Iwasawa Theory for Kummer Extensions

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Classical Iwasawa Theory:

<u>Basic objects:</u>
p prime number, F number field.
K cyclotomic $\mathbf{Z}_p$-extension of F.
$\Gamma = \mathrm{Gal}(K/F) \cong \mathbf{Z}_p$, $\Lambda = \mathbf{Z}_p[[\Gamma]] \cong \mathbf{Z}_p[[T]]$.

An Iwasawa module is a continuous Λ-module. Any Galois group of a pro-p abelian extension of K Galois over F is an Iwasawa module via conjugation (lifting elements of Γ).

<u>Examples:</u>

Module	Galois group over K of its maximal abelian pro-p...
$\mathfrak{X}_K$	unramified outside p extension
X_K	unramified extension
Y_K	unramified extension in which all primes above p split completely

$\mathfrak{X}_K \twoheadrightarrow X_K \twoheadrightarrow Y_K$.

$\mathfrak{X}_K$ is a finitely generated Λ-module.
X_K is a torsion Λ-module.

Conjecture (Iwasawa)**.** X_K *is finitely generated over* $\mathbf{Z}_p$*.*

Iwasawa's conjecture holds at p if and only if the p-torsion in X_K is finite.
Ferrero-Washington proved this for abelian extensions of $\mathbf{Q}$.

If F is abelian, then Iwasawa's Main Conjecture (Mazur-Wiles) describes the even eigenspaces of $\mathfrak{X}_K$ in terms of the p-adic L-functions of the corresponding characters.

The Main Conjecture also describes the odd eigenspaces of X_K.

Iwasawa modules over larger extensions:

L abelian pro-p extension of K unramified outside a finite set of primes and Galois over F.
$G = \mathrm{Gal}(L/K)$, $\mathcal{G} = \mathrm{Gal}(L/F)$.

Examples:
1. $L = \tilde{K}$, the compositum of all $\mathbf{Z}_p$-extensions of F.

Conjecture (Greenberg)**.** *$X_{\tilde{K}}$ has annihilator of height ≥ 2 as a $\mathbf{Z}_p[[\mathcal{G}]]$-module.*

Greenberg's conjecture $\Leftrightarrow$ X_L is $\mathbf{Z}_p[[G]]$-torsion.

F totally real $\Rightarrow$ $\tilde{K} = K$ and Greenberg's conjecture reduces to the finiteness of X_K.

In fact, Vandiver had conjectured that X_K is trivial when $F = \mathbf{Q}(\mu_p)^+$.

2. L is a $\mathbf{Z}_p$-extension of K Galois over F.
Is X_L a $\mathbf{Z}_p[[G]]$-torsion module?

If $\mu_p \subset F$, Kummer theory implies that L can be defined by the p-power roots of an element of the p-completion of $K^\times$.

Conjecture. *If $F = \mathbf{Q}(\mu_p)$ and L is defined by a sequence of p-units of $K = \mathbf{Q}(\mu_{p^\infty})$, then X_L is $\mathbf{Z}_p[[G]]$-torsion.*

Theorem 1. *Let $F = \mathbf{Q}(\mu_p)$ and $L = K(p^{1/p^\infty})$. For $p < 1000$, we have $X_L \cong X_K$.*

Theorem 2 (Hachimori-S.)**.** *If L is a CM-field and the $\mathbf{Z}_p$-rank of X_K is at least 2, then X_L is not $\mathbf{Z}_p[[G]]$-torsion.*

There are unramified outside p examples for $p = 157$, 353, 379, etc., when $F = \mathbf{Q}(\mu_p)$.

Theorem 1 has the following consequence.

Theorem 3. *Greenberg's conjecture holds for* $\mathbf{Q}(\mu_p)$ *for all* $p < 1000$*.*

The above conjecture on those L defined by p-units implies Greenberg's conjecture for $\mathbf{Q}(\mu_p)$.

Another application:

A slightly stronger statement gives us the following.

Theorem 4. *The Galois group* $\mathcal{G}_K^{\mathsf{un}}$ *of the maximal unramified pro-*p *extension of* $K = \mathbf{Q}(\mu_{p^\infty})$ *is abelian for all* $p < 1000$*.*

Note: this contradicts a twice-published result (1994) that $\mathcal{G}_K^{\mathsf{un}}$ is free pro-p under Vandiver's conjecture.

Iwasawa theory for Kummer extensions:

L a $\mathbf{Z}_p$-extension of K Galois over F and unramified outside a finite set of primes.

Theorem 5. *There is a canonical exact sequence of* Λ*-modules:*

$$0 \to Y_L^G \otimes_{\mathbf{Z}_p} G \to \mathcal{U}_{L/K} \otimes_{\mathbf{Z}_p} G \to \ker \Sigma_{L/K} \\ \to (Y_L)_G \to Y_K \to \operatorname{coker} \Sigma_{L/K} \to 0.$$

G_v decomposition group of G at a prime v.
$\Sigma_{L/K}\colon \bigoplus_v G_v \to G$ product map.
$\mathcal{U}_{L/K}$ quotient of universal norm sequences of p-units for K/F by universal norms from L.

$\Sigma_{L/K}$ is an isomorphism if there is a unique prime above p in L outside of which L/K is unramified.
For simplicity, let us assume this.

I_G augmentation ideal in $\mathbf{Z}_p[[G]]$.
Note: $(Y_L)_G = Y_L/I_G Y_L$.

We can consider higher graded quotients.

Theorem 6. *For each $k \geq 1$, there is a canonical isomorphism:*

$$I_G^k Y_L / I_G^{k+1} Y_L \cong Y_K/\mathcal{P}^k_{L/K} \otimes_{\mathbf{Z}_p} G^{\otimes k},$$

with $\mathcal{P}^k_{L/K}$ the group of "$(k+1)$-fold Massey products for L/K."

In particular, $\mathcal{P}^1_{L/K}$ is generated by inverse limits of cup products of a Kummer generator of L with a universal norm of p-units.

Modular Galois representations:

$\mathfrak{h}$ even eigenspace of Hida's ordinary cuspidal Hecke algebra of increasing p-power levels (mild restrictions on the eigenspace).
$\mathfrak{h}$ has Hecke operators U_p and T_l for $l \neq p$.
Eisenstein ideal $\mathcal{I}$ of $\mathfrak{h}$: generated by T_l minus the lth coefficient of an Eisenstein series and $U_p - 1$.

Consider the Galois action on the corresponding eigenspace $\mathfrak{Z}$ of an inverse limit of cohomology groups for $X_1(p^n)$.

Note: M. Ohta used this action to give another proof of the Main Conjecture for $\mathbf{Q}$.

The fixed field of the kernel of $G_{\mathbf{Q}} \to \mathrm{Aut}_{\mathfrak{h}}\mathfrak{Z}$ contains an unramified outside p, totally ramified at p extension L/K with Λ-torsion Galois group G.

$\mathcal{V} = \mathbf{Z}_p$-submodule of $\mathcal{I}$ generated by $U_p - 1$.
$F = \mathbf{Q}(\mu_p)$, $\tilde{\Gamma} = \mathrm{Gal}(K/\mathbf{Q})$.

Theorem 7. *There are canonical isomorphisms*

$$(I_G X_L / I_G^2 X_L)_{\tilde{\Gamma}} \cong \mathcal{I}/\mathcal{I}^2,$$
$$(I_G Y_L / I_G^2 Y_L)_{\tilde{\Gamma}} \cong \mathcal{I}/(\mathcal{V} + \mathcal{I}^2).$$

Outline of proof of Theorem 1. We show that $U_p - 1$ generates $\mathcal{I}/\mathcal{I}^2$ as a pro-p group if and only if $G \cong \mathbf{Z}_p$ and an eigenspace of $\mathcal{P}^1_{L/K}$ and Y_K agree. This eigenspace is generated by the cup product of an element of the p-completion of $K^\times$ defining L with a universal norm in K with value p in F. Set $E = K(p^{1/p^\infty})$. By an argument involving antisymmetry of the cup product, we can deduce that X_E and X_K agree if $U_p - 1$ generates $\mathcal{I}/\mathcal{I}^2$ in all eigenspaces. The latter holds for $p < 1000$ by computation in level p.

Cup Products:

$G_{F,\{p\}}$ Galois group of the maximal unramified outside p extension of F.
$\mathcal{E}_F$ p-units in F, A_F p-part of class group of F.
$\Delta = \mathrm{Gal}(F/\mathbf{Q})$.
Cup product on $H^1(G_{F,\{p\}}, \mu_p)$ induces:

$$(\cdot\,,\,\cdot)\colon \mathcal{E}_F \times \mathcal{E}_F \to A_F \otimes \mu_p.$$

Theorem 1 $\Rightarrow$ $(p, \cdot)$ is surjective for $p < 1000$.

Theorem 8 (McCallum-S.)**.** *For* $p < 10000$, *we have a computation of* $(\cdot\,,\,\cdot)$ *up to scalar in* $\mathbf{Z}/p\mathbf{Z}[\Delta]$ (*or, in terms the value* $(p, 1-\zeta)$ *with* $\langle\zeta\rangle = \mu_p$).

This allows us to prove Theorem 4 on unramified Galois groups.

Rough form of a conjecture: the values of the cup product pairing are p-adic L-values modulo p of cusp forms congruent to Eisenstein series modulo p.