Eisenstein cocycles and their specializations

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Background

Herbrand-Ribet

Let p be an odd prime and k be a positive even integer. Let $A = \mathrm{Cl}(\mathbb{Q}(\mu_p))[p]$, and let $A^{(1-k)}$ denote the part of A upon which $\mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ acts by the pth power of the mod p cyclotomic character. Then

$$A^{(1-k)} \neq 0 \iff p \mid \frac{B_k}{k}.$$

Ribet showed that if $p \nmid \frac{B_k}{k}$ then there is a mod p congruence between a weight k Eisenstein series and a cuspidal eigenform f and finds an unramified p-extension of $\mathbb{Q}(\mu_p)$ in the fixed field of a residual representation of f.

Philosophy

I'll describe a different connection between modular curves and cyclotomic fields in the form of *Eisenstein maps* from the first homology of modular curves to second K-groups of cyclotomic integer rings, which are closely related to class groups. We view this as

$$\text{topology of } \mathrm{GL}_{2/\mathbb{Q}} \text{ modulo Eisenstein} \iff \text{arithmetic of } \mathrm{GL}_{1/\mathbb{Q}}.$$

We'll see some instances of an extension of this philosophy.

Symbols

Manin symbols

For $N \geq 1$, the homology $H_1(X_1(N), C_1(N), \mathbb{Z})$ of the modular curve $X_1(N)$ relative to its cusps $C_1(N)$ has a simple presentation

$$[u : v] = -[-v : u] = [u : v - u] + [u - v : v].$$

on generators $[u : v]$ for relatively prime $u, v \in \mathbb{Z}/N\mathbb{Z}$ known as *Manin symbols*. Here, $[u : v]$ is the geodesic from $\frac{b}{d}$ to $\frac{a}{c}$ in the extended upper half-plane, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $(u, v) = (c, d) \bmod N\mathbb{Z}^2$.

Steinberg symbols

The second K-group $K_2(\mathbb{Z}[\mu_N, \frac{1}{N}])$ contains *Steinberg symbols* $\{1 - \zeta_N^u, 1 - \zeta_N^v\}$ for $u, v \in \mathbb{Z}/N\mathbb{Z} - \{0\}$, where $\zeta_N = e^{2\pi i/N}$. This is the product in K-theory, as K_1 is the unit group. These satisfy the Steinberg relation

$$\{x, 1 - x\} = 0 \text{ for } x, 1 - x \in \mathbb{Z}[\mu_N, \tfrac{1}{N}]^\times.$$

The explicit map

The map (Busuioc, S., 2007)

We have

$$\Pi_N^\circ \colon H_1(X_1(N), C_1(N), \mathbb{Z})^+ \to K_2(\mathbb{Z}[\mu_N, \tfrac{1}{N}])^+$$
$$[u:v] \mapsto \begin{cases} \{1-\zeta_N^u, 1-\zeta_N^v\} & u, v \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $+$ denotes the part fixed by complex conjugation after inverting 2.
We denote by

$$\Pi_N \colon H_1(X_1(N), \mathbb{Z})^+ \to K_2(\mathbb{Z}[\mu_N])^+$$

the restriction of Π_N°.

Well-definedness

The well-definedness of Π_N° can be verified explicitly: for instance,

$$\frac{1-\zeta_N^u}{1-\zeta_N^{u-v}} + \frac{1-\zeta_N^v}{1-\zeta_N^{v-u}} = 1,$$

which corresponds to $[u:v] = [u:v-u] + [u-v:v]$.

Eisenstein property

Eisenstein ideal

The Eisenstein ideal I is generated by $T_\ell - 1 - \ell\langle\ell\rangle$ for primes ℓ, where we take $\langle\ell\rangle = 0$ if $\ell \mid N$. The action is via dual correspondences on $X_1(N)$.

Conjecture (S.)

1. *The map Π_N is Eisenstein, i.e., $\Pi_N \circ (T_\ell - 1 - \ell\langle\ell\rangle) = 0$ for all primes ℓ.*
2. *The resulting map ϖ_N on the quotient by I is an isomorphism.*

Theorem (Fukaya-Kato)

The map $\Pi_N \otimes \mathbb{Z}_p$ induced by Π_N on p-parts is Eisenstein for $p \mid N$, $p \geq 5$.

Theorem (S.-Venkatesh)

The map Π_N is Eisenstein away from the level: $\Pi_N \circ (T_\ell - 1 - \ell\langle\ell\rangle) = 0$ for primes $\ell \nmid N$.

Combining these two results with level compatibilities, Lecouturier and Wang showed that the map $\Pi_N \otimes \mathbb{Z}[\frac{1}{3}]$ is Eisenstein.

Other Eisenstein maps?

Eisenstein maps

Fukaya, Kato, and I speculated that there should be Eisenstein maps

1. over an imaginary quadratic field,

$$H_1(\text{Bianchi space}) \longrightarrow K_2(\text{ray class group})$$
$$\text{Cremona symbol} \mapsto \text{Steinberg symbol of elliptic units},$$

2. over $\mathbb{Q}$,

$$H_1(\text{locally symmetric space for } \mathrm{GL}_3(\mathbb{Z})) \longrightarrow K_3(\text{modular curve})$$
$$\text{Ash-Rudolph symbol} \mapsto \text{Steinberg symbol of three Siegel units},$$

3. over a global function field, away from the characteristic,

$$H_{d-1}\binom{\text{Bruhat-Tits building}}{\text{for } \mathrm{PGL}_d} \longrightarrow K_d\binom{\text{Drinfeld modular variety}}{\text{of rank } d-1}$$
$$\text{Kondo-Yasuda symbol} \mapsto \text{Steinberg symbol of } d \text{ Kato-Siegel units.}$$

Remarks

- If the base ring is non-Euclidean, the symbols do not generate homology.
- Related maps were studied in cases (1) and (2) by Goncharov.

Zeta maps

Zeta maps (Fukaya-Kato, Goncharov, Brunault)

Fukaya and Kato view Π_N as a specialization at ∞ of a Hecke-equivariant *zeta map* having roughly the form

$$H_1(X_1(N), C_1(N), \mathbb{Z}) \to K_2(Y_1(N)), \quad [u:v] \to \{g_{\frac{u}{N}}, g_{\frac{v}{N}}\},$$

where $g_{\frac{u}{N}}$ denotes a Siegel unit on $Y_1(N)_{/\mathbb{Z}[\frac{1}{N}]}$.

Zeta maps for function fields

The Eisenstein maps in (3) above should be specializations of Hecke-equivariant zeta maps

$$H_{d-1}\binom{\text{Bruhat-Tits building}}{\text{for } \mathrm{PGL}_d} \longrightarrow K_d\binom{\text{Drinfeld modular variety}}{\text{of rank } d}$$

Kondo-Yasuda symbol $\mapsto$ Steinberg symbol of d Kato-Siegel units.

Constructions

These maps can all be constructed in some form as specializations of *Eisenstein cocycles*.

1. The maps for function fields are work-in-progress with Peter Xu.
2. The imaginary quadratic setting, which I'll discuss later, is joint work with Lecouturier, S. Shih, and J. Wang.

Motivic cohomology

Motivic cohomology

For S smooth, connected, quasi-projective of finite type over a perfect field F, we have motivic cohomology groups $H^i(S,k)$ for $i \in \mathbb{Z}$ and $k \geq 0$. Some special cases:

$$H^1(S,1) \cong \mathcal{O}(S)^\times \quad \text{and} \quad H^0(S,0) \cong \mathbb{Z}.$$

The groups $H^n(S,n)$ form a graded ring under cup product. For $S = \operatorname{Spec} F$, this is the ring of Milnor K-groups of F.

A toy case: cyclotomic units

Consider $\mathbb{G}_m - \{1\} = \mathbb{P}^1 - \{0,1,\infty\}$ over $\mathbb{Q}$. We have an exact Gysin sequence

$$H^1(\mathbb{G}_m,1) \to H^1(\mathbb{G}_m - \{1\},1) \xrightarrow{\partial} H^0(\{1\},0).$$

For z the coordinate function on $\mathbb{G}_m$, we have $\partial(1-z) = e$, the class of 1. Then $1-z$ is the unique such element that

1. extends to $\mathbb{A}^1 - \{1\}$ and
2. is fixed by all *trace maps* $[m]_*$ for $m \geq 1$, where $[m]_* f(z) = \prod_{i=0}^{n-1} f(\zeta_m^i z)$.

For $\zeta_N = e^{2\pi i/N}$, we can pull back under the corresponding map $\iota\colon \operatorname{Spec} \mathbb{Q}(\mu_N) \to \mathbb{G}_m$ to obtain the cyclotomic N-unit $1 - \zeta_N = \iota^*(1-z)$.

Eisenstein symbols

Theta functions, Siegel units

- For an elliptic curve E/S and $c \geq 2$, we have an exact sequence

$$0 \to H^1(E,1) \to H^1(E - E[c],1) \xrightarrow{\text{div}} H^0(E[c],0) \to H^2(E,1).$$

  The trace-fixed part $H^i(E,1)^{(0)}$ of $H^i(E,1) \otimes \mathbb{Z}[\frac{1}{6}]$ is zero for $i = 1$ and $\mathbb{Z}[\frac{1}{6}]$ for $i = 2$. The last map becomes the degree map on trace-fixed parts. The *theta function* ${}_c\theta \in H^1(E - E[n],1)^{(0)}$ is unique with degree 0 divisor $c^2(0) - E[c]$.
- The universal elliptic curve $\mathcal{E} \to \mathcal{Y} = Y_1(N)_{/\mathbb{Q}}$ has a canonical N-torsion section ι_N. For c prime to N, the pullback $\iota_N^*({}_c\theta)$ is a *Siegel unit* ${}_cg_{\frac{1}{N}} \in \mathcal{O}(\mathcal{Y})^\times$.

Elliptic units

For E with CM by the integer ring $\mathcal{O}$ of F imaginary quadratic, we have theta functions ${}_\mathfrak{c}\theta$ with divisor $N\mathfrak{c}(0) - E[\mathfrak{c}]$, for $\mathfrak{c}$ a nonzero ideal of $\mathcal{O}$. Given a nonzero ideal $\mathcal{N}$ of $\mathcal{O}$ prime to $\mathfrak{c}$ and a point $P \in E[\mathcal{N}]$, the pullback ${}_\mathfrak{c}\theta(P) \in F(\mathcal{N})^\times$ is an *elliptic $\mathcal{N}$-unit*.

Eisenstein symbols (Faltings)

For A abelian of relative dimension g over S, the pullback by a torsion section of an element of $H^{2g-1}(A - A[c], g)^{(0)} \otimes \mathbb{Z}[\frac{1}{(2g+1)!}]$ with residue $c^g(0) - A[c]$ is called an *Eisenstein symbol*.

Equivariant complexes

Explicit complexes

For $(a,c) \in \mathbb{Z}^2 - \{0\}$, let $S_{a,c} = \ker(\mathbb{G}_m^2 \xrightarrow{(x,y)\mapsto x^a y^c} \mathbb{G}_m)$.
For any collection I of such distinct such pairs in $\mathbb{P}^1(\mathbb{Q})$, let

$$S_I = \bigcup_{\alpha\in I} S_\alpha \quad \text{and} \quad T_I = \bigcup_{\substack{\alpha,\beta\in I\\ \alpha\neq\beta}} S_\alpha \cap S_\beta.$$

Each of the homological complexes k_I in degrees $[2,0]$ given by

$$H^2(\mathbb{G}_m^2 - S_I, 2) \to H^1(S_I - T_I, 1) \to H^0(T_I, 0)$$

has homology $H^2(\mathbb{G}_m^2, 2)$ concentrated in degree 2. The limit $\mathsf{k} = \varinjlim_I \mathsf{k}_I$ sits in an even larger quasi-isomorphic complex K given by

$$K_2(\mathbb{Q}(\mathbb{G}_m^2)) \to \bigoplus_D K_1(\mathbb{Q}(D)) \to \bigoplus_x K_0(\mathbb{Q}(x)),$$

where D runs over irreducible divisors and x over closed points.

The semigroup $\Delta = M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ acts on the right on points of $\mathbb{G}_m^2$.
Pullback gives left Δ-actions on k and K.

Symbols in K

Symbols

Let z_1 and z_2 denote the coordinate functions on $\mathbb{G}_m^2$.

- In K_0, let e be the canonical generator of $H^0(\{1\}, 0) \cong \mathbb{Z}$.
- In K_1, for $a, c \in \mathbb{Z}$ with $(a, c) = 1$, let

$$\langle a, c \rangle = 1 - z_1^b z_2^d \in H^1(S_{a,c} - \{1\}, 1),$$

  where $ad - bc = 1$. Then $\langle a, c \rangle = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)^* \langle 1, 0 \rangle$.
- In K_2, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(\mathbb{Z})$, let

$$\langle \gamma \rangle = \langle (a, c), (b, d) \rangle = (1 - z_1^a z_2^c) \cup (1 - z_1^b z_2^d) \in H^2(\mathbb{G}_m^2 - S_{a,c} \cup S_{b,d}, 2).$$

  Then $\langle \gamma \rangle = \gamma^* \langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \rangle$.

Residues

We have

$$\partial \langle a, c \rangle = e \quad \text{and} \quad \partial \langle \gamma \rangle = \begin{cases} \langle a, c \rangle - \langle -b, -d \rangle, & \det \gamma = 1 \\ \langle -a, -c \rangle - \langle b, d \rangle, & \det \gamma = -1. \end{cases}$$

The Eisenstein cocycle for $\mathbb{G}_m^2$

Theorem (S.-Venkatesh)

Set $\overline{\mathsf{K}}_2 = \mathsf{K}_2/H^2(\mathbb{G}_m^2, 2)$. *There exists a unique* 1*-cocycle*

$$\Theta\colon \mathrm{GL}_2(\mathbb{Z}) \to \overline{\mathsf{K}}_2, \quad \gamma \mapsto \Theta_\gamma$$

such that $\partial\Theta_\gamma = (\gamma^* - 1)\langle 0, 1\rangle$ *for all* $\gamma \in \mathrm{GL}_2(\mathbb{Z})$. *Moreover, we have the following.*

1. Θ *is parabolic:* $\Theta|_P$ *is null-cohomologous on all right* $\mathrm{GL}_2(\mathbb{Z})$*-stabilizers* P *of elements of* $\mathbb{P}^1(\mathbb{Q})$.
2. Θ *satisfies an explicit formula: for* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, *we have*

$$\Theta_\gamma = \sum_{i=1}^{k} \left\langle \begin{pmatrix} -b_i & b_{i-1} \\ -d_i & d_{i-1} \end{pmatrix} \right\rangle \in \overline{\mathsf{K}}_2,$$

   where $(b_i, d_i) \in \mathbb{Z}^2$ *for* $0 \le i \le k$ *are such that* $v_0 = (0,1)$, $v_k = \det(\gamma)(b,d)$, *and* $b_{i-1}d_i - b_i d_{i-1} = 1$ *for* $i \ge 1$.
3. Θ *is Eisenstein:* $(T_\ell - \ell - [\ell]^*)\Theta$ *is null-cohomologous for every prime* ℓ, *where* T_ℓ *is a Hecke operator attached to the* $\mathrm{GL}_2(\mathbb{Z})$*-double coset of* $\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Proof of the theorem

Proof.

That a unique Θ exists is the snake lemma. We compute its properties on residues:

1. $\partial\Theta$ vanishes on the stabilizer of $0 \in \mathbb{P}^1(\mathbb{Q})$, and all parabolics are conjugate.
2. We perform telescoping:

$$\partial\left(\sum_{i=1}^{k}\langle v_i, -v_{i-1}\rangle\right) = \sum_{i=1}^{k}(\langle v_i\rangle - \langle v_{i-1}\rangle) = (\gamma^* - 1)\langle 0, 1\rangle = \partial\Theta_\gamma.$$

3. The Hecke-equivariant connecting map

$$\mathsf{K}_0^{\mathrm{GL}_2(\mathbb{Z})} \to H^1(\mathrm{GL}_2(\mathbb{Z}), \overline{\mathsf{K}}_2)$$

sends e to the class of Θ. We have

$$T_\ell e = \sum_{\substack{C \le \mu_\ell^2 \\ |C| = \ell}} C = \mu_\ell^2 + \ell e = (\ell + [\ell]^*)e.$$

□

Specialization

Specialization and trace-fixed parts

If U is an open in $\mathbb{G}_m^2$ with $s = (1, \zeta_N) \in U$, then we have a well-defined pullback map

$$s^* \colon H^2(U, 2) \to H^2(\mathbb{Q}(\mu_N), 2) \cong K_2(\mathbb{Q}(\mu_N)).$$

- Θ_γ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lies in a quotient of $H^2(U, 2)$ for $U = \mathbb{G}_m^2 - (S_{0,1} \cup S_{b,d})$, and $(1, \zeta_N) \in U$ if and only if $N \nmid d$.
- Θ_γ lies in the trace-fixed part $\overline{\mathsf{K}}_2^{(0)} = \mathsf{K}_2^{(0)} / \langle -z_1 \cup -z_2 \rangle$.
- $s^*(-z_1 \cup -z_2) = \{-1, -\zeta_N\}$, which is $\{-1, -1\}$ if N is odd and 0 if N is even.

The specialized cocycle

Set $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) \mid c \in N\mathbb{Z}\}$ and define

$$\Theta_N \colon \Gamma_0(N) \to K_2(\mathbb{Q}(\mu_N)) / \langle \{-1, -\zeta_N\} \rangle, \quad \gamma \mapsto \Theta_{N,\gamma} = s^* \Theta_\gamma.$$

It has the following properties:

1. Θ_N is a cocycle for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acting as $\sigma_d \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ where $\sigma_d(\zeta_N) = \zeta_N^d$.
2. Θ_N satisfies an explicit formula, since $s^* \langle \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \rangle = \{1 - \zeta_N^b, 1 - \zeta_N^d\}$.
3. Θ_N is parabolic: this is somewhat nontrivial!
4. Θ_N is Eisenstein away from the level: the T_ℓ for $\ell \nmid N$ commute with s^*.

Maps on homology

Construction of Π_N

- The restriction of Θ_N to $\Gamma_1(N) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(N) \mid d \in 1+N\mathbb{Z}\}$ is a homomorphism.
- Being parabolic, its further restriction to $\Gamma_1(N) \cap \mathrm{SL}_2(\mathbb{Z})$ induces a map

$$H_1(X_1(N), \mathbb{Z})_+ \to K_2(\mathbb{Z}[\mu_N])/\langle\{-1, \zeta_N\}\rangle$$

  where the subscript $+$ is the maximal quotient on which complex conjugation acts trivially.
- Inverting 2, we obtain a map that is seen by the explicit formula to be the restriction Π_N of Π_N° defined earlier, and which is Eisenstein away from the level.

Remark on the Eisenstein property

The Eisenstein property looks different that our original formulation but is the same. What we show here is that $\Pi_N \circ (T_\ell^* - \ell - \langle\ell\rangle^*) = 0$ for $\ell \nmid N$, where T_ℓ^* and $\langle\ell\rangle^*$ have the dual action to the one we used earlier (i.e., the usual action). Since $T_\ell^* = \langle\ell\rangle^{-1}T_\ell$ and $\langle\ell\rangle^* = \langle\ell\rangle^{-1}$, this amounts to the desired Eisenstein property of Π_N away from the level.

Products of two elliptic curves

Gersten-Kato complex

Let E and E' be elliptic curves over a characteristic 0 field L. Set $\mathcal{E} = E \times_L E'$. Again we have a complex K in homological degrees $[2, 0]$:

$$K_2(L(\mathcal{E})) \xrightarrow{\partial} \bigoplus_D K_1 L(D) \xrightarrow{\partial} \bigoplus_x K_0 L(x)$$

with $H_i(\mathsf{K}) = H^{4-i}(E \times_L E', 2)$.

Trace-fixed parts

None of the groups $M = H^j(\mathcal{E}, 2)$ vanish for $j \leq 4$. Set $\mathbb{Z}' = \mathbb{Z}[\frac{1}{5!}]$. We consider their *trace-fixed parts*

$$M^{(0)} = \{x \in M \otimes_{\mathbb{Z}} \mathbb{Z}' \mid [p]_* x = x \text{ for almost all primes } p\}.$$

Then $H^i(\mathcal{E}, 2)^{(0)} = 0$ unless $i = 4$ and $H^4(\mathcal{E}, 2)^{(0)} \cong \mathbb{Z}'$. The sequence

$$0 \to \mathsf{K}_2^{(0)} \to \mathsf{K}_1^{(0)} \to \mathsf{K}_0^{(0)} \to \mathbb{Z}' \to 0$$

is exact, with the last map a degree map.

CM elliptic curves

Setup

- Let F be an imaginary quadratic field with integer ring $\mathcal{O}$.
- Let $h = |\operatorname{Cl}_F|$ and $I = \{1, \ldots, h\}$. Let $\mathfrak{a}_r$ for $r \in I$ be ideals representing Cl_F.
- Let $\mathfrak{f}$ be an ideal of $\mathcal{O}$ with the property that $\mathcal{O}^\times \hookrightarrow (\mathcal{O}/\mathfrak{f})^\times$.
- Let $\sigma_r = \mathcal{R}(\mathfrak{a}_r^{-1}) \in \operatorname{Gal}(L/F)$, for $\mathcal{R}$ the Artin map and $L = F(\mathfrak{f})$ the ray class field.
- Fix a nonzero ideal $\mathfrak{c}$ of $\mathcal{O}$.

Elliptic curves

- Let $E_1, \ldots, E_h$ be non-isomorphic elliptic curves with CM by $\mathcal{O}$ defined over the ray class field $k = F(\mathfrak{f})$ such that for $r \in I$, we have

$$F(E_r^{\mathrm{tors}}) \subset F^{\mathrm{ab}}, \quad E_r(\mathbb{C}) \cong \mathbb{C}/\mathfrak{a}_r, \quad \text{and } E_r = E_1^{\sigma_r}.$$

  (That these exist follows from the exposition in the book of deShalit.)
- For $r, s \in I$, we have $\operatorname{Hom}(E_r, E_s) = \operatorname{Hom}_L(E_r, E_s) = \mathfrak{a}_{r,s} := \mathfrak{a}_s\mathfrak{a}_r^{-1}$.
- Let $P_1 \in E_1[\mathcal{N}]$ be an $\mathcal{O}$-generator, and set $P_r = P_1^{\sigma_r} \in E_r[\mathcal{N}]$.

Products of CM elliptic curves

Construction of classes

- For $i = (i_1, i_2) \in I^2$, set $\mathcal{E}_i = E_{i_1} \times_k E_{i_2}$. Let $\mathsf{K}(i)$ be the complex $\mathsf{K}^{(0)}$ for $\mathcal{E}_i$.
- For $i, j \in I^2$, let

$$\Delta_{i,j} = \{(a_{u,v})_{u,v} \in M_2(F) \mid a_{u,v} \in \mathfrak{a}_{i_u,j_v} \text{ for } u, v \in \{1,2\}\}.$$

  Then $\delta \in \Delta_{i,j}$ gives $\delta\colon \mathcal{E}_i \to \mathcal{E}_j$ by right multiplication, and $\delta^*\colon \mathsf{K}(j) \to \mathsf{K}(i)$.
- Let $\Gamma_i = \Delta_{i,i} \cap \det^{-1}(\mathcal{O}^\times)$.
- Set ${}_{\mathfrak{c}}e_i = N\mathfrak{c}^2(0) - \mathcal{E}_i[\mathfrak{c}]$. Then ${}_{\mathfrak{c}}e_i \in \mathsf{K}_0(i)^{\Gamma_i}$ has degree zero. The connecting map to $H^1(\Gamma_i, \mathsf{K}_2(i))$ provides a class $[{}_{\mathfrak{c}}\Theta_i]$.

A representative cocycle

For $i \in I^2$, set

$${}_{\mathfrak{c}}f_i = {}_{\mathfrak{c}}\theta_{E_{i_1}} \boxtimes E_{i_2}[\mathfrak{c}] + N\mathfrak{c}(0) \boxtimes {}_{\mathfrak{c}}\theta_{E_{i_2}} \in \mathsf{K}_1(i).$$

Here, $\boxtimes$ indicates the support of the unit in the other variable. We can then take our cocycle ${}_{\mathfrak{c}}\Theta_i$ to have $\partial({}_{\mathfrak{c}}\Theta_i) = d({}_{\mathfrak{c}}f_i)$.

Parabolicity

The two terms in ${}_{\mathfrak{c}}f_i$ are fixed by opposite parabolics, so ${}_{\mathfrak{c}}\Theta_i$ is not parabolic!

Eisenstein property

Hecke operators

Let $i, j \in I^2$. For $g \in \Delta_{i,j}$, we have Hecke operators

$$T(g)\colon H^1(\Gamma_j, \mathsf{K}_n(j)) \to H^1(\Gamma_i, \mathsf{K}_n(i)).$$

(We give a formalism of Hecke operators for the cohomology a Δ-*module system*.)
For $\mathfrak{n}$ an ideal, let $k \in I^2$ with each $\mathfrak{n}\mathfrak{a}_{i_u,k_u} = (\lambda_u)$ principal.

- For $j = (k_1, i_2)$, set $T_\mathfrak{n} = T(\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$.
- For $j = k$, set $[\mathfrak{n}]^* = T(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix})$.

Proposition

The collection ${}_\mathfrak{c}\Theta$ of classes of the ${}_\mathfrak{c}\Theta_i$ for $i \in I^2$ is Eisenstein:

$$T_\mathfrak{p}({}_\mathfrak{c}\Theta) = (N\mathfrak{p} + [\mathfrak{p}]^*){}_\mathfrak{c}\Theta$$

for each prime $\mathfrak{p}$ of $\mathcal{O}$.

Proof.

Checking this on ${}_\mathfrak{c}e = ({}_\mathfrak{c}e_i)_{i \in I^2} = (N\mathfrak{c}^2 - [\mathfrak{c}]^*)(0)$ reduces to checking it on (0). The proof is then just as for $\mathbb{G}_m^2$. □

Specialized cocycles

Specialization

- Fix a nonzero ideal $\mathcal{N}$ of $\mathcal{O}$ prime to $\mathfrak{cf}$. Set $\Gamma_0(\mathcal{N})_i = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_i \mid c \in \mathcal{N}\mathfrak{a}_{i_2,j_1}\}$.
- We have ${}_{\mathfrak{c}}\Theta_i(\gamma) \in H^2(U, 2)$ for U containing $(0, P_{i_2})$ if $\gamma \in \Gamma_0(\mathcal{N})_i$.
- We set $\lambda_i = (0, P_{i_2})^* \colon H^2(U, 2) \to K_2(F(\mathcal{N}\mathfrak{f})) \otimes \mathbb{Z}'$.
- We have $\lambda_i({}_{\mathfrak{c}}\Theta_i(\gamma)) \in K_2(F(\mathcal{N})) \otimes \mathbb{Z}'$ for $\gamma \in \Gamma_0(\mathcal{N})_i$.

Theorem (LSSW)

For $i \in I^2$, set ${}_{\mathfrak{c}}\Theta_{i,\mathcal{N}} = \lambda_i \circ {}_{\mathfrak{c}}\Theta_i \colon \Gamma_0(\mathcal{N})_i \to K_2(F(\mathcal{N})) \otimes \mathbb{Z}'$. Then:

1. *${}_{\mathfrak{c}}\Theta_{i,\mathcal{N}}$ is a cocycle for the action of $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(\mathcal{N})_i$ as $\mathcal{R}(d)$.*
2. *The collection ${}_{\mathfrak{c}}\Theta_{\mathcal{N}}$ of classes is Eisenstein away from the level.*
3. *For any $\mathfrak{d}$ prime to $\mathcal{N}$, we have $(N\mathfrak{d} - [\mathfrak{d}]^*){}_{\mathfrak{c}}\Theta_{\mathcal{N}} = (N\mathfrak{c} - [\mathfrak{c}]^*){}_{\mathfrak{d}}\Theta_{\mathcal{N}}$.*
4. *${}_{\mathfrak{c}}\Theta_i$ is valued in $K_2(\mathcal{O}_{F(\mathcal{N})}[\frac{1}{\mathcal{N}\mathfrak{c}}]) \otimes \mathbb{Z}'$.*

Theorem (LSSW)

Let $p \geq 7$ be a prime with either $p \nmid h$ or $p \nmid \mathcal{N}^2$. Then exists a collection $\Theta_{\mathcal{N}}$ in $(H^1(\Gamma_i, K_2(F(\mathcal{N})) \otimes \mathbb{Z}_p))_{i \in I^2}$ such that $(N\mathfrak{c} - [\mathfrak{c}]^*)\Theta_{\mathcal{N}} = {}_{\mathfrak{c}}\Theta_{\mathcal{N}}$ for all $\mathfrak{c}$ prime to $\mathcal{N}$.

Maps on homology

Bianchi space

- For $r \in I$, set $\Gamma_1(\mathcal{N})_r = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathcal{N})_{(r,1)} \mid (d) + \mathcal{N} = \mathcal{O}\}$.
- The *Bianchi space* $Y_1(\mathcal{N}) = \coprod_{r \in I} Y_1(\mathcal{N})_r$ is given by $Y_1(\mathcal{N})_r = \Gamma_1(\mathcal{N})_r \backslash (\mathbb{C} \times \mathbb{R}_{>0})$.

Proposition

The restrictions of $({}_{\mathfrak{c}}\Theta_{(r,1),\mathcal{N}})_{r \in I}$ *define a homomorphism*

$$ {}_{\mathfrak{c}}\Pi_{\mathcal{N}} \colon H_1(Y_1(\mathcal{N}), \mathbb{Z}) \to K_2(\mathcal{O}_{F(\mathcal{N})}[\tfrac{1}{\mathcal{N}\mathfrak{c}}]) \otimes \mathbb{Z}' $$

that is Eisenstein away from the level: ${}_{\mathfrak{c}}\Pi_{\mathcal{N}} \circ T_{\mathfrak{p}} = (N\mathfrak{p} + \mathcal{R}(\mathfrak{p})) \circ {}_{\mathfrak{c}}\Pi_{\mathcal{N}}$ *for primes* $\mathfrak{p} \nmid \mathcal{N}$.

Remarks

1. The Hecke operators we defined on cohomology are nonstandard but work well with Δ-module systems. We have $[\mathfrak{n}]^* = \langle \mathfrak{n} \rangle^* \circ S_{\mathfrak{n}}$, where $S_{\mathfrak{n}}$ is an adelically defined diagonal operator. The diamond operator preserves ${}_{\mathfrak{c}}\Theta$, and $S_{\mathfrak{n}}$ acts as $\mathcal{R}(\mathfrak{n})$.
2. We show that ${}_{\mathfrak{c}}\Pi_{\mathcal{N}} \otimes \mathbb{Z}_p$ is parabolic outside of certain isotypical components in the case that the p-part of the level $\mathcal{N}$ is square-free.