The Riemann zeta function

**Definition**

The *Riemann zeta function* is the unique meromorphic function on $\mathbb{C}$ satisfying

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

for $\text{Re}(s) > 1$.

**Remarks**

1. $\zeta(s)$ is holomorphic outside $s = 1$, where it has a simple pole of residue 1,
2. $\zeta(s)$ has a *functional equation*: setting

$$\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

we have $\Lambda(s) = \Lambda(1 - s)$ for $s \neq 0, 1$. Here $\Gamma(s)$ is meromorphic on $\mathbb{C}$ with

$$\Gamma(s) = \int_{0}^{\infty} x^{s-1} e^{-x} \, dx$$

for $\text{Re}(s) > 0$. It has simple poles at non-positive integers, and $\Gamma(n) = (n - 1)!$ for $n \geq 1$. 
Values of the zeta function

**Definition**

For \( n \geq 0 \), the \( n \)th **Bernoulli number** \( B_n \in \mathbb{Q} \) is given by

\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.
\]

1. \( B_1 = -\frac{1}{2} \), but \( B_n = 0 \) for odd \( n \geq 3 \), since \( \frac{x}{e^x - 1} - \frac{-x}{e^{-x} - 1} = -x \).
2. \( B_0 = 1 \), \( B_2 = \frac{1}{6} \), \( B_4 = -\frac{1}{30} \), \( B_6 = \frac{1}{42} \), \( B_8 = -\frac{1}{30} \), \( B_{10} = \frac{5}{66} \).
3. \( B_{12} = -\frac{691}{2730} \). Note: 691 is prime, and 2730 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13.

4. \( \zeta(k) = (-1)^{\frac{k}{2} + 1} \frac{(2\pi)^k B_k}{2 \cdot k!} \) for even \( k \geq 0 \), and \( \zeta(3) \) is irrational (Apéry).

2. \( \zeta(1 - n) = -\frac{B_n}{n} \) for \( n \geq 2 \). In particular, \( \zeta(-k) = 0 \) for even \( k \geq 2 \).

**Conjecture**

*The numbers \( \zeta(n) \) for odd \( n \geq 3 \), and \( \pi \), are \( \mathbb{Q} \)-algebraically independent.*
Irregular primes

From now on, we use $k$ to denote a positive even integer.

**Remark (Kummer congruences)**

1. We have $(p - 1) \mid k$ if and only if the denominator of $\frac{B_k}{k}$ is divisible by $p$.
2. If $(p - 1) \nmid k$, then $\frac{B_k}{k} \equiv \frac{B_{k+p-1}}{k+p-1} \mod p$.

**Definition**

A prime $p$ is *regular* if $p \nmid B_2 B_4 \cdots B_{p-3}$, and it is *irregular* otherwise.

**Remark**

There are infinitely many irregular primes (Jensen, 1915).

**Example**

The smallest irregular primes are 37, 59, and 67, dividing $B_{32}$, $B_{44}$, and $B_{58}$, respectively. Note that 691 is irregular, as $691 \mid B_{12}$. 
The class group of a number field $F$ is a finite abelian group that measures the failure of ideals of the ring of integers $\mathcal{O}$ of $F$ to be principal. (The class group consists of equivalence classes of nonzero ideals of $\mathcal{O}$ up to by multiplication by principal ideals.)

The class number of $F$ is the order of the class group of $F$.

**Definition**

For $n \geq 1$, the $n$th cyclotomic field $\mathbb{Q}(\mu_n)$ is the smallest subfield of $\mathbb{C}$ containing the group of $n$th roots of unity $\mu_n$.

**Theorem (Dirichlet, Kummer 1847)**

A prime $p$ divides the class number of $\mathbb{Q}(\mu_p)$ if and only if $p$ is irregular.

**Conjecture (Kummer 1849, known as Vandiver’s conjecture)**

The class number of $\mathbb{Q}(\mu_p) \cap \mathbb{R}$ is not divisible by $p$.

**Theorem (Buhler-Harvey 2011)**

Vandiver’s conjecture holds for $p < 163577356$. 

The $K$-groups $K_n(\mathbb{Z})$ for $n \geq 0$ are abelian groups that tell us in some sense about “higher arithmetic” of $\mathbb{Z}$. We have $K_0(\mathbb{Z}) \cong \mathbb{Z}$ and $K_1(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

**Theorem (Borel 1974)**

For $n \geq 2$, the groups $K_n(\mathbb{Z})$ are finite unless $n \equiv 1 \mod 4$, in which case they are isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

The following is a consequence of the Quillen-Lichtenbaum conjecture, known by the work of many: Soulé, Dwyer-Friedlander, Rognes-Weibel, Rost, Voevodsky....

**Theorem**

For even $k \geq 2$, we have that $K_{2k-1}(\mathbb{Z})$ is a cyclic group, and

$$\frac{|K_{2k-2}(\mathbb{Z})|}{|K_{2k-1}(\mathbb{Z})|} = \frac{B_k}{2^k} = \frac{\zeta(1-k)}{2}.$$  

Aside from a single power of 2 if $4 \nmid k$, the leftmost fraction is reduced.

Vandiver’s conjecture implies that $K_{2k}(\mathbb{Z}) = 0$ and $K_{2k-2}(\mathbb{Z})$ is cyclic for even $k$. It is known that $K_4(\mathbb{Z}) = 0$ (Rognes).
Zeta values and Eisenstein series

Let \( \mathbb{H} \) denote the complex upper half-plane.

**Definition**

For even \( k \geq 4 \), the \( k \)-th normalized *Eisenstein series* \( G_k \) is the holomorphic function on \( \mathbb{H} \) given by

\[
G_k(z) = \frac{(k-1)!}{2 \cdot (2\pi i)^k} \sum_{(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(a + bz)^k}.
\]

It extends to a holomorphic function at \( \infty \), with value \(-\frac{1}{2} \zeta(1 - k)\).

\( G_k \) is a *weight* \( k \) *modular form* for \( \text{SL}_2(\mathbb{Z}) \). That is, if \((a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z})\), then

\[
G_k \left( \frac{az + b}{cz + d} \right) = (cz + d)^k G_k(z).
\]

For \( q = e^{2\pi i z} \) with \( z \in \mathbb{H} \), we have

\[
G_k(q) = -\frac{\zeta(1 - k)}{2} + \sum_{n=1}^{\infty} \left( \sum_{d \mid n} d^{k-1} \right) q^n.
\]
Hecke operators and eigenforms

**Definition**

1. The $m$th *Hecke operator* $T_m$ on the space $M_k$ of modular forms of weight $k$ for $\text{SL}_2(\mathbb{Z})$ takes $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ to

   $$T_m f = \sum_{n=0}^{\infty} \left( \sum_{d | \gcd(m,n)} d^{k-1} a_{mn/d^2} \right) q^n.$$

2. The Hecke algebra $\mathcal{H}_k$ of $M_k$ is the $\mathbb{Z}$-subalgebra of $\text{End}_\mathbb{C}(M_k)$ generated by the operators $T_m$ for $m \geq 1$.

**Definition**

$f \in M_k$ is a (normalized) *eigenform* if $T_m f = a_m f$ for all $m \geq 1$.

**Example**

The Eisenstein series $G_k$ are all eigenforms.

**Remark**

The space $M_k$ has a $\mathbb{C}$-basis of eigenforms.
Cusp forms and the Eisenstein ideal

**Definition**

1. A modular form in $M_k$ is a *cusp form* if it vanishes at $\infty$ (i.e., $a_0 = 0$).
2. The $\mathcal{S}_k$-submodule of weight $k$ cusp forms in $M_k$ is denoted $S_k$.

**Remark**

We have $M_k \cong S_k \oplus \mathbb{C}G_k$ as $\mathcal{S}_k$-modules.

**Definition**

Let $\mathfrak{H}_k$ denote the *cuspidal Hecke algebra*, the image of $\mathcal{S}_k$ in $\text{End}_{\mathbb{C}}(S_k)$.

**Definition**

The *Eisenstein ideal* is the ideal $I_k = (T_n - \sum_{d|n} d^{k-1} | n \geq 1)$ of $\mathfrak{H}_k$.

**Theorem (Kurihara, Harder-Pink)**

We have an isomorphism $\mathfrak{H}_k/I_k \cong \mathbb{Z}/c_k\mathbb{Z}$ defined by $T_n \mapsto \sum_{d|n} d^{k-1} \mod c_k$, where $c_k$ is the numerator of $|\frac{B_k}{2k}|$. 
Definition

The *L-function* of a modular form \( f = \sum_{n=0}^{\infty} a_n q^n \) is the holomorphic function on \( \mathbb{C} \) such that

\[
L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}
\]

for \( \text{Re}(s) > k \).

Remark

The function \( \Lambda(f, s) = (2\pi)^{-s} \Gamma(s) L(f, s) \) satisfies the functional equation

\[
\Lambda(f, s) = (-1)^{\frac{k}{2}} \Lambda(f, k - s).
\]

Example

We have \( L(G_k, s) = \zeta(s) \zeta(s - k + 1) \).
**Theorem (Manin 1973)**

For an eigenform \( f = \sum_{n=1}^{\infty} a_n q^n \in S_k \), there exist periods \( \Omega_{f}^{\pm} \in \mathbb{C} \) such that

\[
\frac{\Lambda(f, i)}{\Omega_{f}^{\pm}} \in K_f = \mathbb{Q}(a_1, a_2, \ldots)
\]

for \( 1 \leq i \leq k - 1 \), where \( \pm \) is the sign of \((-1)^{i-1}\).

**Example**

The space \( S_{12} \) is one-dimensional, spanned by the eigenform

\[
\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.
\]

Note: \( 691 \mid \zeta(-11) \), and in fact, \( \Delta \equiv G_{12} \mod 691 \). Take \( \Omega_{\Delta}^{+} = \Lambda(\Delta, 1) \). Then

\[
\frac{\Lambda(\Delta, 3)}{\Omega_{\Delta}^{+}} = \frac{-691}{2^2 \cdot 3^4 \cdot 5} \quad \text{and} \quad \frac{\Lambda(\Delta, 5)}{\Omega_{\Delta}^{+}} = \frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7}.
\]

The ratio \( \frac{\Lambda(\Delta, 5)}{\Lambda(\Delta, 3)} \) is \( \frac{-9}{14} \equiv -50 \mod 691 \).
Fact

There are product maps $K_i(\mathbb{Z}) \times K_j(\mathbb{Z}) \to K_{i+j}(\mathbb{Z})$ for all $i, j \geq 0$. If $x \in K_i(\mathbb{Z})$ and $y \in K_j(\mathbb{Z})$, then $xy = (-1)^{ij}yx$.

Theorem (Soulé, Beilinson-Deligne, Huber-Wildeshaus)

For odd $i \geq 3$, there is a canonical element $\kappa_i$ of infinite order in $K_{2i-1}(\mathbb{Z})$.

For a finitely generated abelian group $A$, let us set $A' = A/(2$-power torsion$)$.

Remark

The Soulé element $\kappa_i$ generates $K_{2i-1}(\mathbb{Z})' \cong \mathbb{Z}$ if Vandiver’s conjecture holds.

Example (S.)

We have $K_{22}(\mathbb{Z}) \cong \mathbb{Z}/691\mathbb{Z}$. For the product maps

$$K_5(\mathbb{Z}) \times K_{17}(\mathbb{Z}) \to K_{22}(\mathbb{Z}) \quad \text{and} \quad K_9(\mathbb{Z}) \times K_{13}(\mathbb{Z}) \to K_{22}(\mathbb{Z})$$

we have $\kappa_5\kappa_7 = a\kappa_3\kappa_9 \neq 0$ with $a \equiv -50 \mod 691$. 
For odd primes $p < 25,000$ and $k$ with $p \mid \frac{B_k}{k}$, a computation of McCallum-S. (2003), along with results of S. and Fukaya-Kato, provides the values $\kappa_i \kappa_{k-i}$ of

$$K_{2i-1}(\mathbb{Z}) \times K_{2(k-i)-1}(\mathbb{Z}) \to K_{2k-2}(\mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z}.$$ 

For small $k$ and $p$ with $p \mid B_k$, the following table lists $\kappa_i \kappa_{k-i} \mod p$ for $i = 3, 5, \ldots, k - 3$ under an isomorphism of the $p$-part of $K_{2k-2}(\mathbb{Z})$ with $\mathbb{Z}/p\mathbb{Z}$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$p$</th>
<th>$\kappa_i \kappa_{k-i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>691</td>
<td>(222, 647, 44, 469)</td>
</tr>
<tr>
<td>16</td>
<td>3617</td>
<td>(1787, 2884, 3312, 305, 733, 1830)</td>
</tr>
<tr>
<td>20</td>
<td>283</td>
<td>(251, 194, 260, 172, 111, 23, 89, 32)</td>
</tr>
<tr>
<td>20</td>
<td>617</td>
<td>(144, 593, 53, 110, 507, 564, 24, 473)</td>
</tr>
<tr>
<td>22</td>
<td>131</td>
<td>(35, 74, 129, 81, 0, 50, 2, 57, 96)</td>
</tr>
<tr>
<td>22</td>
<td>593</td>
<td>(469, 77, 541, 10, 0, 583, 52, 516, 124)</td>
</tr>
<tr>
<td>24</td>
<td>103</td>
<td>(70, 17, 22, 77, 25, 78, 26, 81, 86, 33)</td>
</tr>
<tr>
<td>32</td>
<td>37</td>
<td>(26, 0, 36, 1, 35, 31, 34, 3, 6, 2, 36, 1, 0, 11)</td>
</tr>
<tr>
<td>44</td>
<td>59</td>
<td>(45, 21, 30, 14, 35, 5, 0, 48, 57, 7, 52, 2, 11, 0, 54, 24, 45, 29, 38, 14)</td>
</tr>
<tr>
<td>58</td>
<td>67</td>
<td>(45, 38, 56, 0, 47, 62, 9, 29, 15, 65, 26, 45, 57, 0, 10, 22, 41, 2, 52, 38, 58, 5, 20, 0, 11, 29, 22)</td>
</tr>
</tbody>
</table>
Relations among multiple zeta values

Definition

For positive integers \( r_1, \ldots, r_d \) with \( r_1 \geq 2 \), the value

\[
\zeta(r_1, \ldots, r_d) = \sum_{n_1 > \cdots > n_d > 0} \frac{1}{n_1^{r_1} n_2^{r_2} \cdots n_d^{r_d}}
\]

is called a *multiple zeta value* (MZV) of weight \( k = r_1 + \ldots + r_d \) and depth \( d \).

Remarks

1. These are values of iterated integrals yielding periods of \( \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \).
2. They satisfy standard relations: e.g., \( \zeta(r, s) + \zeta(s, r) = \zeta(r)\zeta(s) - \zeta(r + s) \).
3. Hoffman (1997) conjectured that the MZVs with \( r_i \in \{2, 3\} \) form a basis of the \( \mathbb{Q} \)-span of MZVs. Using a result of Zagier, Brown (2012) proved that these MZVs span (and motivic versions are linearly independent).

Example (Gangl-Kaneko-Zagier)

We have \( 28\zeta(9, 3) + 150\zeta(7, 5) + 168\zeta(5, 7) = \frac{5197}{691}\zeta(12) \). Note: \( \frac{150 - 168}{28} = -\frac{9}{14} \).
Let $Z_k = \{ P \in \mathbb{Z}[X,Y] \mid P \text{ is homogeneous of degree } k - 2 \}$.

**Definition**

1. The group $\mathcal{M}_k$ of *modular symbols* of weight $k$ is the maximal torsion-free quotient of $Z_k$ by the following relations for $P \in Z_k$:

   $$P(X,Y) + P(-Y,X) \quad \text{and} \quad P(X,Y) - P(X,X+Y) - P(X+Y,Y).$$

2. The group $\mathcal{S}_k$ of *cuspidal modular symbols* in $\mathcal{M}_k$ is generated by the $X^{j-1}Y^{k-1-j}$ with $1 < j < k - 1$.

3. The *plus parts* $\mathcal{M}_k^+$ (and $\mathcal{S}_k^+$) are the subgroups generated by the $X^{j-1}Y^{k-1-j}$ for $j$ odd.

The spaces of (cuspidal) modular symbols are modules for $\mathfrak{H}_k$ (resp., $\mathfrak{h}_k$).

**Theorem (Eichler 1957, Shimura 1959)**

*For modular forms with real coefficients, we have a pairing*

$$\mathcal{M}_k^+ \times M_k(\mathbb{R}) \to \mathbb{R}, \quad (X^{j-1}Y^{k-j-1}, f) = \int_{0}^{i\infty} z^{j-1} f(z) dz = \Lambda(f, j),$$

*inducing an isomorphism $M_k(\mathbb{R}) \cong \text{Hom}(\mathcal{M}_k^+, \mathbb{R})$ of $\mathfrak{H}_k$-modules.*
Theorem (S., Fukaya-Kato)

For $k \geq 2$ even, there exists a map

$$\varpi_k : S_k^+ / I_k S_k^+ \to K_{2k-2}(\mathbb{Z})', \quad X_i^{-1} Y^{k-i-1} \mapsto \kappa_i \kappa_{k-i}.$$ 

Remark

The map on $S_k^+$ was constructed by S. and conjectured to factor through $S_k^+ / I_k S_k^+$. In an equivalent form, this was proven by Fukaya and Kato in 2012.

Conjecture (S.)

The map $\varpi_k$ is an isomorphism for all $k$.

Remark

Surjectivity of $\varpi_k$ is equivalent to a conjecture of McCallum-S. (2003)

Theorem (S. for $p < 1000$, Fukaya-Kato)

The map $\varpi_k$ is an isomorphism on $p$-parts for $p < 136577356$. 
The unramified Iwasawa module

For an odd prime $p$, set $\mathbb{Q}(\mu_{p^\infty}) = \bigcup_{r=1}^{\infty} \mathbb{Q}(\mu_{p^r})$ and $G = \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$.

**Definition**

The $p$-adic cyclotomic character is the isomorphism $\chi: G \to \mathbb{Z}_p^\times$ satisfying $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for all $p$-power roots of unity $\zeta$.

**Remark**

We have $G = \Gamma \times \Delta$ with $\Gamma \cong \mathbb{Z}_p$ and $\Delta \cong (\mathbb{Z}/p\mathbb{Z})^\times$.

**Definition**

1. The Iwasawa algebra is $\Lambda = \mathbb{Z}_p[\Gamma] = \varprojlim_r \mathbb{Z}_p[\Gamma/\Gamma^{p^r}]$.
2. $X^{\infty}$ is the finitely generated, torsion $\Lambda$-module of norm compatible sequences in the $p$-parts of the class groups of the fields $\mathbb{Q}(\mu_{p^r})$ for $r \geq 1$.

**Remarks**

1. $\Lambda$ is noncanonically isomorphic to a power series ring $\mathbb{Z}_p[T]$.
2. By class field theory, $X^{\infty}$ is isomorphic to Galois group of the maximal abelian pro-$p$ extension of $\mathbb{Q}(\mu_{p^\infty})$ in which no prime ramifies.
Let $\omega: (\mathbb{Z}/p\mathbb{Z})^\times \hookrightarrow \mathbb{Z}_p^\times$ be the unique splitting of the reduction-mod-$p$ map.

**Definition**

The *Kubota-Leopoldt* $p$-adic $L$-function $L_p(\omega^k, s)$ is the unique continuous $\mathbb{Q}_p$-valued function on $\mathbb{Z}_p$ that for positive integers $n \equiv k \mod (p - 1)$ satisfies

$$L_p(\omega^k, 1 - n) = -(1 - p^{n-1}) \frac{B_n}{n}.$$

**Remark**

Every finitely generated torsion $\Lambda$-module $M$ is isomorphic up to finite modules to a direct sum of cyclic modules $\Lambda/(f)$, where $f$ is a non-unit in $\Lambda \approx \mathbb{Z}_p[T]$. The product of these elements $f$ is a *characteristic power series* of $M$.

**Theorem (Iwasawa main conjecture, Mazur-Wiles 1984)**

The $\Lambda$-module summand $X_{\infty}^{(1-k)}$ of elements of $X_{\infty}$ on which $\Delta$ acts through $\chi^{1-k}$ has a characteristic power series interpolating $L_p(\omega^k, s)$. 
Mazur and Wiles extended work of Ribet to construct unramified extensions of cyclotomic fields out of two-dimensional Galois representations attached to cusp forms satisfying congruences with Eisenstein series. That is, they studied Galois actions on the quotients of étale cohomology groups of modular curves by Eisenstein ideals (Wiles, Ohta).

Remark

As a consequence of the Quillen-Lichtenbaum conjecture, the maximal quotient of $X_\infty$ on which $G$ acts through $\chi^{1-k}$ is isomorphic to $K_{2k-2}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

A mild refinement of the Mazur-Wiles method gives the following map on $p$-parts.

**Theorem (S. 2011)**

There is a canonical map $\Upsilon_k: K_{2k-2}(\mathbb{Z})' \to S_k^+/I_kS_k^+$.  

The following explicitly refines the main conjecture.

**Conjecture (S. 2011)**

The maps $\varpi_k$ and $\Upsilon_k$ are inverse to each other.

We have \( \xi_k' \gamma_k \circ \varpi_k = \xi_k' \) on \( S_k^+ / I_k S_k^+ \), where \( \xi_k' = \frac{d}{ds} L_p(\omega_k, s)|_{s=1-k} \in \mathbb{Z}_p \).

Remark

Fukaya and Kato proved an analogous result up the cyclotomic tower after taking a tensor product with \( \mathbb{Q}_p \). Our work removed the need to tensor with \( \mathbb{Q}_p \).

Remark

If \( p \nmid \xi_k' \) (or \( p \nmid \frac{B_k}{k} \)), the conjecture holds on \( p \)-parts by the above theorem.

Corollary

The conjecture is true on \( p \)-parts for primes \( p < 163577356 \).
The above generalizes to modular symbols of arbitrary weight and level modulo Eisenstein ideals, describing $K_{2k-2}(\mathcal{O})$ for integer rings $\mathcal{O}$ of cyclotomic fields in terms of products on special elements in $K_{2i-1}(\mathcal{O})$ for odd $i$.

By Hida and Iwasawa theory, it is sufficient to consider weight 2 modular symbols.

**Example (Weight 2, Level $N$)**

- A *Manin symbol* $[u : v]$ for $u, v \in \mathbb{Z}/N\mathbb{Z}$ with $(u, v) = (1)$ is a certain class in first homology of the modular curve $X_1(N)$ relative to its cusps.

- There is a map
  
  $$[u : v]^+ \mapsto \{1 - \zeta_N^u, 1 - \zeta_N^v\}^+$$

  for $u, v \neq 0$, inducing

  $$\varpi : (H_1(X_1(N), \mathbb{Z})^+/I)' \to (K_2(\mathbb{Z}[\mu_N])^+)'$$

  Here, $\{ , \}$ is the Steinberg symbol, $I$ is an Eisenstein ideal, $+$ is fixed part under complex conjugation, and $\zeta_N = e^{2\pi i/N}$. The conjecture (S.) is that $\varpi$ is an isomorphism outside of a trivial $(\mathbb{Z}/N\mathbb{Z})^\times$-component.

- Excluding certain exceptional components, we can construct a conjectural inverse $\Upsilon$ on $p$-parts (S.). The analogous theorems hold (F-K, F-K-S).
Over a global base field $F$, our philosophy is very roughly summarized by:

**Philosophy (Fukaya-Kato-S.)**

geometry of $GL_d$ modulo Eisenstein $\iff$ arithmetic of $GL_{d-1}$

The above is the special case $F = \mathbb{Q}$ and $d = 2$.

**Question ($F = \mathbb{Q}$ and $d = 3$)**

Can we use modular symbols for an arithmetic quotient of $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ to construct generators of the dual of the Selmer group of a modular Galois representation out of products of three Siegel units?

**Work-in-progress ($F$ global function field, arbitrary $d$)**

We are writing a series of papers to formulate a conjecture and prove an analogue of the Fukaya-Kato theorem in this setting. This will describe products of Siegel units on a Drinfeld modular variety (Kondo-Yasuda) in terms of modular symbols in the homology of an arithmetic quotient of a Bruhat-Tits building. Our first preprint in this series (2015) compactifies the latter quotient. We have the analogous conjecture and theorem in the case $F = \mathbb{F}_q(t)$ and $d = 2$ (2014).