X}^{\flat}_v$.

For a nonempty finite set S of places of F, we define the Borel-Serre topology on $\bar{X}_{F,S}$ (resp., $\bar{X}^{\flat}_{F,S}$) as the subspace topology for the product topology on $\prod_{v\in S} \bar{X}_{F,v}$ (resp., $\prod_{v\in S} \bar{X}^{\flat}_{F,v}$) for the Borel-Serre topology on each $\bar{X}_{F,v}$ (resp., $\bar{X}^{\flat}_{F,v}$).

3.4 Satake topology

3.4.1. For a nonempty finite set of places S of F, we define the Satake topology on $\bar{X}_{F,S}$ and, under the assumption S contains all archimedean places, on $\bar{X}^{\flat}_{F,S}$.

The Satake topology is coarser than the Borel-Serre topology of 3.3.10. On the other hand, the Satake topology and the Borel-Serre topology induce the same topology on the quotient space by an arithmetic group (4.1.8). Thus, the Hausdorff compactness of this quotient space can be formulated without using the Satake topology (i.e., using only the Borel-Serre topology). However, arguments involving the Satake topology appear naturally in the proof of this property. One nice aspect of the Satake topology is that each point has an explicit base of neighborhoods (3.2.5, 3.4.9, 4.4.9).

3.4.2. Let H be a finite-dimensional vector space over a local field E. Let H' and H'' be E-subspaces of H such that $H' \supset H''$. Then a norm μ on H induces a norm ν on H'/H'' as follows. Let μ' be the restriction of μ to H'. Let $(\mu')^*$ be the norm on $(H')^*$ dual to μ'. Let ν^* be the restriction of $(\mu')^*$ to the subspace $(H'/H'')^*$ of $(H')^*$. Let ν be the dual of ν^*. This norm ν is given on $x \in H'/H''$ by

$$\nu(x) = \inf\{\mu(\tilde{x}) \mid \tilde{x} \in H' \text{ such that } \tilde{x} + H'' = x\}.$$

3.4.3. For a parabolic subgroup P of PGL_V, let $(V_i)_{-1\le i\le m}$ be the corresponding flag. Set

$$\bar{X}_{F,S}(P) = \{(P',\mu) \in \bar{X}_{F,S} \mid P' \supset P\}.$$

For a place v of F, let us set

$$\mathfrak{Z}_{F,v}(P) = \prod_{i=0}^{m} X_{(V_i/V_{i-1})_v} \quad \text{and} \quad \mathfrak{Z}_{F,S}(P) = \prod_{v\in S} \mathfrak{Z}_{F,v}(P).$$

We let $P(F_v)$ act on $\mathfrak{Z}_{F,v}(P)$ through $P(F_v)/P_u(F_v)$, using the $\mathrm{PGL}_{(V_i/V_{i-1})_v}(F_v)$-action on $X_{(V_i/V_{i-1})_v}$ for $0 \le i \le m$. We define a $P(F_v)$-equivariant map

$$\phi_{P,v} \colon \bar{X}_{F,v}(P) \to \mathfrak{Z}_{F,v}(P)$$

with the product of these over $v \in S$ giving rise to a map $\phi_{P,S} \colon \bar{X}_{F,S}(P) \to \mathfrak{Z}_{F,S}(P)$.

Let $(P',\mu) \in \bar{X}_{F,v}(P)$. Then the spaces in the flag $0 = V'_{-1} \subsetneq V'_0 \subsetneq \cdots \subsetneq V'_{m'} = V$ corresponding to P' form a subset of $\{V_i \mid -1 \le i \le m\}$. The image $\nu = (\nu_i)_{0\le i\le m}$ of (P',μ) under $\phi_{P,v}$ is as follows: there is a unique j with $0 \le j \le m'$ such that

$$V'_j \supset V_i \supsetneq V_{i-1} \supset V'_{j-1},$$

and ν_i is the norm induced from μ_j on the subquotient $(V_i/V_{i-1})_v$ of $(V'_j/V'_{j-1})_v$, in the sense of 3.4.2. The $P(F_v)$-equivariance of $\phi_{P,v}$ is easily seen using the actions on norms of 2.1.5 and 2.1.7.

Though the following map is not used in this subsection, we introduce it here by way of comparison between $\bar{X}_{F,S}$ and $\bar{X}^\flat_{F,S}$.

3.4.4. Let W be a nonzero F-subspace of V, and set

$$\bar{X}^\flat_{F,S}(W) = \{(W',\mu) \in \bar{X}^\flat_{F,S} \mid W' \supset W\}.$$

For a place v of F, we have a map

$$\phi^\flat_{W,v} \colon \bar{X}^\flat_{F,v}(W) \to X_{W_v}$$

which sends $(W',\mu) \in \bar{X}^\flat_{F,v}(W)$ to the restriction of μ to W_v. The map $\phi^\flat_{W,v}$ is $P(F_v)$-equivariant, for P the parabolic subgroup of PGL_V consisting of all elements that preserve W. Setting $\mathfrak{Z}^\flat_{F,S}(W) = \prod_{v\in S} X_{W_v}$, the product of these maps over $v \in S$ provides a map $\phi^\flat_{W,S} \colon \bar{X}^\flat_{F,S}(W) \to \mathfrak{Z}^\flat_{F,S}(W)$.

3.4.5. For a finite-dimensional vector space H over a local field E, a basis $e=(e_i)_{1\le i\le d}$ of H, and a norm μ on H, we define the absolute value $|\mu : e| \in \mathbb{R}_{>0}$ of μ relative to e as follows. Suppose that μ is defined by a basis $e'=(e'_i)_{1\le i\le d}$ and a tuple $(r_i)_{1\le i\le d} \in \mathbb{R}^d_{>0}$. Let $h \in \mathrm{GL}_H(E)$ be the element such that $e' = he$. We then define

$$|\mu : e| = |\det(h)|^{-1} \prod_{i=1}^{d} r_i.$$

This is independent of the choice of e' and $(r_i)_i$. Note that we have

$$|g\mu : e| = |\det(g)|^{-1} |\mu : e|$$

for all $g \in \mathrm{GL}_H(E)$.

3.4.6. Let P and $(V_i)_i$ be as in 3.4.3, and let v be a place of F. Fix a basis $e^{(i)}$ of $(V_i/V_{i-1})_v$ for each $0 \le i \le m$. Then we have a map

$$\phi'_{P,v} : \bar{X}_{F,v}(P) \to \mathbb{R}^m_{\ge 0}, \qquad (P',\mu) \mapsto (t_i)_{1\le i\le m}$$

where $(t_i)_{1\le i\le m}$ is defined as follows. Let $(V'_j)_{-1\le j\le m'}$ be the flag associated to P'. Let $1 \le i \le m$. If V_{i-1} belongs to $(V'_j)_j$, let $t_i = 0$. If V_{i-1} does not belong to the last flag, then there is a unique j such that $V'_j \supset V_i \supset V_{i-2} \supset V'_{j-1}$. Let $\tilde{\mu}_j$ be a norm on $(V'_j/V'_{j-1})_v$ which belongs to the class μ_j, and let $\tilde{\mu}_{j,i}$ and $\tilde{\mu}_{j,i-1}$ be the norms induced by μ_j on the subquotients $(V_i/V_{i-1})_v$ and $(V_{i-1}/V_{i-2})_v$, respectively. We then let

$$t_i = |\tilde{\mu}_{j,i-1} : e^{(i-1)}|^{1/d_{i-1}} \cdot |\tilde{\mu}_{j,i} : e^{(i)}|^{-1/d_i},$$

where $d_i := \dim(V_i/V_{i-1})$.

The map $\phi'_{P,v}$ is $P(F_v)$-equivariant for the following action of $P(F_v)$ on $\mathbb{R}^m_{\ge 0}$. For $g \in P(F_v)$, let $\tilde{g} \in \mathrm{GL}_V(F_v)$ be a lift of g, and for $0 \le i \le m$, let $g_i \in \mathrm{GL}_{V_i/V_{i-1}}(F_v)$ be the element induced by $\tilde{g}$. Then $g \in P(F_v)$ sends $t \in \mathbb{R}^m_{\ge 0}$ to $t' \in \mathbb{R}^m_{\ge 0}$ where

$$t'_i = |\det(g_i)|^{1/d_i} \cdot |\det(g_{i-1})|^{-1/d_{i-1}} \cdot t_i.$$

If we have two families $e = (e^{(i)})_i$ and $f = (f^{(i)})_i$ of bases $e^{(i)}$ and $f^{(i)}$ of $(V_i/V_{i-1})_v$, and if the map $\phi'_{P,v}$ defined by e (resp., f) sends an element to t (resp., t'), then the same formula also describes the relationship between t and t', in this case taking g_i to be the element of $\mathrm{GL}_{V_i/V_{i-1}}$ such that $e^{(i)} = g_i f^{(i)}$.

3.4.7. Fix a basis $e^{(i)}$ of V_i/V_{i-1} for each $0 \le i \le m$. Then we have a map

$$\phi'_{P,S} : \bar{X}_{F,S}(P) \to \mathbb{R}^m_{\ge 0}, \qquad (P',\mu) \mapsto (t_i)_{1\le i\le m}$$

where $t_i = \prod_{v\in S} t_{v,i}$, with $(t_{v,i})_i$ the image of (P',μ_v) under the map $\phi'_{P,v}$ of 3.4.6.

3.4.8. We define the Satake topology on $\bar{X}_{F,S}$ as follows.

For a parabolic subgroup P of PGL_V, consider the map

$$\psi_{P,S} := (\phi_{P,S}, \phi'_{P,S})\colon \bar{X}_{F,S}(P) \to \mathfrak{Z}_{F,S}(P) \times \mathbb{R}^m_{\geq 0}$$

from 3.4.3 and 3.4.7, which we recall depends on a choice of bases of the V_i/V_{i-1}. We say that a subset of $\bar{X}_{F,S}(P)$ is P-open if it is the inverse image of an open subset of $\mathfrak{Z}_{F,S}(P) \times \mathbb{R}^m_{\geq 0}$. By 3.4.6, the property of being P-open is independent of the choice of bases.

We define the Satake topology on $\bar{X}_{F,S}$ to be the coarsest topology for which every P-open set for each parabolic subgroup P of PGL_V is open.

By this definition, we have:

3.4.9. Let $a \in \bar{X}_{F,S}$ be of the form (P, μ) for some μ. As U ranges over neighborhoods of the image $(\mu, 0)$ of a in $\mathfrak{Z}_{F,S}(P) \times \mathbb{R}^m_{\geq 0}$, the inverse images of the U in $\bar{X}_{F,S}(P)$ under $\psi_{P,S}$ form a base of neighborhoods of a in $\bar{X}_{F,S}$.

3.4.10. In §3.5 and §3.6, we explain that the Satake topology on $\bar{X}_{F,S}$ is strictly coarser than the Borel-Serre topology for $d \geq 2$.

3.4.11. The Satake topology on $\bar{X}_{F,S}$ can differ from the subspace topology of the product topology for the Satake topology on each $\bar{X}_{F,v}$ with $v \in S$.

Example. Let F be a real quadratic field, let $V = F^2$, and let $S = \{v_1, v_2\}$ be the set of real places of F. Consider the point $(\infty, \infty) \in (\mathfrak{H} \cup \{\infty\}) \times (\mathfrak{H} \cup \{\infty\}) \subset \bar{X}_{v_1} \times \bar{X}_{v_2}$ (see §3.2), which we regard as an element of $\bar{X}_{F,S}$. Then the sets

$$U_c := \{(x_1 + y_1 i, x_2 + y_2 i) \in \mathfrak{H} \times \mathfrak{H} \mid y_1 y_2 \geq c\} \cup \{(\infty, \infty)\}$$

with $c \in \mathbb{R}_{>0}$ form a base of neighborhoods of (∞, ∞) in $\bar{X}_{F,S}$ for the Satake topology, whereas the sets

$$U'_c := \{(x_1 + y_1 i, x_2 + y_2 i) \in \mathfrak{H} \times \mathfrak{H} \mid y_1 \geq c, y_2 \geq c\} \cup \{(\infty, \infty)\}$$

for $c \in \mathbb{R}_{>0}$ form a base of neighborhoods of (∞, ∞) in $\bar{X}_{F,S}$ for the topology induced by the product of Satake topologies on $\bar{X}_{F,v_1}$ and $\bar{X}_{F,v_2}$.

3.4.12. Let $G = \mathrm{PGL}_V$, and let Γ be a subgroup of $G(F)$.

- For a parabolic subgroup P of G, let $\Gamma_{(P)}$ be the subgroup of $\Gamma \cap P(F)$ consisting of all elements with image in the center of $(P/P_u)(F)$.

- For a nonzero F-subspace W of V, let $\Gamma_{(W)}$ denote the subgroup of elements of Γ that can be lifted to elements of $\mathrm{GL}_V(F)$ which fix every element of W.

3.4.13. We let $\mathbb{A}_F$ denote the adeles of F, let $\mathbb{A}_F^S$ denote the adeles of F outside of S, and let $\mathbb{A}_{F,S} = \prod_{v \in S} F_v$ so that $\mathbb{A}_F = \mathbb{A}_F^S \times \mathbb{A}_{F,S}$. Assume that S contains all archimedean places of F. Let $G = \mathrm{PGL}_V$, let K be a compact open subgroup of $G(\mathbb{A}_F^S)$, and let $\Gamma_K < G(F)$ be the inverse image of K under $G(F) \to G(\mathbb{A}_F^S)$.

The following proposition will be proved in 3.5.15.

Proposition 3.4.14. *For S, G, K and Γ_K as in 3.4.13, the Satake topology on $\bar{X}_{F,S}$ is the coarsest topology such that for every parabolic subgroup P of G, a subset U of $\bar{X}_{F,S}(P)$ is open if*

(i) it is open for Borel-Serre topology, and

(ii) it is stable under the action of $\Gamma_{K,(P)}$ (see 3.4.12).

The following proposition follows easily from the fact that for any two compact open subgroups K and K' of $G(\mathbb{A}_F^S)$, the intersection $\Gamma_K \cap \Gamma_{K'}$ is of finite index in both Γ_K and $\Gamma_{K'}$.

Proposition 3.4.15. *For S, K and Γ_K as in 3.4.13, consider the coarsest topology on $\bar{X}^{\flat}_{F,S}$ such that for every nonzero F-subspace W, a subset U of $\bar{X}^{\flat}_{F,S}(W)$ is open if*

(i) it is open for Borel-Serre topology, and

(ii) it is stable under the action of $\Gamma_{K,(W)}$ (see 3.4.12).

Then this topology is independent of the choice of K.

3.4.16. We call the topology in 3.4.15 the Satake topology on $\bar{X}^{\flat}_{F,S}$.

Proposition 3.4.17.

(1) Let P be a parabolic subgroup of PGL_V. For both the Borel-Serre and Satake topologies on $\bar{X}_{F,S}$, the set $\bar{X}_{F,S}(P)$ is open in $\bar{X}_{F,S}$, and the action of the topological group $P(\mathbb{A}_{F,S})$ on $\bar{X}_{F,S}(P)$ is continuous.

(2) The actions of the discrete group $\mathrm{PGL}_V(F)$ on the following spaces are continuous: $\bar{X}_{F,S}$ and $\bar{X}^{\flat}_{F,S}$ with their Borel-Serre topologies, $\bar{X}_{F,S}$ with the Satake topology, and assuming S contains all archimedean places, $\bar{X}^{\flat}_{F,S}$ with the Satake topology.

Proposition 3.4.18. *Let W be a nonzero F-subspace of V. Then for the Borel-Serre topology, and for the Satake topology if S contains all archimedean places of F, the subset $\bar{X}^{\flat}_{F,S}(W)$ is open in $\bar{X}^{\flat}_{F,S}$.*

Part (1) of 3.4.17 was shown in §3.3 for the Borel-Serre topology, and the result for the Satake topology on $\bar{X}_{F,S}$ follows from it. The rest of 3.4.17 and 3.4.18 is easily proven.

3.5 Properties of $\bar{X}_{F,S}$

Let S be a nonempty finite set of places of F.

3.5.1. Let P and $(V_i)_{-1\le i\le m}$ be as before. Fix a basis $e^{(i)}$ of V_i/V_{i-1} for each i. Set

$$Y_0 = (\mathbb{R}^S_{>0} \cup \{(0)_{v\in S}\})^m \subset (\mathbb{R}^S_{\ge 0})^m.$$

The maps $\psi_{P,v} := (\phi_{P,v}, \phi'_{P,v})\colon \bar{X}_{F,v}(P) \to \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^m_{\ge 0}$ of 3.4.3 and 3.4.6 for $v \in S$ combine to give the map

$$\psi_{P,S}\colon \bar{X}_{F,S}(P) \to \mathfrak{Z}_{F,S}(P) \times Y_0.$$

3.5.2. In addition to the usual topology on Y_0, we consider the weak topology on Y_0 that is the product topology for the topology on $\mathbb{R}^S_{>0} \cup \{(0)_{v\in S}\}$ which extends the usual topology on $\mathbb{R}^S_{>0}$ by taking the sets

$$\left\{(t_v)_{v\in S} \in \mathbb{R}^S_{>0} \mid \prod_{v\in S} t_v \le c\right\} \cup \{(0)_{v\in S}\}$$

for $c \in \mathbb{R}_{>0}$ as a base of neighborhoods of $(0)_{v\in S}$. In the case that S consists of a single place, we have $Y_0 = \mathbb{R}^m_{\ge 0}$, and the natural topology and the weak topology on Y_0 coincide.

Proposition 3.5.3. *The map $\psi_{P,S}$ of 3.5.1 induces a homeomorphism*

$$P_u(\mathbb{A}_{F,S})\backslash \bar{X}_{F,S}(P) \xrightarrow{\sim} \mathfrak{Z}_{F,S}(P) \times Y_0$$

for the Borel-Serre topology (resp., Satake topology) on $\bar{X}_{F,S}$ and the usual (resp., weak) topology on Y_0. This homeomorphism is equivariant for the action of $P(\mathbb{A}_{F,S})$, with the action of $P(\mathbb{A}_{F,S})$ on Y_0 being that of 3.4.6.

This has the following corollary, which is also the main step in the proof.

Corollary 3.5.4. *For any place v of F, the map*

$$P_u(F_v)\backslash \bar{X}_{F,v}(P) \to \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^m_{\ge 0}$$

is a homeomorphism for both the Borel-Serre and Satake topologies on $\bar{X}_{F,v}$.

We state and prove preliminary results towards the proof of 3.5.3.

3.5.5. Fix a basis $(e_i)_i$ of V and a parabolic subgroup P of PGL_V which satisfies the condition in 3.3.2 for this basis. Let $(V_i)_{-1\le i\le m}$ be the flag corresponding to P, and for each i, set $c(i) = \dim(V_i)$. We define two maps

$$\xi, \xi^\star\colon P_u(F_v) \times \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^m_{\ge 0} \to \bar{X}_{F,v}(P).$$

3.5.6. First, we define the map ξ.

Set $\Delta(P) = \{c(0), \dots, c(m-1)\}$. Let $\Delta_i = \{j \in \mathbb{Z} \mid c(i-1) < j < c(i)\}$ for $0 \le i \le m$. We then clearly have

$$\{1, \dots, d-1\} = \Delta(P) \amalg \left(\coprod_{i=0}^{m} \Delta_i \right).$$

For $0 \le i \le m$, let $V^{(i)} = \sum_{c(i-1)<j\le c(i)} F e_j$, so $V_i = V_{i-1} \oplus V^{(i)}$. We have

$$\mathbb{R}^{d-1}_{\ge 0}(P) = \mathbb{R}^{\Delta(P)}_{\ge 0} \times \prod_{i=0}^{m} \mathbb{R}^{\Delta_i}_{>0} \cong \mathbb{R}^{m}_{\ge 0} \times \prod_{i=0}^{m} \mathbb{R}^{\Delta_i}_{>0}.$$

Let B be the Borel subgroup of PGL_V consisting of all upper triangular matrices for the basis $(e_i)_i$. Fix a place v of F. We consider two surjections

$$B_u(F_v) \times \mathbb{R}^{d-1}_{\ge 0}(P) \twoheadrightarrow \bar{X}_{F,v}(P) \quad \text{and} \quad B_u(F_v) \times \mathbb{R}^{d-1}_{\ge 0}(P) \twoheadrightarrow P_u(F_v) \times \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^{m}_{\ge 0}.$$

The first is induced by the surjection $\bar{\pi}_v$ of 3.3.8.

The second map is obtained as follows. For $0 \le i \le m$, let B_i be the image of B in $\mathrm{PGL}_{V^{(i)}}$ under $P \to \mathrm{PGL}_{V_i/V_{i-1}} \cong \mathrm{PGL}_{V^{(i)}}$. Then B_i is a Borel subgroup of $\mathrm{PGL}_{V^{(i)}}$, and we have a canonical bijection

$$P_u(F_v) \times \prod_{i=0}^{m} B_{i,u}(F_v) \xrightarrow{\sim} B_u(F_v).$$

By 3.3.8, we have surjections $B_{i,u}(F_v) \times \mathbb{R}^{\Delta_i}_{>0} \twoheadrightarrow X_{(V_i/V_{i-1})_v}$ for $0 \le i \le m$. The second (continuous) surjection is then the composite

$$\begin{aligned} B_u(F_v) \times \mathbb{R}^{d-1}_{\ge 0}(P) &\xrightarrow{\sim} \left(P_u(F_v) \times \prod_{i=0}^{m} B_{i,u}(F_v) \right) \times \left(\mathbb{R}^{m}_{\ge 0} \times \prod_{i=0}^{m} \mathbb{R}^{\Delta_i}_{>0} \right) \\ &\twoheadrightarrow P_u(F_v) \times \left(\prod_{i=0}^{m} X_{(V_i/V_{i-1})_v} \right) \times \mathbb{R}^{m}_{\ge 0} = P_u(F_v) \times \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^{m}_{\ge 0}. \end{aligned}$$

Proposition 3.5.7. *There is a unique surjective continuous map*

$$\xi \colon P_u(F_v) \times \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^{m}_{\ge 0} \twoheadrightarrow \bar{X}_{F,v}(P)$$

for the Borel-Serre topology on $\bar{X}_{F,v}(P)$ that is compatible with the surjections from $B_u(F_v) \times \mathbb{R}^{d-1}_{\ge 0}(P)$ to these sets. This map induces a homeomorphism

$$\mathfrak{Z}_{F,v}(P) \times \mathbb{R}^{m}_{\ge 0} \xrightarrow{\sim} P_u(F_v) \backslash \bar{X}_{F,v}(P)$$

that restricts to a homeomorphism of $\mathfrak{Z}_{F,v}(P) \times \mathbb{R}^{m}_{>0}$ with $P_u(F_v) \backslash X_v$.

This follows from 3.3.8.

3.5.8. Next, we define the map $\xi^\star$.

For $g \in P_u(F_v)$, $(\mu_i)_i \in (X_{(V_i/V_{i-1})_v})_{0\le i\le m}$, and $(t_i)_{1\le i\le m} \in \mathbb{R}^m_{\ge 0}$, we let

$$\xi^\star(g, (\mu_i)_i, (t_i)_i) = g(P', \nu),$$

where P' and ν are as in (1) and (2) below, respectively.

(1) Let $J = \{c(i-1) \mid 1 \le i \le m,\ t_i = 0\}$. Write $J = \{c'(0), \dots, c'(m'-1)\}$ with $c'(0) < \cdots < c'(m'-1)$. Let $c'(-1) = 0$ and $c'(m') = d$. For $-1 \le i \le m'$, let

$$V'_i = \sum_{j=1}^{c'(i)} F e_j \subset V.$$

Let $P' \supset P$ be the parabolic subgroup of PGL_V corresponding to the flag $(V'_i)_i$.

(2) For $0 \le i \le m'$, set

$$J_i = \{j \mid c'(i-1) < c(j) \le c'(i)\} \subset \{1, \dots, m\}.$$

We identify V'_i/V'_{i-1} with $\bigoplus_{j\in J_i} V^{(j)}$ via the basis $(e_k)_{c'(i-1)<k\le c'(i)}$. We define a norm $\tilde{\nu}_i$ on V'_i/V'_{i-1} as follows. Let $\tilde{\mu}_j$ be the unique norm on $V^{(j)}$ which belongs to μ_j and satisfies $|\tilde{\mu}_j : (e_k)_{c(j-1)<k\le c(j)}| = 1$. For $x = \sum_{j\in J_i} x_j$ with $x_j \in V^{(j)}$, set

$$\tilde{\nu}_i(x) = \begin{cases} \sum_{j\in J_i} (r_j^2 \tilde{\mu}_j(x_j)^2)^{1/2} & \text{if } v \text{ is real,} \\ \sum_{j\in J_i} r_j \tilde{\mu}_j(x_j) & \text{if } v \text{ is complex,} \\ \max_{j\in J_i}(r_j \tilde{\mu}_j(x_j)) & \text{if } v \text{ is non-archimedean,} \end{cases}$$

where for $j \in J_i$, we set

$$r_j = \prod_{\substack{\ell\in J_i \\ \ell<j}} t_\ell^{-1}.$$

Let $\nu_i \in X_{(V'_i/V'_{i-1})_v}$ be the class of the norm $\tilde{\nu}_i$.

We omit the proofs of the following two lemmas.

Lemma 3.5.9. *The composition*

$$P_u(F_v) \times \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^m_{\ge 0} \xrightarrow{\xi^\star} \bar{X}_{F,v}(P) \xrightarrow{\psi_{P,v}} \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^m_{\ge 0}$$

coincides with the canonical projection. Here, the definition of the second arrow uses the basis $(e_j \bmod V_{i-1})_{c(i-1)<j\le c(i)}$ *of* V_i/V_{i-1}.

Lemma 3.5.10. *We have a commutative diagram*

$$\begin{array}{ccc} P_u(F_v)\times \mathfrak{Z}_{F,v}(P)\times \mathbb{R}^m_{\geq 0} & \xrightarrow{\quad\xi\quad} & \bar{X}_{F,v}(P) \\ \downarrow & & \| \\ P_u(F_v)\times \mathfrak{Z}_{F,v}(P)\times \mathbb{R}^m_{\geq 0} & \xrightarrow{\quad\xi^\star\quad} & \bar{X}_{F,v}(P) \end{array}$$

in which the left vertical arrow is $(u,\mu,t)\mapsto(u,\mu,t')$*, for* t' *defined as follows. Let* $I_i\colon X_{(V_i/V_{i-1})_v}\to \mathbb{R}^{\Delta_i}_{>0}$ *be the unique continuous map for which the composition*

$$B_{i,u}(F_v)\times \mathbb{R}^{\Delta_i}_{>0}\to X_{(V_i/V_{i-1})_v}\xrightarrow{I_i}\mathbb{R}^{\Delta_i}_{>0}$$

is projection onto the second factor, and for $j\in\Delta_i$*, let* $I_{i,j}\colon X_{(V_i/V_{i-1})_v}\to\mathbb{R}_{>0}$ *denote the composition of* I_i *with projection onto the factor of* $\mathbb{R}^{\Delta_i}_{>0}$ *corresponding to* j*. Then*

$$t_i' = t_i\cdot \prod_{j\in\Delta_{i-1}} I_{i-1,j}(\mu_i)^{\frac{j-c(i-2)}{c(i-1)-c(i-2)}}\cdot \prod_{j\in\Delta_i} I_{i,j}(\mu_i)^{\frac{c(i)-j}{c(i)-c(i-1)}}$$

for $1\leq i\leq m$*.*

3.5.11. Proposition 3.5.3 is quickly reduced to Corollary 3.5.4, which now follows from 3.5.7, 3.5.9 and 3.5.10.

3.5.12. For two topologies T_1, T_2 on a set Z, we use $T_1\geq T_2$ to denote that the identity map of Z is a continuous map from Z with T_1 to Z with T_2, and $T_1>T_2$ to denote that $T_1\geq T_2$ and $T_1\neq T_2$. In other words, $T_1\geq T_2$ if T_1 is finer than T_2 and $T_1>T_2$ if T_1 is strictly finer than T_2.

By 3.5.3, the map $\psi_{P,S}\colon \bar{X}_{F,S}(P)\to\mathfrak{Z}_{F,S}(P)\times Y_0$ is continuous for the Borel-Serre topology on $\bar{X}_{F,S}$ and usual topology on Y_0. On $\bar{X}_{F,S}$, we therefore have

$$\text{Borel-Serre topology} \geq \text{Satake topology.}$$

Corollary 3.5.13. *For any nonempty finite set* S *of places of* F*, the map* $\phi^\flat_{W,S}\colon \bar{X}^\flat_{F,S}(W)\to\mathfrak{Z}^\flat_{F,S}(W)$ *of 3.4.4 is continuous for the Borel-Serre topology on* $\bar{X}^\flat_{F,S}$*. If* S *contains all archimedean places of* F*, it is continuous for the Satake topology.*

Proof. The continuity for the Borel-Serre topology follows from the continuity of $\psi_{P,S}$, noting that the Borel-Serre topology on $\bar{X}^\flat_{F,S}$ is the quotient topology of the Borel-Serre topology on $\bar{X}_{F,S}$. Suppose that S contains all archimedean places. As $\phi^\flat_{W,S}$ is $\Gamma_{K,(W)}$-equivariant, and $\Gamma_{K,(W)}$ acts trivially on $\mathfrak{Z}^\flat_{F,S}(W)$, the continuity for the Satake topology is reduced to the continuity for the Borel-Serre topology. □

Remark 3.5.14. We remark that the map $\phi_{P,v}\colon \bar{X}_{F,v}(P)\to \mathfrak{Z}_{F,v}(P)$ of 3.4.3 need not be continuous for the topology on $\bar{X}_{F,v}$ as a subspace of $\bar{X}_v$. Similarly, the map $\phi^\flat_{W,v}\colon \bar{X}^\flat_{F,v}(W)\to X_{W_v}$ of 3.4.4 need not be continuous for the subspace topology on $\bar{X}^\flat_{F,v}\subset \bar{X}^\flat_v$. See 3.6.6 and 3.6.7.

3.5.15. We prove Proposition 3.4.14.

Proof. Let $\alpha=(P,\mu)\in \bar{X}_{F,S}$. Let U be a neighborhood of α for the Borel-Serre topology which is stable under the action of $\Gamma_{K,(P)}$. By 3.5.12, it is sufficient to prove that there is a neighborhood W of α for the Satake topology such that $W\subset U$.

Let $(V_i)_{-1\le i\le m}$ be the flag corresponding to P, and let $V^{(i)}$ be as before. Let $\Gamma_1=\Gamma_K\cap P_u(F)$, and let Γ_0 be the subgroup of Γ_K consisting of the elements that preserve $V^{(i)}$ and act on $V^{(i)}$ as a scalar for all i. Then Γ_1 is a normal subgroup of $\Gamma_{K,(P)}$ and $\Gamma_1\Gamma_0$ is a subgroup of $\Gamma_{K,(P)}$ of finite index.

Let

$$Y_1=\left\{(a_v)_{v\in S}\in \mathbb{R}^S_{>0}\mid \prod_{v\in S} a_v=1\right\}^m,$$

and set $s=\sharp S$. We have a surjective continuous map

$$\mathbb{R}^m_{\ge 0}\times Y_1\twoheadrightarrow Y_0,\qquad (t,t')\mapsto (t_i^{1/s}t'_{v,i})_{v,i}.$$

The composition $\mathbb{R}^m_{\ge 0}\times Y_1\to Y_0\to \mathbb{R}^m_{\ge 0}$, where the second arrow is $(t_{v,i})_{v,i}\mapsto (\prod_{v\in S} t_{v,i})_i$, coincides with projection onto the first coordinate.

Let

$$\Phi=P_u(\mathbb{A}_{F,S})\times Y_1 \quad\text{and}\quad \Psi=\mathfrak{Z}_{F,S}(P)\times \mathbb{R}^m_{\ge 0}.$$

Consider the composite map

$$f\colon \Phi\times\Psi\to P_u(\mathbb{A}_{F,S})\times \mathfrak{Z}_{F,S}(P)\times Y_0 \xrightarrow{(\xi^\star_v)_{v\in S}} \bar{X}_{F,S}(P).$$

The map f is $\Gamma_1\Gamma_0$-equivariant for the trivial action on Ψ and the following action on Φ: for $(g,t)\in\Phi$, $\gamma_1\in\Gamma_1$ and $\gamma_0\in\Gamma_0$, we have

$$\gamma_1\gamma_0\cdot(g,t)=(\gamma_1\gamma_0 g\gamma_0^{-1},\gamma_0 t),$$

where γ_0 acts on Y_1 via the embedding $\Gamma_K\to P(\mathbb{A}_{F,S})$ and the actions of the $P(F_v)$ described in 3.4.6. The composition

$$\Phi\times\Psi\xrightarrow{f}\bar{X}_{F,S}(P)\xrightarrow{\psi_{P,S}}\Psi$$

coincides with the canonical projection.

There exists a compact subset C of Φ such that $\Phi = \Gamma_1\Gamma_0 C$ for the above action of $\Gamma_1\Gamma_0$ on Φ. Let $\beta = (\mu, 0) \in \Psi$ be the image of α under $\psi_{P,S}$. For $x \in \Phi$, we have $f(x,\beta) = \alpha$. Hence, there is an open neighborhood $U'(x)$ of x in Φ and an open neighborhood $U''(x)$ of β in Ψ such that $U'(x) \times U''(x) \subset f^{-1}(U)$. Since C is compact, there is a finite subset R of C such that $C \subset \bigcup_{x\in R} U'(x)$. Let U'' be the open subset $\bigcap_{x\in R} U''(x)$ of Ψ, which contains β. The P-open set $W = \psi_{P,S}^{-1}(U'') \subset \bar{X}_{F,S}(P)$ is by definition an open neighborhood of α in the Satake topology on $\bar{X}_{F,S}$. We show that $W \subset f^{-1}(U)$. Since the map $\Phi \times \Psi \to \bar{X}_{F,S}(P)$ is surjective, it is sufficient to prove that the inverse image $\Phi \times U''$ of W in $\Phi \times \Psi$ is contained $f^{-1}(U)$. For this, we note that

$$\Phi \times U'' = \Gamma_1\Gamma_0 C \times U'' = \Gamma_1\Gamma_0 \left(\bigcup_{x\in R} U'(x) \times U'' \right) \subset \Gamma_1\Gamma_0 f^{-1}(U) = f^{-1}(U),$$

the last equality by the stability of U under the action of $\Gamma_{K,(P)} \supset \Gamma_1\Gamma_0$ and the $\Gamma_1\Gamma_0$-equivariance of f. □

3.5.16. In the case $d = 2$, the canonical surjection $\bar{X}_{F,S} \to \bar{X}^{\flat}_{F,S}$ is bijective. It is a homeomorphism for the Borel-Serre topology. If S contains all archimedean places of F, it is a homeomorphism for the Satake topology by 3.4.14.

3.6 Comparison of the topologies

When considering $\bar{X}^{\flat}_{F,v}$, we assume that all places of F other than v are non-archimedean.

3.6.1. For $\bar{X}_{F,v}$ (resp., $\bar{X}^{\flat}_{F,v}$), we have introduced several topologies: the Borel-Serre topology, the Satake topology, and the subspace topology from $\bar{X}_v$ (resp., $\bar{X}^{\flat}_v$), which we call the weak topology. We compare these topologies below; note that we clearly have

Borel-Serre topology ≥ Satake topology and Borel-Serre topology ≥ weak topology.

3.6.2. For both $\bar{X}_{F,v}$ and $\bar{X}^{\flat}_{F,v}$, the following hold:

(1) Borel-Serre topology > Satake topology if $d \geq 2$,

(2) Satake topology > weak topology if $d = 2$,

(3) Satake topology $\not\geq$ weak topology if $d > 2$.

We do not give full proofs of these statements. Instead, we describe some special cases that give clear pictures of the differences between these topologies. The general cases can be proven in a similar manner to these special cases.

Recall from 3.5.16 that in the case $d = 2$, the sets $\bar{X}_{F,v}$ and $\bar{X}^{\flat}_{F,v}$ are equal, their Borel-Serre topologies coincide, and their Satake topologies coincide.

3.6.3. We describe the case $d = 2$ of 3.6.2(1).

Take a basis $(e_i)_{i=1,2}$ of V. Consider the point $\alpha = (B, \mu)$ of $\bar{X}_{F,v}$, where B is the Borel subgroup of upper triangular matrices with respect to $(e_i)_i$, and μ is the unique element of $\mathfrak{Z}_{F,v}(B) = X_{F_v e_1} \times X_{V_v/F_v e_1}$.

Let $\bar{\pi}_v$ be the surjection of 3.3.8(2), and identify $B_u(F_v)$ with F_v in the canonical manner. The images of the sets

$$\{(x,t) \in F_v \times \mathbb{R}_{\geq 0} \mid t \leq c\} \subset B_u(F_v) \times \mathbb{R}_{\geq 0}$$

in $\bar{X}_{F,v}(B)$ for $c \in \mathbb{R}_{>0}$ form a base of neighborhoods of α for the Satake topology. Thus, while the image of the set

$$\{(x,t) \in F_v \times \mathbb{R}_{\geq 0} \mid t < |x|^{-1}\}$$

is a neighborhood of α for Borel-Serre topology, it is not a neighborhood of α for the Satake topology.

3.6.4. We prove 3.6.2(2) in the case that v is non-archimedean. The proof in the archimedean case is similar. Since all boundary points of $\bar{X}_{F,v} = \bar{X}^{\flat}_{F,v}$ are $\mathrm{PGL}_V(F)$-conjugate, to show 3.6.2(2), it is sufficient to consider any one boundary point. We consider α of 3.6.3 for a fixed basis $(e_i)_{i=1,2}$ of V.

For $x \in F_v$ and $y \in \mathbb{R}_{>0}$, let $\mu_{y,x}$ be the norm on V_v defined by

$$\mu_{y,x}(ae_1 + be_2) = \max(|a - xb|, y|b|).$$

The class of $\mu_{y,x}$ is the image of $(x, y^{-1}) \in B_u(F_v) \times \mathbb{R}_{>0}$. Any element of X_v is the class of the norm $\mu_{y,x}$ for some x, y. If we vary $x \in F_\infty$ and $y \in \mathbb{R}_{>0}$, the classes of $\mu_{y,x}$ in $\bar{X}_{F,v}$ converge under the Satake topology to the point α if and only if y approaches ∞. In $\bar{X}_v$, the point α is the class of the semi-norm ν on V_v^* defined by $\nu(ae_1^* + be_2^*) = |a|$. By 2.1.7,

$$\mu_{y,x}^* = \left(\mu_{y,0} \circ \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right)^* = \mu_{y,0}^* \circ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

from which we see that

$$\mu_{y,x}^*(ae_1^* + be_2^*) = \max(|a|, y^{-1}|xa + b|).$$

Then $\mu_{y,x}^*$ is equivalent to the norm $\nu_{y,x}$ on V_v^* defined by

$$\nu_{y,x}(ae_1^* + be_2^*) = \min(1, y|x|^{-1}) \max(|a|, y^{-1}|xa + b|),$$

and the classes of the $\nu_{y,x}$ converge in $\bar{X}_v$ to the class of the semi-norm ν as $y \to \infty$. Therefore, the Satake topology is finer than the weak topology.

Now, the norm $\mu_{1,x}^*$ is equivalent to the norm $\nu_{1,x}$ on V_v^* defined above, which for sufficiently large x satisfies

$$\nu_{1,x}(ae_1 + be_2) = \max(|a/x|, |a + (b/x)|).$$

Thus, as $|x| \to \infty$, the sequence $\mu_{1,x}$ converges in $\bar{X}_v = \bar{X}_v^\flat$ to the class of the semi-norm ν. However, the sequence of classes of the norms $\mu_{1,x}$ does not converge to α in $\bar{X}_{F,v} = \bar{X}_{F,v}^\flat$ for the Satake topology, so the Satake topology is strictly finer than the weak topology.

3.6.5. We explain the case $d = 3$ of 3.6.2(3) in the non-archimedean case.

Take a basis $(e_i)_{1 \le i \le 3}$ of V. For $y \in \mathbb{R}_{>0}$, let μ_y be the norm on V_v defined by

$$\mu_y(ae_1 + be_2 + ce_3) = \max(|a|, y|b|, y^2|c|).$$

For $x \in F_v$, consider the norm $\mu_y \circ g_x$, where

$$g_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

If we vary $x \in F_\infty$ and let $y \in \mathbb{R}_{>0}$ approach ∞, then the class of $\mu_y \circ g_x$ in X_v converges under the Satake topology to the class $\alpha \in \bar{X}_{F,v}$ of the pair that is the Borel subgroup of upper triangular matrices and the unique element of $\prod_{i=0}^2 X_{(V_i/V_{i-1})_v}$, where $(V_i)_{-1 \le i \le 2}$ is the corresponding flag. The quotient topology on $\bar{X}_{F,v}^\flat$ of the Satake topology on $\bar{X}_{F,v}$ is finer than the Satake topology on $\bar{X}_{F,v}^\flat$ by 3.4.14 and 3.4.15. Thus, if the Satake topology is finer than the weak topology on $\bar{X}_{F,v}$ or $\bar{X}_{F,v}^\flat$, then the composite $\mu_y \circ g_x$ should converge in $\bar{X}_v^\flat$ to the class of the semi-norm ν on V_v^* that satisfies $\nu(ae_1^* + be_2^* + ce_3^*) = |a|$. However, if $y \to \infty$ and $y^{-2}|x| \to \infty$, then the class of $\mu_y \circ g_x$ in X_v converges in $\bar{X}_v^\flat$ to the class of the semi-norm $ae_1^* + be_2^* + ce_3^* \mapsto |b|$. In fact, by 2.1.7 we have

$$\begin{aligned}(\mu_y \circ g_x)^*(ae_1^* + be_2^* + ce_3^*) &= \mu_y^* \circ (g_x^*)^{-1}(ae_1^* + be_2^* + ce_3^*) \\ &= \max(|a|,\ y^{-1}|b|,\ y^{-2}|-bx + c|) = y^{-2}|x|\nu_{y,x}\end{aligned}$$

where $\nu_{y,x}$ is the norm

$$ae_1^* + be_2^* + ce_3^* \mapsto \max(y^2|x|^{-1}|a|,\ y|x|^{-1}|b|,\ |-b + x^{-1}c|)$$

on V_v^*. The norms $\nu_{y,x}$ converge to the semi-norm $ae_1^* + be_2^* + ce_3^* \mapsto |b|$.

3.6.6. Let W be a nonzero subspace of V. We demonstrate that the map $\phi^\flat_{W,v}\colon \bar{X}^\flat_{F,v}(W) \to X_{W_v}$ of 3.4.4 given by restriction to W_v need not be continuous for the weak topology, even though by 3.5.13, it is continuous for the Borel-Serre topology and (if all places other than v are non-archimedean) for the Satake topology.

For example, suppose that v is non-archimedean and $d=3$. Fix a basis $(e_i)_{1\le i\le 3}$ of V, and let $W = Fe_1 + Fe_2$. Let μ be the class of the norm

$$ae_1 + be_2 \mapsto \max(|a|,|b|)$$

on W_v, and consider the element $(W,\mu) \in \bar{X}^\flat_{F,v}$. For $x \in F_v$ and $\epsilon \in \mathbb{R}_{>0}$, let $\mu_{x,\epsilon} \in X_v$ be the class of the norm

$$ae_1 + be_2 + ce_3 \mapsto \max(|a|,|b|,\epsilon^{-1}|c+bx|)$$

on V_v. Then $\mu^*_{x,\epsilon}$ is the class of the norm

$$ae_1^* + be_2^* + ce_3^* \mapsto \max(|a|,|b-xc|,\epsilon|c|)$$

on V_v^*. When $x\to 0$ and $\epsilon \to 0$, the last norm converges to the semi-norm

$$ae_1^* + be_2^* + ce_3^* \mapsto \max(|a|,|b|)$$

on V_v^*, and this implies that $\mu_{x,\epsilon}$ converges to (W,μ) for the weak topology. However, the restriction of $\mu_{x,\epsilon}$ to W_v is the class of the norm

$$ae_1 + be_2 \mapsto \max(|a|,|b|,\epsilon^{-1}|x||b|).$$

If $x \to 0$ and $\epsilon = r^{-1}|x| \to 0$ for a fixed $r > 1$, then the latter norms converge to the norm $ae_1 + be_2 \mapsto \max(|a|, r|b|)$, the class of which does not coincide with μ.

3.6.7. Let P be a parabolic subgroup of $\mathrm{PGL}_V(F)$. We demonstrate that the map $\phi_{P,v}\colon \bar{X}_{F,v}(P) \to \mathfrak{Z}_{F,v}(P)$ of 3.4.3 is not necessarily continuous for the weak topology, though by 3.5.4, it is continuous for the Borel-Serre topology and for the Satake topology.

Let $d=3$ and W be as in 3.6.6, and let P be the parabolic subgroup of PGL_V corresponding to the flag

$$0 = V_{-1} \subset V_0 = W \subset V_1 = V.$$

In this case, the canonical map $\bar{X}_{F,v}(P) \to \bar{X}^\flat_{F,v}(W)$ is a homeomorphism for the weak topology on both spaces. It is also a homeomorphism for the Borel-Serre topology, and for the Satake topology if all places other than v are non-archimedean. Since $\mathfrak{Z}_{F,v}(P) = X_{(V_0)_v} \times X_{(V/V_0)_v} \cong X_{W_v}$, the argument of 3.6.6 shows that $\phi_{P,v}$ is not continuous for the weak topology.

3.6.8. For $d \ge 3$, the Satake topology on $\bar{X}^\flat_{F,v}$ does not coincide with the quotient topology for the Satake topology on $\bar{X}_{F,v}$, which is strictly finer. This is explained in 4.4.12.

3.7 Relations with Borel-Serre spaces and reductive Borel-Serre spaces

3.7.1. In this subsection, we describe the relationship between our work and the theory of Borel-Serre and reductive Borel-Serre spaces (see Proposition 3.7.4). We also show that $\bar{X}^{\sharp}_{F,v}$ is not Hausdorff if v is a non-archimedean place.

3.7.2. Let G be a semisimple algebraic group over $\mathbb{Q}$. We recall the definitions of the Borel-Serre and reductive Borel-Serre spaces associated to G from [3] and [26, p. 190], respectively.

Let $\mathscr{Y}$ be the space of all maximal compact subgroups of $G(\mathbb{R})$. Recall from [3, Proposition 1.6] that for $K \in \mathscr{Y}$, the Cartan involution θ_K of $G_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Q}} G$ corresponding to K is the unique homomorphism $G_{\mathbb{R}} \to G_{\mathbb{R}}$ such that

$$K = \{g \in G(\mathbb{R}) \mid \theta_K(g) = g\}.$$

Let P be a parabolic subgroup of G, let S_P be the largest $\mathbb{Q}$-split torus in the center of P/P_u, and let A_P be the connected component of the topological group $S_P(\mathbb{R})$ containing the origin. We have

$$A_P \cong \mathbb{R}^r_{>0} \subset S_P(\mathbb{R}) \cong (\mathbb{R}^\times)^r$$

for some integer r. We define an action of A_P on $\mathscr{Y}$ as follows (see [3, Section 3]). For $K \in \mathscr{Y}$, we have a unique subtorus $S_{P,K}$ of $P_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} P$ over $\mathbb{R}$ such that the projection $P \to P/P_u$ induces an isomorphism

$$S_{P,K} \xrightarrow{\sim} (S_P)_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Q}} S_P$$

and such that the Cartan involution $\theta_K \colon G_{\mathbb{R}} \to G_{\mathbb{R}}$ of K satisfies $\theta_K(t) = t^{-1}$ for all $t \in S_{P,K}(\mathbb{R})$. For $t \in A_P$, let $t_K \in S_{P,K}(\mathbb{R})$ be the inverse image of t. Then A_P acts on $\mathscr{Y}$ by

$$A_P \times \mathscr{Y} \to \mathscr{Y}, \qquad (t, K) \mapsto t_K K t_K^{-1}.$$

The Borel-Serre space is the set of pairs (P, Z) such that P is a parabolic subgroup of G and Z is an A_P-orbit in $\mathscr{Y}$. The reductive Borel-Serre space is the quotient of the Borel-Serre space by the equivalence relation under which two elements (P, Z) and (P', Z') are equivalent if $(P', Z') = g(P, Z)$ (that is, $P = P'$ and $Z' = gZ$) for some $g \in P_u(\mathbb{R})$.

3.7.3. Now assume that $F = \mathbb{Q}$ and $G = \mathrm{PGL}_V$. Let v be the archimedean place of $\mathbb{Q}$.

We have a bijection between X_v and the set $\mathscr{Y}$ of all maximal compact subgroups of $G(\mathbb{R})$, whereby an element of X_v corresponds to its isotropy group in $G(\mathbb{R})$, which is a maximal compact subgroup.

Suppose that $K \in \mathscr{Y}$ corresponds to $\mu \in X_v$, with μ the class of a norm that in turn corresponds to a positive definite symmetric bilinear form (,) on V_v. The Cartan involution $\theta_K : G_{\mathbb{R}} \to G_{\mathbb{R}}$ is induced by the unique homomorphism $\theta_K\colon \mathrm{GL}_{V_v} \to \mathrm{GL}_{V_v}$ satisfying

$$(gx, \theta_K(g)y) = (x, y) \quad \text{for all } g \in \mathrm{GL}_V(\mathbb{R}) \text{ and } x, y \in V_v.$$

For a parabolic subgroup P of G corresponding to a flag $(V_i)_{-1\le i\le m}$, we have

$$S_P = \left(\prod_{i=0}^{m} \mathbb{G}_{\mathrm{m},\mathbb{Q}}\right)/\mathbb{G}_{\mathrm{m},\mathbb{Q}},$$

where the ith term in the product is the group of scalars in $\mathrm{GL}_{V_i/V_{i-1}}$, and where the last $\mathbb{G}_{\mathrm{m},\mathbb{Q}}$ is embedded diagonally in the product. The above description of θ_K shows that $S_{P,K}$ is the lifting of $(S_P)_{\mathbb{R}}$ to $P_{\mathbb{R}}$ obtained through the orthogonal direct sum decomposition

$$V_v \cong \bigoplus_{i=0}^{m} (V_i/V_{i-1})_v$$

with respect to (,).

Proposition 3.7.4. *If v is the archimedean place of $\mathbb{Q}$, then $\bar{X}^{\sharp}_{\mathbb{Q},v}$ (resp., $\bar{X}_{\mathbb{Q},v}$) is the Borel-Serre space (resp., reductive Borel-Serre space) associated to* PGL_V.

Proof. Denote the Borel-Serre space by $(\bar{X}^{\sharp}_{\mathbb{Q},v})'$ in this proof. We define a canonical map

$$\bar{X}^{\sharp}_{\mathbb{Q},v} \to (\bar{X}^{\sharp}_{\mathbb{Q},v})', \qquad (P, \mu, s) \mapsto (P, Z),$$

where Z is the subset of $\mathscr{Y}$ corresponding to the following subset Z' of X_v. Let $(V_i)_{-1\le i\le m}$ be the flag corresponding to P. Recall that s is an isomorphism

$$s\colon \bigoplus_{i=0}^{m} (V_i/V_{i-1})_v \xrightarrow{\sim} V_v.$$

Then Z' is the subset of X_v consisting of classes of the norms

$$\tilde{\mu}^{(s)}\colon x \mapsto \left(\sum_{i=0}^{m} \tilde{\mu}_i (s^{-1}(x)_i)^2\right)^{1/2}$$

on V_v, where $s^{-1}(x)_i \in (V_i/V_{i-1})_v$ denotes the ith component of $s^{-1}(x)$ for $x \in V_v$, and $\tilde{\mu} = (\tilde{\mu}_i)_{0\le i\le m}$ ranges over all families of norms $\tilde{\mu}_i$ on $(V_i/V_{i-1})_v$ with class equal to μ_i. It follows from the description of $S_{P,K}$ in 3.7.3 that Z is an A_P-orbit.

For a parabolic subgroup P of G, let

$$(\bar{X}^{\sharp}_{\mathbb{Q},v})'(P) = \{(Q, Z) \in (\bar{X}^{\sharp}_{\mathbb{Q},v})' \mid Q \supset P\}.$$

By [3, 7.1], the subset $(\bar{X}^{\sharp}_{\mathbb{Q},v})'(P)$ is open in $(X^{\sharp}_{\mathbb{Q},v})'$.

Take a basis of V, and let B denote the Borel subgroup of PGL_V of upper-triangular matrices for this basis. By 3.3.8(3), we have a homeomorphism

$$B_u(\mathbb{R}) \times \mathbb{R}^{d-1}_{\geq 0} \xrightarrow{\sim} \bar{X}^{\sharp}_{\mathbb{Q},v}(B).$$

It follows from [3, 5.4] that the composition

$$B_u(\mathbb{R}) \times \mathbb{R}^{d-1}_{\geq 0} \to (\bar{X}^{\sharp}_{\mathbb{Q},v})'(B)$$

induced by the above map

$$\bar{X}^{\sharp}_{\mathbb{Q},v}(B) \to (\bar{X}^{\sharp}_{\mathbb{Q},v})'(B), \qquad (P,\mu,s) \mapsto (P,Z)$$

is also a homeomorphism. This proves that the map $\bar{X}^{\sharp}_{\mathbb{Q},v} \to (\bar{X}^{\sharp}_{\mathbb{Q},v})'$ restricts to a homeomorphism $\bar{X}^{\sharp}_{\mathbb{Q},v}(B) \xrightarrow{\sim} (\bar{X}^{\sharp}_{\mathbb{Q},v})'(B)$. Therefore, $\bar{X}^{\sharp}_{\mathbb{Q},v} \to (\bar{X}^{\sharp}_{\mathbb{Q},v})'$ is a homeomorphism as well. It then follows directly from the definitions that the reductive Borel-Serre space is identified with $\bar{X}_{\mathbb{Q},v}$. □

3.7.5. Suppose that F is a number field, let S be the set of all archimedean places of F, and let G be the Weil restriction $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{PGL}_V$, which is a semisimple algebraic group over $\mathbb{Q}$. Then $\mathscr{Y}$ is identified with $X_{F,S}$, and $\bar{X}_{F,S}$ is related to the reductive Borel-Serre space associated to G but does not always coincide with it. We explain this below.

Let $(\bar{X}^{\sharp}_{F,S})'$ and $\bar{X}'_{F,S}$ be the Borel-Serre space and the reductive Borel-Serre space associated to G, respectively. Let $\bar{X}^{\sharp}_{F,S}$ be the subspace of $\prod_{v\in S}\bar{X}^{\sharp}_{F,v}$ consisting of all elements $(x_v)_{v\in S}$ such that the parabolic subgroup of G associated to x_v is independent of v. Then by similar arguments to the case $F=\mathbb{Q}$, we see that $\mathscr{Y}$ is canonically homeomorphic to $X_{F,S}$ and this homeomorphism extends uniquely to surjective continuous maps

$$(\bar{X}^{\sharp}_{F,S})' \to \bar{X}^{\sharp}_{F,S}, \qquad \bar{X}'_{F,S} \to \bar{X}_{F,S}.$$

However, these maps are not bijective unless F is $\mathbb{Q}$ or imaginary quadratic. We illustrate the differences between the spaces in the case that F is a real quadratic field and $d=2$.

Fix a basis $(e_i)_{i=1,2}$ of V. Let $\tilde{P}$ be the Borel subgroup of upper triangular matrices in PGL_V for this basis, and let P be the Borel subgroup $\mathrm{Res}_{F/\mathbb{Q}}\tilde{P}$ of G. Then $P/P_u \cong \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$ and $S_P = \mathbb{G}_{m,\mathbb{Q}} \subset P/P_u$. We have the natural identifications $\mathscr{Y} = X_{F,S} = \mathfrak{H}\times\mathfrak{H}$. For $a\in\mathbb{R}_{>0}$, the set

$$Z_a := \{(y\,i, a\,y\,i) \in \mathfrak{H}\times\mathfrak{H} \mid y\in\mathbb{R}_{>0}\}$$

is an A_P-orbit. If $a\neq b$, the images of (P,Z_a) and (P,Z_b) in $(\bar{X}_{F,S})'$ do not coincide. On the other hand, both the images of (P,Z_a) and (P,Z_b) in $\bar{X}^{\sharp}_{F,S}$ coincide with $(x_v)_{v\in S}$, where $x_v=(P,\mu_v,s_v)$ with μ_v the unique element of $X_{F_v e_1}\times X_{V_v/F_v e_1}$ and s_v the splitting given by e_2.

In the case that v is non-archimedean, the space $\bar{X}^{\sharp}_{F,v}$ is not good in the following sense.

Proposition 3.7.6. *If v is non-archimedean, then $\bar{X}^{\sharp}_{F,v}$ is not Hausdorff.*

Proof. Fix $a, b \in B_u(F_v)$ with $a \neq b$, for a Borel subgroup B of PGL_V. When $t \in \mathbb{R}^{d-1}_{>0}$ is sufficiently near to $0 = (0, \dots, 0)$, the images of (a, t) and (b, t) in X_v coincide by 3.3.8(4) applied to $B_u(F_v) \times \mathbb{R}^{d-1}_{>0} \to X_v$. We denote this element of X_v by $c(t)$. The images $f(a)$ of $(a, 0)$ and $f(b)$ of $(b, 0)$ in $\bar{X}^{\sharp}_{F,v}$ are different. However, $c(t)$ converges to both $f(a)$ and $f(b)$ as t tends to 0. Thus, $\bar{X}^{\sharp}_{F,v}$ is not Hausdorff. □

3.7.7. Let F be a number field, S its set of archimedean places, and $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{PGL}_V$, as in 3.7.5. Then $\bar{X}_{F,S}$ may be identified with the maximal Satake space for G of [23]. Its Satake topology was considered by Satake (see also [2, III.3]), and its Borel-Serre topology was considered by Zucker [27] (see also [14, 2.5]). The space $\bar{X}^{\flat}_{F,S}$ is also a Satake space corresponding to the standard projective representation of G on V viewed as a $\mathbb{Q}$-vector space.

4 Quotients by S-arithmetic groups

As in §3, fix a global field F and a finite-dimensional vector space V over F.

4.1 Results on S-arithmetic quotients

4.1.1. Fix a nonempty finite set S_1 of places of F which contains all archimedean places of F, fix a finite set S_2 of places of F which is disjoint from S_1, and let $S = S_1 \cup S_2$.

4.1.2. In the following, we take $\bar{X}$ to be one of the following two spaces:

(i) $\bar{X} := \bar{X}_{F,S_1}$,

(ii) $\bar{X} := \bar{X}^{\flat}_{F,S_1}$.

We endow $\bar{X}$ with either the Borel-Serre or the Satake topology.

4.1.3. Let $G = \mathrm{PGL}_V$, and let K be a compact open subgroup of $G(\mathbb{A}^S_F)$, with $\mathbb{A}^S_F$ as in 3.4.13.

We consider the two situations in which $(\mathfrak{X}, \bar{\mathfrak{X}})$ is taken to be one of the following pairs of spaces (for either choice of $\bar{X}$):

(I) $\mathfrak{X} := X_S \times G(\mathbb{A}^S_F)/K \subset \bar{\mathfrak{X}} := \bar{X} \times X_{S_2} \times G(\mathbb{A}^S_F)/K$,

(II) $\mathfrak{X} := X_S \subset \bar{\mathfrak{X}} := \bar{X} \times X_{S_2}$.

We now come to the main result of this paper.

Theorem 4.1.4. *Let the situations and notation be as in 4.1.1–4.1.3.*

(1) *Assume we are in situation (I). Let* Γ *be a subgroup of* $G(F)$. *Then the quotient space* $\Gamma\backslash\bar{\mathfrak{X}}$ *is Hausdorff. It is compact if* $\Gamma = G(F)$.

(2) *Assume we are in situation (II). Let* $\Gamma_K \subset G(F)$ *be the inverse image of* K *under the canonical map* $G(F) \to G(\mathbb{A}_F^S)$, *and let* Γ *be a subgroup of* Γ_K. *Then the quotient space* $\Gamma\backslash\bar{\mathfrak{X}}$ *is Hausdorff. It is compact if* Γ *is of finite index in* Γ_K.

4.1.5. The case $\Gamma = \{1\}$ of Theorem 4.1.4 shows that $\bar{X}_{F,S}$ and $\bar{X}^{\flat}_{F,S}$ are Hausdorff.

4.1.6. Let O_S be the subring of F consisting of all elements which are integral outside S. Take an O_S-lattice L in V. Then $\mathrm{PGL}_L(O_S)$ coincides with Γ_K for the compact open subgroup $K = \prod_{v\notin S} \mathrm{PGL}_L(O_v)$ of $G(\mathbb{A}_F^S)$. Hence Theorem 1.6 of the introduction follows from Theorem 4.1.4.

4.1.7. In the case that F is a number field and S (resp., S_1) is the set of all archimedean places of F, Theorem 4.1.4 in situation (II) is a special case of results of Satake [23] (resp., of Ji, Murty, Saper, and Scherk [14, Proposition 4.2]).

4.1.8. If in Theorem 4.1.4 we take $\Gamma = G(F)$ in part (1), or Γ of finite index in Γ_K in part (2), then the Borel-Serre and Satake topologies on $\bar{X}$ induce the same topology on the quotient space $\Gamma\backslash\bar{\mathfrak{X}}$. This can be proved directly, but it also follows from the compact Hausdorff property.

4.1.9. We show that some modifications of Theorem 4.1.4 are not good.

Consider the case $F = \mathbb{Q}$, $S = \{p, \infty\}$ for a prime number p, and $V = \mathbb{Q}^2$, and consider the S-arithmetic group $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])$. Note that $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])\backslash(\bar{X}_{\mathbb{Q},\infty} \times X_p)$ is compact Hausdorff, as is well known (and follows from Theorem 4.1.4). We show that some similar spaces are not Hausdorff. That is, we prove the following statements:

(1) $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])\backslash(\bar{X}_{\mathbb{Q},p} \times X_\infty)$ is not Hausdorff.

(2) $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{p}])\backslash(\bar{X}_{\mathbb{Q},\infty} \times \mathrm{PGL}_2(\mathbb{Q}_p))$ is not Hausdorff.

(3) $\mathrm{PGL}_2(\mathbb{Q})\backslash(\bar{X}_{\mathbb{Q},\infty} \times \mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}))$ is not Hausdorff.

Statement (1) shows that it is important to assume in 4.1.4 that S_1, not only S, contains all archimedean places. Statement (3) shows that it is important to take the quotient $G(\mathbb{A}_F^S)/K$ in situation (I) of 4.1.3.

Our proofs of these statements rely on the facts that the quotient spaces $\mathbb{Z}[\frac{1}{p}]\backslash\mathbb{R}$, $\mathbb{Z}[\frac{1}{p}]\backslash\mathbb{Q}_p$, and $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}^{\infty}$ are not Hausdorff.

Proof of statements (1)–(3). For an element x of a ring R, let

$$g_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(R).$$

In (1), for $b \in \mathbb{R}$, let h_b be the point $i+b$ of the upper half plane $\mathfrak{H} = X_\infty$. In (2), for $b \in \mathbb{Q}_p$, let $h_b = g_b \in \mathrm{PGL}_2(\mathbb{Q}_p)$. In (3), for $b \in \mathbb{A}_{\mathbb{Q}}^\infty$, let $h_b = g_b \in \mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$. In (1) (resp., (2) and (3)), let $\infty \in \bar{X}_{\mathbb{Q},p}$ (resp., $\bar{X}_{\mathbb{Q},\infty}$) be the boundary point corresponding to the the Borel subgroup of upper triangular matrices.

In (1) (resp., (2), resp., (3)), take an element b of $\mathbb{R}$ (resp., $\mathbb{Q}_p$, resp., $\mathbb{A}_{\mathbb{Q}}^\infty$) which does not belong to $\mathbb{Z}[\frac{1}{p}]$ (resp., $\mathbb{Z}[\frac{1}{p}]$, resp., $\mathbb{Q}$). Then the images of (∞, h_0) and (∞, h_b) in the quotient space are different, but they are not separated. Indeed, in (1) and (2) (resp., (3)), some sequence of elements x of $\mathbb{Z}[\frac{1}{p}]$ (resp., $\mathbb{Q}$) will converge to b, in which case $g_x(\infty, h_0)$ converges to (∞, h_b) since $g_x\infty = \infty$. □

4.2 Review of reduction theory

We review important results in the reduction theory of algebraic groups: 4.2.2, 4.2.4, and a variant 4.2.6 of 4.2.2. More details may be found in the work of Borel [1] and Godement [8] in the number field case and Harder [12, 13] in the function field case.

Fix a basis $(e_i)_{1\le i\le d}$ of V. Let B be the Borel subgroup of $G = \mathrm{PGL}_V$ consisting of all upper triangular matrices for this basis. Let S be a nonempty finite set of places of F containing all archimedean places.

4.2.1. For $b = (b_v) \in \mathbb{A}_F^\times$, set $|b| = \prod_v |b_v|$. Let A_v be as in 3.3.4. We let $a \in \prod_v A_v$ denote the image of a diagonal matrix $\mathrm{diag}(a_1, \dots, a_d)$ in $\mathrm{GL}_d(\mathbb{A}_F)$. The ratios $a_i a_{i+1}^{-1}$ are independent of the choice. For $c \in \mathbb{R}_{>0}$, we let $B(c) = B_u(\mathbb{A}_F)A(c)$, where

$$A(c) = \left\{ a \in \prod_v A_v \cap \mathrm{PGL}_d(\mathbb{A}_F) \mid |a_i a_{i+1}^{-1}| \ge c \text{ for all } 1 \le i \le d-1 \right\}.$$

Let $K^0 = \prod_v K_v^0 < G(\mathbb{A}_F)$, where K_v^0 is identified via $(e_i)_i$ with the standard maximal compact subgroup of $\mathrm{PGL}_d(F_v)$ of 3.3.4. Note that $B_u(F_v)A_v K_v^0 = B(F_v)K_v^0 = G(F_v)$ for all v.

We recall the following known result in reduction theory: see [8, Theorem 7] and [12, Satz 2.1.1].

Lemma 4.2.2. *For sufficiently small $c \in \mathbb{R}_{>0}$, one has $G(\mathbb{A}_F) = G(F)B(c)K^0$.*

4.2.3. Let the notation be as in 4.2.1. For a subset I of $\{1,\dots,d-1\}$, let P_I be the parabolic subgroup of G corresponding to the flag consisting of 0, the F-subspaces $\sum_{1\le j\le i} Fe_j$ for $i\in I$, and V. Hence $P_I \supset B$ for all I, with $P_\varnothing = G$ and $P_{\{1,\dots,d-1\}} = B$.

For $c'\in\mathbb{R}_{>0}$, let $B_I(c,c') = B_u(\mathbb{A}_F)A_I(c,c')$, where

$$A_I(c,c') = \{a\in A(c) \mid |a_i a_{i+1}^{-1}| \ge c' \text{ for all } i\in I\}.$$

Note that $B_I(c,c') = B(c)$ if $c\ge c'$.

The following is also known [12, Satz 2.1.2] (see also [8, Lemma 3]):

Lemma 4.2.4. *Fix $c\in\mathbb{R}_{>0}$ and a subset I of $\{1,\dots,d-1\}$. Then there exists $c'\in\mathbb{R}_{>0}$ such that*

$$\{\gamma\in G(F) \mid B_I(c,c')K^0 \cap \gamma^{-1}B(c)K^0 \neq \varnothing\} \subset P_I(F).$$

4.2.5. We will use the following variant of 4.2.3.

Let $A_S = \prod_{v\in S} A_v$. For $c\in\mathbb{R}_{>0}$, let

$$A(c)_S = A_S \cap A(c) \quad\text{and}\quad B(c)_S = B_u(\mathbb{A}_{F,S})A(c)_S.$$

For $c_1, c_2 \in \mathbb{R}_{>0}$, set

$$A(c_1,c_2)_S = \{a\in A_S \mid \text{for all } v\in S \text{ and } 1\le i\le d-1, \\ |a_{v,i}a_{v,i+1}^{-1}| \ge c_1 \text{ and } |a_{v,i}a_{v,i+1}^{-1}| \ge c_2 |a_{w,i}a_{w,i+1}^{-1}| \text{ for all } w\in S\}.$$

Note that $A(c_1,c_2)_S$ is empty if $c_2 > 1$. For a compact subset C of $B_u(\mathbb{A}_{F,S})$, we then set

$$B(C;c_1,c_2)_S = C\cdot A(c_1,c_2)_S.$$

Let $D_S = \prod_{v\in S} D_v$, where $D_v = K_v^0 < G(F_v)$ if v is archimedean, and $D_v < G(F_v)$ is identified with $S_d\,\mathrm{Iw}(O_v) < \mathrm{PGL}_d(F_v)$ using the basis $(e_i)_i$ otherwise. Here, S_d is the symmetric group of degree d and $\mathrm{Iw}(O_v)$ is the Iwahori subgroup of $\mathrm{PGL}_d(O_v)$, as in 3.3.6.

Lemma 4.2.6. *Let K be a compact open subgroup of $G(\mathbb{A}_F^S)$, let Γ_K be the inverse image of K under $G(F)\to G(\mathbb{A}_F^S)$, and let Γ be a subgroup of Γ_K of finite index. Then there exist c_1, c_2, C as above and a finite subset R of $G(F)$ such that*

$$G(\mathbb{A}_{F,S}) = \Gamma R\cdot B(C;c_1,c_2)_S D_S.$$

Proof. This can be deduced from 4.2.2 by standard arguments in the following manner. By the Iwasawa decomposition 3.3.4, we have $G(\mathbb{A}_F^S) = B(\mathbb{A}_F^S)K^{0,S}$ where $K^{0,S}$ is the non-S-component of K^0. Choose a set E of representatives in $B(\mathbb{A}_F^S)$ of the finite set

$$B(F)\backslash B(\mathbb{A}_F^S)/(B(\mathbb{A}_F^S)\cap K^{0,S}).$$

Let $D^0 = D_S \times K^{0,S}$, and note that since $A_S \cap D_S = 1$, we can (by the Bruhat decomposition 3.3.6) replace K^0 by D^0 in Lemma 4.2.2. Using the facts that E is finite, $|a| = 1$ for all $a \in F^\times$, and D^0 is compact, we then have that there exists $c \in \mathbb{R}_{>0}$ such that

$$G(\mathbb{A}_F) = G(F)(B(c)_S \times E)D^0.$$

For any finite subset R of $G(F)$ consisting of one element from each of those sets $G(F)\cap K^{0,S}e^{-1}$ with $e \in E$ that are nonempty, we obtain from this that

$$G(\mathbb{A}_{F,S}) = \Gamma_K R \cdot B(c)_S D_S.$$

As Γ_K is a finite union of right Γ-cosets, we may enlarge R and replace Γ_K by Γ. Finally, we can replace $B(c)_S$ by $C \cdot A(c)_S$ for some C by the compactness of the image of

$$B_u(\mathbb{A}_{F,S}) \to \Gamma\backslash G(\mathbb{A}_{F,S})/D_S$$

and then by $B(C; c_1, c_2)_S$ for some $c_1, c_2 \in \mathbb{R}_{>0}$ by the compactness of the cokernel of

$$\Gamma \cap B(\mathbb{A}_{F,S}) \to (B/B_u)(\mathbb{A}_{F,S})_1,$$

where $(B/B_u)(\mathbb{A}_{F,S})_1$ denotes the kernel of the homomorphism

$$(B/B_u)(\mathbb{A}_{F,S}) \to \mathbb{R}_{>0}^{d-1}, \qquad aB_u(\mathbb{A}_{F,S}) \mapsto \left(\prod_{v\in S}\left|\frac{a_{v,i}}{a_{v,i+1}}\right|\right)_{1\le i\le d-1}.$$

□

4.3 $\bar{X}_{F,S}$ and reduction theory

4.3.1. Let S be a nonempty finite set of places of F containing all archimedean places. We consider $\bar{X}_{F,S}$. From the results 4.2.6 and 4.2.4 of reduction theory, we will deduce results 4.3.4 and 4.3.10 on $\bar{X}_{F,S}$, respectively. We will also discuss other properties of $\bar{X}_{F,S}$ related to reduction theory. Let G, $(e_i)_i$, and B be as in §4.2.

For $c_1, c_2 \in \mathbb{R}_{>0}$ with $c_2 \ge 1$, we define a subset $\mathfrak{T}(c_1, c_2)$ of $(\mathbb{R}_{\ge 0}^S)^{d-1}$ by

$$\mathfrak{T}(c_1, c_2) = \{t \in (\mathbb{R}_{\ge 0}^S)^{d-1} \mid t_{v,i} \le c_1,\ t_{v,i} \le c_2 t_{w,i} \text{ for all } v, w \in S \text{ and } 1 \le i \le d-1\}.$$

Let $Y_0 = (\mathbb{R}^S_{>0} \cup \{(0)_{v\in S}\})^{d-1}$ as in 3.5.1 (for the parabolic B), and note that $\mathfrak{T}(c_1, c_2) \subset Y_0$. Define the subset $\mathfrak{S}(c_1, c_2)$ of $\bar{X}_{F,S}(B)$ as the image of $B_u(\mathbb{A}_{F,S}) \times \mathfrak{T}(c_1, c_2)$ under the map

$$\pi_S = (\pi_v)_{v\in S} : B_u(\mathbb{A}_{F,S}) \times Y_0 \to \bar{X}_{F,S}(B),$$

with π_v as in 3.3.3. For a compact subset C of $B_u(\mathbb{A}_{F,S})$, we let $\mathfrak{S}(C; c_1, c_2) \subset \mathfrak{S}(c_1, c_2)$ denote the image of $C \times \mathfrak{T}(c_1, c_2)$ under π_S.

4.3.2. We give an example of the sets of 4.3.1.

Example. Consider the case that $F = \mathbb{Q}$, the set S contains only the real place, and $d = 2$, as in §3.2. Fix a basis $(e_i)_{1\le i\le 2}$ of V. Identify $B_u(\mathbb{R})$ with $\mathbb{R}$ in the natural manner. We have

$$\mathfrak{S}(C; c_1, c_2) = \{x + yi \in \mathfrak{H} \mid x \in C, y \ge c_1^{-1}\} \cup \{\infty\},$$

which is contained in

$$\mathfrak{S}(c_1, c_2) = \{x + yi \in \mathfrak{H} \mid x \in \mathbb{R}, y \ge c_1^{-1}\} \cup \{\infty\}.$$

4.3.3. Fix a compact open subgroup K of $G(\mathbb{A}^S_F)$, and let $\Gamma_K \subset G(F)$ be the inverse image of K under $G(F) \to G(\mathbb{A}^S_F)$.

Proposition 4.3.4. *Let Γ be a subgroup of Γ_K of finite index. Then there exist c_1, c_2, C as in 4.3.1 and a finite subset R of $G(F)$ such that*

$$\bar{X}_{F,S} = \Gamma R \cdot \mathfrak{S}(C; c_1, c_2).$$

Proof. It suffices to prove the weaker statement that there are c_1, c_2, C and R such that

$$X_S = \Gamma R \cdot (X_S \cap \mathfrak{S}(C; c_1, c_2)).$$

Indeed, we claim that the proposition follows from this weaker statement for the spaces in the product $\prod_{v\in S} X_{(V_i/V_{i-1})_v}$, where P_I is as in 4.2.3 for a subset I of $\{1, \dots, d-1\}$ and $(V_i)_{-1\le i\le m}$ is the corresponding flag. To see this, first note that there is a finite subset R' of $G(F)$ such that every parabolic subgroup of G has the form $\gamma P_I \gamma^{-1}$ for some I and $\gamma \in \Gamma R'$. It then suffices to consider $a = (P, \mu) \in \bar{X}_{F,S}$, where $P = P_I$ for some I, and $\mu \in \mathfrak{Z}_{F,S}(P)$. We use the notation of 3.5.6 and 3.5.1. By Proposition 3.5.7, the set $\bar{X}_{F,S}(P) \cap \mathfrak{S}(C; c_1, c_2)$ is the image under ξ of the image of $C \times \mathfrak{T}(c_1, c_2)$ in $P_u(\mathbb{A}_{F,S}) \times \mathfrak{Z}_{F,S}(P) \times Y_0$. Note that a has image $(1, \mu, 0)$ in the latter set (for 1 the identity matrix of $P_u(\mathbb{A}_{F,S})$), and $\xi(1, \mu, 0) = a$. Since the projection of $\mathfrak{T}(c_1, c_2)$ (resp., C) to $(\mathbb{R}^S_{>0})^{\Delta_i}$ (resp., $B_{i,u}(\mathbb{A}_{F,S})$) is the analogous set for c_1 and c_2 (resp., a compact subset), the claim follows.

For $v \in S$, we define subsets Q_v and Q'_v of X_v as follows. If v is archimedean, let $Q_v = Q'_v$ be the one point set consisting of the element of X_v given by the basis $(e_i)_i$ and $(r_i)_i$ with $r_i = 1$ for all i. If v is non-archimedean, let Q_v (resp., Q'_v) be the subset of X_v consisting of elements given by $(e_i)_i$ and $(r_i)_i$ such that $1 = r_1 \le \cdots \le r_d \le q_v$ (resp., $r_1 = 1$ and $1 \le r_i \le q_v$ for $1 \le i \le d$). Then $X_v = G(F_v)Q_v$ for each $v \in S$. Hence by 4.2.6, there exist c'_1, c'_2, C as in 4.3.1 and a finite subset R of $G(F)$ such that

$$X_S = \Gamma R \cdot B(C; c'_1, c'_2)_S \cdot D_S Q_S,$$

where $Q_S = \prod_{v \in S} Q_v$.

We have $D_S Q_S = Q'_S$ for $Q'_S = \prod_{v \in S} Q'_v$, noting for archimedean (resp., non-archimedean) v that K_v^0 (resp., $\mathrm{Iw}(O_v)$) stabilizes all elements of Q_v. We have $B(C; c'_1, c'_2)_S Q'_S \subset \mathfrak{S}(C; c_1, c_2)$, where

$$c_1 = \max\{q_v \mid v \in S_f\}(c'_1)^{-1} \quad \text{and} \quad c_2 = \max\{q_v^2 \mid v \in S_f\}(c'_2)^{-1},$$

with S_f the set of all non-archimedean places in S (and taking the maxima to be 1 if $S_f = \varnothing$). □

4.3.5. For $v \in S$ and $1 \le i \le d-1$, let $t_{v,i} : \mathfrak{S}(c_1, c_2) \to \mathbb{R}_{\ge 0}$ be the map induced by $\phi'_{B,v} : \bar{X}_{F,v}(B) \to \mathbb{R}_{\ge 0}^{d-1}$ (see 3.4.6) and the ith projection $\mathbb{R}_{\ge 0}^{d-1} \to \mathbb{R}_{\ge 0}$. Note that $t_{v,i}$ is continuous.

4.3.6. Fix a subset I of $\{1, \dots, d-1\}$, and let P_I be the parabolic subgroup of G defined in 4.2.3. For $c_1, c_2, c_3 \in \mathbb{R}_{>0}$, let

$$\mathfrak{S}_I(c_1, c_2, c_3) = \{x \in \mathfrak{S}(c_1, c_2) \mid \min\{t_{v,i}(x) \mid v \in S\} \le c_3 \text{ for each } i \in I\}.$$

4.3.7. For an element $a \in \bar{X}_{F,S}$, we define the parabolic type of a to be the subset

$$\{\dim(V_i) \mid 0 \le i \le m-1\}$$

of $\{1, \dots, d-1\}$, where $(V_i)_{-1 \le i \le m}$ is the flag corresponding to the parabolic subgroup of G associated to a.

Lemma 4.3.8. *Let $a \in \bar{X}_{F,S}(B)$, and let J be the parabolic type of a. Then the parabolic subgroup of G associated to a is P_J.*

This is easily proved.

4.3.9. In the following, we will often consider subsets of $G(F)$ of the form $R_1 \Gamma_K R_2$, $\Gamma_K R$, or $R \Gamma_K$, where R_1, R_2, R are finite subsets of $G(F)$. These three types of cosets are essentially the same thing when we vary K. For finite subsets R_1, R_2 of $G(F)$, we have $R_1 \Gamma_K R_2 = R' \Gamma_{K'} = \Gamma_{K''} R''$ for some compact open subgroups K' and K'' of $G(\mathbb{A}_F^S)$ contained in K and finite subsets R' and R'' of $G(F)$.

Proposition 4.3.10. *Given $c_1 \in \mathbb{R}_{>0}$ and finite subsets R_1, R_2 of $G(F)$, there exists $c_3 \in \mathbb{R}_{>0}$ such that for all $c_2 \in \mathbb{R}_{>0}$ we have*

$$\{\gamma \in R_1 \Gamma_K R_2 \mid \gamma \mathfrak{S}_I(c_1, c_2, c_3) \cap \mathfrak{S}(c_1, c_2) \neq \varnothing\} \subset P_I(F).$$

Proof. First we prove the weaker version that c_3 exists if the condition on $\gamma \in R_1 \Gamma_K R_2$ is replaced by $\gamma \mathfrak{S}_I(c_1, c_2, c_3) \cap \mathfrak{S}(c_1, c_2) \cap X_S \neq \varnothing$.

Let Q'_v for $v \in S$ and Q'_S be as in the proof of 4.3.4.

Claim 1. If $c'_1 \in \mathbb{R}_{>0}$ is sufficiently small (independent of c_2), then we have

$$X_S \cap \mathfrak{S}(c_1, c_2) \subset B(c'_1)_S Q'_S.$$

Proof of Claim 1. Any $x \in X_S \cap \mathfrak{S}(c_1, c_2)$ satisfies $t_{v,i}(x) \leq c_1$ for $1 \leq i \leq d-1$. Moreover, if $\prod_{v \in S} t_{v,i}(x)$ is sufficiently small relative to $(c'_1)^{-1}$ for all such i, then $x \in B(c'_1)_S Q'_S$. The claim follows.

Let C_v denote the compact set

$$C_v = \{g \in G(F_v) \mid g Q'_v \cap Q'_v \neq \varnothing\}.$$

If v is archimedean, then C_v is the maximal compact open subgroup K^0_v of 4.2.1. Set $C_S = \prod_{v \in S} C_v$. We use the decomposition $G(\mathbb{A}_F) = G(\mathbb{A}_{F,S}) \times G(\mathbb{A}^S_F)$ to write elements of $G(\mathbb{A}_F)$ as pairs.

Claim 2. Fix $c'_1 \in \mathbb{R}_{>0}$. The subset $B(c'_1)_S C_S \times R_1 K R_2$ of $G(\mathbb{A}_F)$ is contained in $B(c) K^0$ for sufficiently small $c \in \mathbb{R}_{>0}$.

Proof of Claim 2. This follows from the compactness of the C_v for $v \in S$ and the Iwasawa decomposition $G(\mathbb{A}_F) = B(\mathbb{A}_F) K^0$.

Claim 3. Let c'_1 be as in Claim 1, and let $c \leq c'_1$. Let $c' \in \mathbb{R}_{>0}$. If $c_3 \in \mathbb{R}_{>0}$ is sufficiently small (independent of c_2), we have

$$X_S \cap \mathfrak{S}_I(c_1, c_2, c_3) \subset B_I(c, c')_S Q'_S,$$

where $B_I(c, c')_S = B(\mathbb{A}_{F,S}) \cap B_I(c, c')$.

Proof of Claim 3. An element $x \in B(c)_S Q'_S$ lies in $B_I(c, c')_S Q'_S$ if $\prod_{v \in S} t_{v,i}(x) \leq (c')^{-1}$ for all $i \in I$. An element $x \in X_S \cap \mathfrak{S}(c_1, c_2)$ lies in $X_S \cap \mathfrak{S}_I(c_1, c_2, c_3)$ if $\min\{t_{v,i}(x) \mid v \in S\} \leq c_3$ for all $i \in I$. In this case, x will lie in $B_I(c, c')_S Q'_S$ if $c_3 \leq (c')^{-1} c_1^{1-s}$, with $s = \sharp S$.

Let c_1' be as in Claim 1, take c of Claim 2 for this c_1' such that $c \le c_1'$, and let $c' \in \mathbb{R}_{>0}$. Take c_3 satisfying the condition of Claim 3 for these c_1', c, and c'.

Claim 4. If $X_S \cap \mathfrak{S}_I(c_1, c_2, c_3) \cap \gamma^{-1}\mathfrak{S}(c_1, c_2)$ is nonempty for some $\gamma \in R_1\Gamma R_2 \subset G(F)$, then $B_I(c, c') \cap \gamma^{-1}B(c)K^0$ contains an element of $G(\mathbb{A}_{F,S}) \times \{1\}$.

Proof of Claim 4. By Claim 3, any $x \in X_S \cap \mathfrak{S}_I(c_1, c_2, c_3) \cap \gamma^{-1}\mathfrak{S}(c_1, c_2)$ lies in gQ_S' for some $g \in B_I(c, c')_S$. By Claim 1, we have $\gamma x \in g'Q_S'$ for some $g' \in B(c_1')_S$. Since $\gamma x \in \gamma g Q_S' \cap g'Q_S'$, we have $(g')^{-1}\gamma g \in C_S$. Hence $\gamma g \in B(c_1')_S C_S$, and therefore $\gamma(g, 1) = (\gamma g, \gamma) \in B(c)K^0$ by Claim 2.

We prove the weaker version of 4.3.10: let $x \in X_S \cap \mathfrak{S}_I(c_1, c_2, c_3) \cap \gamma^{-1}\mathfrak{S}(c_1, c_2)$ for some $\gamma \in R_1\Gamma R_2$. Then by Claim 4 and Lemma 4.2.4, with c' satisfying the condition of 4.2.4 for the given c, we have $\gamma \in P_I(F)$.

We next reduce the proposition to the weaker version, beginning with the following.

Claim 5. Let $c_1, c_2 \in \mathbb{R}_{>0}$. If $\gamma \in G(F)$ and $x \in \mathfrak{S}(c_1, c_2) \cap \gamma^{-1}\mathfrak{S}(c_1, c_2)$, then $\gamma \in P_J(F)$, where J is the parabolic type of x.

Proof of Claim 5. By Lemma 4.3.8, the parabolic subgroup associated to x is P_J and that associated to γx is P_J. Hence $\gamma P_J \gamma^{-1} = P_J$. Since a parabolic subgroup coincides with its normalizer, we have $\gamma \in P_J(F)$.

Fix $J \subset \{1, \dots, d-1\}$, $\xi \in R_1$, and $\eta \in R_2$.

Claim 6. There exists $c_3 \in \mathbb{R}_{>0}$ such that if $\gamma \in \Gamma_K$ and $x \in \mathfrak{S}_I(c_1, c_2, c_3) \cap (\xi\gamma\eta)^{-1}\mathfrak{S}(c_1, c_2)$ is of parabolic type J, then $\xi\gamma\eta \in P_I(F)$.

Proof of Claim 6. Let $(V_i)_{-1 \le i \le m}$ be the flag corresponding to P_J. Suppose that we have $\gamma_0 \in \Gamma_K$ and $x_0 \in \mathfrak{S}(c_1, c_2) \cap (\xi\gamma_0\eta)^{-1}\mathfrak{S}(c_1, c_2)$ of parabolic type J. By Claim 5, we have $\xi\gamma_0\eta, \xi\gamma\eta \in P_J(F)$. Hence

$$\xi\gamma\eta = \xi(\gamma\gamma_0^{-1})\xi^{-1}\xi\gamma_0\eta \in \Gamma_{K'}\eta',$$

where K' is the compact open subgroup $\xi K \xi^{-1} \cap P_J(\mathbb{A}_F^S)$ of $P_J(\mathbb{A}_F^S)$, and $\eta' = \xi\gamma_0\eta \in P_J(F)$. The claim follows from the weaker version of the proposition in which V is replaced by V_i/V_{i-1} (for $0 \le i \le m$), the group G is replaced by $\mathrm{PGL}_{V_i/V_{i-1}}$, the compact open subgroup K is replaced by the image of $\xi K \xi^{-1} \cap P_J(\mathbb{A}_F^S)$ in $\mathrm{PGL}_{V_i/V_{i-1}}(\mathbb{A}_F^S)$, the set R_1 is replaced by $\{1\}$, the set R_2 is replaced by the image of $\{\eta'\}$ in $\mathrm{PGL}_{V_i/V_{i-1}}(F)$, and $P_I(F)$ is replaced by the image of $P_I(F) \cap P_J(F)$ in $\mathrm{PGL}_{V_i/V_{i-1}}(F)$.

By Claim 6 for all J, ξ, and η, the result is proven. $\square$

Lemma 4.3.11. *Let $1 \le i \le d-1$, and let $V' = \sum_{j=1}^{i} F e_j$. Let $x \in X_v$ for some $v \in S$, and let $g \in \mathrm{GL}_V(F_v)$ be such that $gV' = V'$. For $1 \le i \le d-1$, we have*

$$\prod_{j=1}^{d-1} \left(\frac{t_{v,j}(gx)}{t_{v,j}(x)} \right)^{e(i,j)} = \frac{|\det(g_v \colon V'_v \to V'_v)|^{(d-i)/i}}{|\det(g_v \colon V_v/V'_v \to V_v/V'_v)|},$$

where

$$e(i,j) = \begin{cases} \frac{j(d-i)}{i} & \text{if } j \le i, \\ d-j & \text{if } j \ge i. \end{cases}$$

Proof. By the Iwasawa decomposition 3.3.4 and 3.4.5, it suffices to check this in the case that g is represented by a diagonal matrix $\mathrm{diag}(a_1, \dots, a_d)$. It follows from the definitions that $t_{v,j}(gx) t_{v,j}(x)^{-1} = |a_j a_{j+1}^{-1}|$, and the rest of the verification is a simple computation. □

Lemma 4.3.12. *Let i and V' be as in 4.3.11. Let R_1 and R_2 be finite subsets of $G(F)$. Then there exist $A, B \in \mathbb{R}_{>0}$ such that for all $\gamma \in \mathrm{GL}_V(F)$ with image in $R_1 \Gamma_K R_2 \subset G(F)$ and for which $\gamma V' = V'$, we have*

$$A \le \prod_{v \in S} \frac{|\det(\gamma \colon V'_v \to V'_v)|^{(d-i)/i}}{|\det(\gamma \colon V_v/V'_v \to V_v/V'_v)|} \le B.$$

Proof. We may assume that R_1 and R_2 are one point sets $\{\xi\}$ and $\{\eta\}$, respectively. Suppose that an element γ_0 with the stated properties of γ exists. Then for any such γ, the image of $\gamma\gamma_0^{-1}$ in $G(F)$ belongs to $\xi \Gamma_K \xi^{-1} \cap P_{\{i\}}(F)$, and hence the image of $\gamma\gamma_0^{-1}$ in $G(\mathbb{A}_F^S)$ belongs to the compact subgroup $\xi K \xi^{-1} \cap P_{\{i\}}(\mathbb{A}_F^S)$ of $P_{\{i\}}(\mathbb{A}_F^S)$. Hence

$$|\det(\gamma\gamma_0^{-1} : V'_v \to V'_v)| = |\det(\gamma\gamma_0^{-1} : V_v/V'_v \to V_v/V'_v)| = 1$$

for every place v of F which does not belong to S. By Lemma 4.3.11 and the product formula, we have

$$\prod_{v \in S} (|\det(\gamma\gamma_0^{-1} \colon V'_v \to V'_v)|^{(d-i)/i} \cdot |\det(\gamma\gamma_0^{-1} \colon V_v/V'_v \to V_v/V'_v)|^{-1}) = 1,$$

so the value of the product in the statement is constant under our assumptions, proving the result. □

Proposition 4.3.13. *Fix $c_1, c_2 \in \mathbb{R}_{>0}$ and finite subsets R_1, R_2 of $G(F)$. Then there exists $A > 1$ such that if $x \in \mathfrak{S}(c_1, c_2) \cap \gamma^{-1} \mathfrak{S}(c_1, c_2)$ for some $\gamma \in R_1 \Gamma_K R_2$, then*

$$A^{-1} t_{v,i}(x) \le t_{v,i}(\gamma x) \le A t_{v,i}(x)$$

for all $v \in S$ and $1 \le i \le d-1$.

Proof. By a limit argument, it is enough to consider $x \in X_S \cap \mathfrak{S}(c_1, c_2)$. Fix $v \in S$. For $x, x' \in X_S \cap \mathfrak{S}(c_1, c_2)$ and $1 \le i \le d-1$, let $s_i(x, x') = t_{v,i}(x') t_{v,i}(x)^{-1}$.

For each $1 \le i \le d-1$, take $c_3(i) \in \mathbb{R}_{>0}$ satisfying the condition in 4.3.10 for the set $I = \{i\}$ and both pairs of finite subsets R_1, R_2 and R_2^{-1}, R_1^{-1} of $G(F)$. Let

$$c_3 = \min\{c_3(i) \mid 1 \le i \le d-1\}.$$

For a subset I of $\{1, \dots, d-1\}$, let $Y(I)$ be the subset of $(X_S \cap \mathfrak{S}(c_1, c_2))^2$ consisting of all pairs (x, x') such that $x' = \gamma x$ for some $\gamma \in R_1 \Gamma_K R_2$ and such that

$$I = \{1 \le i \le d-1 \mid \min(t_{v,i}(x), t_{v,i}(x')) \le c_3\}.$$

For the proof of 4.3.13, it is sufficient to prove the following statement (S_d), fixing I.

(S_d) There exists $A > 1$ such that $A^{-1} \le s_i(x, x') \le A$ for all $(x, x') \in Y(I)$ and $1 \le i \le d-1$.

By Proposition 4.3.10, if $\gamma \in R_1 \Gamma_K R_2$ is such that there exists $x \in X_S$ with $(x, \gamma x) \in Y(I)$, then $\gamma \in P_{\{i\}}(F)$ for all $i \in I$. Lemmas 4.3.11 and 4.3.12 then imply the following for all $i \in I$, noting that $c_2^{-1} t_{w,i}(y) \le t_{v,i}(y) \le c_2 t_{w,i}(y)$ for all $w \in S$ and $y \in X_S \cap \mathfrak{S}(c_1, c_2)$.

(T_i) There exists $B_i > 1$ such that for all $(x, x') \in Y(I)$, we have

$$B_i^{-1} \le \prod_{j=1}^{d-1} s_j(x, x')^{e(i,j)} \le B_i,$$

where $e(i, j)$ is as in 4.3.11.

We prove the following statement (S_i) for $0 \le i \le d-1$ by induction on i.

(S_i) There exists $A_i > 1$ such that $A_i^{-1} \le s_j(x, x') \le A_i$ for all $(x, x') \in Y(I)$ and all j such that $1 \le j \le i$ and j is not the largest element of $I \cap \{1, \dots, i\}$ (if it is nonempty).

That (S_0) holds is clear. Assume that (S_{i-1}) holds for some $i \ge 1$. If $i \notin I$, then since $c_3 \le t_{v,i}(x) \le c_1$ and $c_3 \le t_{v,i}(x') \le c_1$, we have

$$\frac{c_3}{c_1} \le s_i(x, x') \le \frac{c_1}{c_3},$$

and hence (S_i) holds with $A_i := \max(A_{i-1}, c_1 c_3^{-1})$.

Assume that $i \in I$. If $I \cap \{1, \dots, i-1\} = \varnothing$, then (S_i) is evidently true with $A_i := A_{i-1}$. If $I \cap \{1, \dots, i-1\} \ne \varnothing$, then let i' be the largest element of this intersection. We compare (T_i) and $(T_{i'})$. We have $e(i, j) = e(i', j)$ if $j \ge i$ and $e(i, j) < e(i', j)$ if $j < i$, so taking the quotient of the equations in $(T_{i'})$ and (T_i), we have

$$(B_i B_{i'})^{-1} \le \prod_{j=1}^{i-1} s_j(x, x')^{e(i',j)-e(i,j)} \le B_i B_{i'}.$$

Since (S_{i-1}) is assumed to hold, there then exists $a \in \mathbb{R}_{>0}$ such that

$$(B_i B_{i'})^{-1} A_{i-1}^{-a} \le s_{i'}(x, x')^{e(i',i')-e(i,i')} \le B_i B_{i'} A_{i-1}^{a}$$

As the exponent $e(i', i') - e(i, i')$ is nonzero, this implies that (S_i) holds.

By induction, we have (S_{d-1}). To deduce (S_d) from it, we may assume that I is nonempty, and let i be the largest element of I. Then (S_{d-1}) and (T_i) imply (S_d). □

Proposition 4.3.14. *Let $c_1, c_2 \in \mathbb{R}_{>0}$ and $a \in \bar{X}_{F,S}$. Let I be the parabolic type (4.3.7) of a. Fix a finite subset R of $G(F)$ and $1 \le i \le d-1$.*

(1) If $i \in I$, then for any $\epsilon > 0$, there exists a neighborhood U of a in $\bar{X}_{F,S}$ for the Satake topology such that $\max\{t_{v,i}(x) \mid v \in S\} < \epsilon$ for all $x \in (\Gamma_K R)^{-1} U \cap \mathfrak{S}(c_1, c_2)$.

(2) If $i \notin I$, then there exist a neighborhood U of a in $\bar{X}_{F,S}$ for the Satake topology and $c \in \mathbb{R}_{>0}$ such that $\min\{t_{v,i}(x) \mid v \in S\} \ge c$ for all $x \in (\Gamma_K R)^{-1} U \cap \mathfrak{S}(c_1, c_2)$.

Proof. The first statement is clear by continuity of $t_{v,i}$ and the fact that $t_{v,i}(\gamma^{-1} a) = 0$ for all $\gamma \in G(F)$, and the second follows from 4.3.13, noting 4.3.4. □

Proposition 4.3.15. *Let $a \in \bar{X}_{F,S}$, and let P be the parabolic subgroup of PGL_V associated to a. Let $\Gamma_{K,(P)} \subset \Gamma_K$ be as in 3.4.12. Then there are $c_1, c_2 \in \mathbb{R}_{>0}$ and $\varphi \in G(F)$ such that $\Gamma_{K,(P)} \varphi \mathfrak{S}(c_1, c_2)$ is a neighborhood of a in $\bar{X}_{F,S}$ for the Satake topology.*

Proof. This holds by definition of the Satake topology with $\varphi = 1$ if $a \in \bar{X}_{F,S}(B)$. In general, let I be the parabolic type of a. Then the parabolic subgroup associated to a has the form $\varphi P_I \varphi^{-1}$ for some $\varphi \in G(F)$. We have $\varphi^{-1} a \in \bar{X}_{F,S}(P_I) \subset \bar{X}_{F,S}(B)$. By that already proven case, there exists $\gamma \in \Gamma_{K,(P_I)}$ such that $\Gamma_{K,(P)} \varphi \gamma \mathfrak{S}(c_1, c_2)$ is a neighborhood of a for the Satake topoology. □

The following result can be proved in the manner of 4.4.8 for $\bar{X}_{F,S}^{\flat}$ below, replacing R by $\{\varphi\}$, and $\Gamma_{K,(W)}$ by $\Gamma_{K,(P)}$.

Lemma 4.3.16. *Let the notation be as in 4.3.15. Let U' be a neighborhood of $\varphi^{-1} a$ in $\bar{X}_{F,S}$ for the Satake topology. Then there is a neighborhood U of a in $\bar{X}_{F,S}$ for the Satake topology such that*

$$U \subset \Gamma_{K,(P)} \varphi(\mathfrak{S}(c_1, c_2) \cap U').$$

4.4 $\bar{X}^{\flat}_{F,S}$ and reduction theory

4.4.1. Let S be a finite set of places of F containing the archimedean places. In this subsection, we consider $\bar{X}^{\flat}_{F,S}$. Fix a basis $(e_i)_{1\leq i\leq d}$ of V. Let $B\subset G=\mathrm{PGL}_V$ be the Borel subgroup of upper triangular matrices for $(e_i)_i$. Let K be a compact open subgroup of $G(\mathbb{A}_F^S)$.

4.4.2. Let $c_1,c_2\in\mathbb{R}_{>0}$. We let $\mathfrak{S}^{\flat}(c_1,c_2)$ denote the image of $\mathfrak{S}(c_1,c_2)$ under $\bar{X}_{F,S}\to\bar{X}^{\flat}_{F,S}$. For $r\in\{1,\dots,d-1\}$, we then define

$$\mathfrak{S}^{\flat}_r(c_1,c_2)=\{(W,\mu)\in\mathfrak{S}^{\flat}(c_1,c_2)\mid \dim(W)\geq r\}.$$

Then the maps $t_{v,i}$ of 4.3.5 for $v\in S$ and $1\leq i\leq r$ induce maps

$$t_{v,i}\colon\mathfrak{S}^{\flat}_r(c_1,c_2)\to\mathbb{R}_{>0}\ \ (1\leq i\leq r-1)\quad\text{and}\quad t_{v,r}\colon\mathfrak{S}^{\flat}_r(c_1,c_2)\to\mathbb{R}_{\geq 0}.$$

For $c_3\in\mathbb{R}_{>0}$, we also set

$$\mathfrak{S}^{\flat}_r(c_1,c_2,c_3)=\{x\in\mathfrak{S}^{\flat}_r(c_1,c_2)\mid \min\{t_{v,r}(x)\mid v\in S\}\leq c_3\}.$$

Proposition 4.4.3. *Fix $c_1\in\mathbb{R}_{>0}$ and finite subsets R_1,R_2 of $G(F)$. Then there exists $c_3\in\mathbb{R}_{>0}$ such that for all $c_2\in\mathbb{R}_{>0}$, we have*

$$\{\gamma\in R_1\Gamma_K R_2\mid \gamma\mathfrak{S}^{\flat}_r(c_1,c_2,c_3)\cap\mathfrak{S}^{\flat}_r(c_1,c_2)\neq\emptyset\}\subset P_{\{r\}}.$$

Proof. Take $(W,\mu)\in\gamma\mathfrak{S}^{\flat}_r(c_1,c_2,c_3)\cap\mathfrak{S}^{\flat}_r(c_1,c_2)$, and let $r'=\dim W$. Let P be the parabolic subgroup of V corresponding to the flag $(V_i)_{-1\leq i\leq d-r'}$ with $V_i=W+\sum_{j=r'+1}^{r'+i}Fe_i$ for $0\leq i\leq d-r'$. Let $\mu'\in\mathfrak{Z}_{F,S}(P)$ be the unique element such that $a=(P,\mu')\in\bar{X}_{F,S}$ maps to (W,μ). Then $a\in\gamma\mathfrak{S}_{\{r\}}(c_1,c_2,c_3)\cap\mathfrak{S}(c_1,c_2)$, so we can apply 4.3.10. □

Proposition 4.4.4. *Fix $c_1,c_2\in\mathbb{R}_{>0}$ and finite subsets R_1,R_2 of $G(F)$. Then there exists $A>1$ such that if $x\in\mathfrak{S}^{\flat}_r(c_1,c_2)\cap\gamma^{-1}\mathfrak{S}^{\flat}_r(c_1,c_2)$ for some $\gamma\in R_1\Gamma_K R_2$, then*

$$A^{-1}t_{v,i}(x)\leq t_{v,i}(\gamma x)\leq A\,t_{v,i}(x)$$

for all $v\in S$ and $1\leq i\leq r$.

Proof. This follows from 4.3.13. □

We also have the following easy consequence of Lemma 4.3.8.

Lemma 4.4.5. *Let a be in the image of $\bar{X}_{F,S}(B)\to\bar{X}^{\flat}_{F,S}$, and let r be the dimension of the F-subspace of V associated to a. Then the F-subspace of V associated to a is $\sum_{i=1}^{r}Fe_i$.*

Proposition 4.4.6. *Let $a \in \bar{X}^{\flat}_{F,S}$ and let r be the dimension of the F-subspace of V associated to a. Let $c_1, c_2 \in \mathbb{R}_{>0}$. Fix a finite subset R of $G(F)$.*

(1) For any $\epsilon > 0$, there exists a neighborhood U of a in $\bar{X}^{\flat}_{F,S}$ for the Satake topology such that $\max\{t_{v,r}(x) \mid v \in S\} < \epsilon$ for all $x \in (\Gamma_K R)^{-1} U \cap \mathfrak{S}^{\flat}_r(c_1, c_2)$.

(2) If $1 \le i < r$, then there exist a neighborhood U of a in $\bar{X}^{\flat}_{F,S}$ for the Satake topology and $c \in \mathbb{R}_{>0}$ such that $\min\{t_{v,i}(x) \mid v \in S\} \ge c$ for all $x \in (\Gamma_K R)^{-1} U \cap \mathfrak{S}^{\flat}_r(c_1, c_2)$.

Proof. This follows from 4.4.4, as in the proof of 4.3.14. □

Proposition 4.4.7. *Let W be an F-subspace of V of dimension $r \ge 1$. Let Φ be set of $\varphi \in G(F)$ such that $\varphi(\sum_{i=1}^{r} Fe_i) = W$.*

(1) There exists a finite subset R of Φ such that for any $a \in \bar{X}^{\flat}_{F,S}(W)$, there exist $c_1, c_2 \in \mathbb{R}_{>0}$ for which the set $\Gamma_{K,(W)} R \mathfrak{S}^{\flat}_r(c_1, c_2)$ is a neighborhood of a in the Satake topology.

(2) For any $\varphi \in \Phi$ and $a \in \bar{X}^{\flat}_{F,S}$ with associated subspace W, there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that $a \in \varphi \mathfrak{S}^{\flat}_r(c_1, c_2)$ and $\Gamma_{K,(W)} \varphi \mathfrak{S}^{\flat}_r(c_1, c_2)$ is a neighborhood of a in the Satake topology.

Proof. We may suppose without loss of generality that $W = \sum_{j=1}^{r} Fe_j$, in which case $\Phi = G(F)_{(W)}$ (see 3.4.12). Consider the set $\mathcal{Q}$ of all parabolic subgroups Q of G such that W is contained in the smallest nonzero subspace of V preserved by Q. Any $Q \in \mathcal{Q}$ has the form $Q = \varphi P_I \varphi^{-1}$ for some $\varphi \in G(F)_{(W)}$ and subset I of $J := \{i \in \mathbb{Z} \mid r \le i \le d-1\}$. There exists a finite subset R of $G(F)_{(W)}$ such that we may always choose $\varphi \in \Gamma_{K,(W)} R$.

By 4.4.5, an element of $\bar{X}_{F,S}(B)$ has image in $\bar{X}^{\flat}_{F,S}(W)$ if and only if the parabolic subgroup associated to it has the form P_I for some $I \subset J$. The intersection of the image of $\bar{X}_{F,S}(B) \to \bar{X}^{\flat}_{F,S}$ with $\bar{X}^{\flat}_{F,S}(W)$ is the union of the $\mathfrak{S}^{\flat}_r(c_1, c_2)$ with $c_1, c_2 \in \mathbb{R}_{>0}$. By the above, for any $a \in \bar{X}^{\flat}_{F,S}(W)$, we may choose $\xi \in \Gamma_{K,(W)} R$ such that $\xi^{-1} a$ is in this intersection, and part (1) follows. Moreover, if W is the subspace associated to a, then $\varphi^{-1} a \in \bar{X}^{\flat}_{F,S}(W)$ is in the image of $\bar{X}_{F,S}(B)$ for all $\varphi \in G(F)_{(W)}$, from which (2) follows. □

Lemma 4.4.8. *Let W, Φ, R be as in 4.4.7, fix $a \in \bar{X}^{\flat}_{F,S}(W)$, and let $c_1, c_2 \in \mathbb{R}_{>0}$ be as in 4.4.7(1) for this a. For each $\varphi \in R$, let U_φ be a neighborhood of $\varphi^{-1} a$ in $\bar{X}^{\flat}_{F,S}$ for the Satake topology. Then there is a neighborhood U of a in $\bar{X}^{\flat}_{F,S}$ for the Satake topology such that*

$$U \subset \bigcup_{\varphi \in R} \Gamma_{K,(W)} \varphi(\mathfrak{S}^{\flat}_r(c_1, c_2) \cap U_\varphi).$$

Proof. We may assume that each $\varphi(U_\varphi)$ is stable under the action of $\Gamma_{K,(W)}$. Let

$$U = \Gamma_{K,(W)} R \mathfrak{S}^{\flat}_r(c_1, c_2) \cap \bigcap_{\varphi \in R} \varphi(U_\varphi).$$

Then U is a neighborhood of a by 4.4.7(1). Let $x \in U$. Take $\gamma \in \Gamma_{K,(W)}$ and $\varphi \in R$ such that $x \in \gamma\varphi\mathfrak{S}^\flat_r(c_1, c_2)$. Since $\varphi(U_\varphi)$ is $\Gamma_{K,(W)}$-stable, $\gamma^{-1}x \in \varphi(U_\varphi)$ and hence $\varphi^{-1}\gamma^{-1}x \in \mathfrak{S}^\flat_r(c_1, c_2) \cap U_\varphi$. □

Proposition 4.4.9. *Let $a = (W,\mu) \in \bar{X}^\flat_{F,S}$, and let $r = \dim(W)$. Take $\varphi \in G(F)$ and $c_1, c_2 \in \mathbb{R}_{>0}$ as in 4.4.7(2) such that $\Gamma_{K,(W)}\varphi\mathfrak{S}^\flat_r(c_1, c_2)$ is a neighborhood of a. Let $\phi^\flat_{W,S}\colon \bar{X}^\flat_{F,S}(W) \to \mathfrak{Z}^\flat_{F,S}(W)$ be as in 3.4.4. For any neighborhood U of $\mu = \phi^\flat_{W,S}(a)$ in $\mathfrak{Z}^\flat_{F,S}(W)$ and any $\epsilon \in \mathbb{R}_{>0}$, set*

$$\Phi(U,\epsilon) = (\phi^\flat_{W,S})^{-1}(U) \cap \Gamma_{K,(W)}\varphi\{x \in \mathfrak{S}^\flat_r(c_1, c_2) \mid t_{v,r}(x) < \epsilon \text{ for all } v \in S\}.$$

Then the set of all $\Phi(U,\epsilon)$ forms a base of neighborhoods of a in $\bar{X}^\flat_{F,S}$ under the Satake topology.

Proof. We may suppose that $W = \sum_{i=1}^r Fe_i$ without loss of generality, in which case $\varphi \in G(F)_{(W)}$. Let P be the smallest parabolic subgroup containing B with flag $(V_i)_{-1\le i\le m}$ such that $V_0 = W$ and $m = d - r$. Let Q be the parabolic of all elements that preserve W. We then have $G \supset Q \supset P \supset B$. Let B' be the Borel subgroup of $\mathrm{PGL}_{V/W}$ that is the image of P and which we regard as a subgroup of G using $(e_{r+i})_{1\le i\le m}$ to split $V \to V/W$.

Let

$$f_v\colon Q_u(F_v) \times \bar{X}_{V/W,F,v}(B') \times X_{W_v} \times \mathbb{R}_{\ge 0} \to \bar{X}_{F,v}(P)$$

be the unique surjective continuous map such that $\xi = f_v \circ h$, where ξ is as in 3.5.6 and h is defined as the composition

$$P_u(F_v) \times X_{W_v} \times \mathbb{R}^m_{\ge 0} \xrightarrow{\sim} Q_u(F_v) \times B'_u(F_v) \times X_{W_v} \times \mathbb{R}_{\ge 0} \times \mathbb{R}^{m-1}_{\ge 0} \to Q_u(F_v) \times \bar{X}_{V/W,F,v}(B') \times X_{W_v} \times \mathbb{R}_{\ge 0}$$

of the map induced by the isomorphism $P_u(F_v) \xrightarrow{\sim} Q_u(F_v) \times B'_u(F_v)$ and the map induced by the surjection $\bar{\pi}_{B',v}\colon B'_u(F_v) \times \mathbb{R}^{m-1}_{\ge 0} \to \bar{X}_{V/W,F,v}(B')$ of 3.3.8(2). The existence of f_v follows from 3.3.8(4).

Set $Y_0 = \mathbb{R}^S_{>0} \cup \{(0)_{v\in S}\}$, and let

$$f_S\colon Q_u(\mathbb{A}_{F,S}) \times \bar{X}_{V/W,F,S}(B') \times \mathfrak{Z}^\flat_{F,S}(W) \times Y_0 \to \bar{X}_{F,S}(P)$$

be the product of the maps f_v. Let $t_{v,r}\colon \bar{X}_{F,v}(P) \to \mathbb{R}_{\ge 0}$ denote the composition

$$\bar{X}_{F,v}(P) \to \bar{X}_{F,v}(B) \xrightarrow{\phi'_{B,v}} \mathbb{R}^{d-1}_{\ge 0} \to \mathbb{R}_{\ge 0},$$

where the last arrow is the rth projection. The composition of f_S with $(t_{v,r})_{v\in S}$ is projection onto Y_0 by 3.5.7 and 3.4.6.

Let $\bar{X}_{F,S}(W)$ denote the inverse image of $\bar{X}^\flat_{F,S}(W)$ under the canonical surjection $\Pi_S\colon \bar{X}_{F,S} \to \bar{X}^\flat_{F,S}$. Combining f_S with the action of $G(F)_{(W)}$, we obtain a surjective map

$$f'_S\colon G(F)_{(W)} \times (Q_u(\mathbb{A}_{F,S}) \times \bar{X}_{V/W,F,S}(B') \times \mathfrak{Z}^\flat_{F,S}(W) \times Y_0) \to \bar{X}_{F,S}(W), \qquad f'_S(g,z) = g\, f_S(z).$$

The composition of f'_S with $\phi^\flat_{W,S} \circ \Pi_S$ is projection onto $\mathfrak{Z}^\flat_{F,S}(W)$ by 3.5.9 and 3.5.10.

Applying 4.3.4 with V/W in place of V, there exists a compact subset C of $Q_u(\mathbb{A}_{F,S}) \times \bar{X}_{V/W,F,S}(B')$ and a finite subset R of $G(F)_{(W)}$ such that $f'_S(\Gamma_{K,(W)}R \times C \times \mathfrak{Z}^\flat_{F,S}(W) \times Y_0) = \bar{X}_{F,S}(W)$. Consider the restriction of $\Pi_S \circ f'_S$ to a surjective map

$$\lambda_S \colon \Gamma_{K,(W)}R \times C \times \mathfrak{Z}^\flat_{F,S}(W) \times Y_0 \to \bar{X}^\flat_{F,S}(W).$$

We may suppose that R contains φ, since it lies in $G(F)_{(W)}$.

Now, let U' be a neighborhood of a in $\bar{X}^\flat_{F,S}(W)$ for the Satake topology. It is sufficient to prove that there exist an open neighborhood U of μ in $\mathfrak{Z}^\flat_{F,S}(W)$ and $\epsilon \in \mathbb{R}_{>0}$ such that $\Phi(U,\epsilon) \subset U'$. For $\epsilon \in \mathbb{R}_{>0}$, set $Y_\epsilon = \{(t_v)_{v \in S} \in Y_0 \mid t_v < \epsilon \text{ for all } v \in S\}$.

For any $x \in C$, we have $\lambda_S(\alpha, x, \mu, 0) = (W,\mu) \in U'$ for all $\alpha \in R$. By the continuity of λ_S, there exist a neighborhood $D(x) \subset Q_u(\mathbb{A}_{F,S}) \times \bar{X}_{V/W,F,S}(B')$ of x, a neighborhood $U(x) \subset \mathfrak{Z}^\flat_{F,v}(W)$ of μ, and $\epsilon(x) \in \mathbb{R}_{>0}$ such that

$$\lambda_S(R \times D(x) \times U(x) \times Y_{\epsilon(x)}) \subset U'.$$

Since C is compact, some finite collection of the sets $D(x)$ cover C. Thus, there exist a neighborhood U of μ in $\mathfrak{Z}^\flat_{F,v}(W)$ and $\epsilon \in \mathbb{R}_{>0}$ such that $\lambda_S(R \times C \times U \times Y_\epsilon) \subset U'$. Since U' is $\Gamma_{K,(W)}$-stable by 3.4.15, we have $\lambda_S(\Gamma_{K,(W)}R \times C \times U \times Y_\epsilon) \subset U'$.

Let $y \in \Phi(U,\epsilon)$, and write $y = gx$ with $g \in \Gamma_{K,(W)}\varphi$ and $x \in \mathfrak{S}^\flat_r(c_1,c_2)$ such that $t_{v,r}(x) < \epsilon$ for all $v \in S$. Since $\Phi(U,\epsilon) \subset \bar{X}^\flat_{F,S}(W)$, we may by our above remarks write $y = \lambda_S(g,c,\nu,t) = g\Pi_S(f_S(c,\nu,t))$, where $c \in C$, $\nu = \phi^\flat_{W,S}(y)$, and $t = (t_{v,r}(x))_{v \in S}$. Since $\nu \in U$ and $t \in Y_\epsilon$ by definition, y is contained in U'. Therefore, we have $\Phi(U,\epsilon) \subset U'$. □

Example 4.4.10. Consider the case $F = \mathbb{Q}$, $S = \{v\}$ with v the archimedean place, and $d = 3$. We construct a base of neighborhoods of a point in $\bar{X}^\flat_{\mathbb{Q},v}$ for the Satake topology.

Fix a basis $(e_i)_{1 \le i \le 3}$ of V. Let $a = (W,\mu) \in \bar{X}^\flat_{\mathbb{Q},v}$, where $W = \mathbb{Q}e_1$, and μ is the unique element of X_{W_v}.

For $c \in \mathbb{R}_{>0}$, let U_c be the subset of $X_v = \mathrm{PGL}_3(\mathbb{R})/\mathrm{PO}_3(\mathbb{R})$ consisting of the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\gamma \in \mathrm{PGL}_2(\mathbb{Z})$, $x_{ij} \in \mathbb{R}$, $y_1 \ge c$, and $y_2 \ge \frac{\sqrt{3}}{2}$. When γ, x_{ij} and y_2 are fixed and $y_1 \to \infty$, these elements converge to a in $\bar{X}^\flat_{\mathbb{Q},v}$ under the Satake topology. When γ, x_{ij}, and y_1 are fixed and $y_2 \to \infty$, they converge in the Satake topology to

$$\mu(\gamma, x_{12}, y_1) := \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & x_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mu(y_1),$$

where $\mu(y_1)$ is the class in $\bar{X}^{\flat}_{\mathbb{Q},v}$ of the semi-norm $a_1 e_1^* + a_2 e_2^* + a_3 e_3^* \mapsto (a_1^2 y_1^2 + a_2^2)^{1/2}$ on V_v^*.

The set of

$$\bar{U}_c := \{a\} \cup \{\mu(\gamma, x, y) \mid \gamma \in \mathrm{PGL}_2(\mathbb{Z}), x \in \mathbb{R}, y \geq c\} \cup U_c.$$

is a base of neighborhoods for a in $\bar{X}^{\flat}_{\mathbb{Q},v}$ under the Satake topology. Note that $\mathfrak{H} = \mathrm{SL}_2(\mathbb{Z})\{z \in \mathfrak{H} \mid \mathrm{Im}(z) \geq \frac{\sqrt{3}}{2}\}$, which is the reason for the appearance of $\frac{\sqrt{3}}{2}$. It can of course be replaced by any $b \in \mathbb{R}_{>0}$ such that $b \leq \frac{\sqrt{3}}{2}$.

4.4.11. We continue with Example 4.4.10. Under the canonical surjection $\bar{X}_{\mathbb{Q},v} \to \bar{X}^{\flat}_{\mathbb{Q},v}$, the inverse image of $a = (W, \mu)$ in $\bar{X}_{\mathbb{Q},v}$ is canonically homeomorphic to $\bar{X}_{(V/W)_v} = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ under the Satake topology on both spaces. This homeomorphism sends $x + y_2 i \in \mathfrak{H}$ ($x \in \mathbb{R}$, $y_2 \in \mathbb{R}_{>0}$) to the limit for the Satake topology of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{PGL}_3(\mathbb{R}) / \mathrm{PO}_3(\mathbb{R})$$

as $y_1 \to \infty$. (This limit in $\bar{X}_{\mathbb{Q},v}$ depends on x and y_2, but the limit in $\bar{X}^{\flat}_{\mathbb{Q},v}$ is a.)

4.4.12. In the example of 4.4.10, we explain that the quotient topology on $\bar{X}^{\flat}_{\mathbb{Q},v}$ of the Satake topology on $\bar{X}_{\mathbb{Q},v}$ is different from the Satake topology on $\bar{X}^{\flat}_{\mathbb{Q},v}$.

For a map

$$f\colon \mathrm{PGL}_2(\mathbb{Z}) / \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \to \mathbb{R}_{>0},$$

define a subset U_f of X_v as in the definition of U_c but replacing the condition on γ, x_{ij}, y_i by $\gamma \in \mathrm{PGL}_2(\mathbb{Z})$, $x_{ij} \in \mathbb{R}$, $y_1 \geq f(\gamma)$, and $y_2 \geq \frac{\sqrt{3}}{2}$. Let

$$\bar{U}_f = \{a\} \cup \{\mu(\gamma, x, y) \mid \gamma \in \mathrm{PGL}_2(\mathbb{Z}), x \in \mathbb{R}, y \geq f(\gamma)\} \cup U_f.$$

When f varies, the $\bar{U}_f$ form a base of neighborhoods of a in $\bar{X}^{\flat}_{\mathbb{Q},v}$ for the quotient topology of the Satake topology on $\bar{X}_{\mathbb{Q},v}$. On the other hand, if $\inf\{f(\gamma) \mid \gamma \in \mathrm{PGL}_2(\mathbb{Z})\} = 0$, then $\bar{U}_f$ is not a neighborhood of a for the Satake topology on $\bar{X}^{\flat}_{\mathbb{Q},v}$.

4.5 Proof of the main theorem

In this subsection, we prove Theorem 4.1.4. We begin with the quasi-compactness asserted therein. Throughout this subsection, we set $Z = X_{S_2} \times G(\mathbb{A}_F^S)/K$ in situation (I) and $Z = X_{S_2}$ in situation (II), so $\bar{\mathfrak{X}} = \bar{X} \times Z$.

Proposition 4.5.1. *In situation (I) of 4.1.3, the quotient $G(F)\backslash\bar{\mathfrak{X}}$ is quasi-compact. In situation (II), the quotient $\Gamma\backslash\bar{\mathfrak{X}}$ is quasi-compact for any subgroup Γ of Γ_K of finite index.*

Proof. We may restrict to case (i) of 4.1.2 that $\bar{X} = \bar{X}_{F,S_1}$, as $\bar{X}^{\flat}_{F,S_1}$ of case (ii) is a quotient of $\bar{X}_{F,S_1}$ (under the Borel-Serre topology). In situation (I), we claim that there exist $c_1, c_2 \in \mathbb{R}_{>0}$, a compact subset C of $B_u(\mathbb{A}_{F,S})$, and a compact subset C' of Z such that $\bar{\mathfrak{X}} = G(F)(\mathfrak{S}(C; c_1, c_2) \times C')$. In situation (II), we claim that there exist c_1, c_2, C, C' as above and a finite subset R of $G(F)$ such that $\bar{\mathfrak{X}} = \Gamma R(\mathfrak{S}(C; c_1, c_2) \times C')$. It follows that in situation (I) (resp., (II)), there is a surjective continuous map from the compact space $C \times \mathfrak{T}(c_1, c_2) \times C'$ (resp., $R \times C \times \mathfrak{T}(c_1, c_2) \times C'$) onto the quotient space under consideration, which yields the proposition.

For any compact open subgroup K' of $G(\mathbb{A}_F^{S_1})$, the set $G(F)\backslash G(\mathbb{A}_F^{S_1})/K'$ is finite. Each X_v for $v \in S_2$ may be identified with the geometric realization of the Bruhat-Tits building for PGL_{V_v}, the set of i-simplices of which for a fixed i can be identified with $G(F_v)/K'_v$ for some K'. So, we see that in situation (I) (resp., (II)), there is a compact subset D of Z such that $Z = G(F)D$ (resp., $Z = \Gamma D$).

Now fix such a compact open subgroup K' of $G(\mathbb{A}_F^{S_1})$. By 4.3.4, there are $c_1, c_2 \in \mathbb{R}_{>0}$, a compact subset C of $P_u(\mathbb{A}_{F,S_1})$, and a finite subset R' of $G(F)$ such that $\bar{X}_{F,S} = \Gamma_{K'} R' \mathfrak{S}(C; c_1, c_2)$. We consider the compact subset $C' := (R')^{-1} K' D$ of Z.

Let $(x, y) \in \bar{\mathfrak{X}}$, where $x \in \bar{X}_{F,S}$ and $y \in Z$. Write $y = \gamma z$ for some $z \in D$ and $\gamma \in G(F)$ (resp., $\gamma \in \Gamma$) in situation (I) (resp., (II)). In situation (II), we write $\Gamma\Gamma_{K'} R' = \Gamma R$ for some finite subset R of $G(F)$. Write $\gamma^{-1} x = \gamma' \varphi s$ where $\gamma' \in \Gamma_{K'}$, $\varphi \in R'$, $s \in \mathfrak{S}(C; c_1, c_2)$. We have

$$(x, y) = \gamma(\gamma^{-1} x, z) = \gamma(\gamma' \varphi s, z) = (\gamma\gamma'\varphi)(s, \varphi^{-1}(\gamma')^{-1} z).$$

As $\gamma\gamma'\varphi$ lies in $G(F)$ in situation (I) and in ΓR in situation (II), we have the claim. □

4.5.2. To prove Theorem 4.1.4, it remains only to verify the Hausdorff property. For this, it is sufficient to prove the following.

Proposition 4.5.3. *Let $\Gamma = G(F)$ in situation (I) of 4.1.3, and let $\Gamma = \Gamma_K$ in situation (II). For every $a, a' \in \bar{\mathfrak{X}}$, there exist neighborhoods U of a and U' of a' such that if $\gamma \in \Gamma$ and $\gamma U \cap U' \neq \varnothing$, then $\gamma a = a'$.*

In the rest of this subsection, let the notation be as in 4.5.3. It is sufficient to prove 4.5.3 for the Satake topology on $\bar{\mathfrak{X}}$. In 4.5.4–4.5.8, we prove 4.5.3 in situation (II) for $S = S_1$. That is, we suppose that $\bar{\mathfrak{X}} = \bar{X}$. In 4.5.9 and 4.5.10, we deduce 4.5.3 in general from this case.

Lemma 4.5.4. *Assume that $\bar{\mathfrak{X}} = \bar{X}_{F,S}$. Suppose that $a, a' \in \bar{\mathfrak{X}}$ have distinct parabolic types (4.3.7). Then there exist neighborhoods U of a and U' of a' such that $\gamma U \cap U' = \varnothing$ for all $\gamma \in \Gamma$.*

Proof. Let I (resp., I') be the parabolic type of a (resp., a'). We may assume that there exists an $i \in I$ with $i \notin I'$.

By 4.3.15, there exist $\varphi, \psi \in G(F)$ and $c_1, c_2 \in \mathbb{R}_{>0}$ such that $\Gamma_K \varphi \mathfrak{S}(c_1, c_2)$ is a neighborhood of a and $\Gamma_K \psi \mathfrak{S}(c_1, c_2)$ is a neighborhood of a'. By 4.3.14(2), there exist a neighborhood $U' \subset \Gamma_K \psi \mathfrak{S}(c_1, c_2)$ of a' and $c \in \mathbb{R}_{>0}$ with the property that $\min\{t_{v,i}(x) \mid v \in S\} \geq c$ for all $x \in (\Gamma_K \psi)^{-1} U' \cap \mathfrak{S}(c_1, c_2)$. Let $A \in \mathbb{R}_{>1}$ be as in 4.3.13 for these c_1, c_2 for $R_1 = \{\varphi^{-1}\}$ and $R_2 = \{\psi\}$. Take $\epsilon \in \mathbb{R}_{>0}$ such that $A\epsilon \leq c$. By 4.3.14(1), there exists a neighborhood $U \subset \Gamma_K \varphi \mathfrak{S}(c_1, c_2)$ of a such that $\max\{t_{v,i}(x) \mid v \in S\} < \epsilon$ for all $x \in (\Gamma_K \varphi)^{-1} U \cap \mathfrak{S}(c_1, c_2)$.

We prove that $\gamma U \cap U' = \varnothing$ for all $\gamma \in \Gamma_K$. If $x \in \gamma U \cap U'$, then we may take $\delta, \delta' \in \Gamma_K$ such that $(\delta\varphi)^{-1}\gamma^{-1} x \in \mathfrak{S}(c_1, c_2)$ and $(\delta'\psi)^{-1} x \in \mathfrak{S}(c_1, c_2)$. Since

$$(\delta\varphi)^{-1}\gamma^{-1} x = \varphi^{-1}(\delta^{-1}\gamma^{-1}\delta')\psi(\delta'\psi)^{-1} x \in \varphi^{-1}\Gamma_K \psi \cdot (\delta'\psi)^{-1} x,$$

we have by 4.3.13 that

$$c \leq t_{v,i}((\delta'\psi)^{-1} x) \leq A\, t_{v,i}((\delta\varphi)^{-1}\gamma^{-1} x) < A\epsilon,$$

for all $v \in S$ and hence $c < A\epsilon$, a contradiction. □

Lemma 4.5.5. *Assume that $\bar{\mathfrak{X}} = \bar{X}^{\flat}_{F,S}$. Let $a, a' \in \bar{\mathfrak{X}}$ and assume that the dimension of the F-subspace associated to a is different from that of a'. Then there exist neighborhoods U of a and U' of a' such that $\gamma U \cap U' = \varnothing$ for all $\gamma \in \Gamma$.*

Proof. The proof is similar to that of 4.5.4. In place of 4.3.13, 4.3.14, and 4.3.15, we use 4.4.4, 4.4.6, and 4.4.7, respectively. □

Lemma 4.5.6. *Let P be a parabolic subgroup of G. Let $a, a' \in \mathfrak{Z}_{F,S}(P)$ (see 3.4.3), and let R_1 and R_2 be finite subsets of $G(F)$. Then there exist neighborhoods U of a and U' of a' in $\mathfrak{Z}_{F,S}(P)$ such that $\gamma a = a'$ for every $\gamma \in R_1 \Gamma_K R_2 \cap P(F)$ for which $\gamma U \cap U' \neq \varnothing$.*

Proof. For each $\xi \in R_1$ and $\eta \in R_2$, the set $\xi \Gamma_K \eta \cap P(F)$ is a $\xi \Gamma_K \xi^{-1} \cap P(F)$-orbit for the left action of $\xi \Gamma_K \xi^{-1}$. Hence its image in $\prod_{i=0}^{m} \mathrm{PGL}_{V_i/V_{i-1}}(\mathbb{A}_{F,S})$ is discrete, for $(V_i)_{-1 \leq i \leq m}$ the flag corresponding to P, and thus the image of $R_1\Gamma_K R_2 \cap P(F)$ in $\prod_{i=0}^m \mathrm{PGL}_{V_i/V_{i-1}}(\mathbb{A}_{F,S})$ is discrete as well. On the other hand, for any compact neighborhoods U of a and U' of a', the set

$$\left\{g \in \prod_{i=0}^m \mathrm{PGL}_{V_i/V_{i-1}}(\mathbb{A}_{F,S}) \mid gU \cap U' \neq \varnothing\right\}$$

is compact. Hence the intersection $M := \{\gamma \in R_1\Gamma_K R_2 \cap P(F) \mid \gamma U \cap U' \neq \varnothing\}$ is finite. If $\gamma \in M$ and $\gamma a \neq a'$, then replacing U and U' by smaller neighborhoods of a and a', respectively, we have $\gamma U \cap U' = \varnothing$. Hence for sufficiently small neighborhoods U and U' of a and a', respectively, we have that if $\gamma \in M$, then $\gamma a = a'$. □

Lemma 4.5.7. *Let W be an F-subspace of V. Let $a, a' \in \mathfrak{Z}^\flat_{F,S}(W)$ (see 3.4.4), and let R_1 and R_2 be finite subsets of $G(F)$. Let P be the parabolic subgroup of G consisting of all elements which preserve W. Then there exist neighborhoods U of a and U' of a' in $\mathfrak{Z}^\flat_{F,S}(W)$ such that $\gamma a = a'$ for every $\gamma \in R_1\Gamma_K R_2 \cap P(F)$ for which $\gamma U \cap U' \neq \varnothing$.*

Proof. This is proven in the same way as 4.5.6. □

4.5.8. We prove 4.5.3 in situation (II), supposing that $S = S_1$.

In case (i) (that is, $\bar{X} = \bar{\mathfrak{X}} = \bar{X}_{F,S}$), we may assume by 4.5.4 that a and a' have the same parabolic type I. In case (ii) (that is, $\bar{X} = \bar{\mathfrak{X}} = \bar{X}^\flat_{F,S}$), we may assume by 4.5.5 that the dimension r of the F-subspace of V associated to a coincides with that of a'. In case (i) (resp., (ii)), take $c_1, c_2 \in \mathbb{R}_{>0}$ and elements φ and ψ (resp., finite subsets R and R') of $G(F)$ such that c_1, c_2, φ (resp., c_1, c_2, R) satisfy the condition in 4.3.15 (resp., 4.4.7) for a and c_1, c_2, ψ (resp., c_1, c_2, R') satisfy the condition in 4.3.15 (resp., 4.4.7) for a'. In case (i), we set $R = \{\varphi\}$ and $R' = \{\psi\}$.

Fix a basis $(e_i)_{1\le i\le d}$ of V. In case (i) (resp., (ii)), denote $\mathfrak{S}(c_1, c_2)$ (resp., $\mathfrak{S}^\flat_r(c_1, c_2)$) by $\mathfrak{S}$. In case (i), let $P = P_I$, and let $(V_i)_{-1\le i\le m}$ be the associated flag. In case (ii), let $W = \sum_{i=1}^r Fe_i$, and let P be the parabolic subgroup of G consisting of all elements which preserve W.

Note that in case (i) (resp., (ii)), for all $\varphi \in R$ and $\psi \in R'$, the parabolic subgroup P is associated to $\varphi^{-1}a$ and to $\psi^{-1}a'$ (resp., W is associated to $\varphi^{-1}a$ and to $\psi^{-1}a'$) and hence these elements are determined by their images in $\mathfrak{Z}_{F,S}(P)$ (resp. $\mathfrak{Z}^\flat_{F,S}(W)$).

In case (i) (resp., case (ii)), apply 4.5.6 (resp., 4.5.7) to the images of $\varphi^{-1}a$ and $\psi^{-1}a'$ for $\varphi \in R$, $\psi \in R'$ in $\mathfrak{Z}_{F,S}(P)$ (resp., $\mathfrak{Z}^\flat_{F,S}(W)$). By this, and by 4.3.10 for case (i) and 4.4.3 for case (ii), we see that there exist neighborhoods U_φ of $\varphi^{-1}a$ for each $\varphi \in R$ and U'_ψ of $\psi^{-1}a'$ for each $\psi \in R'$ for the Satake topology with the following two properties:

(A) $\{\gamma \in (R')^{-1}\Gamma_K R \mid \gamma(\mathfrak{S} \cap U_\varphi) \cap (\mathfrak{S} \cap U'_\psi) \neq \varnothing \text{ for some } \varphi \in R, \psi \in R'\} \subset P(F)$,

(B) if $\gamma \in (R')^{-1}\Gamma_K R \cap P(F)$ and $\gamma U_\varphi \cap U'_\psi \neq \varnothing$ for $\varphi \in R$ and $\psi \in R'$, then $\gamma\varphi^{-1}a = \psi^{-1}a'$.

In case (i) (resp., (ii)), take a neighborhood U of a satisfying the condition in 4.3.16 (resp., 4.4.8) for $(U_\varphi)_{\varphi\in R}$, and take a neighborhood U' of a' satisfying the condition in 4.3.16 (resp., 4.4.8) for $(U'_\psi)_{\psi\in R'}$. Let $\gamma\in\Gamma_K$ and assume $\gamma U\cap U'\neq\emptyset$. We prove $\gamma a=a'$. Take $x\in U$ and $x'\in U'$ such that $\gamma x=x'$. By 4.3.16 (resp., 4.4.8), there are $\varphi\in R$, $\psi\in R'$, and $\epsilon\in\Gamma_{K,(\varphi P\varphi^{-1})}$ and $\delta\in\Gamma_{K,(\psi P\psi^{-1})}$ in case (i) (resp., $\epsilon\in\Gamma_{K,(\varphi W)}$ and $\delta\in\Gamma_{K,(\psi W)}$ in case (ii)) such that $\varphi^{-1}\epsilon^{-1}x\in\mathfrak{S}\cap U_\varphi$ and $\psi^{-1}\delta^{-1}x'\in\mathfrak{S}\cap U'_\psi$. Since

$$(\psi^{-1}\delta^{-1}\gamma\epsilon\varphi)\varphi^{-1}\epsilon^{-1}x=\psi^{-1}\delta^{-1}x',$$

we have $\psi^{-1}\delta^{-1}\gamma\epsilon\varphi\in P(F)$ by property (A). By property (B), we have

$$(\psi^{-1}\delta^{-1}\gamma\epsilon\varphi)\varphi^{-1}a=\psi^{-1}a'.$$

Since $\epsilon a=a$ and $\delta a'=a'$, this proves $\gamma a=a'$.

We have proved 4.5.3 in situation (II) under the assumption $S=S_1$. In the following 4.5.9 and 4.5.10, we reduce the general case to that case.

Lemma 4.5.9. *Let $a,a'\in Z$. In situation (I) (resp., (II)), let $H=G(\mathbb{A}_F^{S_1})$ (resp., $H=G(\mathbb{A}_{F,S_2})$). Then there exist neighborhoods U of a and U' of a' in Z such that $ga=a'$ for all $g\in H$ for which $gU\cap U'\neq\emptyset$.*

Proof. For any compact neighborhoods U of a and U' of a', the set $M:=\{g\in H\mid gU\cap U'\neq\emptyset\}$ is compact. By definition of Z, there exist a compact open subgroup N of H and a compact neighborhood U of a such that $gx=x$ for all $g\in N$ and $x\in U$. For such a choice of U, the set M is stable under the right translation by N, and M/N is finite because M is compact and N is an open subgroup of H. If $g\in M$ and if $ga\neq a'$, then by shrinking the neighborhoods U and U', we have that $gU\cap U'=\emptyset$. As M/N is finite, we have sufficiently small neighborhoods U and U' such that if $g\in M$ and $gU\cap U'\neq\emptyset$, then $ga=a'$. □

4.5.10. We prove Proposition 4.5.3.

Let H be as in Lemma 4.5.9. Write $a=(a_{S_1},a_Z)$ and $a'=(a'_{S_1},a'_Z)$ as elements of $\bar{X}\times Z$. By 4.5.9, there exist neighborhoods U_Z of a_Z and U'_Z of a'_Z in Z such that if $g\in H$ and $gU_Z\cap U'_Z\neq\emptyset$, then $ga=a'$. The set $K':=\{g\in H\mid ga_Z=a_Z\}$ is a compact open subgroup of H. Let Γ' be the inverse image of K' under $\Gamma\to H$, where $\Gamma=G(F)$ in situation (I). In situation (II), the group Γ' is of finite index in the inverse image of the compact open subgroup $K'\times K$ under $G(F)\to G(\mathbb{A}_F^{S_1})$. In both situations, the set $M:=\{\gamma\in\Gamma\mid\gamma a_Z=a'_Z\}$ is either empty or a Γ'-torsor for the right action of Γ'.

Assume first that $M\neq\emptyset$, in which case we may choose $\theta\in\Gamma$ such that $M=\theta\Gamma'$. Since we have proven 4.5.3 in situation (II) for $S_1=S$, there exist neighborhoods U_{S_1} of a_{S_1} and U'_{S_1}

of $\theta^{-1}a'_{S_1}$ such that if $\gamma \in \Gamma'$ satisfies $\gamma U_{S_1} \cap U'_{S_1} \neq \varnothing$, then $\gamma a_{S_1} = \theta^{-1}a'_{S_1}$. Let $U = U_{S_1} \times U_Z$ and $U' = \theta U'_{S_1} \times U'_Z$, which are neighborhoods of a and a' in $\bar{\mathfrak{X}}$, respectively. Suppose that $\gamma \in \Gamma$ satisfies $\gamma U \cap U' \neq \varnothing$. Then, since $\gamma U_Z \cap U'_Z \neq \varnothing$, we have $\gamma a_Z = a'_Z$ and hence $\gamma = \theta\gamma'$ for some $\gamma' \in \Gamma'$. Since $\theta\gamma' U_{S_1} \cap \theta U'_{S_1} \neq \varnothing$, we have $\gamma' U_{S_1} \cap U'_{S_1} \neq \varnothing$, and hence $\gamma' a_{S_1} = \theta^{-1}a'_{S_1}$. That is, we have $\gamma a_{S_1} = a'_{S_1}$, so $\gamma a = a'$.

In the case that $M = \varnothing$, take any neighborhoods U_{S_1} of a_{S_1} and U'_{S_1} of a'_{S_1}, and set $U = U_{S_1} \times U_Z$ and $U' = U'_{S_1} \times U'_Z$. Any $\gamma \in \Gamma$ such that $\gamma U \cap U' \neq \varnothing$ is contained in M, so no such γ exists.

4.6 Supplements to the main theorem

We use the notation of §4.1 throughout this subsection. We suppose that $\Gamma = G(F)$ in situation (I), and we let Γ be a subgroup of Γ_K of finite index in situation (II). For $a \in \bar{\mathfrak{X}}$, let $\Gamma_a < \Gamma$ denote the stabilizer of a.

Theorem 4.6.1. *For $a \in \bar{\mathfrak{X}}$ (with either the Borel-Serre or the Satake topology), there is an open neighborhood U of the image of a in $\Gamma_a \backslash \bar{\mathfrak{X}}$ such that the image U' of U under the quotient map $\Gamma_a \backslash \bar{\mathfrak{X}} \to \Gamma \backslash \bar{\mathfrak{X}}$ is open and the map $U \to U'$ is a homeomorphism.*

Proof. By the case $a = a'$ of Proposition 4.5.3, there is an open neighborhood $U'' \subset \bar{\mathfrak{X}}$ of a such that if $\gamma \in \Gamma_K$ and $\gamma U'' \cap U'' \neq \varnothing$, then $\gamma a = a$. Then the subset $U := \Gamma_a \backslash \Gamma_a U''$ of $\Gamma_a \backslash \bar{\mathfrak{X}}$ is open and has the desired property. □

Proposition 4.6.2. *Suppose that $S = S_1$, and let $a \in \bar{\mathfrak{X}}$.*

(1) Take $\bar{X} = \bar{X}_{F,S}$, and let P be the parabolic subgroup associated to a. Then $\Gamma_{(P)}$ (as in 3.4.12) is a normal subgroup of Γ_a of finite index.

(2) Take $\bar{X} = \bar{X}^{\flat}_{F,S}$, and let W be the F-subspace of V associated to a. Then $\Gamma_{(W)}$ (as in 3.4.12) is a normal subgroup of Γ_a of finite index.

Proof. We prove (1), the proof of (2) being similar. Let $(V_i)_{-1 \le i \le m}$ be the flag corresponding to P. The image of $\Gamma \cap P(F)$ in $\prod_{i=0}^m \mathrm{PGL}_{V_i/V_{i-1}}(\mathbb{A}_{F,S})$ is discrete. On the other hand, the stabilizer in $\prod_{i=0}^m \mathrm{PGL}_{V_i/V_{i-1}}(\mathbb{A}_{F,S})$ of the image of a in $\mathfrak{Z}_{F,S}(P)$ is compact. Hence the image of Γ_a in $\prod_{i=0}^m \mathrm{PGL}_{V_i/V_{i-1}}(F)$, which is isomorphic to $\Gamma_a/\Gamma_{(P)}$, is finite. □

Theorem 4.6.3. *Assume that F is a function field and $\bar{X} = \bar{X}_{F,S_1}$, where S_1 consists of a single place v. Then the inclusion map $\Gamma \backslash \mathfrak{X} \hookrightarrow \Gamma \backslash \bar{\mathfrak{X}}$ is a homotopy equivalence.*

Proof. Let $a \in \bar{\mathfrak{X}}$. In situation (I) (resp., (II)), write $a = (a_v, a^v)$ with $a_v \in \bar{X}_{F,v}$ and $a^v \in X_{S_2} \times G(\mathbb{A}_F^S)/K$ (resp., X_{S_2}). Let K' be the isotropy subgroup of a^v in $G(\mathbb{A}_F^v)$ (resp., $\prod_{w \in S_2} G(F_w)$), and let $\Gamma' < \Gamma$ be the inverse image of K' under the map $\Gamma \to G(\mathbb{A}_F^v)$ (resp., $\Gamma \to \prod_{w \in S_2} G(F_w)$).

Let P be the parabolic subgroup associated to a. Let Γ_a be the isotropy subgroup of a in Γ, which is contained in $P(F)$ and equal to the isotropy subgroup Γ'_{a_v} of a_v in Γ'. In situation (I) (resp., (II)), take a Γ_a-stable open neighborhood D of a^v in $X_{S_2} \times G(\mathbb{A}_F^S)/K$ (resp., X_{S_2}) that has compact closure.

Claim 1. The subgroup $\Gamma_D := \{\gamma \in \Gamma_a \mid \gamma x = x \text{ for all } x \in D\}$ of Γ_a is normal of finite index.

Proof of Claim 1. Normality follows from the Γ_a-stability of D. For any x in the closure $\bar{D}$ of D, there exists an open neighborhood V_x of x and a compact open subgroup N_x of $G(\mathbb{A}_F^v)$ (resp., $\prod_{w \in S_2} G(F_w)$) in situation (I) (resp., (II)) such that $gy = y$ for all $g \in N_x$ and $y \in V_x$. For a finite subcover $\{V_{x_1}, \dots, V_{x_n}\}$ of $\bar{D}$, the group Γ_D is the inverse image in Γ_a of $\bigcap_{i=1}^n N_{x_i}$, so is of finite index.

Claim 2. The subgroup $H := \Gamma_D \cap P_u(F)$ of Γ_a is normal of finite index.

Proof of Claim 2. Normality is immediate from Claim 1 as $P_u(F)$ is normal in $P(F)$. Let $H' = \Gamma'_{(P)} \cap P_u(F)$, which has finite index in $\Gamma'_{(P)}$ and equals $\Gamma' \cap P_u(F)$ by definition of $\Gamma'_{(P)}$. Since $\Gamma'_{(P)} \subset \Gamma'_{a_v} \subset \Gamma'$ and $\Gamma'_{a_v} = \Gamma_a$, we have $H' = \Gamma_a \cap P_u(F)$ as well. By Claim 1, we then have that H' contains H with finite index, so H has finite index in $\Gamma'_{(P)}$. Proposition 4.6.2(1) tells us that $\Gamma'_{(P)}$ is of finite index in $\Gamma'_{a_v} = \Gamma_a$.

Let $(V_i)_{-1 \le i \le m}$ be the flag corresponding to P. By Corollary 3.5.4, we have a homeomorphism

$$\chi : P_u(F_v)\backslash \bar{X}_{F,v}(P) \xrightarrow{\sim} \mathfrak{Z}_{F,v}(P) \times \mathbb{R}^m_{\ge 0}$$

on quotient spaces arising from the $P(F_v)$-equivariant homeomorphism $\psi_{P,v} = (\phi_{P,v}, \phi'_{P,v})$ of 3.5.1 (see 3.4.3 and 3.4.6).

Claim 3. For a sufficiently small open neighborhood U of $0 = (0, \dots, 0)$ in $\mathbb{R}^m_{\ge 0}$, the map χ induces a homeomorphism

$$\chi_U : H\backslash \bar{X}_{F,v}(P)_U \xrightarrow{\sim} \mathfrak{Z}_{F,v}(P) \times U,$$

where $\bar{X}_{F,v}(P)_U$ denotes the inverse image of U under $\phi'_{P,v} : \bar{X}_{F,v}(P) \to \mathbb{R}^m_{\ge 0}$.

Proof of Claim 3. By definition, χ restricts to a homeomorphism

$$P_u(F_v)\backslash \bar{X}_{F,v}(P)_U \xrightarrow{\sim} \mathfrak{Z}_{F,v}(P) \times U$$

for any open neighborhood U of 0. For a sufficiently large compact open subset C of $P_u(F_v)$, we have $P_u(F_v) = HC$. For U sufficiently small, every $g \in C$ fixes all $x \in \bar{X}_{F,v}(P)_U$, which yields the claim.

Claim 4. The map χ_U and the identity map on D induce a homeomorphism

$$\chi_{U,a}\colon \Gamma_a\backslash(\bar{X}_{F,v}(P)_U \times D) \xrightarrow{\sim} (\Gamma_a\backslash(\mathfrak{Z}_{F,v}(P) \times D)) \times U.$$

Proof of Claim 4. The quotient group Γ_a/H is finite by Claim 2. Since the determinant of an automorphism of V_i/V_{i-1} of finite order has trivial absolute value at v, the Γ_a-action on $\mathbb{R}^m_{\geq 0}$ is trivial. Since H acts trivially on D, the claim follows from Claim 3.

Now let $c \in \mathbb{R}^m_{>0}$, and set $U = \{t \in \mathbb{R}^m_{\geq 0} \mid t_i < c \text{ for all } 1 \leq i \leq m\}$. Set $(X_v)_U = X_v \cap \bar{X}_{F,v}(P)_U$. If c is sufficiently small, then

$$(\Gamma_a\backslash(\mathfrak{Z}_{F,v}(P) \times D)) \times (U \cap \mathbb{R}^m_{>0}) \hookrightarrow (\Gamma_a\backslash(\mathfrak{Z}_{F,v}(P) \times D)) \times U$$

is a homotopy equivalence, and we can apply $\chi_{U,a}^{-1}$ to both sides to see that the inclusion map

$$\Gamma_a\backslash((X_v)_U \times D) \hookrightarrow \Gamma_a\backslash(\bar{X}_{F,v}(P)_U \times D)$$

is also a homotopy equivalence. By Theorem 4.6.1, this proves Theorem 4.6.3. □

Remark 4.6.4. Theorem 4.6.3 is well-viewed as a function field analogue of the homotopy equivalence for Borel-Serre spaces of [3].

4.6.5. Theorem 4.1.4 remains true if we replace $G = \mathrm{PGL}_V$ by $G = \mathrm{SL}_V$ in 4.1.3 and 4.1.4. It also remains true if we replace $G = \mathrm{PGL}_V$ by $G = \mathrm{GL}_V$ and we replace $\bar{\mathfrak{X}}$ in 4.1.4 in situation (I) (resp., (II)) by

$$\bar{X} \times X_{S_2} \times (\mathbb{R}^S_{>0} \times G(\mathbb{A}^S_F)/K)_1 \quad (\text{resp., } \bar{X} \times X_{S_2} \times (\mathbb{R}^S_{>0})_1),$$

where $(\ )_1$ denotes the kernel of

$$((a_v)_{v\in S}, g) \mapsto |\det(g)| \prod_{v\in S} a_v \quad (\text{resp., } (a_v)_{v\in S} \mapsto \prod_{v\in S} a_v),$$

and $\gamma \in \mathrm{GL}_V(F)$ (resp., $\gamma \in \Gamma_K$) acts on this kernel by multiplication by $((|\det(\gamma)|_v)_{v\in S}, \gamma)$ (resp., $(|\det(\gamma)|_v)_{v\in S}$).

Theorems 4.6.1 and 4.6.3 also remain true under these modifications. These modified versions of the results are easily reduced to the original case $G = \mathrm{PGL}_V$.

4.7 Subjects related to this paper

4.7.1. In this subsection, as possibilities of future applications of this paper, we describe connections with the study of

- toroidal compactifications of moduli spaces of Drinfeld modules (4.7.2–4.7.5)
- the asymptotic behavior of Hodge structures and p-adic Hodge structures associated to a degenerating family of motives over a number field (4.7.6, 4.7.7), and
- modular symbols over function fields (4.7.8, 4.7.9).

4.7.2. In [21], Pink constructed a compactification of the moduli space of Drinfeld modules that is similar to the Satake-Baily-Borel compactification of the moduli space of polarized abalian varieties. In a special case, it had been previously constructed by Kapranov [15].

In [20], Pink, sketched a method for constructing a compactification of the moduli space of Drinfeld modules that is similar to the toroidal compactification of the moduli space of polarized abelian varieties (in part, using ideas of K. Fujiwara). However, the details of the construction have not been published. Our plan for constructing toroidal compactifications seems to be different from that of [20].

4.7.3. We give a rough outline of the relationship that we envision between this paper and the analytic theory of toroidal compactifications. Suppose that F is a function field, and fix a place v of F. Let O be the ring of all elements of F which are integral outside v. In [6], the notion of a Drinfeld O-module of rank d is defined, and the moduli space of such Drinfeld modules is constructed.

Let $\mathbb{C}_v$ be the completion of an algebraic closure of F_v and let $|\ |: \mathbb{C}_v \to \mathbb{R}_{\geq 0}$ be the absolute value which extends the normalized absolute value of F_v. Let $\Omega \subset \mathbb{P}^{d-1}(\mathbb{C}_v)$ be the $(d-1)$-dimensional Drinfeld upper half-space consisting of all points $(z_1 : \cdots : z_d) \in \mathbb{P}^{d-1}(\mathbb{C}_v)$ such that $(z_i)_{1\leq i\leq d}$ is linearly independent over F_v.

For a compact open subgroup K of $\mathrm{GL}_d(\mathbb{A}_F^v)$, the set of $\mathbb{C}_v$-points of the moduli space M_K of Drinfeld O-modules of rank d with K-level structure is expressed as

$$M_K(\mathbb{C}_v) = \mathrm{GL}_d(F)\backslash(\Omega \times \mathrm{GL}_d(\mathbb{A}_F^v)/K)$$

(see [6]).

Consider the case $V = F^d$ in §3 and §4. We have a map $\Omega \to X_v$ which sends $(z_1 : \cdots : z_d) \in \Omega$ to the class of the the norm $V_v = F_v^d \to \mathbb{R}_{\geq 0}$ given by $(a_1, \ldots, a_d) \mapsto |\sum_{i=1}^d a_i z_i|$ for $a_i \in F_v$. This map induces a canonical continuous map

(1) $M_K(\mathbb{C}_v) = \mathrm{GL}_d(F)\backslash(\Omega \times \mathrm{GL}_d(\mathbb{A}_F^v)/K) \to \mathrm{GL}_d(F)\backslash(X_v \times \mathrm{GL}_d(\mathbb{A}_F^v)/K)$.

The map (1) extends to a canonical continuous map

(2) $\bar{M}_K^{\mathrm{KP}}(\mathbb{C}_v) \to \mathrm{GL}_d(F)\backslash(\bar{X}^{\flat}_{F,v} \times \mathrm{GL}_d(\mathbb{A}_F^v)/K)$,

where $\bar{M}_K^{\mathrm{KP}}$ denotes the compactification of Kapranov and Pink of M_K. In particular, $\bar{M}_K^{\mathrm{KP}}$ is related to $\bar{X}^{\flat}_{F,v}$. On the other hand, the toroidal compactifications of M_K should be related to $\bar{X}_{F,v}$. If we denote by $\bar{M}_K^{\mathrm{tor}}$ the projective limit of all toroidal compactifications of M_K, then the map of (1) should extend to a canonical continuous map

(3) $\bar{M}_K^{\mathrm{tor}}(\mathbb{C}_v) \to \mathrm{GL}_d(F)\backslash(\bar{X}_{F,v} \times \mathrm{GL}_d(\mathbb{A}_F^v)/K)$.

that fits in a commutative diagram

$$\begin{array}{ccc} \bar{M}_K^{\mathrm{tor}}(\mathbb{C}_v) & \longrightarrow & \mathrm{GL}_d(F)\backslash(\bar{X}_{F,v} \times \mathrm{GL}(\mathbb{A}_F^v)/K) \\ \downarrow & & \downarrow \\ M_K^{\mathrm{KP}}(\mathbb{C}_v) & \longrightarrow & \mathrm{GL}_d(F)\backslash(\bar{X}^{\flat}_{F,v} \times \mathrm{GL}_d(\mathbb{A}_F^v)/K). \end{array}$$

4.7.4. The expected map of 4.7.3(3) is the analogue of the canonical continuous map from the projective limit of all toroidal compactifications of the moduli space of polarized abelian varieties to the reductive Borel-Serre compactification (see [10, 16]).

From the point of view of our study, the reductive Borel-Serre compactification and $\bar{X}_{F,v}$ are enlargements of spaces of norms. A polarized abelian variety A gives a norm on the polarized Hodge structure associated to A (the Hodge metric). This relationship between a polarized abelian variety and a norm forms the foundation of the relationship between the toroidal compactifications of a moduli space of polarized abelian varieties and the reductive Borel-Serre compactification. This is similar to the relationship between M_K and the space of norms X_v given by the map of 4.7.3(1), as well as the relationship between $\bar{M}_K^{\mathrm{tor}}$ and $\bar{X}_{F,v}$ given by 4.7.3(3).

4.7.5. In the usual theory of toroidal compactifications, cone decompositions play an important role. In the toroidal compactifications of 4.7.3, the simplices of Bruhat-Tits buildings (more precisely, the simplices contained in the fibers of $\bar{X}_{F,v} \to \bar{X}^{\flat}_{F,v}$) should play the role of the cones in cone decompositions.

4.7.6. We are preparing a paper in which our space $\bar{X}_{F,S}$ with F a number field and with S containing a non-archimedean place appears in the following way.

Let F be a number field, and let Y be a polarized projective smooth variety over F. Let $m \geq 0$, and let $V = H^m_{\mathrm{dR}}(Y)$ be the de Rham cohomology. For a place v of F, let $V_v = F_v \otimes_F V$.

For an archimedean place v of F, it is well known that V_v has a Hodge metric. For v non-archimedean, we can under certain assumptions define a Hodge metric on V_v by the method illustrated in the example of 4.7.7 below. The $[F_v : \mathbb{Q}_v]$-powers of these Hodge metrics for $v \in S$ are norms and therefore provide an element of $\prod_{v\in S} X_{V_v}$. When Y degenerates, this element of $\prod_{v\in S} X_{V_v}$ can converge to a boundary point of $\bar{X}_{F,S}$.

4.7.7. Let Y be an elliptic curve over a number field F, and take $m = 1$.

Let v be a non-archimedean place of F, and assume that $F_v \otimes_F Y$ is a Tate elliptic curve of q-invariant $q_v \in F_v^\times$ with $|q_v| < 1$. Then the first log-crystalline cohomology group D of the special fiber of this elliptic curve is a free module of rank 2 over the Witt vectors $W(k_v)$ with a basis (e_1, e_2) on which the Frobenius φ acts as $\varphi(e_1) = e_1$ and $\varphi(e_2) = pe_2$, where $p = \operatorname{char} k_v$. The first ℓ-adic étale cohomology group of this elliptic curve is a free module of rank 2 over $\mathbb{Z}_\ell$ with a basis $(e_{1,\ell}, e_{2,\ell})$ such that the inertia subgroup of $\operatorname{Gal}(\bar{F}_v/F_v)$ fixes e_1. The monodromy operator N satisfies

$$Ne_2 = \xi'_v e_1, \quad Ne_1 = 0, \quad Ne_{2,\ell} = \xi'_v e_{1,\ell}, \quad Ne_{1,\ell} = 0$$

where $\xi'_v = \operatorname{ord}_{F_v}(q_v) > 0$. The standard polarization $\langle\ ,\ \rangle$ of the elliptic curve satisfies $\langle e_1, e_2\rangle = 1$ and hence

$$\langle Ne_2, e_2\rangle = \xi'_v, \quad \langle e_1, N^{-1}e_1\rangle = 1/\xi'_v \quad \langle Ne_{2,\ell}, e_{2,\ell}\rangle = \xi'_v, \quad \langle e_{1,\ell}, N^{-1}e_{1,\ell}\rangle = 1/\xi'_v.$$

For $V = H^1_{\mathrm{dR}}(Y)$, we have an isomorphism $V_v \cong F_v \otimes_{W(k_v)} D$. The Hodge metric $|\ |_v$ on V_v is defined by

$$|a_1e_1 + a_2e_2|_v = \max(\xi_v^{-1/2}|a_1|_p, \xi_v^{1/2}|a_2|_p)$$

for $a_1, a_2 \in F_v$, where $|\ |_p$ denotes the absolute value on F_v satisfying $|p|_p = p^{-1}$ and

$$\xi_v := -\xi'_v \log(|\varpi_v|_p) = -\log(|q_v|_p) > 0,$$

where ϖ_v is a prime element of F_v. That is, to define the Hodge metric on V_v, we modify the naive metric (coming from the integral structure of the log-crystalline cohomology) by using ξ_v which is determined by the polarization $\langle\ ,\ \rangle$, the monodromy operator N, and the integral structures of the log-crystalline and ℓ-adic cohomology groups (for $\ell \neq p$).

This is similar to what happens at an archimedean place v. We have $Y(\mathbb{C}) \cong \mathbb{C}^\times/q_v^{\mathbb{Z}}$ with $q_v \in F_v^\times$. Assume for simplicity that we can take $|q_v| < e^{-2\pi}$ where $|\ \ |$ denotes the usual absolute value. Then q_v is determined by $F_v \otimes_F Y$ uniquely. Let $\xi := -\log(|q_v|) > 2\pi$. If v is real, we further assume that $q_v > 0$ and that we have an isomorphism $Y(F_v) \cong F_v^\times/q_v^{\mathbb{Z}}$ which is compatible with $Y(\mathbb{C}) \cong \mathbb{C}^\times/q_v^{\mathbb{Z}}$. Then in the case v is real (resp., complex), there is a basis (e_1, e_2) of V_v such that $(e_1, (2\pi i)^{-1}e_2)$ is a $\mathbb{Z}$-basis of $H^1(Y(\mathbb{C}), \mathbb{Z})$ and such that the Hodge metric $|\ |_v$ on V_v satisfies $|e_1|_v = \xi_v^{-1/2}$ and $|e_2|_v = \xi_v^{1/2}$ (resp., $||e_2|_v - \xi_v^{1/2}| \le \pi\xi_v^{-1/2}$).

Consider for example the elliptic curves $y^2 = x(x-1)(x-t)$ with $t \in F = \mathbb{Q}, t \neq 0, 1$. As t approaches $1 \in \mathbb{Q}_v$ for all $v \in S$, the elliptic curves $F_v \otimes_F Y$ satisfy the above assumptions for all $v \in S$, and each q_v approaches 0, so ξ_v tends to ∞. The corresponding elements of $\prod_{v \in S} X_{V_v}$ defined by the classes of the $|\ |_v$ for $v \in S$ converge to the unique boundary point of $\bar{X}_{F,S}$ with associated parabolic equal to the Borel subgroup of upper triangular matrices in PGL_V for the basis (e_1, e_2).

We hope that this subject about $\bar{X}_{F,S}$ is an interesting direction to be studied. It may be related to the asymptotic behaviors of heights of motives in degeneration studied in [18].

4.7.8. Suppose that F is a function field and let v be a place of F. Let Γ be as in 1.3.

Kondo and Yasuda [17] proved that the image of $H_{d-1}(\Gamma\backslash X_v, \mathbb{Q}) \to H^{\mathrm{BM}}_{d-1}(\Gamma\backslash X_v, \mathbb{Q})$ is generated by modular symbols, where H^{BM}_* denotes Borel-Moore homology. Our hope is that the compactification $\Gamma\backslash\bar{X}_{F,v}$ of $\Gamma\backslash X_v$ is useful in further studies of modular symbols.

Let $\partial := \bar{X}_{F,v} \setminus X_v$. Then we have an isomorphism

$$H^{\mathrm{BM}}_*(\Gamma\backslash X_v, \mathbb{Q}) \cong H_*(\Gamma\backslash\bar{X}_{F,v}, \Gamma\backslash\partial, \mathbb{Q})$$

and an exact sequence

$$\cdots \to H_i(\Gamma\backslash\bar{X}_{F,v}, \mathbb{Q}) \to H_i(\Gamma\backslash\bar{X}_{F,v}, \Gamma\backslash\partial, \mathbb{Q}) \to H_{i-1}(\Gamma\backslash\partial, \mathbb{Q}) \to H_{i-1}(\Gamma\backslash\bar{X}_{F,v}, \mathbb{Q}) \to \dots.$$

Since $\Gamma\backslash X_v \to \Gamma\backslash\bar{X}_{F,v}$ is a homotopy equivalence by Thm. 4.6.3, we have

$$H_*(\Gamma\backslash X_v, \mathbb{Q}) \xrightarrow{\sim} H_*(\Gamma\backslash\bar{X}_{F,v}, \mathbb{Q}).$$

Hence the result of Kondo and Yasuda shows that the kernel of

$$H^{\mathrm{BM}}_{d-1}(\Gamma\backslash X_v, \mathbb{Q}) \cong H_{d-1}(\Gamma\backslash\bar{X}_{F,v}, \Gamma\backslash\partial, \mathbb{Q}) \to H_{d-2}(\Gamma\backslash\partial, \mathbb{Q})$$

is generated by modular symbols.

If we want to prove that $H^{\mathrm{BM}}_{d-1}(\Gamma\backslash X_v, \mathbb{Q})$ is generated by modular symbols, it is now sufficient to prove that the kernel of $H_{d-2}(\Gamma\backslash\partial, \mathbb{Q}) \to H_{d-2}(\Gamma\backslash\bar{X}_{F,v}, \mathbb{Q})$ is generated by the images (i.e., boundaries) of modular symbols.

4.7.9. In 4.7.8, assume $d = 2$. Then we can prove that $H^{\mathrm{BM}}_1(\Gamma\backslash X_v, \mathbb{Q})$ is generated by modular symbols. In this case, the map $H_0(\Gamma\backslash\partial, \mathbb{Q}) = \mathrm{Map}(\Gamma\backslash\partial, \mathbb{Q}) \to H_0(\bar{X}_{F,v}, \mathbb{Q}) = \mathbb{Q}$ is just summation, and it is clear that the kernel of it is generated by the boundaries of modular symbols.

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