The arithmetic of modular symbols

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Modular curves

Set $Z = (\mathbb{C} - \mathbb{R}) \cup \mathbb{Q} \cup \{\infty\}$. $\mathrm{GL}_2(\mathbb{Z})$ acts on Z by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$.
Fix $N \geq 4$.

Definition (Congruence subgroups)

Let $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N\}$.
Let $\Gamma_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \mid d \equiv 1 \bmod N\}$.

Definition (Complex modular curves)

Let $\mathfrak{Y} = \Gamma_1(N) \backslash (\mathbb{C} - \mathbb{R})$ and $\mathfrak{X} = \Gamma_1(N) \backslash Z$.

Definition (Modular schemes)

$Y_1(N)$ (resp., $X_1(N)$) is the fine moduli scheme over $\mathbb{Z}[\frac{1}{N}]$ for isomorphism classes of pairs (E, P) with E a (generalized) elliptic curve and a "point P of order N on E."

Remark

We have $\mathfrak{Y} = Y_1(N)(\mathbb{C})$ and $\mathfrak{X} = X_1(N)(\mathbb{C})$.

Modular symbols

Definition (Cusps)

The cusps on $\mathfrak{X}$ are $C = \Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q})$.
The nonzero cusps $C^0 \subset C$ are those not lying over $0 \in \Gamma_0(N)\backslash Z$.

Definition (Plus-minus parts)

Set $\mathbb{Z}' = \mathbb{Z}[\frac{1}{2}]$. For a $\mathbb{Z}'[\tau]$-module M, where τ is a complex conjugation, let $M^\pm = \{m \in M \mid \tau m = \pm m\}$.

Definition (Modular symbols)

Modular symbols are elements of $\mathcal{M} = H_1(X, C, \mathbb{Z}')^+$.
Partial modular symbols are elements of $\mathcal{M}^0 = H_1(X, C^0, \mathbb{Z}')^+$.
Cuspidal modular symbols are elements of $\mathcal{S} = H_1(X, \mathbb{Z}')^+$.

Remark

$0 \to \mathcal{S} \to \mathcal{M}^0 \to \mathfrak{C}^0 \to 0$, where $\mathfrak{C}^0 = \ker(\bigoplus_{x\in C^0} \mathbb{Z}' \to \mathbb{Z}')^+$, is exact.

Manin symbols

Remark

The modular symbols are generated by the plus parts $\{\alpha \to \beta\} \in \mathcal{M}$ of the classes of the geodesics on $\mathfrak{X}$ from some $\alpha \in C$ to some $\beta \in C$.

Definition (Manin symbols)

For $u, v \in \mathbb{Z}/N\mathbb{Z}$ with $(u, v) = (1)$, the Manin symbol $[u : v] \in \mathcal{M}$ is

$$[u : v] = \{\tfrac{d}{Nb} \to \tfrac{c}{Na}\},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $(u, v) = (c, d) \bmod N\mathbb{Z}^2$.

Theorem (Manin)

The Manin symbols $[u : v]$ generate $\mathcal{M}$, and those with $u, v \neq 0$ generate $\mathcal{M}^0$, and the relations $[u : v] = [-u : v] = -[v : u]$ along with

$$[u : v] = [u : u + v] + [u + v : v]$$

(with $u, v, u + v \neq 0$ for $\mathcal{M}^0$) present these $\mathbb{Z}'$-modules.

Hecke operators

Definition (Double coset operators)

For $g \in M_2(\mathbb{Z})$, $\det g > 0$, the double coset operator $t(g)$ on $\mathcal{M}$ acts as

$$t(g)\{\alpha \to \beta\} = \sum_{h \in S_g} \{h\alpha, h\beta\},$$

where $\Gamma_1(N) g \Gamma_1(N) = \coprod_{h \in S_g} \Gamma_1(N) h$.

Definition (Hecke operators)

For $n \geq 1$, we have the nth Hecke operator $T(n) = t\left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right)$.
For $j \in \mathbb{Z}$ prime to N, the diamond operator $\langle j \rangle$ is $t\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ with $d \equiv j \bmod N$.

Remark

$\langle j \rangle$ depends only on the image of j in $\Delta = (\mathbb{Z}/N\mathbb{Z})^\times / \langle -1 \rangle$.

Hecke algebras and the Eisenstein ideal

Definition (Hecke algebras)

The modular Hecke algebra $\mathfrak{H} \subset \mathrm{End}_{\mathbb{Z}'}(\mathcal{M})$ is the commutative $\mathbb{Z}'$-subalgebra generated by the Hecke and diamond operators.
We likewise define the cuspidal Hecke algebra $\mathfrak{h} \subset \mathrm{End}_{\mathbb{Z}'}(\mathcal{S})$.

Remark

Both $\mathfrak{H}$ and $\mathfrak{h}$ are $\mathbb{Z}'[\Delta]$-algebras for inverses of diamond operators.

Definition (The Eisenstein ideal)

Let I be the ideal of $\mathfrak{H}$ (or its image in $\mathfrak{h}$) generated by $T(\ell) - 1 - \ell\langle\ell\rangle$ for primes $\ell \nmid N$ and $T(\ell) - 1$ for primes $\ell \mid N$.

Remark

The quotient $\mathfrak{H}/\mathcal{I}$ is isomorphic to $\mathbb{Z}'[\Delta]$. The quotient $\mathfrak{h}/I$ is finite.

Cyclotomic units

Let $\zeta_N = e^{2\pi i/N} \in \mathbb{C}$ denote a primitive Nth root of 1. Let $\mu_N = \langle \zeta_N \rangle$.

Definition (Cyclotomic fields)

$F_N = \mathbb{Q}(\zeta_N)$ is the Nth cyclotomic field with integer ring $\mathcal{O} = \mathbb{Z}[\zeta_N]$.
$F_N^+ = \mathbb{Q}(\zeta_N + \zeta_N^{-1})$ is the maximal totally real subfield of F_N.

Remark

Note that $\mathrm{Gal}(F_N/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ and $\mathrm{Gal}(F_N^+/\mathbb{Q}) \cong \Delta$.

Definition (Cyclotomic units)

The group $\mathcal{C}$ of cyclotomic N-units is the subgroup of $\mathcal{E} = \mathcal{O}[\frac{1}{N}]^\times$ generated by $1 - \zeta_N^i$ for $i \in \mathbb{Z} - N\mathbb{Z}$.

Remark

The index $[\mathcal{E} : \mathcal{C}]$ is the class number of F_N^+ (up to a power of 2).

The second K-group

The groups $K_2(\mathcal{O}[\frac{1}{M}])$ are finite. We list some results.

Theorem (Tate)

There is a canonical isomorphism

$$K_2(\mathcal{O}[\tfrac{1}{M}]) \otimes \mathbb{Z}_p \xrightarrow{\sim} H^2(G_{F_N,M}, \mathbb{Z}_p(2)),$$

for every prime p dividing M, where $G_{F_N,M}$ is the Galois group of the maximal extension of F_N unramified outside the set of primes over M.

Remark

There are canonical exact sequences:

$$0 \to \mathrm{Cl}(\mathcal{O}[\tfrac{1}{N}]) \otimes \mu_N \to K_2(\mathcal{O}[\tfrac{1}{N}]) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \to \bigoplus_{\mathfrak{p}|N} \mu_N \to \mu_N \to 0,$$

where $\mathrm{Cl}(\mathcal{O}[\frac{1}{N}])$ is the class group of $\mathcal{O}[\frac{1}{N}]$, and

$$0 \to K_2(\mathcal{O}) \to K_2(\mathcal{O}[\tfrac{1}{N}]) \to \bigoplus_{\mathfrak{p}|N} (\mathcal{O}/\mathfrak{p})^\times \to 0.$$

Symbols in K_2

The product on $K_1(\mathcal{O}[\frac{1}{N}]) \cong \mathcal{E}$ is a map

$$\{\ ,\ \}\colon \mathcal{E} \times \mathcal{E} \to K_2(\mathcal{O}[\tfrac{1}{N}])$$

that takes (x, y) to the Steinberg symbol $\{x, y\}$.

Notation

Let $\mathcal{K}_2 = (K_2(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{Z}')^+$ and $\mathcal{K}_2^0 = (K_2(\mathcal{O}[\frac{1}{N}]) \otimes_{\mathbb{Z}} \mathbb{Z}')^+$.

Definition (Symbol map)

The symbol map is the map

$$\{\ ,\ \}^+\colon \mathcal{C} \times \mathcal{C} \to \mathcal{K}_2^0$$

induced by $\{\ ,\ \}$ via restriction and projection.

Conjecture (McCallum-S.)

If $N = p$ is prime, then the image of $\{\ ,\ \}^+$ contains $\mathcal{K}_2^0 \otimes_{\mathbb{Z}'} \mathbb{Z}_p$.

The map ϖ

Theorem (S., Busuioc)

There is a well-defined map of $\mathbb{Z}'[\Delta]$*-modules*

$$\Pi \colon \mathcal{M}^0 \to \mathcal{K}_2^0, \quad \Pi([u:v]) = \{1-\zeta_N^u, 1-\zeta_N^v\}^+,$$

where $j \in \Delta$ *acts by* $\langle j \rangle$ *on* $\mathcal{M}^0$ *and by* σ_j *with* $\sigma_j(\zeta_N) = \zeta_N^j$ *on* $\mathcal{K}_2^0$ *.*

Theorem (Fukaya-Kato, Conj. of S.)

The map Π *factors through a map* $\varpi^0 \colon \mathcal{M}^0/I\mathcal{M}^0 \to \mathcal{K}_2^0$*.*

Proposition (Fukaya-Kato)

There is a commutative diagram

$$\begin{tikzcd}
 & \mathcal{S}/I\mathcal{S} \arrow[r] \arrow[d, "\varpi"] & \mathcal{M}^0/I\mathcal{M}^0 \arrow[r] \arrow[d, "\varpi^0"] & \mathfrak{C}^0/I\mathfrak{C}^0 \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathcal{K}_2 \arrow[r] & \mathcal{K}_2^0 \arrow[r] & \bigoplus_{\mathfrak{p}|N} (\mathcal{O}/\mathfrak{p})^\times \arrow[r] & 0
\end{tikzcd}$$

A conjecture on ϖ

Question

How close are ϖ and ϖ^0 to being isomorphisms?

Conjecture (S.)

Write $N = Mp^r$ with p an odd prime, $p \nmid M$, and $r \geq 0$. On p-parts, ϖ and ϖ^0 are split surjections that are isomorphisms

1. *if $p \nmid |(\mathbb{Z}/M\mathbb{Z})^\times|$ and*
2. *on primitive parts for the $(\mathbb{Z}/Mp\mathbb{Z})^\times$-actions on both sides.*

Idea

The homology groups of modular curves modulo Eisenstein ideals should tell us the structure of the minus part of $\mathrm{Cl}(\mathcal{O})$.

p-adic characters

Notation

- Let $p \geq 5$ be prime with $N = Mp^r$ for $r \geq 1$ and $p \nmid M\varphi(M)$.
- Fix an primitive even character $\theta\colon (\mathbb{Z}/Mp\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p^\times$.
- Let $\omega\colon (\mathbb{Z}/Mp\mathbb{Z})^\times \to \mathbb{Z}_p^\times$ denote the Teichmüller character.

Conditions

$\theta\omega^{-1}|_{(\mathbb{Z}/p\mathbb{Z})^\times} \neq 1$ or $\theta\omega^{-1}|_{(\mathbb{Z}/M\mathbb{Z})^\times}(p) \neq 1$ (and $\theta \neq 1, \omega^2$).

Definition (θ-eigenspace)

Let R_θ denote the ring of values of θ over $\mathbb{Z}_p$. For a $\mathbb{Z}_p[(\mathbb{Z}/Mp\mathbb{Z})^\times]$-module M, let $M_\theta = M \otimes_{\mathbb{Z}_p[(\mathbb{Z}/Mp\mathbb{Z})^\times]} R_\theta$, where $\mathbb{Z}_p[(\mathbb{Z}/Mp\mathbb{Z})^\times] \to R_\theta$ is induced by θ.

A map in the other direction

Theorem (Ohta, S., Fukaya-Kato)

Let $\mathcal{T}$ denote the $\mathfrak{h}[G_{\mathbb{Q},N}]$-module $H^1_{\acute{e}t}(X_1(N)_{/\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))$. There is a canonical exact sequence

$$0 \to P \to \mathcal{T}_\theta / I\mathcal{T}_\theta \to Q \to 0$$

of $\mathfrak{h}_\theta[G_{\mathbb{Q},N}]$-modules and canonical isomorphisms

1. $P \cong \mathcal{S}_\theta / I\mathcal{S}_\theta$ *with trivial Galois action, and*
2. $Q \cong (\mathfrak{h}/I)^\iota_\theta(1)$, *where σ_j acts by multiplication by $\langle j \rangle$ on $(\mathfrak{h}/I)^\iota$.*

Remark

The 1-cocycle $b \colon G_{\mathbb{Q}} \to \mathrm{Hom}_{\mathfrak{h}}(Q, P)$ attached to the sequence is a map which restricts to an unramified homomorphism on $G_{\mathbb{Q}(\mu_{Np^\infty})}$.
We compose the restriction with evaluation at the canonical generator of Q and use class field theory and Tate's theorem to obtain a map

$$\Upsilon_\theta \colon (\mathcal{K}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\theta \to \mathcal{S}_\theta / I\mathcal{S}_\theta.$$

The conjecture

We now have maps

$$\varpi_\theta\colon \mathcal{S}_\theta/I\mathcal{S}_\theta \to (\mathcal{K}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\theta$$

and

$$\Upsilon_\theta\colon (\mathcal{K}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\theta \to \mathcal{S}_\theta/I\mathcal{S}_\theta.$$

Conjecture (S.)

The maps ϖ_θ and Υ_θ are inverse to each other.

Remark

The quotient $\mathcal{S}_\theta/I\mathcal{S}_\theta$ is canonically isomorphic to an Eisenstein quotient of the space of cusp forms with $\mathbb{Z}_p$-coefficients of weight 2, level N, and character θ^{-1} on $(\mathbb{Z}/Mp\mathbb{Z})^\times$.

Remark

Fukaya and Kato have proven a major result towards this conjecture, which is best stated up the modular and cyclotomic towers $X_1(Mp^r)$ and $\mathbb{Q}(\mu_{Mp^r})$ for increasing r, in which our conjecture was originally formulated.

Passing up the modular and cyclotomic towers

Definition (Iwasawa algebra)

$\Lambda = \mathbb{Z}_p[\![\varprojlim_r (\mathbb{Z}/Mp^r\mathbb{Z})^\times]\!]$ is the Iwasawa algebra, which we view both as an algebra of Galois elements and of inverses of diamond operators.

Definition (The unramified Iwasawa module)

Let X_∞ denote the Galois group of the maximal unramified abelian pro-p extension of $\mathbb{Q}(\mu_{Mp^\infty})$.

Definition

Let $\mathcal{S}_\infty = \varprojlim_r H_1(X_1(Mp^r), \mathbb{Z}_p)$.
Let I_∞ denote the Eisenstein ideal in Hida's cuspidal Hecke algebra.

Remark

Inverse limits over r of ϖ_θ and Υ_θ in towers for $N = Mp^r$ are maps

$$\varpi_{\infty,\theta} \colon (\mathcal{S}_\infty/I_\infty\mathcal{S}_\infty)_\theta \to X_\infty(1)_\theta, \qquad \Upsilon_{\infty,\theta} \colon X_\infty(1)_\theta \to (\mathcal{S}_\infty/I_\infty\mathcal{S}_\infty)_\theta$$

of Λ_θ-modules.

The result of Fukaya-Kato

Definition (power series interpolating p-adic L-function)

Let $\xi_\theta \in \Lambda_\theta$ be the unique element such that

$$\xi_\theta(u^s - 1) = L_p(\omega^2\theta^{-1}, s-1)$$

for all $s \in \mathbb{Z}_p$.

Theorem (Fukaya-Kato)

One has

$$\xi'_\theta \Upsilon_{\infty,\theta} \circ \varpi_{\infty,\theta} = \xi'_\theta$$

on $(\mathcal{S}_\infty / I_\infty \mathcal{S}_\infty)_\theta \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$*, where* ξ'_θ *denotes the derivative of* ξ_θ *in the* s*-variable.*

Remark

If we can remove $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ or know that $\mathcal{S}_\infty / I_\infty \mathcal{S}_\infty$ is p-torsion free, then this descends immediately to an interesting statement for ϖ_θ and Υ_θ, where the modules are finite.

Some consequences

Remark (Iwasawa, Ferrero-Washington)

X_∞^- is p-torsion free.

Corollary

If $L_p(\omega^2\theta^{-1}, s)$ has no multiple zeros, then $\varpi_{\infty,\theta}$ and $\Upsilon_{\infty,\theta}$ are inverse modulo torsion in $(\mathcal{S}_\infty/I_\infty\mathcal{S}_\infty)_\theta$.

Corollary

If $L_p(\omega^2\theta^{-1}, s)$ has no multiple zeros, then $\varpi_{\infty,\theta} \circ \Upsilon_{\infty,\theta} = 1$. In particular, $\varpi_{\infty,\theta}$ is surjective, so the conjecture of McCallum-S. holds in the θ-eigenspace.

Remark

The most important of many keys to the proofs of the results of Fukaya-Kato is the fact that Beilinson-Kato elements specialize to Steinberg symbols of cyclotomic N-units at the cusps.

The rational function field

For the remainder of the talk, we turn to joint work with Fukaya and Kato on an analogue of the conjecture for global function fields.

Notation

Let q be a power of a prime ℓ.
Set $F = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$.
Let ∞ be the infinite place of F, corresponding to $\pi = t^{-1}$.
Let $F_\infty = \mathbb{F}_q(\!(\pi)\!)$ and $\mathcal{O}_\infty = \mathbb{F}_q[\![\pi]\!]$.
Let $N \in A$ be a nonconstant polynomial.

Table: Analogy with rational numbers

Ground field	$\mathbb{Q}$	$\mathbb{F}_q(t)$
Modular group	$\mathrm{GL}_2(\mathbb{Z})$	$\mathrm{GL}_2(A)$
Level	$N \geq 1$	$N \in A$ nonconstant
Homology	$H_1(X_1(N), \mathbb{Z}')^+$	H_1 of Bruhat-Tits tree quot.
Modular symbols	paths between cusps	paths between ends
Cyclotomic units	$1 - \zeta_N^u$	roots λ_u of Carlitz polynomial
Cyclotomic field	$\mathrm{Gal} = (\mathbb{Z}/N\mathbb{Z})^\times$	$\mathrm{Gal} = (A/NA)^\times$
Real subfield	quotient by $\langle -1 \rangle$	quotient by $\mathbb{F}_q^\times$
Galois module	$H^1_{\text{ét}}$ of modular curve	$H^1_{\text{ét}}$ of Drinfeld modular curve

The Bruhat-Tits tree

Definition (Bruhat-Tits tree)

The Bruhat-Tits tree B for F is a $(q+1)$-valent tree with vertices given by homothety classes of $\mathcal{O}_\infty$-lattices in F_∞^2. There is an edge between classes of lattices L and L' if and only if there exists $i \in \mathbb{Z}$ such that $\pi^i L \subset L'$ with $[L' : \pi^i L] = q$.

Remark

There is a canonical action of $\mathrm{PGL}_2(F_\infty)$ on B induced by the action of $\mathrm{GL}_2(F_\infty)$ on lattices.

Definition (End)

An end is a ray in B consisting of a union of edges starting from the class of $\mathcal{O}_\infty^2$. These correspond to elements of $\mathbb{P}^1(F_\infty)$.

Notation

Let $B^* = B \cup \mathbb{P}^1(F)$ (adding a point at the end of each rational edge).

Modular symbols

Definition (Congruence subgroups)

Let $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A) \mid c \equiv 0 \bmod N \}$.
Let $\Gamma_1(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \mid d \equiv 1 \bmod N \}$.

Definition (Modular quotients and ends)

Let $\mathfrak{Y} = \Gamma_1(N)\backslash B$ and $\mathfrak{X} = \Gamma_1(N)\backslash B^*$. Let $C = \Gamma_1(N)\backslash \mathbb{P}^1(F)$.
Let C^0 denote the ends in C which do not lie over $0 \in \Gamma_0(N)\backslash \mathbb{P}^1(F)$.

Notation (Modular symbols)

Let $\mathcal{S} = H_1(\mathfrak{X}, \mathbb{Z}_p)$, $\mathcal{M}^0 = H_1(\mathfrak{X}, C^0, \mathbb{Z}_p)$, and $\mathcal{M} = H_1(\mathfrak{X}, C, \mathbb{Z}_p)$.

Definition (Modular symbol)

For $\alpha, \beta \in C$, the modular symbol $\{\alpha \to \beta\} \in \mathcal{M}$ is the class of the image of a path from α to β in B^*.

Manin-Teitelbaum symbols

Remark

The modular symbols $\{\alpha \to \beta\}$ generate $\mathcal{M}$.

Definition (Manin-Teitelbaum symbols)

For $u, v \in A/NA$ with $(u, v) = (1)$, the Manin-Teitelbaum symbol $[u : v] \in \mathcal{M}$ is

$$[u : v] = \{\tfrac{d}{Nb} \to \tfrac{c}{Na}\},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(A)$ with $(u, v) = (c, d) \bmod NA^2$.

Theorem (Teitelbaum)

The symbols $[u : v]$ generate $\mathcal{M}$, those with $u, v \neq 0$ generate $\mathcal{M}^0$, and the relations $[u : v] = [-u : v] = -[v : u] = [u : u + v] + [u + v : v]$ present these groups.

Remark

There are Hecke and diamond operators acting on $\mathcal{M}$, $\mathcal{M}^0$, and $\mathcal{S}$.

Drinfeld modules

Definition

For an A-algebra R, let $R\{\tau\}$ denote the ring of power series in τ with R-coefficients with multiplication the usual multiplication on R and powers of τ and $\tau^k \cdot r = r^{q^k}\tau^k$ for $r \in R$ and $k \geq 1$.

Definition (Drinfeld module over a field)

For an A-field L (given by $\iota \colon A \to L$) and $n \geq 1$, a rank n Drinfeld module is a ring homomorphism $\Phi \colon A \to L\{\tau\}$ such that, writing $\Phi(a) = \sum_{i=0}^{n \deg a} c_i(a)\tau^i$ for $a \in A$, we have $c_0 = \iota$ and $c_n(t) \neq 0$.

Example (Carlitz module)

The Carlitz module is the rank 1 Drinfeld module with $\Phi(t) = t + \tau$.

Remark

There is a notion of a Drinfeld module Φ over an A-scheme S, which in particular is a Drinfeld module at each point. We also have a notion of generalized Drinfeld module, where the condition on $c_n(t)$ is relaxed.

Cyclotomic extensions

Let $\mathbb{C}_\infty$ denote the completion of the algebraic closure of F_∞.

Definition (Carlitz exponential)

The Carlitz exponential is the additive map $\exp\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ with

$$\exp(x) = x \prod_{a \in A - \{0\}} (1 - \tfrac{x}{a}),$$

with $\exp(ax) = \Phi(a)(\exp(x))$ for $a \in A$ and $x \in \mathbb{C}_\infty$.

Definition (Cyclotomic unit)

For $u \in A - NA$, set $\lambda_u = \exp(\frac{u\pi}{N})$, where $\pi \in \mathbb{C}_\infty$ with π an A-module generator of the kernel of exp.

Theorem (Carlitz)

The λ_u are N-units in an abelian, unramified outside $N\infty$ extension $F_N = F(\lambda_1)$ of F with constant field $\mathbb{F}_q$, Galois group $(A/NA)^\times$, and integer ring $A_N = A[\lambda_1]$. The maximal subextension F_N^+ in which ∞ splits completely satisfies $\mathrm{Gal}(F_N/F_N^+) \cong \mathbb{F}_q^\times$.

The map ϖ

Fix a prime $p \neq \ell$ with $p \nmid (q^2-1)$. Use $+$ to denote the $\mathbb{F}_q^\times$-fixed part.

Definition (K-groups)

Set $\mathcal{K}_2^0 = (K_2(A_N[\frac{1}{N}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^+$ and $\mathcal{K}_2 = (K_2(A_N) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^+$.

From now on, we write $\mathcal{S}$ for $\mathcal{S} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and so on.

Theorem

There exists a map $\Pi\colon \mathcal{M}^0 \to \mathcal{K}_2^0$ such that $\Pi([u:v]) = \{\lambda_u, \lambda_v\}^+$, where $\{\ ,\ \}^+$ denotes the projection of the symbol map to the plus part.

Let $\mathfrak{H}$ and $\mathfrak{h}$ denote the Hecke algebras for $\mathcal{M}$ and $\mathcal{S}$, respectively. These have Eisenstein ideals I, defined as before but for elements of A.

Theorem

The map Π factors through a map $\varpi^0\colon \mathcal{M}^0/I\mathcal{M}^0 \to \mathcal{K}_2^0$, where I is an Eisenstein ideal in $\mathfrak{H}$. It restricts to a map $\varpi\colon \mathcal{S}/I\mathcal{S} \to \mathcal{K}_2$.

The two theorems are proven simultaneously using products of Siegel units on $Y_1(N)$.

Drinfeld modular curves

Definition (Drinfeld modular curves)

The Drinfeld modular curve $Y_1(N)$ (resp., $X_1(N)$) is the $A[\frac{1}{N}]$-scheme parameterizing isomorphism classes of rank 2 Drinfeld modules (resp., generalized Drinfeld modules) $\mathcal{E}$ together with a "point of order N."

Remark

We have identifications $Y_1(N)(\mathbb{C}_\infty) = \Gamma_1(N)\backslash(\mathbb{C}_\infty - F_\infty)$ and $X_1(N)(\mathbb{C}_\infty) = \Gamma_1(N)\backslash((\mathbb{C}_\infty - F_\infty) \cup F)$.

Remark

There is a $\mathrm{GL}_2(F_\infty)$-equivariant "building map" $X_1(N)(\mathbb{C}_\infty) \to \mathfrak{X}$ restricting to $Y_1(N)(\mathbb{C}_\infty) \to \mathfrak{Y}$.

Notation

Let $\mathcal{T} = H^1_{\text{ét}}(X_1(N)_{/\bar{F}_\infty}, \mathbb{Z}_p(1))$.

The map Υ

Notation

- Fix a primitive character $\theta\colon (A/NA)^\times/\mathbb{F}_q^\times \to \bar{\mathbb{Q}}_p^\times$.
- Let R_θ the value ring of θ over $\mathbb{Z}_p$, and define θ-parts as before.

Theorem

There is an exact sequence $0 \to P \to \mathcal{T}_\theta/I\mathcal{T}_\theta \to Q \to 0$ *of* $\mathfrak{h}[G_{A,N}]$*-modules, where* $P \cong \mathcal{S}/I\mathcal{S}$ *and* $Q \cong (\mathfrak{h}/I)^\iota(1)$*.*

Via a connecting homomorphism in the Galois cohomology of the Tate twist of this sequence, we can construct a map $\Upsilon_\theta\colon (\mathcal{S}/I\mathcal{S})_\theta \to (\mathcal{K}_2)_\theta$.

Conjecture

The maps ϖ_θ *and* Υ_θ *are inverse to each other.*

Consider $L(\theta^{-1}, s)$ with Euler product $\prod_{r\nmid N}(1-\theta(r)^{-1}\mathfrak{N}(r)^{-s})^{-1}$.

Theorem

We have $\xi_\theta'\Upsilon_\theta \circ \varpi_\theta = \xi_\theta'$*, where* $\xi_\theta' = L'(\theta^{-1}, -1)$*.*

The big picture

Idea

Our philosophy is this:

$$\text{geometric theory of } \mathrm{GL}_n \text{ modulo an Eisenstein ideal}$$
$$\Longleftrightarrow$$
$$\text{arithmetic theory of } \mathrm{GL}_{n-1}$$

Remark

For function fields, many of the tools required to formulate and attempt to prove a conjecture validating this philosophy are available, in particular due to recent work of Kondo-Yasuda.