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Sorin Popa

Mathematical Sciences Research Institute
1000 Centennial Drive Berkeley, California 94720–5070

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SORIN POPA

University of California, Los Angeles

ABSTRACT. Let σ be the action of an infinite property T group G on the hyperfinite type II$_1$ factor $R = \overline{\bigotimes_{g \in G}}(M_{2\times 2}(\mathbb{C}), tr)_g$, by Bernoulli shifts. We prove that the cocycle actions obtained by reducing σ to the algebras pRp, for p non-trivial projections in R, cannot be perturbed to actions. We also prove that any 1-cocycle for σ vanishes. More generally, we calculate all 1-cocycles for actions of property T groups G by Bernoulli shifts of Connes-Størmer type and use this to provide an invariant that distinguishes these actions up to cocycle conjugacy.

0. INTRODUCTION.

The study of automorphisms in the theory of von Neumann algebras has often followed problems and phenomena already present in commutative ergodic theory. But starting with the early 70's, with the advent of Tomita-Takesaki theory and Connes' ground breaking work on the classification of factors, the full importance and the distinguishing features of this "non-commutative ergodic theory" have become increasingly evident.

A typically non-commutative aspect related to the study of automorphisms of a von Neumann algebra N is that such automorphisms are regarded both as elements in the automorphism group of N, Aut$N$, and modulo perturbation by inner automorphisms, in the quotient group Aut$N$/Int$N$. This leads to the study of two types of classification up to conjugacy for automorphisms, in Aut$N$ and respectively in Aut$N$/Int$N$. It also leads to considering cocycles for the corresponding actions.

Along these lines, the more general problem is to study morphisms from groups G into Aut$N$ and into Aut$N$/Int$N$, up to conjugacy. The former are called *actions*

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of G on N. Not all faithful group morphisms σ from G into $\text{Aut}N/\text{Int}N$ can be lifted to actions. Thus, Nakamura and Takeda have already pointed out in ([**NT**]) that to any such morphism σ one can associate a scalar 3-cocycle $\mu \in H^3(G, \mathbb{T})$ of the group G, and that μ gives an obtruction for σ to be liftable to a genuine action. Also, it was shown in ([**NT**]) that the morphism σ has trivial obstruction μ if and only if it can be lifted to a *cocycle action*. It is a most interesting problem to decide whether any cocycle action of a group G can be perturbed by inner automorphisms to a genuine action of G. In other words, whether the 3-cocycle μ is the only obstruction for a σ to be liftable to an action.

One of the results in ([**C3**]) showed that this is indeed the case if $G = \mathbb{Z}/n\mathbb{Z}$ and N is an arbitrary type II_1 factor. Then in ([**Su**]) it was proved that this is still the case for all finite groups G. It was further proved in ([**Oc**]) that the result actually holds true for all countable amenable groups G, but when N is equal to the hyperfinite type II_1 factor R. Finally, the condition that N be hyperfinite was removed in ([**Po1,9**]). Thus, all cocycle actions of an amenable group G on an arbitrary type II_1 factor N can be perturbed to genuine actions.

At the oposite end, Connes and Jones have found an example of a cocycle action of a property T group G ([**K**]) on the free group factor $N = L(\mathbb{F}_\infty)$ that cannot be perturbed to an action ([**CJ**]). It remained however as an open problem of whether such examples exist or not in the case N is the hyperfinite type II_1 factor.

In this paper we solve this problem, by providing a class of examples of cocycle actions on the hyperfinite type II_1 factor that cannot be perturbed to actions. More precisely, we prove that if σ is an action of an infinite disctere group G with the property T of Kazhdan on the hyperfinite type II_1 factor $R = \overline{\bigotimes_{g\in G}}(M_{2\times 2}(\mathbb{C}), tr)_g$, by Bernoulli shifts, and if p is an arbitrary projection in R, $p \neq 0, 1$, then the cocycle actions obtained by reducing σ by p cannot be perturbed to genuine actions. The same result is proved for the group $G = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$, which doesn't have the property T.

Our result also implies that given any two projections of different trace p, q in R the cocycle actions on $pRp \simeq R \simeq qRq$, obtained by reducing σ by p, respectively q, are not outer conjugate, i.e., they are not conjugate in $\text{Aut}R/\text{Int}R$.

At the same time, we also prove a vanishing 1-cohomology result for the action σ, showing that any 1-cocycle (modulo scalars) w for σ is a coboundary, modulo the scalars. More precisely, if $w : G \to \mathcal{U}(R)$ satisfies $w_g\sigma_g(w_h) = w_{gh}, \forall g, h \in G$, modulo $\mathbb{C}$, then there exists a unitary element $v \in \mathcal{U}(R)$ such that $w_g = v^*\sigma_g(v), \forall g \in G$, modulo $\mathbb{C}$. Related to this, recall that by ([**C2**]) any unitary 1-cocycle for an arbitrary action of a finite group G on a type II_1 factor vanishes. Our result shows that such a rigidity result may hold true for some special actions of infinite discrete groups as well.

This vanishing cohomology result for 1-cocycles also implies that if an action of G on R is cocycle conjugate to the Bernoulli shift σ, then it is actually conjugate to it.

More generally, we explicitly calculate all the 1-cocycles for actions of an infinite property T group G by Bernoulli shifts of Connes-Stormer type. These are actions σ_λ on the hyperfinite type II_1 factor R obtained by restricting the Bernoulli shift of G on the hyperfinite type III_λ factor $\mathcal{N} = \overline{\bigotimes}_{g\in G}(M_{2\times 2}(\mathbb{C}), \varphi_\lambda)_g$ to its type II_1 core. We prove that all 1-cocycles for σ_λ are locally of the form $v\sigma_{\lambda,g}(v^*)$, with $v \in \mathcal{N}$ isometries normalizing the core R of $\mathcal{N}$. As a consequence, for each 1-cocycle w the fixed point algebra of the action $\text{Ad}w \circ \sigma$ follows atomic. Moreover, the scalar λ can be recuperated from the range of the trace on the set of minimal projections of these fixed point algebras, thus being an outer conjugacy invariant for σ. As a consequence, we obtain that the actions $\sigma_\lambda, 0 < \lambda < 1$, of the group G on R are mutually non outer conjugate.

Similarily, we consider arbitrary Connes-Størmer Bernoulli shifts σ of G on the core factor of $\mathcal{N} = \overline{\bigotimes}_{g\in G}(M_{k\times k}(\mathbb{C}), \varphi_0)_g$ and show that the multiplicative subgroup of $\mathbb{R}_+^*$ generated by $\{t_i/t_j\}_{i,j}$, where $t_i > 0$ are the diagonal elements that determine the faithful state φ_0 of $M_{k\times k}(\mathbb{C})$, is an outer conjugacy invariant for σ.

Note that if in the above construction the group G one starts with is $\mathbb{Z}$, then the actions σ_λ are the usual Connes-Størmer Bernoulli shifts in ([**CS**]) and they have Connes-Størmer entropy equal to $-t\log t - (1-t)\log(1-t)$, where $t/1-t = \lambda$. Thus, the entropy does distinguish the actions $\sigma_\lambda, 0 < \lambda < 1$, up to conjugacy, but by ([**C2**]) these actions are all outer conjugate. Similarily for G an arbitrary amenable group, by ([**Ka**]) and respectively ([**Oc**]). Our result shows that in the case G has the property T, the Connes-Størmer Bernoulli shifts σ_λ become much more rigid, as they are not even outer conjugate. Also, rather than entropical, our invariant comes from the cohomology properties of the action.

We also obtain some results similar to the above for the free Bernoulli shifts of an infinite property T group G on the free group factor $L(\mathbb{F}_\infty) \simeq L(\mathbb{F}_G)$.

To prove these results we use the fact that if σ is a Bernoulli shift of an infinite property T group G on R then the identity automorphism on $R\overline{\otimes}R$ can be approximated by product-type automorphisms that are commuting with the action $\sigma \otimes \sigma$ on $R\overline{\otimes}R$. Using this deformation of the identity and the property T of the group G, and assuming by contradiction that the reduced cocycle action by some projection $p \in R$ can be perturbed to an action, one obtains non-zero intertwiners that implement the flip automorphism on appropriate elements of $R\overline{\otimes}R$. When suitably interpreted, this forces $p = 1$ and the cocycle to be trivial.

The paper is organized as follows: In Section 1 we recall some basic definitions of actions, cocycle actions and 1-cocycles. In Section 2 we give some examples of actions, that are to be used in the paper, amply discussing the Bernoulli shifts and their generalized version considered by Connes and Stormer. In Section 3 we construct the cocycle actions obtained by reducing actions by projections (Proposition 3.1). Also, we introduce a class of non-trivial 1-cocycles for Connes-Størmer Bernoulli shifts that are needed in the sequel (Theorem 3.2).

In Section 4 we prove the main result of the paper, showing that the 1-cocycles constructed in Section 3 give the list of all 1-cocycles of the actions of property T groups by Connes-Størmer Bernoulli shifts (Theorem 4.1). In Section 5 we derive some consequences of the main theorem, proving the existence of cocycle actions that cannot be perturbed to actions (Corollary 5.7), the vanishing cohomology result for Bernoulli shifts of property T groups (Corollary 5.6) and showing that the outer conjugacy classes of the Connes-Størmer Bernoulli shifts can be distinguished by the "cohomology picture" of those actions, a structure that comes from the spectrum of the discrete decomposition of the type III_λ factors they are defined on (Theorem 5.3, 5.4). Also, we make several remarks. In Section 6 we prove the analogue result for free shifts. We include some remarks here as well.

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1. Preliminaries and notations.

Although the main results in this paper are stated in the framework of type II_1 factors, their proofs will require some considerations on von Neumann factors of type III_λ, $0 < \lambda \leq 1$, as well. Thus, most of the definitions and notations on actions of groups and their cocycles that we recall in this section will be stated in the context of arbitrary von Neumann factors.

1.1. Actions and cross products. We denote by $(\mathcal{N}, \varphi)$ a pair consisting of a von Neumann factor $\mathcal{N}$ (typically of type II_1 or of type $\mathrm{III}_\lambda, 0 < \lambda \leq 1$), with a normal faithful state φ on it (typically a trace or a generalized trace). The *centralizer* of φ is the set $\mathcal{N}_\varphi = \{x \in \mathcal{N} \mid \varphi(xy) = \varphi(yx), \forall x \in \mathcal{N}\}$.

In case the von Neumann algebra $\mathcal{N}$ is specified to be a type II_1 factor, we will use the notation N instead of $\mathcal{N}$. In this case the state φ will always be taken to be the unique trace τ on N.

An *automorphism* of $(\mathcal{N}, \varphi)$ is an automorphism of $\mathcal{N}$ that preserves φ. We sometimes denote the group of all such automorphisms of $\mathcal{N}$ by $\mathrm{Aut}(\mathcal{N}, \varphi)$. Note that any automorphism of $(\mathcal{N}, \varphi)$ normalizes $\mathcal{N}_\varphi$.

An automorphism ρ of $\mathcal{N}$ is *inner* if there exists u in the unitary group of $\mathcal{N}$, $\mathcal{U}(\mathcal{N})$, such that $\rho(x) = \mathrm{Ad}u(x) = uxu^*, \forall x \in \mathcal{N}$. Note that the set Int $(\mathcal{N}, \varphi)$ of inner automorphisms of $\mathcal{N}$ that preserve φ coincides with the set of automorphisms Adu with $u \in \mathcal{U}(\mathcal{N}_\varphi)$. Int $(\mathcal{N}, \varphi)$ is clearly a normal subgroup of Aut $(\mathcal{N}, \varphi)$.

An automorphism ρ of the factor $\mathcal{N}$ is *properly outer* (or simply *outer*) if it is not inner.

Let G be a discrete group. An *action* of G on $(\mathcal{N}, \varphi)$ is a group morphism $\sigma : G \to \mathrm{Aut}(\mathcal{N}, \varphi)$. Note that in case $\mathcal{N} = N$ is a type II_1 factor, any automorphism of N preserves the trace τ. We will simply denote by Aut$N$ the set of all automorphisms of N.

Recall that the *cross product algebra* associated to an action σ on $(\mathcal{N}, \varphi)$ denoted $(\mathcal{N} \rtimes_\sigma G, \varphi)$, is the von Neumann algebra generated inside $\mathcal{B}(\ell^2(G,\ L^2(\mathcal{N}, \varphi)))$ by the unitaries $u_g \in \mathcal{B}(\ell^2(G,\ L^2(\mathcal{N}, \varphi)))$, $g \in G$, where

$$u_g(f)(h) = f(g^{-1}h), \text{ for } f \in \ell^2(G, L^2(\mathcal{N}, \varphi)),$$

and by a copy of the algebra $\mathcal{N}$ given by

$$(b \cdot f)(g) = \sigma_g^{-1}(b) f(g), \text{ for } b \in N,\ f \in \ell^2(G, L^2(\mathcal{N}, \varphi)),\ g \in G.$$

together with the vector state $\varphi(X) = \langle X\xi_\varphi, \xi_\varphi\rangle$, where $\xi_\varphi \in \ell^2(G, L^2(\mathcal{N}, \varphi))$ is the function on G that takes the value ξ_φ at e and 0 elsewhere.

The cross product algebra $\mathcal{N} \rtimes_\sigma G$ can alternatively be viewed as the completion (on bounded sequences) of the Hilbert algebra of finite formal sums $\sum_{g \in G} u_g b_g$, $b_g \in \mathcal{N}$, with multiplication rules $u_g u_h = u_{gh}$, $bu_g = u_g \sigma_g^{-1}(b)$, $b = u_e b = 1b$, for $g, h \in G$, $b \in \mathcal{N}$, and $*$-operation $(u_g b)^* = u_{g^{-1}} \sigma_g(b^*)$, and with $\mathcal{N}$-valued expectation $E\left(\sum_g u_g b_g\right) \overset{\text{def}}{=} b_e$ and scalar expectation

$$\varphi\left(\sum u_g b_g\right) \overset{\text{def}}{=} \varphi\left(E\left(\sum u_g b_g\right)\right) = \varphi(b_e).$$

Recall that $(\mathcal{N} \rtimes_\sigma G, \varphi)$ this way defined is itself a von Neumann algebra, with φ a faithful normal state. In case $\mathcal{N}$ is a finite von Neumann factor with φ a trace, then this cross product algebra is a finite von Neumann algebra itself, with $\varphi = \varphi \circ E$ a faithful trace on it.

If the action σ is properly outer, i.e., if σ_g is a properly outer automorphism, $\forall g \neq e$, then $\mathcal{N}' \cap \mathcal{N} \rtimes_\sigma G = \mathbb{C}$. In particular, if σ is properly outer then $\mathcal{M} = \mathcal{N} \rtimes_\sigma G$ is a factor.

1.2. Cocycle actions. A cocycle action σ of G on $(\mathcal{N}, \varphi)$ is a map $\sigma : G \to \mathrm{Aut}(\mathcal{N}, \varphi)$ with the property that there exists a map $v : G \times G \to \mathcal{U}(\mathcal{N}_\varphi)$ such that:

$$\sigma_e = \mathrm{id} \text{ and } \sigma_g \sigma_h = \mathrm{Ad}\, v_{g,h} \sigma_{gh},\ \forall g, h \in G \tag{1.2.1}$$

$$v_{g,h} v_{gh,k} = \sigma_g(v_{h,k}) v_{g,hk},\ \forall g, h, k \in G. \tag{1.2.2}$$

A map v satisfying (1.2.2) is called a *2-cocycle* for σ. The 2-cocycle is *normalized* if $v_{g,e} = v_{e,g} = 1$, $\forall g \in G$. Note that, since $\mathcal{N}$ is a factor, any 2-cocycle satisfies $v_{e,e} \in \mathbb{C}$. Thus any 2-cocycle v can be normalized by replacing it, if necessary, by $v'_{g,h} = v^*_{e,e}\, v_{g,h}$, $g, h \in G$.

All 2-cocycles considered from now on will be normalized. Furthermore, all cocycle actions (in particular all actions) that we will consider in this paper are assumed to be properly outer, i.e., σ_g cannot be implemented by unitary elements in $\mathcal{N}, \forall g \neq e$.

Also, when given a cocycle action σ, we will sometimes specify from the beginning the 2-cocycle it comes with, thus considering it as a pair (σ, v).

Note that the 2-cocycle v is unique modulo perturbation by a *scalar* 2-cocycle μ. More precisely, $v' : G \times G \to \mathcal{U}(\mathcal{N}_\varphi)$, with $v'_{e,e} = 1$, satisfies conditions (1.2.1), (1.2.2) if and only if $v' = \mu v$ for some scalar valued function $\mu : G \times G \to \mathbb{T}$ satisfying $\mu_{e,e} = 1$ and

$$\mu_{g,h}\mu_{gh,k} = \mu_{h,k}\mu_{g,hk}, \quad \forall g, h, k \in G \tag{1.2.3}$$

A 2-cocycle v for the action σ is *trivial* (or it is a *coboundary*) if there exists a map $w : G \to \mathcal{U}(\mathcal{N})$ such that $w_e = 1$ and $v = \partial w$, i.e.:

$$v_{g,h} = (\partial w)_{g,h} \overset{\text{def}}{=} \sigma_g(w_h^*) w_g^* w_{gh}, \ \forall g, h \in G. \tag{1.2.3}$$

The 2-cocycle v is *weakly trivial* if there exists $w : G \to \mathcal{U}(\mathcal{N})$ such that $w_e = 1$ and $v = \partial w$ modulo scalars, i.e.:

$$w_g \sigma_g(w_h) v_{g,h} w_{gh}^* \in \mathbb{C}1, \ \forall g, h \in G. \tag{1.2.4}$$

Note that this is equivalent to

$$(\text{Ad } w_g\sigma_g)\,(\text{Ad } w_h\sigma_h) = \text{ Ad } w_{gh}\sigma_{gh}, \ \forall g, h \tag{1.2.4$'$}$$

i.e., to $\sigma'_g \overset{\text{def}}{=} \text{ Ad } w_g\sigma_g$ being an action.

A weakly trivial 2-cocycle is not necessarily trivial. In fact, if we take any scalar valued 2-cocycle, then the conditions (1.2.1), (1.2.2) and (1.2.4$'$) are satisfied for any genuine action σ of G (taking μ for v and $w = 1$). Thus μ is a weakly trivial 2-cocycle, in the above sense. But if we take $\mathcal{N} = \mathbb{C}1$ it is not always true that given any μ like this there exists some $w : G \to \mathbb{T}$ such that $\mu = \partial w$. In fact, for most groups G there do exist scalar 2-cocycles μ that are not trivial (or coboundary).

Let us also recall here some well known vanishing cohomology results. The first result along these lines is due to Connes, who considered the vanishing cohomology and classification problems for actions of the cyclic groups $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $n \geq 2$, on factors ([**C2, 3**]).

A general vanishing cohomology result for arbitrary finite groups was obtained by Sutherland ([**Su**] see also [**J1**]), who proved that any 2-cocycle from a cocycle action of a finite group G on an arbitrary type II_1 factor N is coboundary.

In the same vein, Ocneanu proved that any 2-cocycle from a cocycle action of a countable amenable group G on the hyperfinite type II_1 factor R is coboundary ([**Oc**]). It was then proved in ([**Po1,9**]) that the same vanishing cohomology result holds true for cocycle actions of amenable groups G on arbitrary type II_1 factors N. Thus, in particular, any cocycle action of a countable amenable group on an arbitrary type II_1 factor can be perturbed to a genuine action.

Connes and Jones provided the first example of a cocycle action of a countable discrete group G on a type II_1 factor N that cannot be perturbed to an action ([**CJ**]). In their example, G had the property T of Kazhdan, with N being the free group factor $L(\mathbb{F}_\infty)$ (see the Appendix 2.1 at the end of this paper).

1.3. 1-cocycles for actions. Let us now take σ to be a genuine action of G on $(\mathcal{N}, \varphi)$. A map $w : G \to \mathcal{U}(\mathcal{N}_\varphi)$ satisfying condition

$$w_g \sigma_g(w_h) = w_{gh}, \ \forall g, h \tag{1.3.1}$$

is called a 1-*cocycle for* σ. Such a 1-cocycle for σ is a *coboundary* (or it is *trivial*) if there exists a unitary element $w \in \mathcal{U}(\mathcal{N})$ such that $w_g = w^* \sigma_g(w)$, $\forall g$. (Clearly, such maps w_g do satisfy the 1-cocycle condition (1.3.1)).

The map w is called a *weak* 1-*cocycle* if it satisfies the relation (1.3.1) modulo the scalars, i.e.,

$$w_g \sigma_g(w_h) w_{gh}^* \in \mathbb{T}1, \forall g, h \in G \tag{1.3.1'}$$

Note that this is equivalent to $\mathrm{Ad} w_g \circ \sigma_g$ being an action. Note also that if w is a weak 1-cocycle then $\mu_{g,h} = w_g \sigma_g(w_h) w_{gh}^*$ is a scalar 2-cocycle.

A (weak) 1-cocycle w is *weakly trivial* (or *weak cobouboundary*) if there exists a unitary element $w \in \mathcal{U}(\mathcal{N})$ such that $w w_g \sigma_g(w)^* \in \mathbb{T}1, \forall g$.

Two (weak) 1-cocycles w, w' of the action σ are equivalent if there exists a unitary element $v \in \mathcal{N}$ such that $w'_g = v w_g \sigma_g(v)^*, \forall g \in G$ (resp. modulo scalars). Thus, a weak 1-cocycle is weakly trivial iff it is equivalent to a scalar valued weak 1-cocycle (N.B.: this are just plain scalar functions on G). Note that the scalar valued genuine 1-cocycles are just characters of G.

Recall that by Connes 2 by 2 matrix trick ([**C3**]) any 1-cocycle for an action of a finite group G on an arbitrary type II_1 factor is trivial (the result in [**C3**] is for $G = \mathbb{Z}/n\mathbb{Z}$, but the proof there goes the same way for G finite; for all the results on finite groups see also [**J1**]).

1.4. Generalized 1-cocycles. Like in 1.3, let σ be a genuine action of the discrete group G on the factor N. We consider the following general version of 1-cocycles: Let p be a non-zero projection in $\mathcal{N}_\varphi$ and $w : G \to \mathcal{N}_\varphi$ be so that $w_g \in \mathcal{N}_\varphi$ are partial isometries satisfying $w_g w_g^* = p, w_g^* w_g = \sigma_g(p), g \in G$, with $w_e = p$. If w satisfies the condition

$$w_g \sigma_g(w_h) = w_{gh}, \forall g, h \in G \tag{1.4.1}$$

then w is called a *generalized* 1-*cocycle for* σ. If w satisfies the weaker condition

$$w_g \sigma_g(w_h) w_{gh}^* \in \mathbb{T}p, \forall g, h \in G \tag{1.4.1'}$$

then it is called a *generalized weak* 1-*cocycle.* Note that (1.4.1') is equivalent to $\sigma'_g = Ad(w_g)\sigma_{g|p\mathcal{N}p}$ being an action of G on $p\mathcal{N}p$. The projection p is called *the support* of w. Also, like for weak 1-cocycles, note that if w satisfies (1.4.1') then the scalar valued function $\mu_{g,h}$ satisfying $w_g\sigma_g(w_h) = \mu_{g,h} w_{gh}$ is a 2-cocycle.

Note that in case its support is equal to 1, a generalized (weak) 1-cocycle w is a (weak) cocycle.

The generalized 1-cocycle w is *trivial* if there exists a partial isometry $v \in \mathcal{N}$ such that $vv^* = \sigma_g(vv^*)$ and $w_g = v^*\sigma_g(v)$, for all $g \in G$. Similarily, if w is a weak generalized cocycle and the above condition is satisfied modulo scalars then the w is *weakly trivial.*

Note that if σ is ergodic then it cannot have generalized 1-cocycles of support $\neq 1$. Also, if w is a generalized (weak) 1-cocycle of support p and $q \in pNp$ is a projection in the fixed point algebra of $\sigma' = \mathrm{Ad}w \circ \sigma$ then $\{qw_g\}_g$ is a generalized (weak) 1-cocycle for σ as well.

1.5. Cocycle conjugacy of actions. Two cocycle actions (σ, v), (σ', v') of G on $(\mathcal{N}, \varphi)$ are *cocycle conjugate* if there exists an automorphism ρ of $(\mathcal{N}, \varphi)$ such that the following conditions are satisfied:

$$\rho\sigma_g\, \rho^{-1} = \mathrm{Ad}\ w_g \sigma'_g, \quad \forall g. \tag{1.5.1}$$

$$\rho(v_{g,h}) = w_g \sigma'_g(w_h) v'_{g,h}\ w_{gh}^*, \ \forall g, h. \tag{1.5.2}$$

The cocycle actions σ, σ' are *outer conjugate* (or *weakly cocycle conjugate*) if condition (1.5.1) is satisfied. Note that outer conjugacy is equivalent to the image morphisms of σ, σ' in Aut $(\mathcal{N}, \varphi)$/Int $(\mathcal{N}, \varphi)$ being conjugate in Aut $(\mathcal{N}, \varphi)$/Int $(\mathcal{N}, \varphi)$ by an automorphism of $(\mathcal{N}, \varphi)$.

The (cocycle) actions σ, σ' are *conjugate* if there exists an automorphism ρ of $(\mathcal{N}, \varphi)$ such that conditions (1.5.1) is satisfied with $w = 1$. We then write $\sigma \sim \sigma'$.

Recall that Connes proved in ([**C3**]) that in the case $\mathcal{N}$ is isomorphic to the hyperfinite type II_1 factor R, any two actions of $\mathbb{Z}/n\mathbb{Z}$ on it are conjugate. Then Jones proved that any two actions of an arbitrary finite group G on R are conjugate. Also, it was proved in ([**C2**]) that any two actions of $\mathbb{Z}$ on R are cocycle conjugate. Finally, Ocneanu proved in ([**Oc**]) that any two actions of an arbitrary amenable group G on R are cocycle conjugate.

2. Examples of actions.

Let us mention some interesting classes of examples of actions and cocycle actions, that will intervene in the sequel.

2.1. Bernoulli shifts. Let (N_0, τ_0) be a finite von Neumann factor with a faithful normal trace τ_0, $\tau_0(1) = 1$. Let G be a discrete group and $(N, \tau) \stackrel{\text{def}}{=} \overline{\bigotimes_{g\in G}}(N_g, \tau_g)$, where $(N_g, \tau_g) = (N_0, \tau_0)$, $\forall g \in G$. We let $\sigma : G \to \text{Aut } N$ be defined by $\sigma_g \left(\bigotimes_{h\in G} b_h\right) = \bigotimes_{h\in G} b'_h$, where $\bigotimes_{h\in G} b_h \in \overline{\bigotimes_{h\in G}} N_h$ has only finitely many entries $\neq 1$ and $b'_h = b_{g^{-1}h}$.

An action σ defined this way is called a *Bernoulli shift*. When we need to be more specific, we'll say that σ is a (N_0, τ_0)-Bernoulli shift. Note that, with this terminology, we have that $\sigma \otimes \sigma$ is a $N_0 \overline{\otimes} N_0$-Bernoulli shift.

It is well known that if either G is infinite and $N_0 \neq \mathbb{C}$, or if G is finite and N_0 has no atoms, then the Bernoulli shift σ is a properly outer action of G on N. Moreover, if G is infinite then σ is also ergodic (it is even strongly mixing: see 2.4 hereafter).

Also, note that if N_0 is a factor (respectively an approximately finite dimensional finite von Neumann algebra) then so is N.

In particular, by taking N_0 to be isomorphic to the hyperfinite type II_1 factor, this shows that any discrete countable group G admits a properly outer (+ ergodic, if $|G| = \infty$) action on $R \left(\simeq \overline{\bigotimes_{g\in G}} R_g\right)$.

2.2. Connes-Størmer Bernoulli shifts. Let us now consider a generalized version of the above example, as introduced in ([**CS**]). To this end, we first need to recall some basic facts about ITPF1 factors and, more generally, about factors having a discrete decomposition ([**AW, C4,5**]).

2.2.0. Factors with discrete decomposition. We let $(\mathcal{N}, \varphi)$ be a von Neumann factor with the centralizer $N = \mathcal{N}_\varphi$ satisfying $N' \cap \mathcal{N} = \mathbb{C}$. We denote

$$\mathcal{V} = \{v \in \mathcal{N} \mid v^*v = 1, vNv^* = vv^*Nvv^*, \varphi(vX) = \varphi(vv^*)\varphi(Xv), \forall X \in \mathcal{N}\}.$$

Remark that in fact for an isometry $v \in \mathcal{N}$ to be in $\mathcal{V}$ it is sufficient to satisfy $\varphi(vX) = \varphi(vv^*)\varphi(Xv), \forall X \in \mathcal{N}$ (see e.g. [**C4,5**]).

We denote by $H_1 = \{\varphi(vv^* \mid v \in \mathcal{V}\}$ and for each $\beta \in H_1$, denote $\mathcal{V}_\beta = \{v \in \mathcal{V} \mid \varphi(vv^*) = \beta\}$. Thus, $\mathcal{V} = \cup_\beta \mathcal{V}_\beta$. Also, we clearly have $\mathcal{V}_\beta \mathcal{V}_\gamma = \mathcal{V}_{\beta\gamma}$ and $\mathcal{V}_1 = \mathcal{U}(N)$. Actually, for any two isometries $v, v' \in \mathcal{V}_\beta$ the element $w = v'v^*$ is a partial isometry in N with $v' = wv$. Also, it is easy to see that if $v \in \mathcal{V}_\beta, v' \in \mathcal{V}_{\beta'}$, with $\beta \neq \beta'$, then $E(v^*v') = 0 = E(v'v^*)$, where $E = E_N^\varphi$ is the φ-preserving conditional expectation of $\mathcal{N}$ onto N.

We further assume that the set $\mathcal{V}$ generates $\mathcal{N}$ as a von Neumann algebra. By ([**C5**]), this is equivalent to $(\mathcal{N}, \varphi)$ having a *discrete decomposition* over N. More precisely, if for each $\beta \in H_1$ we choose an isometry $v_\beta \in \mathcal{V}_\beta$ then each $X \in \mathcal{N}$ has a cross product type decomposition $X = \Sigma_\beta E(Xv_\beta^*)v_\beta + \Sigma_\beta v_\beta^* E(v_\beta X)$.

In fact, if we denote by H the multiplicative group generated by H_1, i.e., $H = H_1 \cup H_1^{-1}$, then $(N\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N})) \subset \mathcal{N}\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N}))$ is isomorphic to $(N\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N})) \subset (N\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N})) \rtimes H)$. Moreover, if we denote by $p_0 \in N\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N}))$ a projection of the form $1_N \otimes q_0$, where q_0 is a one dimensional projection, and by $\{u_\beta\}_{\beta \in H}$ the canonical unitaries in $\mathcal{N}\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N})) \simeq N\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N})) \rtimes H$ that implement the action of the group H then $p_0 u_\beta p_0 = v_\beta$ and $p_0 u_{\beta^{-1}} p_0 = v_\beta^*, \forall \beta \in H_1$.

In terms of modular theory, it is easy to see that $(\mathcal{N}, \varphi)$ has a discrete decomposition as above iff the modular automorphism group associated with φ is almost periodic and has pure point spectrum. We call the group H the *spectrum of the discrete decomposition.*

Examples of such situations are, for instance, when $(\mathcal{N}, \varphi)$ is a type II_1 factor N with its trace, because then $\mathcal{V}$ equals the set of unitary elements in N, with $H = \{1\}$. Another example is when $(\mathcal{N}, \varphi)$ is a type III_λ factor, $0 < \lambda < 1$, with φ its generalized trace (cf. [**C4, 5**]). In this case $H = \{\lambda^n \mid n \in \mathbb{Z}\}$ and $H_1 = \{\lambda^n \mid n \geq 0\}$. The type III_1 factors that can be obtained as tensor products of some type III_{λ_i} factors, $0 < \lambda_i < 1, i = 1, 2, ..., k$, have discrete decompositions as well. In this case H is the multilicative subgroup of $\mathbb{R}_+^*$ generated by $\{\lambda_i\}_i$.

2.2.1. ITPF1 *factors.* An important class of concrete factors having a discrete decomposition is as follows: Let (N_0, φ_0) to be a finite dimensional factor with a faithful state on it. Let G be an infinite discrete group as before and $(\mathcal{N}, \varphi) \stackrel{\text{def}}{=}$

$\overline{\bigotimes}_{g \in G}(N_g, \varphi_g)$, where $(N_g, \varphi_g) = (N_0, \varphi_0)$, $\forall g \in G$, be the corresponding *infinite tensor product*, or ITPF1 *factors* ([**P, AW**]).

In case $N_0 = M_{2\times 2}(\mathbb{C})$ and φ_0 is given by the diagonal element $(t, 1-t)$ with $t < 1/2$, then $\mathcal{N}$ is a type III_λ factor, where $\lambda = t/1-t$, and φ is its generalized trace. Thus, if $N = \mathcal{N}_\varphi$ then $(\mathcal{N}, \varphi)$ has a discrete decomposition over N.

In general, if $N_0 = M_{k\times k}(\mathbb{C})$ and the state φ_0 is given by a diagonal element with spectrum $0 < t_1 \leq t_2 ... \leq t_k$ then $\mathcal{N}$ is a type II_1 factor if $t_1 = ... = t_k$, it is a type III_λ factor, for some $0 < \lambda < 1$, if the multiplicative group generated by $\{t_i/t_j\}_{i,j}$ in $\mathbb{R}_+^*$ is equal to $\{\lambda^n \mid n \in \mathbb{Z}\}$ and it is a type III_1 factor if the the multiplicative group generated by $\{t_i/t_j\}_{i,j}$ is not single generated. (Thus, generically, the IPTF1 factors $\mathcal{N}$ are of type III_1.)

In this general case the centralizer $N = \mathcal{N}_\varphi$ of the state φ still satisfies $N' \cap \mathcal{N} = \mathbb{C}$ and in fact $(\mathcal{N}, \varphi)$ has a disrete decomposition over N, with spectrum $H \subset \mathbb{R}_+^*$ given by the multiplicative group generated by $\{t_i/t_j \mid 1 \leq i,j \leq k\} \subset \mathbb{R}_+^*$.

2.2.2. Generalized Bernoulli shifts. With $\mathcal{N} = \overline{\bigotimes}_g(M_{k\times k}(\mathbb{C}), \varphi_0)_g$ as above, let $\sigma : G \to \mathrm{Aut}\mathcal{N}$ be defined as follows: first define σ to be the Bernoulli shift on the algebraic infinite tensor product $\otimes_g N_g$, as before; then note that on this dense sbalgebra of $\mathcal{N}$ we have $\varphi \circ \sigma_g = \varphi, \forall g$; thus each σ_g can be extended to a φ-preserving automorphism σ_g on all $\mathcal{N}$. Note that when G is infinite σ is properly outer and ergodic ([**CS**]). Such an action is called a *generalized Bernoulli shift* or a (N_0, φ_0)*-Bernoulli shift.*

Note that the generalized Bernoulli shift σ leaves each one of the sets of isomteries $\mathcal{V}_\beta$ invariant, i.e., $\sigma_g(\mathcal{V}_\beta) = \mathcal{V}_\beta, \forall g \in G, \forall \beta \in H_1$. In fact, if $v \in \mathcal{V}$ then $w_g = v\sigma_g(v^*)$ are partial isometries in N with $\sigma_g(v) = w_g^* v$.

2.2.3 Connes-Størmer Bernoulli shifts. Let $(\mathcal{N}, \varphi)$, σ, $N = \mathcal{N}_\varphi \subset \mathcal{N}$ be as before. Since φ is invariant to σ, $\sigma_g(N) = N, \forall g$. Moreover, by ([**AW, C1,4**]), the von Neumann algebra N (which satisfies $N' \cap \mathcal{N} = \mathbb{C}$) is isomorphic to the hyperfinite type II_1 factor R.

Moreover, the action $\sigma_g = \sigma_{g|N}$ is properly outer and ergodic on N ([**CS**]). Such actions are called *Connes-Størmer Bernoulli shifts* of G on the hyperfinite type II_1 factor.

Note that if σ is a (N_0, φ_0)-Bernoulli shift on $(\mathcal{N}, \varphi)$ as before then $\sigma \otimes \sigma$ is a $(N_0 \otimes N_0, \varphi_0 \otimes \varphi_0)$-Bernoulli shift on on $(\mathcal{N}\overline{\otimes}\mathcal{N}, \varphi \otimes \varphi)$. However, if we let $\tilde{N}$ be the centralizer of $\varphi \otimes \varphi$ in $\mathcal{N}\overline{\otimes}\mathcal{N}$ and denote by θ_g the restriction of $\sigma_g \otimes \sigma_g$ to $\tilde{N}$, then θ is a Connes-Størmer Bernoulli shift, with $\tilde{N} \supset N\overline{\otimes}N$ and the restriction of θ to $N\overline{\otimes}N$ equals $\sigma_{|N} \otimes \sigma_{|N}$, but in general $\tilde{N}$ is different from $N\overline{\otimes}N$ and afortiori $\theta \neq \sigma \otimes \sigma$ as well.

2.3. Free Bernoulli shifts. Let (B_0, τ_0) be a finite von Neumann algebra with a finite faithful normal trace on it. Let $\{(B_g, \tau_g)\}_{g\in G}$ be copies of (B_0, τ_0) indexed by the discrete group G. Define $(N, \tau) \stackrel{\text{def}}{=} \underset{g\in G}{*}(N_g, \tau_g)$ to be the free product of the algebras (B_g, τ_g) (see [**V1**]). Then we define $\sigma_g\left(\underset{h}{*}b_h\right) = \underset{h}{*}b'_h$, where $b'_h = b_{g^{-1}h}$ (see e.g. [**S, Po3**]). Such an action is called a *free Bernoulli shift* of the group G.

Note that by ([**Dy1**]), if G is infinite and $B_0 \neq \mathbb{C}$ then N is a factor. Moreover, σ is then easily seen to be properly outer and ergodic.

Also, by ([**Dy1**]), if B_0 is AFD, or if B_0 is a free group factor, then N is isomorphic to the free group factor $L(\mathbb{F}_\infty)$.

Thus, any discrete countable group G acts properly outer (+ ergodic, if $|G| = \infty$) on the free group factor $L(\mathbb{F}_\infty)$.

Note that one can also define a Connes-Stormer version of the free Bernoulli shifts, by using Dykema's free version of the IPTF1 factors ([**Dy2**]). Thus, one starts with a fixed finite dimensional von Neumann algebra with a faithful state on it, (B_0, φ_0). One defines $(\mathcal{N}, \varphi) \stackrel{\text{def}}{=} \underset{g\in G}{*}(B_g, \varphi_g)$, where $(B_g, \varphi_g) = (B_0, \varphi_0), \forall g \in G$, noting that φ is quasi-periodic on $\mathcal{N}$ ([**Sh**]), being a generalized trace if $B_0 = M_{2\times 2}(\mathbb{C})$. One takes N to be the centralizer of the state φ. Then one considers the free Bernoulli shift σ on $\mathcal{N}$ (which we could call a *generalized* free Bernoulli shift), noting that it preserves φ. Finally, one takes the *free Connes-Stormer Bernoulli shift* to be the restriction of σ to N, which by ([**Dy2**]) is isomorphic to $L(\mathbb{F}_\infty)$ (see also [**Sh**]).

2.4. Bernoulli shifts are mixing. Both the "hyperfinite" (2.1, 2.2) and the "free" (2.3) Bernoulli shifts have the property that they are "very ergodic". More precisely, they satisfy the following property.

2.4.1. Definition. Let $\mathcal{N}$ be a von Neumann algebra with a faithful normal state φ. Let $\sigma : G \to \text{Aut}\mathcal{N}$ be a properly outer action of the discrete group G on $\mathcal{N}$ that preserves φ. The action σ is *strongly mixing* if

$$\lim_{g\to\infty} \varphi(y_1\sigma_g(x)y_2) = \varphi(x)\varphi(y_1y_2), \forall x, y_1, y_2 \in \mathcal{N}.$$

Strongly mixing actions are clearly ergodic. In fact their ergodicity is so "strong" that it goes through to tensor products. More precisely, one has:

2.4.2. Lemma. *Let σ_1 be a strongly mixing action of a group G on the von Neumann algebra $(\mathcal{N}, \varphi)$ and σ_0 an arbitrary action of G on a von Neumann algebra $(\mathcal{N}_0, \varphi_0)$. Denote by θ' the action of G on $\mathcal{N}\overline{\otimes}\mathcal{N}_0$ given by $\theta'_g = \sigma_g \otimes \sigma_{0,g}, g \in G$. Then $(\mathcal{N}\overline{\otimes}\mathcal{N}_0)^{\theta'} = \mathbb{C} \otimes \mathcal{N}_0^{\sigma_0}$.*

Proof. Let $\varphi' = \varphi \otimes \varphi_0$. Let $x \in (\mathcal{N}\overline{\otimes}\mathcal{N}_0)^{\theta'}, x \neq 0$ and $\varepsilon > 0$. Denote by Y_0 the φ'-preserving expectation of x onto $\mathbb{C} \otimes \mathcal{N}_0$ and $X_0 = x - Y_0$. Note that $Y_0 \in \mathbb{C} \otimes \mathcal{N}_0^{\sigma_0} \subset (\mathcal{N}\overline{\otimes}\mathcal{N}_0)^{\theta'}$. Thus X_0 belongs to $(\mathcal{N}\overline{\otimes}\mathcal{N}_0)^{\theta'}$ as well.

By the density of the algebra $\mathcal{N} \otimes \mathcal{N}_0$ in $\mathcal{N}\overline{\otimes}\mathcal{N}_0$, there exists an orthonormal system $x_0 = 1, x_1, x_2, ..., x_n \in \mathcal{N}$ and elements $y_0 = Y_0, y_1, y_2, ..., y_n \in \mathcal{N}_0$ such that if we denote $x' = \sum_{i=0}^{n} x_i \otimes y_i$ then we have

$$\|x - x'\|_{\varphi'} < \varepsilon/(3\|x\|_{\varphi'}) \text{ and} \|x'\|_{\varphi'} \leq \|x\|_{\varphi'}.$$

Since σ is mixing, there exists $g \in G$ such that

$$\sum_{i,j=1}^{n} |\varphi(\sigma_g(x_i)x_j^*)| \; |\varphi_0(\sigma_{0,g}(y_i)y_j^*)| < \varepsilon/3.$$

Thus, if we denote $X_0' = \sum_{i=1}^{n} x_i \otimes y_i = x' - Y_0$ then we have

$$|\varphi'(\theta_g'(X_0')X_0'^*)| < \varepsilon/3 \text{ and} \|X_0 - X_0'\|_{\varphi'} = \|x_0 - x_0'\|_{\varphi'} < \varepsilon/(3\|x\|_{\varphi'}).$$

As a consequence we get

$$\|X_0\|_{\varphi'}^2 = \varphi'(\theta_g'(X_0)X_0^*)$$

$$\leq |\varphi'(\theta_g'(X_0')X_0'^*)| + 2\|X_0 - X_0'\|_{\varphi'}\|X_0\|_{\varphi'} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $X_0 = 0$ so that $x = Y_0$ belongs to $\mathbb{C}\otimes\mathcal{N}_0^{\sigma_0}$. Q.E.D.

2.4.3. Corollary. *If σ is a strongly mixing action of a group G on the von Neumann algebra $(\mathcal{N}, \varphi)$ then $\mathbb{C}1 \subset \mathcal{N}$ is the only finite dimensional subspace of $L^2(\mathcal{N}, \varphi)$ which is invariant to $\sigma_g, \forall g \in G$.*

Proof. By applying 2.4.2 to $\sigma_0 = \sigma$ and by taking into account that the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathcal{N}, \varphi)$, with the action implemented by σ on it, can be naturally identified with $L^2(\mathcal{N}, \varphi)\overline{\otimes}L^2(\mathcal{N}, \varphi)$, with the action $\sigma \otimes \sigma$ on it, it follows that the one dimensional projection of $L^2(\mathcal{N}, \varphi)$ onto $\mathbb{C}1$ is the only fixed point for the former. Q.E.D.

As mentioned before, let us point out that the hyperfinite and the free Bernoulli shifts provide examples of strongly mixing actions.

2.4.4. Lemma. *All hyperfinite Bernoulli shifts (including the "generalized" and the "Connes-Størmer" ones) and the free Bernoulli shifts are strongly mixing.*

Proof. Let (N_0, φ_0) be either the hyperfinite type II_1 factor with $\varphi_0 = \tau$ its trace, or a finite dimensional factor with φ_0 a faithful state on it. Let G be a discrete group and $(\mathcal{N}, \varphi) \stackrel{\text{def}}{=} \overline{\underset{g \in G}{\otimes}} (N_g, \varphi_g)$, where $(N_g, \varphi_g) = (N_0, \varphi_0)$, $\forall g \in G$.

By using first the Kaplansky density theorem then the Cauchy-Schwartz inequality, it follows that it is suficient to prove the statement for $y_{1,2}$ in the algebraic tensor product $\otimes_g (N_g, \varphi_g)$.

Assume that in this infinite algebraic tensor product $y_{1,2}$ are supported by a finite subset $S \subset G$. Since each such element is a linear combination of elements of the form $\underset{g \in S}{\otimes} e^g_{i_g, j_g}$, where $e^g_{k,l} \in N_g \simeq N_0$ are matrix units chosen so that φ_g is supported on the diagonal $\{e^g_{kk}\}_k$.

It follows that it is suficient to prove the statement for $y_{1,2}$ of the form $\underset{g \in S}{\otimes} e^g_{i_g, j_g}$. But for each such element y_i there exists a non-zero scalar c_i such that $\varphi(x' y_i) = c_i \varphi(y_i x'), \forall x' \in \mathcal{N}$.

Thus $\varphi(y_1 \sigma_g(x) y_2) = c_2 \varphi(y_2 y_1 \sigma_g(x))$. By approximating x with elements in the algebraic tensor product $\otimes_g (N_g, \varphi_g)$ and by applying again Cauchy-Schwartz inequality, it follows that in calculating the limit $\lim_{n \to \infty} \varphi(y_2 y_1 \sigma_g(x))$, it is sufficient to take $x \in \otimes_g (N_g, \varphi_g)$. But for such x, y_1, y_2 we clearly have $\lim_{n \to \infty} \varphi(y_2 y_1 \sigma_g(x)) = \varphi(y_2 y_1) \varphi(x) = c_2^{-1} \varphi(y_1 y_2) \varphi(x)$. Altogether, it follows that $\lim_{n \to \infty} \varphi(y_1 \sigma_g(x) y_2) = \varphi(y_1 y_2) \varphi(x)$.

The proof of the free group case is similar, so we leave it as an exercise for the reader (see e.g. [**S**]). Q.E.D.

2.5. Product type actions. Let us also mention a class of examples of properly outer actions that are not ergodic. Thus, with the notations of the examples 2.1-2.2, let $G_0 \subset \text{Aut} N_0$ be a subgroup of automorphisms with $\alpha_{0,v} \neq id, \forall v \neq 1$, and $\varphi_0 \circ \alpha_{0,v} = \varphi_0$, $\forall v \in G_0$, where $\alpha_{0,v}$ is the image of $v \in G_0$ in $\text{Aut} N_0$. G_0 will be considered with the topology inherited from $\text{Aut} N_0$.

Let $\mathcal{N}$ denote the von Neumann algebra constructed from (N_0, φ_0) by infinite tensor product indexed by the infinite discrete group G (or simply indexed by $\mathbb{N}$), as in 2.2. We let $\alpha : G_0 \to \text{Aut}\, \mathcal{N}$ be defined by $\alpha_v \left(\underset{h \in G}{\otimes} b_h \right) = \underset{h \in G}{\otimes} b'_h$, where $b'_h = \alpha_{0,v}(b_h), \forall h \in G$.

It is easy to see that α is a continuous, properly outer action of G_0 on $(\mathcal{N}, \varphi)$. In particular, since α preserves φ, it leaves the centralizer $N = \mathcal{N}_\varphi$ of φ invariant, thus implementing on it an action. Moreover, if N_0 is finite dimensional then the action α has fixed point algebra $\mathcal{N}^{\alpha}$ satisfying $\mathcal{N}^{\alpha\prime} \cap \mathcal{N} = \mathbb{C}1$ ([**W**]). Thus, α is far from being ergodic.

Note that the action α commutes with the (N_0, φ_0)-Bernoulli shift σ on $(\mathcal{N}, \varphi)$ considered in 2.1, 2.2. Thus, $\alpha_{|N}$ commutes with $\sigma_{|N}$ as well.

A particular case of product type automorphism is the *flip* authomorphism on $\mathcal{N}\overline{\otimes}\mathcal{N}$, that takes $x \otimes y$ into $y \otimes x$. This automorphism will play an important role in Section 3. In fact, we will need the existence of some continuous path of automorphisms that relate the identity to the flip automorphism.

2.5.1. Lemma. *With the notations used in 2.2, let (N_0, φ_0) be a finite dimensional factor and σ be the (N_0, φ_0)- Bernoulli shift of the group G on the factor $(\mathcal{N}, \varphi)$. Let $\tilde{N}$ be the centralizer of $\varphi \otimes \varphi$ on $\mathcal{N}\overline{\otimes}\mathcal{N}$ and θ be the Connes-Stormer Bernoulli shift on $\tilde{N}$ obtained by restricting the $(N_0 \otimes N_0, \varphi_0 \otimes \varphi_0)$-Bernoulli shift $\sigma \otimes \sigma$ from $\mathcal{N}\overline{\otimes}\mathcal{N}$ to $\tilde{N}$. Then there exists a continuous product type action α of $\mathbb{R}$ on $(\mathcal{N}\overline{\otimes}\mathcal{N}, \varphi \otimes \varphi)$ such that:*

1°. α is $\varphi \otimes \varphi$-preserving and leaves $\tilde{N}$ invariant, thus implementing on it an action still denoted by α.

2°. α commutes with the generalized Bernoulli shift $\sigma \otimes \sigma$ and its restriction to $\tilde{N}$ commutes with θ.

3°. $\alpha_{\pm 1}(\mathcal{N} \otimes \mathbb{C}) = \mathbb{C} \otimes \mathcal{N}$.

If in addition we assume $(N_0, \varphi_0) = (M_{2^n \times 2^n}(\mathbb{C}), \varphi_0) \simeq \bigotimes_{1 \le i \le n} (M_{2\times 2}(\mathbb{C}), \varphi_{0,i})_i$, for some $n \geq 1$, then α can be constructed so that there exists a period 2 product type automorphism β on $\mathcal{N}\overline{\otimes}\mathcal{N}$ that preserves $\varphi \otimes \varphi$, leaves $\tilde{N}$ invariant and satisfies the conditions:

4°. $\mathcal{N} \otimes \mathbb{C}1 \subset \tilde{N}^{\beta}$.

5°. $\beta\alpha_t\beta^{-1} = \alpha_{-t}, \forall t$.

Proof. Let us first assume σ is of step 2. We use the notations of 2.2.

We let $\{e_{ij}\}_{i,j=1,2}$ be a matrix unit for $N_0 = M_{2\times 2}(\mathbb{C})$ such that the state φ_0 on N_0 is given by a dyagonal operator $\Sigma_i \lambda_i e_{ii}$.

We first take the action α_0 of $\mathbb{R}$ on $N_0 \otimes N_0$ to be defined by $\alpha_{0,t}(x) = \mathrm{Ad}(v_t)(x)$, $t \in \mathbb{R}$, with the unitary $v_t \in N_0 \otimes N_0$ being given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi t/2 & \sin \pi t/2 & 0 \\ 0 & -\sin \pi t/2 & \cos \pi t/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then we define the period 2-automorphism β_0 on $N_0 \otimes N_0$ as $\mathrm{Ad}u_0$, where the unitary u_0 is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that the unitaries $v_t, t \in \mathbb{R}$, and u_0 belong to the centralizer of the state $\varphi_0 \otimes \varphi_0$ on $N_0 \otimes N_0$. Moreover, u_0 lies in $\mathbb{C} \otimes N_0$. An easy calculation shows that $\beta_0 \alpha_{0,t} \beta_0^{-1} = \alpha_{0,-t}$ and that $\mathrm{Ad}(v_{\pm 1})(N_0 \otimes \mathbb{C}) = \mathbb{C} \otimes N_0$.

Define α_t to be the product-type action $\otimes_g (\alpha_{0,t})_g$ and β to be the automorphism $\otimes_g (\beta_0)_g$. Then both α and β leave the centralizer $\tilde{N}$ of the product state $\varphi \otimes \varphi = \otimes_g (\varphi_0 \otimes \varphi_0)_g$ invariant.

Moreover, by the way α, β are defined, conditions $1° - 5°$ are trivially satisfied.

Note that by taking tensor products of the above contsruction, this also proves the case $(N_0, \varphi_0) = \underset{1 \le i \le n}{\otimes} ((M_{2\times 2}(\mathbb{C}), \varphi_{0,i})_i$.

Now, in the general case when $N_0 = M_{k\times k}(\mathbb{C})$, let $\{e_{ij}\}_{i,j}$ be a matrix unit with φ_0 given by a dyagonal operator in $\mathrm{Alg}\{e_{ii}\}_i$, as before. Let also

$$v_t = \Sigma_i e_{ii} \otimes e_{ii}$$

$$+ \sum_{i<j} \cos \pi t/2 (e_{ii} \otimes e_{jj} + e_{jj} \otimes e_{ii}) + \sin \pi t/2 (e_{ij} \otimes e_{ji} - e_{ji} \otimes e_{ij}).$$

Then clearly $1° - 3°$ are satisfied. Q.E.D.

3. Examples of cocycles.

We first explain a method for producing a family of cocycle actions from a given action. Then we construct a class of examples of non-trvial 1-cocycles for actions on type II_1 factors that come from a discrete decomposition.

Thus, we first let $(\mathcal{N}, \varphi)$ be a von Neumann factor with centralizer $N = \mathcal{N}_\varphi$. We assume N to be a factor itself. We consider properly outer cocycle actions (σ, v) of the discrete group G on $(\mathcal{N}, \varphi)$. Thus, N is invariant to any such σ.

Let $p \in \mathcal{P}(N)$ be a non-zero projection. Since N is a factor and σ_g are tace preserving on N, for each $g \in G$ the projections p and $\sigma_g(p)$ are equivalent in N. Choose partial isometries $w_g \in \mathcal{N}_\varphi$ of left support p such that $w_g \sigma_g(p) w_g^* = p$ and $w_e = p$.

3.1. Proposition. *Let $\sigma^p : G \to \mathrm{Aut}(\mathcal{N}_p)$ be defined by $\sigma_g^p(pxp) = w_g \sigma_g(pxp) w_g^*$, for $x \in N$. Let also $v_{g,h}^p = w_g \sigma_g(w_h) v_{g,h} w_{gh}^*, g, h \in G$, which we regard as an element in $(\mathcal{N}_\varphi)_p = p\mathcal{N}_\varphi p$. Then (σ^p, v^p) is a properly outer cocycle action of G on*

$(\mathcal{N}_p, \varphi_p$, where $\varphi_p(x) = \varphi(pxp)/\varphi(p), x \in p\mathcal{N}p$. Moreover, up to cocycle conjugacy, the cocycle action (σ^p, v^p) does not depend on the choice of the partial isometries w_g.

Proof. A straightforward calculation shows that v^p checks the conditions (1.2.1), (1.2.2) with respect to σ^p. Q.E.D.

Assume now $\mathcal{N}$ is the type II_1 factor N. Since $(pNp, \sigma^p) \simeq (qNq, \sigma^q)$ if $p, q \in \mathcal{P}(N)$ have same trace, from Lemma 3.1 we see that the isomorphism class of (N_p, σ^p), up to cocycle conjugacy, only depends on $\tau(p)$. Thus, we can denote it by (N_t, σ^t), for each $0 < t \leq 1$. More generally. we consider the following:

3.1.1. Notation. If $t > 0$ then we denote by (N_t, σ^t) the reduced of $(M_{n\times n}(N)$, id $\otimes \sigma)$, for some $n \geq t$, by a projection p with normalized trace $\tau(p)$ satisfying $n\tau(p) = t$. Again, by the above Lemma, up to cocycle conjugacy this action only depends on t. We call (N_t, σ^t) the amplification of (N, σ) by t.

Note that if in 3.1 we take σ to be a genuine action (so $v = 1$) and take the projection p in the fixed point algebra N^σ then the 2-cocycle v^p is vanishing, in other words the cocycle action (σ^p, v^p) can be perturbed to an actual action. Indeed, because we can take $w_g = p, \forall g$. Thus, if N^σ contains projections of any trace, then all 2-cocycle as in 3.1, obtained by reducing σ by projections, vanish.

However, we will see in the next section that for certain ergodic actions σ of property T groups, these cocycle actions cannot be perturbed to actions.

We now consider actions σ of groups G on type II_1 factors N with the property that N is the core of a discrete decomposition of some type III factor $(\mathcal{N}, \varphi)$, in a way that the action σ itself be the restriction to N of an action on $(\mathcal{N}, \varphi)$. An example of actions satisfying this property is provided by the Connes-Stormer Bernoulli shifts defined in 2.2.3.

The next result provides a class of examples of non-trivial 1-cocycles for such actions. We'll show in Section 4 that if G is infinite and has the property T, with σ being an action of G as in 2.2.3, then in fact these cocycles give the complete list of 1- cocycles for σ. But the construction may have an independent interest for general groups G as well.

To simplify notations, for each scalar 2-cocycle μ for G (see 1.2) we denote by $\mathcal{F}_\mu$ the set of finite dimensional, unitary, projective representations of G with scalar 2-cocycle μ, i.e., $\pi \in \mathcal{F}_\mu$ if $\pi : G \to \mathcal{U}(n)$ for some $n \geq 0$ and it satisfies $\pi(g)\pi(h) = \mu_{g,h}\pi(gh), \forall g, h \in G$. The 0 representation is contained in $\mathcal{F}_\mu, \forall \mu$. Note that if $\mu = 1$ then $\mathcal{F}_1$ is just the set of finite dimensional representations $\mathcal{F}$ of G. This set contains the trivial representation of the group G and in fact any finite multiple of this representation.

Let $(\mathcal{N}, \varphi)$ be a von Neumann factor with a discrete decomposition, as in 2.2.0.

Let $N = \mathcal{N}_\varphi$ and H be the spectrum of this discrete decomposition, i.e., H is the multiplicative subgroup of $\mathbb{R}^*_+$ such that $(N \subset \mathcal{N})\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{N})) = (N\otimes\mathcal{B}(\ell^2(\mathbb{N})) \subset N\otimes\mathcal{B}(\ell^2(\mathbb{N})) \rtimes H)$. Let $H_1 = \{\beta \in H \mid \beta \leq 1\}$. We denote by $\mathcal{F}_\mu^{H_1}$ the set of families of elements π_β in $\mathcal{F}_\mu$ indexed by $\beta \in H_1$ such that $\Sigma_\beta n_\beta \beta \leq 1$, where $n_\beta = \dim \pi_\beta$. Note that some of the representations π_β may be equal to 0. In fact, if $\pi_1 \neq 0$ then this forces π_1 to be a one dimensional representation, thus a character of G, and all other $\pi_\beta, \beta < 1$, must be zero.

Let G be an infinite group and σ an action of G on $(\mathcal{N}, \varphi)$, with its restriction to $N = \mathcal{N}_\varphi$ still denoted by σ.

3.2. Theorem. *Let $\{\pi_\beta\}_\beta \in \mathcal{F}_\mu^{H_1}$. For each β let $v_i^\beta, 1 \leq i \leq n_\beta$, be isometries in $\mathcal{V}_\beta$ such that $\{v_i^\beta\}_{\beta,i}$ have mutually orthogonal left supports. Let $(\pi_\beta(g))_{ij}$ denote the coefficients of the representation π_β with respect to some orthonormal basis. Let $w : G \to \mathcal{N}$ be defined by:*

$$w_g = \Sigma_\beta \Sigma_{i,j} (\pi_\beta(g))_{ij} v_i^\beta \sigma_g(v_j^{\beta^*}), g \in G \tag{3.2.1}$$

1°. *w_g are partial isometries in $N = \mathcal{N}_\varphi$ and in fact w is a generalized weak 1-cocycle for σ with support $p = \Sigma_{\beta,i} v_i^\beta v_i^{\beta^*}$ and scalar 2-cocycle μ.*

2°. *Let $\sigma'_g = \mathrm{Ad}(w_g) \circ \sigma_g$ and denote by $\mathcal{B}$ the von Neumann subalgebra of $p\mathcal{N}p$ generated by the matrix unit*

$$\{v_i^\beta v_{i'}^{\beta'^*} \mid 1 \leq i \leq n_\beta, 1 \leq i' \leq n_{\beta'}, \beta, \beta' \in H_1\}.$$

Let also B be the von Neumann subalgebra of $\mathcal{B}$ generated by

$$\{v_i^\beta v_{i'}^{\beta^*} \mid 1 \leq i, i' \leq n_\beta, \beta \in H_1\}.$$

Then $B = \mathcal{B} \cap N = (\mathcal{B})_\varphi = E_N^\varphi(\mathcal{B})$ and $\sigma'_g(\mathcal{B}) = \mathcal{B}, \sigma'_g(B) = B$, with $\sigma'_{g|\mathcal{B}} \simeq \mathrm{Ad}(\pi(g)), \forall g \in G$, where $\pi = \oplus_\beta \pi_\beta$ is viewed as a representation on the Hilbert space $\oplus_\beta \ell^2(n_\beta)$, with $\mathcal{B}$ being identified with the algebra $\mathcal{B}(\oplus_\beta \pi_\beta)$ of all bounded operators on this Hilbert space.

3°. *Assume in addition that σ is strongly mixing on $\mathcal{N}$. Then $(p\mathcal{N}p)^{\sigma'} = \mathcal{B}^{\sigma'} = \pi(G)' \cap \mathcal{B}, (pNp)^{\sigma'} = B^{\sigma'} = \pi(G)' \cap B$. Moreover, any finite dimensional vector subspace of $p\mathcal{N}p$ (resp. pNp) which is invariant to σ' is contained in $\mathcal{B}$ (resp. B) and in fact $\mathcal{B}$ (resp. B) is the closure of the span of finite dimensional invariant subspaces of $p\mathcal{N}p$ (resp. pNp).*

4°. *Let w respectively w' be cocycles constructed out of $\{\pi_\beta\}_\beta \in \mathcal{F}_\mu^{H_1}$ and $\{v_i^\beta\}_{\beta,i}$, respectively $\{\pi'_\beta\}_\beta \in \mathcal{F}_{\mu'}^{H_1}$ and $\{v_i'^\beta\}_{\beta,i}$, as in (3.2.1). If $\mu = \mu'$ and $\pi_\beta \sim \pi'_\beta, \forall \beta \in$*

H_1 then w and w' are equivalent. If in addition we assume σ is strongly ergodic then, conversely, $w \sim w'$ implies $\mu = \mu'$ and $\pi_\beta \sim \pi'_\beta, \forall \beta$.

Proof. 1°. An easy calculation shows that if we denote $p_\beta = \Sigma_i v_i^{\beta^*} v_i^\beta$ then p_β is fixed by $\sigma'_g, \forall g$. Note also that p_β this way defined is in the center of B_0. Thus, by replacing w_g by $p_\beta w_g$, we see that in order to prove 1° in its full generality it is sufficient to prove it in the case all but one of the representations π_β are equal to zero, i.e., for $\pi = \pi_\beta$, for some $\beta \in H_1$.

Thus, we may assume w_g is of the form $\Sigma_{i,j}(\pi(g))_{ij} v_i^\beta \sigma_g(v_j^{\beta^*})$. For simplicity, denote $v_i = v_i^\beta$ and $c_{ij}^g = (\pi(g))_{ij}$. Taking into account that

$$\Sigma_j c_{ij}^g c_{jl}^h = \mu_{g,h} c_{il}^{gh}$$

and that v_i have mutually orthogonal left supports, we get:

$$w_g \sigma_g(w_h) = (\Sigma_{i,j} c_{ij}^g v_i \sigma_g(v_j{}^*)) \sigma_g(\Sigma_{k,l} c_{kl}^h v_k \sigma_h(v_l{}^*))$$

$$= \Sigma_{i,l}(\Sigma_j c_{ij}^g c_{jl}^h) v_i \sigma_{gh}(v_l{}^*)$$

$$= \mu_{g,h} \Sigma_{i,l} c_{il}^{gh} v_i \sigma_{gh}(v_l{}^*) = \mu_{g,h} w_{gh}.$$

2°. Since $E_N^\varphi(v'v^*) = 0$ for $v \in \mathcal{V}_\beta$ and $v' \in \mathcal{V}_{\beta'}$ with $\beta \neq \beta'$, we have $E_N^\varphi(\mathcal{B}) = B$.

To check that $\sigma'_g(\mathcal{B}) = \mathcal{B}$, we need to show that $\sigma'_g(v_i^\beta v_{i'}^{\beta'^*}) \in \mathcal{B}, \forall i, i', \beta, \beta'$. We have:

$$\sigma'_g(v_i^\beta v_{i'}^{\beta'^*}) = w_g \sigma_g(v_i^\beta v_{i'}^{\beta'^*}) w_g^*$$

$$= (\Sigma_{\alpha,k,l}(\pi_\alpha(g))_{kl} v_k^\alpha \sigma(v_l^{\alpha*})) \sigma_g(v_i v_{i'}^{\beta'^*})(\Sigma_{\alpha',k',l'}(\pi_{\alpha'}(g))_{k'l'} v_{k'}^{\alpha'} \sigma_g(v_{l'}^{\alpha'^*}))^*$$

$$= \Sigma_{k,k'}(\pi_\beta(g))_{ki} \overline{\pi_{\beta'}(g)_{k'i'}} v_k^\beta \sigma_g(v_i^{\beta^*})) \sigma_g(v_i^\beta v_{i'}^{\beta'^*}) \sigma_g(v_{i'}^{\beta'}) v_{k'}^{\beta'^*}$$

$$= \Sigma_{k,k'}(\pi_\beta(g))_{ki} \overline{(\pi_{\beta'}(g))_{k'i'}} v_{i'} v_{k'}^{\beta'^*}.$$

Since this latter element lies in $\mathcal{B}$, we are done. The above computation also shows that if we take $\beta = \beta'$ then $\sigma'(v_i^\beta v_{i'}^{\beta^*})$ lies in $\text{sp}\{v_k^\beta v_{k'}^{\beta^*}\}_{k,k'}$, thus in B.

Moreover, it shows that if we identify the von Neumann algebra $\mathcal{B}$ generated by the matrix unit $\{v_i^\beta v_{i'}^{\beta'^*} \mid i, i', \beta, \beta'\}$ with $\mathcal{B}(\oplus_\beta \ell^2(n_\beta))$, then for each $x = v_i^\beta v_{i'}^{\beta'^*} \in \mathcal{B}$ we have $\sigma'_g(x) = \pi(g) x \pi(g)^*$.

3°. Recall that

$$w_g = \Sigma_\alpha \Sigma_{k,l}(\pi_\alpha(g))_{k,l} v_k^\alpha \sigma_g(v_l^{\alpha*}).$$

Hence

$$w_g^* = \Sigma_{\alpha'}\Sigma_{k',l'}\overline{(\pi_{\alpha'}(g))_{k',l'}}\sigma_g(v_{l'}^{\alpha'})v_{k'}^{\alpha'^*}$$

Any element $x \in \mathcal{B}$ can be expressed in the form

$$x = \Sigma_{\beta,\beta'}\Sigma_{i,i'}v_i^\beta x_{ii'}^{\beta\beta'} v_{i'}^{\beta'^*},$$

for some $x_{ii'}^{\beta\beta'} \in \mathcal{B}$. For such x we have:

$$\sigma_g(x) = \Sigma_{\beta,\beta'}\Sigma_{i,i'}\sigma_g(v_i^\beta)\sigma_g(x_{ii'}^{\beta\beta'})\sigma_g(v_{i'}^{\beta'^*})$$

By taking into account that $v_l^{\alpha*}v_i^\beta = \delta_{\alpha\beta}\delta_{li}$ and $v_{i'}^{\beta'^*} v_{l'}^{\alpha'} = \delta_{\beta'\alpha'}\delta_{i'l'}$ it follows that $\sigma_g(v_l^{\alpha*})\sigma_g(v_i^\beta) = \delta_{\alpha\beta}\delta_{li}$ and $\sigma_g(v_{i'}^{\beta'^*})\sigma_g(v_{l'}^{\alpha'}) = \delta_{\beta'\alpha'}\delta_{i'l'}$ as well. Thus we have:

$$w_g\sigma_g(x)w_g^* = \Sigma_{\beta,\beta'}\Sigma_{k,k'}\Sigma_{i,i'}(\pi_\alpha(g))_{ki}\overline{(\pi_{\beta'}(g)_{k'i'}}v_k^\beta\sigma_g(x_{ii'}^{\beta\beta'})v_{k'}^{\beta'^*} \tag{3.2.2}$$

The equation $w_g\sigma_g(x)w_g^* = x$ is then equivalent to the set of equations

$$\Sigma_{i,i'}(\pi_\alpha(g))_{ki}\overline{(\pi_{\beta'}(g)_{k'i'}}v_k^\beta\sigma_g(x_{ii'}^{\beta\beta'})v_{k'}^{\beta'^*} = x_{kk'}^{\beta\beta'}, \forall i,i',\beta,\beta' \tag{3.2.3}$$

Letting $y_{ii'}^{\beta\beta'} = \sigma_g(x_{ii'}^{\beta\beta'})$ and $x_{ii'}^{\beta\beta'} = \sigma_{g^{-1}}(y_{ii'}^{\beta\beta'})$, this shows in particular that the finite dimensional space $\text{sp}\{y_{ii'}^{\beta\beta'} \mid 1 \leq i \leq n_\beta, 1 \leq i' \leq n_{\beta'}\}$ is invariant to $\sigma_{g^{-1}}, \forall g \in G$, thus to $\sigma_h, \forall h \in G$. Since σ is strongly mixing on $\mathcal{N}$, by 2.4.3 this implies $y_{ii'}^{\beta\beta'}$ are all scalars. Thus, $x_{ii'}^{\beta\beta'}$ are scalars as well implying that x lies in $\mathcal{B}$. By part 2° it follows that $x \in \mathcal{B}^{\sigma'} = \pi(G)' \cap \mathcal{B}$. Restricting to elements $x \in pNp$, we also get $(pNp)^{\sigma'} = B^{\sigma'} = \pi(G)' \cap B$.

Let now $\mathcal{H}_0 \subset p\mathcal{N}p$ be a finite dimensional vector subspace invariant to σ'. Since the projection $p_\beta\Sigma_i v_i^\beta v_i^{\beta^*}$ are fixed by $\sigma', \forall\beta$, it follows that

$$\mathcal{X} = p_\beta\mathcal{H}_0p_{\beta'} = \Sigma_{ii'}v_i^\beta\mathcal{X}_{ii'}^{\beta\beta'}v_{i'}^{\beta'^*} \tag{3.2.4}$$

is invariant to σ' as well, where $\mathcal{X}_{ii'}^{\beta\beta'} = v_i^{\beta^*}\mathcal{H}_0v_{i'}^{\beta'}$.

But the calculation (3.2.2) above shows that

$$w_g\sigma_g(\mathcal{X})w_g^* = \Sigma_{k,k'}\Sigma_{i,i'}(\pi_\alpha(g))_{ki}\overline{(\pi_{\beta'}(g)_{k'i'}}v_k^\beta\sigma_g(\mathcal{X}_{ii'}^{\beta\beta'})v_{k'}^{\beta'^*} \tag{3.2.5}$$

Equating (3.2.4) with (3.2.5) and letting $\mathcal{Y}_{ii'}^{\beta\beta'} = \sigma_g(\mathcal{X}_{ii'}^{\beta\beta'})$, $\mathcal{X}_{ii'}^{\beta\beta'} = \sigma_{g^{-1}}(\mathcal{Y}_{ii'}^{\beta\beta'})$, it follows that the finite dimensional vector space $\mathrm{sp}\{\mathcal{Y}_{ii'}^{\beta\beta'} \mid 1 \leq i \leq n_\beta, 1 \leq i' \leq n_{\beta'}\}$ is invariant to $\sigma_{g^{-1}}, \forall g \in G$, thus to $\sigma_h, \forall h \in G$.

By 2.4.3 it follows again that all the spaces $\mathcal{Y}_{ii'}^{\beta\beta'}$ are equal to $\mathbb{C}$. Thus $\mathcal{X}_{ii'}^{\beta\beta'} = \mathbb{C}$ as well, implying that $\mathcal{X} = \Sigma_{i,i'}\mathbb{C}v_i^\beta v_{i'}^{\beta'^*}$. This proves that $\mathcal{H}_0 \subset \mathcal{B}$.

Since $p_\beta \mathcal{B} p_{\beta'}$ are all invariant subspaces and they span $\mathcal{B}$, the last part of the statement follows as well.

4°. Note first that the equivalence class of the cocycle w does not depend on the choice of the isometries v_i^β, once the family of representations π_β is fixed.

If the two given families of representations satisfy $\pi_\beta \sim \pi'_\beta, \forall \beta \in H_1$, then in particular $\dim(\pi_\beta) = \dim(\pi'_\beta), \forall\beta$. It follows that in the contruction of w and w' we can take the same set of isometries $\{v_i^\beta\}_{\beta,i}$. Thus, both π_β, π'_β "live" in $p_\beta B p_\beta$, where $p_\beta = \Sigma_i v_i^\beta v_i^{\beta^*}$ as usual.

If for each $\beta \in H_1$ with $n_\beta \neq 0$ we take $u_\beta \in p_\beta B p_\beta$ to be the unitary element that implements the equivalence of π_β, π'_β then an immediate calculation shows that $u = \oplus_\beta u_\beta$ also implements the equivalence of w with w'.

Note that this same computation shows that the converse also holds true, provided $\dim(\pi_\beta) = \dim(\pi'_\beta)$. But this equality does hold true if we assume σ is strongly ergodic, by the last part of 3°. Indeed, because by that part we have that the fixed point algebras B, respectively B', of the actions $\mathrm{Ad}(w)\circ\sigma$ and respectively $\mathrm{Ad}(w') \circ \sigma$, are conjugate in pNp. Since the traces of the minimal projections in $p_\beta B p_\beta$ and $p'_\beta B' p'_\beta$ must both be equal to β, thus being different for distinct β, it follows that the the unitary element $u \in pNp$ that satisfies $uBu^* = B'$ must carry projections of trace β onto projections of trace β, implying that $up_\beta B p_\beta u^* = p'_\beta B' p'_\beta$ and thus $n_\beta = n'_\beta, \forall \beta$. Q.E.D.

An immediate consequence of Theorem 3.2 is that any cocycle action constructed by reduction-amplification of an action σ as in 3.2, but with the factor $\mathcal{N}$ being of type II, can in fact be perturbed to an action. We will later see that in case $\mathcal{N}$ is of type II_1, then this may not be the case.

3.3 Corollary. *With σ as in* 3.2 *above, assume in addition that $H_1 \neq \{1\}$ (equivalently, the factor $(\mathcal{N}, \varphi)$ is of type III). For each $t > 0$, let σ^t be a choice of a cocycle action on N^t obtained by reducing, or amplifying the action σ on $N = \mathcal{N}_\varphi$, as in* 3.1.. *Then σ^t can be perturbed to an action.*

Proof. Let m be an integer such that $m \geq t$. Since $H_1 \neq \{1\}$, it contains some $\beta < 1$. Thus, $\beta^n \in H_1, \forall n \geq 1$ as well. But then there exist some non-negative integers k_n such that $\Sigma_n k_n \beta^n = t/m$ and we can apply Theorem 3.2 to the action

$\sigma \otimes id_m$ of G on $N \otimes M_{m\times m}(\mathbb{C})$ (which is the core of the discrete decomposition of $(\mathcal{N} \otimes M_{m\times m}(\mathbb{C}), \varphi \otimes tr))$, to provide a generalized 1-cocycle w for σ with support p, where $p \in N \otimes M_{m\times m}(\mathbb{C})$ is a projection of trace t/m. Q.E.D.

Our next result gives an abstract characterization of the cocycles w constructed in the previous Theorem. Namely, we show that w is locally of the form (3.2.1) iff the projective representation $\xi \mapsto \sigma_g(\xi)w_g^*$ has finite dimensional invariant subspaces. This observation will be needed in the next Section.

3.4. Proposition. *With the same notations and hypothesis as in* 3.2, *assume the action* $\sigma : G \to \mathrm{Aut}(\mathcal{N}, \varphi)$ *is strongly mixing. Let* w *be a generalized weak 1-cocycle for* σ *with support* $p \in \mathcal{P}(N)$ *and scalar 2-cocycle* μ.

1°. *For* $g \in G$ *and* $\xi \in L^2(\mathcal{N}p, \varphi)$ *denote* $\sigma_g^w(\xi) = \sigma_g(\xi)w_g^*$. *Then* σ^w *is a projective unitary representation of* G *on* $L^2(\mathcal{N}p, \varphi)$ *with scalar 2-cocycle* μ.

2°. *Let* $\mathcal{H}$ *be the Hilbert space of Hilbert-Schmidt operators on* $L^2(\mathcal{N}p, \varphi)$. *For each* $X \in \mathcal{H}$ *let* $\tilde{\sigma}_g^w(X) = \sigma_g^w X \sigma_g^{w*}$. *Then* $\tilde{\sigma}^w$ *is a unitary representation of* G *on* $\mathcal{H}$.

3°. *The following conditions are equivalent:*

(i). $\tilde{\sigma}^w$ *contains a copy of the trivial representation.*

(ii). σ^w *has a non-trivial, invariant, finite dimensional subspace* $\mathcal{H}_0 \subset L^2(\mathcal{N}p, \varphi)$.

(iii). There exist some $\beta \in H_1$ *and a finite set of isometries* $v_1, v_2, ..., v_n \in \mathcal{V}_\beta$ *with mutually orthogonal left supports such that* $\mathcal{H}_0 = \Sigma_i \mathbb{C} v_i^*$ *is invariant to* σ^w *and* $q = \Sigma_i v_i v_i^* \leq p$ *is fixed by the action* $\sigma' = \mathrm{Ad}w \circ \sigma$.

(iv). There exists a non-zero projection $q \in pNp$ *fixed by* σ' *such that* qw *is of the form* $qw_g = \Sigma_{i,j}(\pi_0(g))_{ij} v_i \sigma_g(v_j{}^*)), g \in G$, *for some* $\pi_0 \in \mathcal{F}_\mu$ *and some isometries* $v_i, 1 \leq i \leq \dim(\pi_\beta)$, *lying all in some* $\mathcal{V}_\beta$ *and having mutually orthogonal left supports.*

(v). There exists a non-zero projection $q_0 \in pNp$ *fixed by* σ' *such that* $q_0 w$ *is of the form* (3.2.1), *for some family* $\{\pi_\beta\}_\beta \in \mathcal{F}_\mu^{H_1}$, *and such that there are no non-zero projections* $q \leq (p - q_0), q \in B_0$ *with the property that* qw *is of the form* (3.2.1).

Proof. 1°. By the definitions, σ_g^w are clearly unitary operators acting on $L^2(\mathcal{N}p, \varphi)$. Also, we have

$$\sigma_g^w(\sigma_h^w(\xi)) = \sigma_g^w(\sigma_h(\xi)w_h^*)$$
$$= \sigma_g(\sigma_h(\xi)w_h^*)w_g^* = \sigma_{gh}(\xi)\sigma_g(w_h^*)w_g^*$$
$$= \mu_{g,h}\sigma_{gh}(\xi)w_{gh}^*.$$

proving 1°.

2°. By 1° it follows that

$$\tilde{\sigma}_g^w(\tilde{\sigma}_h^w(X)) = \sigma_g^w \sigma_h^w X \sigma_h^{w*} \sigma_g^{w*}$$

$$= \mu_{g,h}\overline{\mu_{g,h}}\sigma^w_{gh}X\sigma^{w}_{gh}{}^* = \tilde{\sigma}^w_{gh}(X).$$

3°. Noticing that $\tilde{\sigma}^w$ extends from the ideal of Hilbert-Schmidt operators to all $\mathcal{B}(L^2(\mathcal{N}p,\varphi))$, it follows that if $X \in \mathcal{H}$ is a fixed point for $\tilde{\sigma}^w$ then the trace class operator X^*X is also a fixed point for $\tilde{\sigma}^w$. Thus all spectral projections P of X^*X, are fixed points for $\tilde{\sigma}^w$ as well. Since P have finite trace, they are finite dimensional. Let $\mathcal{H}_0 \subset L^2(\mathcal{N}p,\varphi)$ be a non-zero finite dimensional space corresponding to some spectral projection $P \neq 0$. By the definitions, $\tilde{\sigma}^w_g(P)$ is the orthogonal projection onto $\sigma^w_g(\mathcal{H}_0)$. Since $\tilde{\sigma}^w_g(P) = P$, it follows that $\sigma^w_g(\mathcal{H}_0) = \mathcal{H}_0$. This proves $(i) \implies (ii)$.

To prove $(ii) \implies (iii)$, note first that if $v \in \mathcal{V}$ then each of the subspaces $L^2(v^*Np,\varphi)$ and $L^2(Nvp,\varphi)$ of $L^2(\mathcal{N},\varphi)$ are invariant to σ^w. Thus, if σ^w has a non-trivial finite dimensional invariant subspace $\mathcal{H}_0$ then by compressing it to one of these spaces and taking into account that they span all $L^2(\mathcal{N}p,\varphi)$ it follows that we may assume $\mathcal{H}_0$ is contained either in some $L^2(v^*Np,\varphi)$ or in some $L^2(Nvp,\varphi)$.

In either case, denote by $\{\xi_i\}_i$ an orthonormal basis of $\mathcal{H}_0$. By the definition of σ^w it follows that $\Sigma_i\xi_i\xi_i^* \in L^1(\mathcal{N},\varphi)$ is fixed by σ. Since σ is strongly mixing, it is ergodic. Thus $\Sigma_i\xi_i\xi_i^* \in \mathbb{C}1$ implying that ξ_i are actually in $\mathcal{N}p$.

Similarily, the finite dimensional vector space $\mathcal{H}_0' = \text{sp}\mathcal{H}_0\mathcal{H}_0^* \subset \mathcal{N}$ is invariant to σ. It follows that $\mathcal{H}_0' = \mathbb{C}$ implying that $\xi_i\xi_j^* \in \mathbb{C}1, \forall i,j$. This in turn implies that each ξ_i^* is a scalar multiple of an isometry v_i and that $A = \text{sp}\{v_iv_j^* \mid 1 \leq i,j \leq n\}$ is a finite dimensional factor. Thus, by replacing if necessary the elements $\{v_i\}_i$ by some elements $\{a_jv_j\}_j$ for some appropriate partial isometries $a_j \in A$, we may assume the isometries v_i have mutually orthogonal left supports, yet still generate $\mathcal{H}_0$. Since all elements in $L^2(Nvp,\varphi)$ have left supports $\neq 1$ while the left support of $\xi_i = v_i^*$ is 1, it follows that $\mathcal{H}_0$ cannot be a subspace of some $L^2(Nvp,\varphi)$, forcing it to be a susbspace of some $L^2(v^*Np,\varphi), v \in \mathcal{V}$. This implies $v_i \in \mathcal{V}_\beta, \forall i$, where $\beta = \varphi(vv^*) \in H_1$. Moreover, since $\text{sp}\mathcal{H}_0^*\mathcal{H}_0$ is invariant to

$$p\mathcal{N}p \ni x \mapsto w_g\sigma_g(x)w_g^* = \sigma_g'(x)$$

and the support projection q of the elements in this vector space is $\Sigma_iv_iv_i^*$, it follows that $\sigma_g'(q) = q, \forall g \in G$.

Assuming (iii) holds true, let $\beta \in H_1$ and $v_1, v_2, ..., v_n \in \mathcal{V}_\beta$ be such that $\mathcal{H}_0 = \Sigma_i\mathbb{C}v_i^*$ is invariant to σ^w. It follows that for each $1 \leq i \leq n$ and $g \in G$ there exist some scalars $\{b^g_{ij}\}_{i,j}$ such that $\sigma_g(v_i^*)w_g^* = \Sigma_jb^g_{ij}v_j^*, \forall i$.

Denote by $c^g_{ij} = \overline{b^g_{ji}}$. We have to prove that c^g_{ij} are the coefficients of a projective unitary representation of G with the same scalar 2-cocycle μ as w.

Since $\sigma_g(v_i^*)w_g^* = \Sigma_j b_{ij}^g v_j^*, \forall i$, we have $qw_g = \Sigma_{i,j} c_{ji}^g v_j \sigma_g(v_i^*)$. Similarily $qw_h = \Sigma_{i,j} c_{ji}^h v_j \sigma_h(v_i^*)$. It follows that

$$w_g\sigma_g(w_h) = (\Sigma_{i,j} c_{ji}^g v_j \sigma_g(v_i^*))(\Sigma_{k,l} c_{lk}^h \sigma_g(v_l)\sigma_{gh}(v_k^*))$$
$$= \Sigma_{j,k}(\Sigma_i c_{ji}^g c_{ik}^h) v_j \sigma_{gh}(v_k^*).$$

Smilarily,we have

$$w_{gh} = \Sigma_{j,k} c_{jk}^{gh} v_j \sigma_{gh}(v_k^*).$$

Replacing the above in the equation $w_g\sigma_h(w_g) = \mu_{g,h} w_{gh}$ and multiplying on the left by $v_j v_j^*$ and to the right by $v_k v_k^*$, we get

$$\Sigma_i c_{ji}^g c_{ik}^h = \mu_{g,h} c_{j,k}^{gh}, \forall j, k, g.$$

But this means c_{ij}^g are the coefficients of a projective representation π_0 of G on $\ell^2(n)$ with 2-cocycle μ.

To prove $(iv) \implies (v)$ we use a maximality argument. Thus, we let $\mathcal{S}$ denote the set of families of mutually orthogonal projections $(q^i)_i \subset pNp$ such that $q^i \in (pNp)^{\sigma'}$ and $q^i w$ is of the form (3.2.1). We endow $\mathcal{S}$ with the obvious order $\leq$ given by inclusion. $(\mathcal{S}, \leq)$ this way defined is clearly inductively ordered. Let $(q_0^i)_i$ be a maximal element, which by (iv) we may suppose non-zero. Let $q_0 = \Sigma_i q_0^i \in (pNp)^{\sigma'}$. Then q_0 clearly satisfies the required conditions, or else the maximality of $(q_i^0)_i$ would be contradicted.

The implication $(v) \implies (iv)$ is trivial. Finally, $(iv) \implies (i)$ follows by taking $P \in \mathcal{H}$ to be the orthogonal projection of $L^2(\mathcal{N}p, \varphi)$ onto $\mathcal{H}_0 = \Sigma_i \mathbb{C} v_i^*$. Indeed, because by hypothesis we have

$$\sigma_g(v_i^*)w_g^* = \Sigma_j \overline{(\pi_0(g))_{ij}} v_j^*$$

showing that $\sigma_g^w(\mathcal{H}_0) = \mathcal{H}_0$, equivalently $\tilde{\sigma}_g^w(P) = P, \forall g \in G$. Q.E.D.

4. Cohomology of Connes-Størmer Bernoulli shifts.

Let $N_0 = M_{k\times k}(\mathbb{C})$ and φ_0 be a faithful state on N_0. Let G be an infinite property T group. Let $(\mathcal{N}, \varphi) = \overline{\bigotimes_{g\in G}}(N_g, \varphi_g)$, where $(N_g, \varphi_g) = (N_0, \varphi_0), \forall g \in G$. As explained in 2.2.0, the ITPF1 factor $(\mathcal{N}, \varphi)$ has a discrete decomposition over $N = \mathcal{N}_\varphi$, with the type II$_1$ factor N being isomorphic to the hyperfinite factor.

We denote by σ both the (N_0, φ_0)-Bernoulli shift of G on $(\mathcal{N}, \varphi)$ and the Connes-Størmer Bernoulli shift obtained by restricting this action to N. In Theorem 3.2 of the previous section we constructed a family of generalized weak 1-cocycles for such Connes-Størmer Bernoulli shifts σ. In this section we prove that the cocycles in 3.2 give a complete list of cocycles for σ. The notations $\mathcal{V}, \mathcal{V}_\beta, H_1, \mathcal{F}_\mu, \mathcal{F}_\mu^{H_1}$ are the ones used in 2.2.0 and 3.2.

4.1. Theorem. *Let w be a generalized weak 1-cocycles of the Connes-Størmer Bernoulli shift σ, with support p and 2-cocycle μ. Let B_0 be the fixed point algebra of the action $\sigma'_g = \mathrm{Ad}(w_g) \circ \sigma_g, g \in G$, on pNp. Then we have:*

1°. If $z \in B_0$ is the maximal central projection of B_0 such that $B_0 z$ is atomic, then there exist a family of finite dimensional, projective, unitary representations $\{\pi_\beta \mid \beta \in H_1\} \subset \mathcal{F}_\mu^{H_1}$ of the property T group G and some isometries $\{v_i^\beta \mid 1 \leq i \leq \dim\pi_\beta\} \subset \mathcal{V}_\beta, \beta \in H_1$, such that $\Sigma_{\beta,i} v_i^\beta v_i^{\beta *} = z$ and such that the generalized weak 1-cocycle zw is given by the formula:

$$zw_g = \Sigma_\beta(\Sigma_{i,j}(\pi_\beta(g))_{ij} v_i^\beta \sigma_g(v_j^{\beta^*})), g \in G \tag{4.1.1}$$

2°. Assume in addition that $(N_0, \varphi_0) = \overset{k}{\underset{i=1}{\otimes}} (N_{0,i}, \varphi_{0,i})$, where $N_{0,i} \simeq M_{2\times 2}(\mathbb{C})$ and $\varphi_{0,i}$ are faithful states on $N_{0,i}$, $1 \leq i \leq k$. Then B_0 is atomic, i.e., $z = 1_{B_o} = p$, so the above formula (4.1.1) *holds true for the given cocycle $w(= pw)$ itself.*

To prove the theorem we first need some lemmas. To state them, recall some notations from 2.2.1-2.2.2. Thus, we consider the generalized $(N_0 \otimes N_0, \varphi_0 \otimes \varphi_0)$ -Bernoulli shift of G on $\mathcal{N}\overline{\otimes}\mathcal{N}$ which we identify in the obvious way with $\sigma \otimes \sigma$. We denote by θ the associated Connes-Størmer Bernoulli shift of G on the centralizer $\tilde{N}$ of $\tilde{\varphi} = \varphi \otimes \varphi$ on $\mathcal{N}\overline{\otimes}\mathcal{N}$. Thus, $\theta_g = (\sigma_g \otimes \sigma_g)_{|\tilde{N}}, g \in G$.

We denote by $\alpha : \mathbb{R} \to \mathrm{Aut}\tilde{N}$ the product type action considered in 1∘-3∘ of 2.5.1. Note that α commutes with the Bernoulli shift action θ defined above. Thus, θ and α implement an action $\theta \oplus \alpha$ of the group $G \oplus \mathbb{R}$ on $\tilde{N}$.

For the next two lemmas, the discrete group G can be arbitrary. Other than that, we are under the general assumptions of 4.1.

4.2. Lemma. *Let M denote the type II_1 factor obtained by taking the cross product of $\tilde{N}$ by the action $\theta \oplus \alpha$ of the group $\mathcal{G} = G \oplus \mathbb{R}$ on it, in which $\mathbb{R}$ is regarded as a discrete group. Let $(U_h)_{h\in\mathcal{G}} \subset M$ be the canonical unitaries implementing this action. Let also $U'_g = w_g U_g, g \in G$, and $P = \{U'_g\}''_{g\in G} \subset pMp$ be the factor with support $1_P = p$ they generate in pMp. Fix $x \in N = N \otimes \mathbb{C}$. For each $t \in \mathbb{R}$ put $x_t \overset{\text{def}}{=} E_{P'\cap pMp}(pxU_t p)U_t^*$. Then we have:*

1°. x_t belongs to $\tilde{N}$ being in fact the unique element of minimal norm-2 in the weakly closed convex subset of $\tilde{N}$

$$K'_t(x) = \overline{\mathrm{co}}^{\mathrm{w}}\{U'_g x U_t U'^*_g U_t^*\}_{g\in G} = \overline{\mathrm{co}}^{\mathrm{w}}\{w_g \sigma_g(x) \alpha_t(w_g^*)\}_{g\in G}.$$

2°. x_t satisfies the equivalent conditions

$$U'_g x_t = x_t \mathrm{Ad}U_t(U'_g), \forall g \in G \tag{a}$$

(b) $$w_g\theta_g(x_t) = x_t\alpha_t(w_g), \forall g \in G$$

Also, x_t is the unique element in $K_t'(x)$ that satisfies these equivalent conditions.

3°. $x_tx_t^* \in B_0 \otimes \mathbb{C}$ *and* $x_t^*x_t \in \alpha_t(B_0 \otimes \mathbb{C})$.

4°. $p_{t^*} = \alpha_{-t}(p_t)^*$ *and* $(bx)_t = b(x_t), \forall b \in B_0$.

Proof. Since the set $\{U_g'\}_{g\in G}$ is total in P, $E_{P'\cap pMp}(xU_t)$ is the unique element of minimal norm-2 in $K_t(x) = \overline{\text{co}}^{\text{w}}\{U_g'xU_tU_g'^*\}_{g\in G}$ (see e.g., [**Po4**]). But $U_g'xU_tU_g'^* = w_g\sigma_g(x)U_tw_g^* = w_g\sigma_g(x)\alpha_t(w_g^*)U_t$, implying that $K_t'(x) = K_t(x)U_t^*$ and that $x_t = E_{P'\cap pMp}(xU_t)U_t^*$ is the unique element of minimal norm-2 in the set $K_t'(x)$. This proves 1°.

The commutation relation $U_g'(x_tU_t) = x_tU_tU_g'$, which holds true for all $g \in G$, is equivalent to the condition $U_g'x_t = x_t\text{Ad}U_t(U_g')$ (by multiplying the former to the right by U_t^*). This shows that (a) in 2° holds true. But condition (a) amounts to $w_gU_gx_t = x_tU_tw_gU_gU_t^*$, which multiplied from the right by U_g^* gives condition (b) (after apropriate simplifications).

Moreover, since

$$uE_{P'\cap pMp}(y)u^* = E_{P'\cap pMp}(uyu^*), \forall u \in P, y \in pMp,$$

it follows that $E_{P'\cap pMp}(y) = E_{P'\cap pMp}(pxU_tp), \forall y \in K_t(x)$, so that x_tU_t is the unique element in $K_t(x) = K_t'(x)U_t$ that commutes with all $U_g', g \in G$. By the equivalence between the commutation relation and the conditions (a) and (b), this proves the uniqueness in 2°.

Since by the way it is defined the element x_tU_t commutes with the *-algebra P, 2.4.2 implies that

$$x_tx_t^* = (x_tU_t)(x_tU_t)^* \in P' \cap p\tilde{N}p$$
$$= (p\tilde{N}p)^{\theta'} = (pNp)^{\sigma'} \otimes \mathbb{C} = B_0 \otimes \mathbb{C},$$

where θ' is the action of G on $p\tilde{N}p$ given by $\theta_g' = \sigma_g' \otimes \sigma_g, g \in G$.

Similarly, we get

$$\alpha_{-t}(x_t^*x_t) = (x_tU_t)^*(x_tU_t) \in P' \cap p\tilde{N}p = B_0 \otimes \mathbb{C},$$

implying that $x_t^*x_t \in \alpha_t(B_0 \otimes \mathbb{C})$. This proves 3°.

Taking $-t$ for t, by the definitions we get $p_{-t}U_{-t} = E_{P'\cap pMp}(pU_{-t}p)$. Thus

$$U_tp_{-t}^*U_t^* = E_{P'\cap pMp}(pU_tp)U_{-t} = p_t,$$

showing that $\alpha_t(p_{-t})^* = p_t$, which by taking adjoints and applying $(\alpha_t)^{-1} = \alpha_{-t}$ gives the first part of 4°. The second part is trivial by the definitions. Q.E.D.

4.3. Lemma. *Let $u(t) = (p_t p_t^*)^{-1/2} p_t \in \tilde{N}$ denote the partial isometry in the polar decomposition of p_t and denote by $l(t) = u(t)u(t)^*, r(t) = u(t)^*u(t)$ its left support and right support. Then we have:*

1°. $U'_g u(t) = u(t) U_t U'_g U_t^*, \forall g \in G$.

2°. $u(-t) = \alpha_{-t}(u(t))^*$.

3°. $l(t) \in B_0$, $r(t) \in \alpha_t(B_0)$ *and* $u(t)^* B_0 u(t) = r(t)\alpha_t(B_0)r(t)$.

Proof. By multiplying the relation 4.2.2° by $(p_t p_t^*)^{-1/2} \in B_0$ from the left, we get 2°.

Since by 4.2.4° we have $p_{-t} = \alpha_{-t}(p_t)^*$, by the definitions of $u(t)$ and $u(-t)$ we immediately get $u(-t) = \alpha_{-t}(u(t))^*$. This proves 2°.

Taking $(p_t p_t^*)^{-1/2}$ to be the element b in 4.2.4°, by 4.2.3∘ and the definitions we get $l(t) = u(t)u(t)^* = b_t b_t^* \in B_0$ and $r(t) = u(t)^*u(t) = b_t^* b_t \in \alpha_t(B_0)$.

Similarily, if we let q be an arbitrary projection in $l(t)B_0 l(t)$ and take this time $q(p_t p_t^*)^{-1/2}$ to be the element b in 4.2.4∘, then we get $u(t)^* q u(t) = b_t^* b_t \in \alpha_t(B_0)$. Since any element in B can be approximated in uniform norm by a linear combination of projections, it follows that $u(t)^* B_0 u(t) \subset r(t)\alpha_t(B_0)r(t)$, proving one inclusion in 1°.

Taking $-t$ for t in this inclusion, we also have $u(-t)B_0 u(-t)^* \subset r(-t)\alpha_{-t}(B_0)r(-t)$. Applying $\alpha(t)$ on both sides we get:

$$\alpha_t(u(-t))\alpha_t(B_0)\alpha_t(u(-t)^*) \subset \alpha_t(r(-t))B_0\alpha_t(r(-t)).$$

But by part 2∘ we have $\alpha_t(u(-t)) = u(t)^*$ and $l(t) = l(\alpha_t(u(-t))^*) = \alpha_t(r(-t))$, so the above implies

$$u(t)^*\alpha_t(B_0)u(t) \subset l(t)B_0 l(t).$$

Equivalently,

$$\alpha_t(B_0) \subset u(t)^* B_0 u(t).$$

This proves the opposite inclusion in 3°. Q.E.D.

For the next three lemmas we will assume that the group G has the property T, as in the hypothesis of 4.1. All other assumptions and notations are as in 4.1-4.3.

4.4. Lemma. *For each $b \in B_0$, the function $t \mapsto b_t$ from $\mathbb{R}$ into the unit ball of $\tilde{N} \subset M$ is continuous at $t = 0$, with respect to the norm-2 topology. Moreover, $\lim_{t \to 0} \|u(t) - p\|_2 = 0$.*

Proof. By 4.2.4°, to prove the continuity at $t = 0$ of the function $t \mapsto b_t$ it is sufficient to prove it for the function $t \mapsto p_t$.

Since $t \mapsto \alpha_t$ is pointwise continuous in the norm-2 topology, there exists $\varepsilon_1 > 0$ such that if $t \in \mathbb{R}$ satisfies $|t| \leq \varepsilon_1$ then

$$\|w_{g_i}\alpha_t(w_{g_i})^* - p\alpha_t(p)\|_2 \leq \varepsilon_0\|p - p\alpha_t(p)\|_2, \forall 1 \leq i \leq n.$$

Consider the representation π of G on $L^2(pMp)$ given by $\pi(g)\xi = U'_g\xi U'^*_g$. Then for $t \in \mathbb{R}$ and $\xi = pU_tp$, we have

$$\|\pi(g)\xi - \xi\| = \|U'_gU_tU'^*_g - pU_tp\|_2$$
$$= \|w_gU_vw_g^* - pU_tp\|_2 = \|w_g\alpha_t(w_g)^* - p\alpha_t(p)\|_2, \forall g \in G.$$

Applying this to $g = g_i, i = 1, 2, ..., n$, we get

$$\|\pi(g_i)\xi - \xi\| = \|w_{g_i}\alpha_t(w_{g_i})^* - p\alpha_t(p)\|_2.$$

By Lemma A.1 it thus follows that

$$\|w_g\alpha_t(w_g)^* - p\alpha_t(p)\|_2 \leq K\mathrm{max}_i\|w_{g_i}\alpha_t(w_{g_i})^* - p\alpha_t(p)\|_2, \forall g \in G.$$

By part 2° it follows that

$$\|p_t - p\alpha_t(p)\|_2 \leq K\mathrm{max}_i\|w_{g_i}\alpha_t(w_{g_i})^* - p\alpha_t(p)\|_2$$

forall $t \in \mathbb{R}$ with $|t| \leq \varepsilon_1$. In particular, since $p_0 = p$ and $\|p\alpha_t(p) - p\|_2 \to 0$ as $t \to 0$, this implies the continuity at $t = 0$ of the function p_t, finishing the proof of the statement.

The continuity of $u(t)$ at $t = 0$ follows trivially from the first part and 4.3. Q.E.D.

4.5. Lemma. *Let $z_0 \in B_0$ be a central projection of B_0 such that B_0z_0 is a finite dimensional algebra. Then there exists $\varepsilon > 0$ such that if $|t| < \varepsilon$ then $z_0 \leq u(t)u(t)^*$ and $u(t)^*B_0z_0u(t) = \alpha_t(B_0z_0)$.*

Proof. By Lemma 4.4 we have $\lim_{t\to 0} u(t)u(t)^* = 1_{B_0} = p$. Since z_0 is central in B_0 and B_0z_0 is a finite von Neumann algebra, it follows that for $|t|$ small enough we have $z_0 \leq u(t)u(t)^*$.

Let $c > 0$ be the minimal trace of a non-zero projection in B_0z_0. Note that for $|t|$ small enough, $\alpha_t(z_0)$ is close to z_0. By 4.4, $u(t)^*z_0u(t)$ is also close to z_0. Thus, by 4.3 it follows that $\alpha_t(z_0)$ and $u(t)^*z_0u(t)$ are projections of $\alpha_t(B_0)$ which are close one to another for $|t|$ small. Moreover, $\alpha_t(z_0)$ is central in $\alpha_t(B_0)$.

Since the minimal trace of a non-zero projection in both

$$u(t)^*(B_0z_0)u(t) = u(t)^*z_0u(t)\alpha_t(B_0)u(t)^*z_0u(t)$$

and in

$$\alpha_t(B_0z_0) = \alpha_t(B_0)\alpha_t(z_0)$$

is c, it follows that if $\|u(t)^*z_0u(t) - \alpha_t(z_0)\|_2^2 < c$ then $u(t)^*z_0u(t) = \alpha_t(z_0)$. Thus, for $|t|$ small enough we have $u(t)^*B_0z_0u(t) = \alpha_t(B_0z_0)$. Q.E.D.

4.6. Lemma. *Let $z \in B_0$ be the maximal atomic projection of B_0, as in the statement of* Theorem 4.1.. *There exists a partial isometry $u \in \tilde{N}$ such that $uu^* = z \in B_0$, $u^*u = \alpha_1(z) \in \alpha_1(B_0)$ and*

$$(w_g \otimes 1)U_g u = u(1 \otimes w_g)U_g, \forall g \in G.$$

Proof. Let $\{z_i\}_i$ be a partition of z with central projections in B_0 such that each $B_0 z_i$ is finite dimensional. It is suficient to prove that for each z_i there exists a partial isometry $u_i \in \tilde{N}$ such that $z_i = u_i u_i^*$ and $(w_g \otimes 1)U_g u_i = u_i(1 \otimes w_g)U_g, \forall g \in G$. Indeed, because then we can just define $u = \oplus_i u_i$ and use the fact that for this u we have $q_i(w_g \otimes 1)U_g u = (w_g \otimes 1)U_g u_i$. Summing up this gives $(w_g \otimes 1)U_g u = u(1 \otimes w_g)U_g$.

This shows that by replacing w by $z_i w$ and B_0 by $B_0 z_i$ we may suppose B_0 is finite dimensional. Denote $z_0 = 1_{B_0}$. By 4.5 it follows that there exists $n \geq 1$ such that $u(t_n)^* z_0 u(t_n) = \alpha_{t_n}(z_0)$, where $t_n = 1/2^n$.

By applying recursively $(\alpha_{t_n})^k$ to this equation, for $k = 1, 2, ..., n-1$, it follows that the element

$$u = u(t_n)(\alpha_{t_n}(u_{t_n}))(\alpha_{t_n}^2(u(t_n)))...(\alpha_{t_n}^{n-1}((u(t_n)))$$

is a partial isometry in $\tilde{N}$ which satisfies $u^* z_0 u = \alpha_{t_n}^n(z_0) = \alpha_1(z_0)$. In particular, $uu^* = z_0$.

Moreover, by multiplying the relation

$$U'_g u(t_n) = u(t_n) U_{t_n} U'_g U_{t_n}^*, \forall g \in G$$

(which holds true by 4.3.1°) from the right by $\alpha_{t_n}(u(t_n))$, it follows that

$$U'_g(u(t_n)\alpha_{t_n}(u(t_n))) = u(t_n)U_{t_n}U'_g U_{t_n}^* \alpha_{t_n}(u(t_n))$$
$$= u(t_n)U_{t_n}U'_g U_{t_n}^*(U_{t_n}u(t_n)U_{t_n}^*) = u(t_n)U_{t_n}(U'_g u(t_n))U_{t_n}^*$$
$$= u(t_n)U_{t_n}(u(t_n)U_{t_n}U'_g U_{t_n}^*)U_{t_n}^*$$
$$= (u(t_n)\alpha_{t_n}(u(t_n)))(U_{t_n}^2 U'_g U_{t_n}^{2*})$$

Recursively, we obtain that $u = \prod_{k=0}^{n-1} \alpha_{t_n}^k$ satisfies $U'_g u = u(U_{t_n}^n)U'_g(U_{t_n}^n)^*$, which by taking into account that $U_{t_n}^n = U_1$ gives $U'_g u = uU_1 U'_g U_1^*$.

Since $U'_g = (w_g \otimes 1)U_g$ and $U_1(w_g \otimes 1)U_1^* = 1 \otimes w_g$, from this last relation we get $(w_g \otimes 1)U_g u = u(1 \otimes w_g)U_g$, as desired. Q.E.D.

For the next lemma and the end of the proof of Theorem 4.1 we'll use the notations in Proposition 3.4. Thus, we denote by $\mathcal{H}$ the Hilbert space of Hilbert-Schmidt operators on $L^2(\mathcal{N}p, \varphi)$, by σ^w the projective representation of G on $L^2(\mathcal{N}p, \varphi)$ given by $\xi \mapsto \sigma_g(\xi)w_g^*$ and by $\tilde{\sigma}^w$ the representation of G on $\mathcal{H}$ implemented by σ^w (see 3.4.2°).

4.7. Lemma. *Assume the representation $\tilde{\sigma}^w$ of G on $\mathcal{H}$ does not contain the trivial representation. Given any finite dimensional subspace $\mathcal{H}_0 \subset L^2(\mathcal{N}p, \varphi)$ and any $\varepsilon > 0$, there exists $g \in G$ such that if P_0 denotes the orthogonal projection of $L^2(\mathcal{N}p, \varphi)$ onto $\mathcal{H}_0$, regarded as an element in $\mathcal{H}$, then $0 \leq Tr(P_0\tilde{\sigma}_g^w(P_0)) < \varepsilon$.*

Proof. This is quite standard, but we prove it in details for convenience. We denote

$$K(P_0) = \overline{\text{co}}^w\{\tilde{\sigma}_g^w(P_0) \mid g \in G\} \subset \mathcal{H}.$$

Since $K(P_0)$ is convex and weakly closed, by the inferior semicontinuity of the norm in $\mathcal{H}$ with respect to the weak topology, it follows that there exists an element $Y_0 \in K(P_0)$ with minimal Hilbert-Schmidt norm. Also, by the convexity of $K(P_0)$, Y_0 is the unique element with this property. Since $\tilde{\sigma}_g^w(Y_0)$ belongs to $K(P_0)$ and has the same norm in $\mathcal{H}$ as Y_0, it follows that $\tilde{\sigma}_g^w(Y_0) = Y_0, \forall g \in G$. Since $\tilde{\sigma}^w$ does not contain the trivial representation, it follows that $Y_0 = 0$.

But if for some $\varepsilon > 0$ we would have $\text{Re}Tr(P_0\tilde{\sigma}_g^w(P_0)) \geq \varepsilon, \forall g \in G$, then by taking convex combinations and weak closure, we would have $\text{Re}Tr(P_0Y) \geq \varepsilon, \forall Y \in K(P_0)$. Taking $Y = Y_0 = 0$, this gives a contradiction. Thus, there exists $g \in G$ such that $\text{Re}Tr(P_0\tilde{\sigma}_g^w(P_0)) < \varepsilon$.

Since P_0 is a projection and $\tilde{\sigma}_g^w(P_0) = \sigma^w(g)P_0\sigma^w(g)^*$, it follows that $\tilde{\sigma}_g^w(P_0)$ is a projection as well. Thus, $Tr(P_0\tilde{\sigma}_g^w(P_0)) \geq 0$. Q.E.D.

Proof of 4.1.1°. By 3.4.3°(v) there exists a projection $q_0 \in B_0$ such that q_0w is of the form (4.1.1) and such that there are no non-zero projections $q_1 \leq (p - q_0), q_1 \in B_0$ with q_1w of the form (4.1.1). Note right away that $q_0 \leq z$, where z is the maximal atomic projection of B_0. Indeed, this is because q_0w is a generalized weak 1-cocycle with $\text{Ad}(q_0w) \circ \sigma$ having atomic fixed point algebra (cf. 3.2.3°).

We have to prove that $q_0 = z$, i.e., $(z - q_0) = 0$. We will proceed by contradiction, assuming that $(z - q_0) \neq 0$. By replacing if necessary w by $(z - q_0)w$, which is still a generalized weak 1-cocycle for σ, with the same scalar 2-cocycle as w, we may assume that the algebra $B_0 = (pNp)^{\sigma'}$ is atomic and, furthermore, that there exists no non-zero projection $q_1 \leq p = 1_{B_0}, q_1 \in B_0$, such that q_1w is of the form (4.1.1).

By Proposition 3.4 the latter condition is equivalent to the fact that the representation $\tilde{\sigma}^w$ of G on the Hilbert space $\mathcal{H}$ of Hilbert-Schmidt operators on $L^2(\mathcal{N}p, \varphi)$ does not contain the trivial representation.

But $\mathcal{H}$ can be identified with the space $L^2(\mathcal{N}p, \varphi)\overline{\otimes}L^2(p\mathcal{N}, \varphi)$. Under this identification, the representation $\tilde{\sigma}^w$ becomes:

$$\tilde{\sigma}^w(g)(x' \otimes x''^*) = (\sigma_g(x')w_g^*) \otimes (w_g\sigma_g(x'')^*), g \in G, x', x'' \in \mathcal{N}p.$$

By Lemma 4.6 there exists a partial isometry u in $\tilde{N}$ with left support $p = 1_{B_0} \in B_0 = B_0 \otimes \mathbb{C} \subset N \otimes \mathbb{C}$ and right support $\alpha_1(p) = 1 \otimes p$ and satisfying:

$$(w_g \otimes 1)U_g u = u(1 \otimes w_g)U_g, \forall g \in G.$$

Note that u actually belongs to $(p \otimes 1)\tilde{N}(1 \otimes p) \subset (p\mathcal{N})\overline{\otimes}(\mathcal{N}p)$. We want to prove that $p = 0$. To obtain this, we'll prove that

$$(*) \qquad \varphi(u(x \otimes y^*)) = 0, \forall x, y \in \mathcal{N}$$

Since the elements of the form $x \otimes y^*$, with $x, y \in \mathcal{N}$, are total in $\mathcal{N}\overline{\otimes}\mathcal{N}$, this will show that $u \in \tilde{N}$ is equal to zero, implying that $p = uu^* = 0$. Since $\varphi(u(x \otimes y^*)) = \varphi(u((xq) \otimes (yq)^*))$, it is sufficient to establish (*) for $x, y \in \mathcal{N}p$.

For each $g \in G$, by Lemma 4.6 we get:

$$\begin{aligned}
\varphi(u(x \otimes y^*)) &= \varphi(U'_g(u(x \otimes y^*))U'^*_g) \\
&= \varphi((w_g \otimes 1)U_g(u(x \otimes y^*))U_g^*(w_g^* \otimes 1)) \\
&= \varphi((u(1 \otimes w_g)U_g(x \otimes y^*))U_g^*(w_g^* \otimes 1)) \\
&= \varphi((u(1 \otimes w_g)(\sigma_g(x) \otimes \sigma_g(y^*))(w_g^* \otimes 1)) \\
&= \varphi(u(\sigma_g(x)w_g^* \otimes w_g\sigma_g(y^*))).
\end{aligned}$$

But if we identify the Hilbert spaces $\mathcal{H}$ and $L^2(\mathcal{N}p, \varphi)\overline{\otimes}L^2(\mathcal{N}p, \varphi)^*$ in the usual way, then we have

$$(\sigma_g(x)w_g^*) \otimes (w_g\sigma_g(y^*)) = \tilde{\sigma}^w(g)(x \otimes y^*)$$

so that by Lemma 4.7, for any given $\delta > 0$ we can find recursively $g_1, g_2, ..., g_m \in G$ such that $\pi(g_{k+1})(x \otimes y^*)$ is δ orthogonal (with respect to the scalar product implemented by $\tilde{\varphi} = \varphi \otimes \varphi$) to each of the vectors $\pi(g_i)(x \otimes y^*), i = 1, 2, ..., k$. Thus, the vectors $\{(1 \otimes w_{g_j})U_{g_j}(x \otimes y^*)U_{g_j}^*(w_{g_j}^* \otimes 1)\}_{1 \leq j \leq m}$, are mutually δ-orthogonal. By the Cauchy-Schwartz inequality we thus obtain:

$$\begin{aligned}
|\varphi(u(x \otimes y^*))|^2 &= |\varphi(u1/m(\Sigma_j(1 \otimes w_{g_j})U_{g_j}x \otimes y^*)U_{g_j}^*(w_{g_j}^* \otimes 1)))|^2 \\
&\leq \|u\|_2^2 \; \|1/m(\Sigma_j(1 \otimes w_{g_j})U_{g_j}(x \otimes y^*)U_{g_j}^*(w_{g_j}^* \otimes 1))\|_{\tilde{\varphi}}^2 \\
&\leq \|u\|_2^2 \, \|x \otimes y\|_{\tilde{\varphi}}^2/m + m(m-1)\delta.
\end{aligned}$$

Since $\delta > 0$ was arbitrary, independently of m, taking first $\delta \to 0$ then $m \to \infty$, we are done. Q.E.D.

In order to prove 4.1.2° (and thus end the proof of 4.1) we need to show that in case (N_0, φ_0) is taken to be of the special form $\overset{k}{\underset{i=1}{\otimes}} (N_{0,i}, \varphi_{0,i})$, where $N_{0,i} \simeq M_{2\times 2}(\mathbb{C})$ and $\varphi_{0,i}$ are faithful states on $N_{0,i}$, $1 \leq i \leq k$, then the fixed point algebra B_0 must be atomic.

We do this in the next two lemmas.

4.8. Lemma. *With the assumptions of* 2.5.1, *let* α *be as in the first part of* 2.5.1 *and* $\beta \in \mathrm{Aut}\tilde{N}$ *be a period* 2 *automorphism like in the last part of* 2.5.1, *i.e., such that* $\beta\alpha_t\beta^{-1} = \alpha_{-t}$ *and such that* $N \otimes \mathbb{C} \subset \tilde{N}^{\beta}$. *Assume there exists a diffuse weakly closed* **-subalgebra* $B^0 \subset N = N \otimes \mathbb{C}$ *and a partial isometry* $v_0 \in \tilde{N}$ *such that* $v_0^* v_0 = 1_{B^0}$ *and* $v_0 B^0 v_0^* \subset \alpha_{1/2^n}(N \otimes \mathbb{C})$, *for some* $n \geq 1$. *Then* $v_0 = 0$.

Proof. We will construct by induction over k some partial isometries $v_k \in \tilde{N}$ and diffuse weakly closed von Neumann subalgebras $B_k \subset N \otimes \mathbb{C}$ such that

$$\tau(v_k v_k^*) = \tau(vv^*), v_k^* v_k = 1_{B_k}, v_k B_k v_k^* \subset \alpha_{1/2^{n-k}}(N \otimes \mathbb{C}) \tag{4.8.1}$$

Letting $B_0 = B^0, v_0 = v_0$, we see that the relation holds true for $k = 0$. Assume we have constructed v_j, B_j for $j = 0, 1, \dots, k$. By applying the automorphism β of 2.5.1 to the inclusion in (4.8.1) and using the properties of β, it follows that

$$\beta(v_k) B_k \beta(v_k)^* \subset \alpha_{-1/2^{n-k}}(N \otimes \mathbb{C}) \tag{4.8.2}$$

By further applying $\alpha_{1/2^{n-k}}$ to this latter inclusiuon it follows that

$$B_{k+1} \overset{\text{def}}{=} \alpha_{1/2^{n-k}}(\beta(v_k))\alpha_{1/2^{n-k}}(B_k)\alpha_{1/2^{n-k}}(\beta(v_k^*)) \subset N \otimes \mathbb{C} \tag{4.8.3}$$

By conjugating (4.8.3) with $\alpha_{1/2^{n-k}}(\beta(v_k^*))$ we thus get:

$$\alpha_{1/2^{n-k}}(\beta(v_k^*))B_{k+1}\alpha_{1/2^{n-k}}(\beta(v_k)) = \alpha_{1/2^{n-k}}(B_k) \subset \alpha_{1/2^{n-k}}(N \otimes \mathbb{C}) \tag{4.8.4}$$

On the other hand, by applying $\alpha_{1/2^{n-k}}$ to (4.8.1) we also have:

$$\alpha_{1/2^{n-k}}(v_k)\alpha_{1/2^{n-k}}(B_k)\alpha_{1/2^{n-k}}(v_k^*) \subset \alpha_{1/2^{n-k-1}}(N \otimes \mathbb{C}) \tag{4.8.5}$$

Altogether, it follows that if we let $v_{k+1} = \alpha_{1/2^{n-k}}(v_k)\alpha_{1/2^{n-k}}(\beta(v_k^*))$ then by (4.8.4) and (4.8.5) we get:

$$v_{k+1} B_{k+1} v_{k+1} \subset \alpha_{1/2^{n-k-1}}(N \otimes \mathbb{C})$$

Moreover, since $\beta(v_k^* v_k) = v_k^* v_k$, we also have $v_{k+1}v_{k+1}^* = \alpha_{1/2^{n-k}}(v_k v_k^*)$, so that $\tau(v_{k+1}v_{k+1}^*) = \tau(v_k v_k^*)$.

This ends the induction argument. Taking $k = n$, it follows that $v_n B_n v_n \subset \alpha_1(N \otimes \mathbb{C}) = \mathbb{C} \otimes N$, because α_1 is the flip automorphism.

But by (Lemma 4.3 in [**Po3**], or 2.6 in [**Po6**]), this implies that $v_n = 0$. Since $\tau(v_n v_n^*) = \tau(vv^*)$, it follows that $v = 0$. Q.E.D.

4.9. Corollary. *Under the assumptions of* 4.1.2°, *the algebra* $B_0 = N^{\sigma'}$ *follows atomic.*

Proof. By 4.3 and 4.4 it follows that if $n \geq 1$ is sufficiently large then the partial isometry $v' = u(1/2^n)^*$ is non-zero, has right support in B_0 and satisfies $v'B_0v'^* \subset \alpha_{1/2^n}(N \otimes \mathbb{C})$. But if we assume $B_0v'^*v'$ has a non-zero diffuse part B^0 and we let $v_0 = v'1_{B^0}$, then 4.8 applies to infer that $v_0 = 0$, a contradiction. Thus, $v'^*v'B_0v'^*v'$ is atomic.

Since $v'^*v' = u(1/2^n)u(1/2^n)^*$ tends to p as n tends to ∞, it follows that B_0 is atomic. Q.E.D.

Proof of 4.1.2°. By Lemma 4.9 the algebra B_0 follows atomic. But then, 4.1.1° applies to obtain that w is necessarily of the form (3.2.1). Q.E.D.

5. Some consequences and comments.

The first application of 4.1 that we are going to emphasize is that, for an action of the property T group G by Connes-Størmer Bernoulli shifts σ on $N = \mathcal{N}_\varphi$, the multiplicative group $H \subset \mathbb{R}_+^*$ arising as the spectrum of the discrete decomposition of the associated ITPF1 factor $(\mathcal{N}, \varphi)$ is an outer conjugacy invariant for σ. In fact, we define a certain outer conjugacy invariant for arbitrary actions σ of groups G, and then use 4.1.1° to prove that if the group G has the property T then this invariant can be explicitly calculated, being equal to $H_1 = \{\beta \in H \mid \beta \leq 1\}$.

5.1. Definition. Let σ be an action of a group G on a type II_1 factor N. Let $\mathcal{H}_\sigma^1$ be the set of generalized weak 1-cocycles w of σ satisfying the following properties:

1°. The fixed point algebra of the action $\mathrm{Ad}(w_g) \circ \sigma_g, g \in G$, on pNp is atomic, where p is the support of w.

2°. There exists an atomic von Neumann subalgebra $B_\sigma^w \subset pNp$ such that $\mathrm{Ad}(w_g) \circ \sigma_g(B_\sigma^w) = B_\sigma^w, \forall g \in G$ and such that if $B \subset pNp$ is atomic and $\mathrm{Ad}(w_g) \circ \sigma_g(B) = B, \forall g \in G$ then $B \subset B_\sigma^w$.

For each $w \in \mathcal{H}_\sigma^1$ let $\mathcal{P}_\sigma^w$ be the set of minimal (non-zero) projections of B_σ^w. Define

$$H(\sigma) = \{\tau(q) \mid \exists w \in \mathcal{H}_\sigma^1, q \in \mathcal{P}_\sigma^w\},$$

with the convention that $H(\sigma) = \{0\}$ if $\mathcal{H}_\sigma^1 = \emptyset$

5.2. Proposition. $H(\sigma)$ *is an outer conjugacy invariant for* σ.

Proof. By the definitions, it follows that if w is a weak 1-cocycle for σ then $\mathcal{H}_\sigma^1 = \mathcal{H}_{\sigma'}^1 w^*$, where $\sigma' = \mathrm{Ad}(w) \circ \sigma$. Moreover, if $w' \in \mathcal{H}_\sigma^1$ then $B_\sigma^{w'} = B_{\sigma'}^{w'w^*}$. This implies $H(\sigma) = H(\sigma')$. Q.E.D.

5.3. Theorem. 2°. *Assume G is an infinite group with the property T of Kazhdan and σ is an action of G on the hyperfinite type II_1 factor by Connes-Størmer Bernoulli shifts, as in the hypothesis of* 4.1. *Let $H_1 \subset (0,1]$ be the set associated with the discrete decomposition of the* ITPF1 *factor $(\mathcal{N}, \varphi)$ on which σ is defined, as in* $2.2.0 - 2.2.2$ *(thus, $H_1 = \{\beta \in H \mid \beta \leq 1\}$, with H being the spectrum of the discrete decomposition of $(\mathcal{N}, \varphi)$). Then $H(\sigma) = H_1$.*

Moreover, if $\mathcal{N}$ is of type III (equivalently, $H \neq \{1\}$), $0 < t \leq 1$ and σ^t is chosen to be a genuine action (cf. 3.3*), then $H(\sigma^t) = \{\beta/t \mid \beta \in H_1, \beta \leq t\}$.*

Proof. To prove the first part, note that by Theorem 4.1.1°, any generalized weak 1-cocycle $w \in \mathcal{H}^1_\sigma$ is of the form (3.2.1). But for such cocycles, Proposition 3.4 shows that there exists indeed an atomic von Neumann subalgebra invariant to $\mathrm{Ad}w \circ \sigma$ satisfying the maximality condition 5.1.2°. Moreover, by 3.2, 3.4 the traces of the minimal projections in this atomic algebra are in the set H_1. Thus, $H(\sigma) \subset H_1$.

Conversely, if $\beta_0 \in H_1$ then there always exist some non-negative integers $n_\beta, \beta \in H_1$, such that $n_{\beta_0} \neq 0$ and $\Sigma_\beta n_\beta \beta = 1$ (exercise !). For each $\beta \in H_1$ take π_β to be a n_β-multiple of the trivial representation of G, on the Hilbert space $\ell^2(n_\beta)$. Let w be a 1-cocycle constructed out of $\{\pi_\beta\}_\beta$, for some appropriate choice of isometries in $\mathcal{V}$, as in (3.2.1). By 3.4 it follows that $w \in \mathcal{H}^1_\sigma$ and by 3.2 the corresponding B^w_σ is given by the algebra B constructed in 3.2.2°. By construction, it follows that if q is a minimal projection in B^w_σ then $\tau(q) = \beta_0$. Thus, $H_1 \subset H(\sigma)$ as well.

To prove the last part of the statement, let w be a generalized 1-cocycle for σ of support p, with $\tau(p) = t$ such that $\sigma^t = \mathrm{Ad}(w) \circ \sigma$. Note that any 1-cocycle w' for σ^t gives rise to a generalized 1-cocycle $w'w$ for σ. Thus, by 4.1.1°, $w'w$ is of the form (3.2.1), having an atomic invariant subalgebra B described by 3.2.2°. Moreover, by 3.2.3°, B contains all other atomic subalgebras of N invariant to σ^t. Then 3.2.3° applies to show that the traces of the minimal projections on B are of the form β/t with $\beta \in H_1, \beta \leq t$. The other inclusion is proved in the same way as the similar inclusion above. Q.E.D.

5.4. Corollary. *Let σ, σ' be two Connes-Størmer Bernoulli shifts of the property T group G on the hyperfinite type II_1 factor. Let H, H' be the corresponding multiplicative groups, arising as the spectra of the discrete decompositions of the* ITPF1 *factors $(\mathcal{N}, \varphi), (\mathcal{N}', \varphi')$ on which σ, σ' are defined. Let $t, t' > 0$ and $\sigma^t, \sigma^{t'}$ be cocycle actions defined as in* 3.1.1, *out of σ, σ'. If σ^t is outer conjugate to ${\sigma'}^{t'}$ then $H = H'$ and $t/t' \in H$. If in addition $\sigma^t, {\sigma'}^{t'}$ are genuine actions, then they follow conjugate.*

Proof. Let $s = \max\{t, t'\}$. Then we have $(\sigma^t)^{1/s}$ is outer conjugate to $({\sigma'}^{t'})^{1/s}$. This shows that we may assume $t' = 1$ and $t \leq 1$.

There are two possibilities: Either $\mathcal{N}$ is of type II_1 and $H = \{1\}$ or $\mathcal{N}$ is of type III.

If $\mathcal{N} = N$ is of type II_1 and $t < 1$ then σ^t outer conjugate to σ' implies there exists a generalized 1-cocycle w for σ of support p such that $\tau(p) = t$ and such that $\text{Ad}(w) \circ \sigma$ is conjugate to σ'. In particular, $\text{Ad}(w) \circ \sigma$ is ergodic, so it has trivial (in particular atomic) fixed point algebra. By 4.1.1° it follows that w is given by (3.2.1). But $H = \{1\}$, so p is forced to be equal to 1 and w to be a weakly trivial weak 1-cocycle. Thus $t = 1$ and σ follows conjugate to σ'. In particular, by 4.1.1°, $H = H = \{1\}$.

If $\mathcal{N}$ is of type III, then by 3.3 we can choose σ^t to be a genuine action. Since $1 \in H_1'$ and $H_1' = H(\sigma')$ (cf. 5.3) it follows that $1 \in H(\sigma')$. But by 5.3, we also have $H(\sigma^t) = \{\beta/t \mid \beta \in N, \beta \leq t\}$. Since σ is outer conjugate to σ^t and $H(\sigma)$ is an outer conjugacy invariant, we have $1 \in H(\sigma^t)$. The only situation when 1 can be contained in $H(\sigma^t)$ is when t belongs to $H_1 = H(\sigma)$. But then (N^t, σ^t) is conjugate to (N, σ) by $\text{Ad}v$, where v is an isometry in $\mathcal{V}_t$. Thus, σ follows outer conjugate to σ', implying that $H(\sigma) = H(\sigma')$.

Moreover, if σ' (which is strongly mixing) is conjugate to $\text{Ad}(w) \circ \sigma$ for some weak 1-cocycle w for σ, then the fixed point algebra of $\text{Ad}(w) \circ \sigma$ is equal to $\mathbb{C}1$ and by 4.1.1° it follows that w is of the form (3.2.1). Also, since $\text{Ad}(w) \circ \sigma$ follows strongly mixing itself (as being conjugate to σ'), it leaves no non-trivial atomic von Neumann subalgebra invariant. By 4.1.1°, this implies the weak 1-cocycle w is weakly trivial. Thus, $\sigma' \sim \sigma$. Q.E.D.

We now derive some consequences of Theorem 4.1 and of the above results, in the case of "classical" Bernoulli shifts. The first such consequence is derived from 4.1.1°, being a particular case of 5.4 and dealing with arbitrary Bernoulli shifts. The other two Corollaries are derived from 4.1.2° and they are proved for $M_{2^n \times 2^n}(\mathbb{C})$-Bernoulli shifts only.

5.5. Corollary. *Let σ_i be $M_{n_i \times n_i}(\mathbb{C})$- Bernoulli shifts, $i = 1, 2$, of the infinite property T group G on the hyperfinite type II_1 factor, as in 2.1. Let also $t_1, t_2 > 0$. If $\sigma_1^{t_1}$ is outer conjugate to $\sigma_2^{t_2}$ then $t_1 = t_2$ and σ_1 is conjugate to σ_2. In particular, taking $n_1 = n_2 = n$, $\sigma_1 = \sigma_2 = \sigma$, $t_1 = 1$ and $t_2 = m$ it follows that the genuine actions $\{\sigma \otimes id_m\}_m$ are mutually non-outer conjugate.*

Proof. This is just a particular case of 5.6. Q.E.D.

5.6. Corollary. *Let σ be a $M_{2^n \times 2^n}(\mathbb{C})$-Bernoulli shift action of the infinite property T group G on the hyperfinite type II_1 factor N. Then any weak 1-cocycle w for σ is weakly trivial, i.e., if $w : G \to \mathcal{U}(N)$ is so that $\text{Ad}(w_g) \circ \sigma_g$ is an action of G on N, then there exists a unitary element $v \in \mathcal{U}(N)$ such that $w_g =$*

$v^*\sigma_g(v)$, mod$\mathbb{C}$, $\forall g \in G$. *Moreover, the set of* 1*-cocycles for* σ*, modulo equivalence, coincides with the set of characters of* G*, thus being finite.*

Proof. By 4.1.2°, since in the case of classical Bernoulli shifts the multiplicative group H is equal to $\{1\}$ and $\mathcal{V} = \mathcal{U}(N)$, it follows that any weak 1-cocycle w with scalar 2-cocycle μ is of the form $w_g = \lambda_g v^* \sigma_g(v)$ for some unitary element $v \in N$ and projective one dimensional representation $\lambda : G \to \mathbb{T}$ having scalar 2-cocycle μ. That implies $\lambda_e = 1$ and $\mu_{g,h} = \lambda_g \lambda_h \overline{\lambda_{gh}}, \forall g, h \in G$.

In particular, if w is a 1-cocycle then $\mu = 1$ and λ follows a character. Moreover, if λ is a character and v is a unitary element in N such that $\lambda_g = v^*\sigma_g(v), \forall g \in G$ then $\sigma_g(v) \in \mathbb{C}v, \forall g$. Since σ is strongly mixing, by 2.4.3 this implies $v \in \mathbb{C}$ itself. Thus, different characters give different 1-cocycles. The number of characters is finite by ([**K**]). In fact, by ([**V2**]), even the number of equivalence classes of representations of a fixed dimension of a group with the property T is finite. Q.E.D.

5.7. Corollary. *Assume* σ, G *are as in* 5.6. *For each* $1 > t > 0$*, let* σ^t *be a choice of a cocycle action on* $N^t \simeq R$ *obtained by reducing, or amplifying the action* σ*, as in* 3.1.1. *Then* σ^t *cannot be perturbed by inner automorphisms to a genuine action.*

Proof. If σ^t would be perturbable to a genuine action then σ would have a generalized weak 1-cocycle w of support p, with $p \in N$ a projection of trace $t < 1$. But in the case of classical Bernoulli shifts the group H is equal to $\{1\}$ and by 4.1.2°, w must be of the form (3.2.1), thus it must have support equal to 1. Q.E.D.

Let us point out that even milder rigidity properties on a group G can be suficient to ensure that results similar to 5.5-5.7 hold true:

5.8. Theorem. *All the conclusions in* $5.5 - 5.7$ *hold true for the group* $G = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ *as well.*

Proof. The only modifications needed, from the proof of the case G has the property T, are as follows:

Assume w is a generalized 1-cocycle for the $M_{2^k \times 2^k}(\mathbb{C})$-Bernoulli shift σ of the group $G = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ on $N = \overline{\otimes}_g (M_{2^k \times 2^k}(\mathbb{C}), tr)_g$. Then in Lemma 4.2 one defines the set $K'_t(x)$ as the weak closure of the convex hull of elements of the form $U'_g x U_t U'^*_g U^*_t$, with g running over the subgroup $\mathbb{Z}^2$ (rather than over all G). The proofs of 4.2, 4.3 do work in this case as well (cf. observation before statement of lemma 4.2).

Next, Lemma 4.4 holds true, due the rigidity of the inclusion $\mathbb{Z}^2 \subset SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ (cf. [**K**], [**DKi**]). All the proofs of 4.5-4.7 (+ the proof of 4.1.1°) are the same, to conclude that if z denotes the maximal atomic projection in B_0 then the restriction of zw to $\mathbb{Z}^2$ is a weak coboundary, if nozero, i.e., for some apropriate $v \in R$ one has $zw_g = v^*\sigma_g(v)$, mod$\mathbb{C}$, $\forall g \in \mathbb{Z}^2$. This already implies that the support of zw is equal to 1, due to the ergodicity of the action $\sigma_{|\mathbb{Z}^2}$ on N (this restriction being a R-Bernoulli shift).

Moreover, by replacing w_g by $w'_g = vw_g\sigma_g(v^*), g \in G$ one gets a weak cocycle for σ which is scalar valued when restricted to $\mathbb{Z}^2$. Let a, b be the generators of $SL(2, \mathbb{Z})$. It follows that $\text{Ad}w'_a\sigma_a$ and $\text{Ad}w'_b\sigma_b$ normalize $\sigma(\mathbb{Z}^2)$, implying that $w'_a\sigma_g({w'_a}^*)$ and $w'_b\sigma_g({w'_b}^*)$ are scalars $\forall g \in \mathbb{Z}^2$. Thus, $\sigma_g(w'_a) \in \mathbb{C}w'_a$ and similarily for b. Since the only finite dimensional vector subspace of $L^2(N)$ that is invariant to $\sigma_{|\mathbb{Z}^2}$ is $\mathbb{C}1$ (cf. 2.4.3), it follows that w'_a, w'_b are scalars. Thus, $w_g = v^*\sigma_g(v), \text{mod}\mathbb{C}, \forall g \in G$.

This proves 4.1.1° in the case of classical $M_{n\times n}(\mathbb{C})$- Bernoulli shifts. But then 4.1.2° follows by simply noticing that Lemmas 4.8 did not require G to have the property T and deducing 4.9 from this lemma and the version of 4.4.3°3, 4.5 that are valid for $G = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ (as explained bove). Q.E.D.

5.9. Remarks. 1°. The results obtained in 4.1.2°, 5.6.2°, 5.7 and (part of) 5.8 are for (N_0, φ_0)- Connes-Størmer Bernoulli shifts, in which (N_0, φ_0) is a tensor product of 2 by 2 matrix algebras with faithful states on them. The only time we need this special type of algebra (N_0, φ_0) is for the construction of the period 2 automorphism β in Lemma 2.5.1. This automorphism, in turn, is being used only in the proof of Lemma 4.8. In fact, one can avoid using the existence of β in the proof of 4.8.

More precisely, one can prove 4.8 for (N_0, φ_0)-Connes-Størmer Bernoulli shifts with $(N_0, \varphi_0) = (M_{k\times k}(\mathbb{C}), \varphi_0)$ for arbitrary k and arbitrary faithful state φ_0. Unfortunately though, the argument we have is rather long and complicated, due to the more involved geometry of the corresponding unitary groups $\mathcal{U}(2k)$, especially when taking into account the centralizer of $\varphi_0 \otimes \varphi_0$ in this group. With this, however, the conclusions in the above mentioned results follow for all Connes-Størmer Bernoulli shifts of a property T group G or for $G = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$.

5.9.2°. We only needed the assumption that σ is a Bernoulli shift for the ergodicity properties that such actions have (strongly mixing) and for the rich structure of the commutant of σ in $\text{Aut}N$. Thus, the following assumptions on an action σ on a type II_1 factor N are in fact sufficient to insure that a statement similar to 4.1 holds true:

(a). G is an infinite property T group.

(b). N is a type II_1 factor and $(\mathcal{N}, \varphi)$ is a von Neumann factor with $\mathcal{N}_\varphi = N$, such that $(\mathcal{N}, \varphi)$ has a discrete decomposition over N.

(c). σ is a strongly mixing action of G on $(\mathcal{N}, \varphi)$.

(d). There exists a type II_1 factor $\tilde{N}$ containing N such that σ extends to an action θ of G on $\tilde{N}$ and such that there exists a one parameter group of automorphisms α of $\mathbb{R}$ on $\tilde{N}$, commuting with the action θ and with the property that if there exists a partial isometry $u \in \tilde{N}$ satisfying $w_g\theta_g(u) = u\alpha_1(w_g), \forall g \in G$, for some generalized weak 1-cocycle w for σ then w is of the form (3.2.1).

(e). There exists a period 2 automorphism β of $\tilde{N}$ such that $N \subset \tilde{N}^\beta, \beta\alpha_t\beta = \alpha_{-t}, \forall t$.

(f). If a partial isometry $u \in \tilde{N}$ satisfies $u^*u = 1_{B^0}, uB^0u^* \subset \alpha_1(B^0)$ for some diffuse von Neumann algebra $B^0 \subset N$ then $u = 0$.

Thus, conditions (a)-(d) imply 4.1.1° and together with the conditions (e)-(f) they imply 4.1.2° as well. For Bernoulli shifts in Sections 4 and 5 we have taken $\tilde{N} = N\overline{\otimes}N$ and $\theta = \sigma \otimes \sigma$. In Section 6 we will give another example when this abstract framework can be realized.

5.10. Remarks. Let us mention some more remarks related to the vanishing cohomology results in this section, as well as some problems:

5.10.1. Note that Corollary 5.5 cannot distinguish between the $M_{n\times n}(\mathbb{C})$-Bernoulli shifts, for $n = 2, 3, \ldots$. This points towards the following problem:

Let G be an infinite group and σ_i the $M_{n_i\times n_i}(\mathbb{C})$-Bernoulli shifts, $i = 1, 2$, of the group G. Is it true that σ_1 conjugate to σ_2 implies $n_1 = n_2$?

If solved in the affirmative then, together with the Corollary 5.5, the $M_{n\times n}(\mathbb{C})$-Bernoulli shifts by a property T group G would follow mutually non-outer conjugate, $n \geq 2$.

Note that the case G is amenable of this problem was proved in the affirmative by Kawahigashi in ([**Ka**]). But the invariant he used, which is an addaption to the case of actions by amenable groups of the Connes-Størmer entropy, doesn't work beyond that case, at least not in an evident way. Note in this respect that if an entropy invariant could be defined for non-amenable groups as well, along the lines of ([**CS, Ka**]), then by using 5.3 one could easily find Connes-Størmer Bernoulli shifts σ with distinct invariant $H(\sigma)$ yet same entropy.

5.10.2. One can also prove a (slightly modified) version of 3.2, 3.4 and 4.1 for actions $\sigma \otimes id_n$ on $N \otimes M_{m\times m}(\mathbb{C})$ with σ as in 4.1. As a result, one obtains a calculation of the invariant $H(\sigma \otimes id_m)$ for these actions. As a consequence, one gets some versions of 5.2.4°, 5.5.1° for all $t > 0$ (not just for $1 \geq t > 0$).

5.10.3. Any finitely generated subgroup H of $\mathbb{R}_+^*$ can be realized as the invariant $H(\sigma)$ of some Connes-Størmer Bernoulli shift of the type considered in 5.3.2°.

5.10.4. Note that M. Choda has already constructed in ([**Ch**]) a continuous family of mutually non-conjugate actions by the property T groups $G = SL(n, \mathbb{Z})$ on R, for $n \geq 3$. Moreover, these actions were proved to be mutually non-cocycle conjugate, modulo countable sets, in ([**Ka**]). The argument in ([**Ka**]) relies on first proving that the set of ergodic 1-cocycles of an ergodic action σ by a property T group G is countable, then using ([**Ch**]). Note that in this respect, our Corollary 5.6 gives a list of all 1-cocycles (not just the "ergodic" ones), in the special case when σ is a $M_{2^n \times 2^n}(\mathbb{C})$-Bernoulli shift, but for arbitrary property T groups G.

Note that the actions of $G = SL(n, \mathbb{Z})$ considered in ([**Ch, Ka**]) are not Bernoulli shifts. Indeed, those actions are so that the corresponding cross products $R \rtimes G$ have the property T as von Neumann algebras (in the sense of [**CJ**]), while the cross product algebras associated to Bernoulli shift actions can never have the property T (cf. [**Ch**]). This is because the automorphism group of such a cross product is large (for instance, $\text{Aut}R/\text{Int}R$ contains a copy of $\mathbb{T}$, see 2.5.1).

5.10.5. It seems to us that some of the results 4.1, 5.2-5.5 could be true for a larger class of non-amenable groups, than the property T and $SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ we considered here. Related to this, the following question is of obvious interest:

What is the class $\mathbb{G}(R)$ of all groups G for which any cocycle action on the hyperfinite type II_1 factor R can be perturbed to a genuine action ?

Note that besides the amenable groups (cf. [**Oc**]), the class $\mathbb{G}(R)$ of such groups contains the free groups F_n (which have no 2-cocycles) and more generally any group G of the form $G = G_1 *_H G_2 *_H G_3 ...$, where G_i are amenable groups and $H \subset G_i$ is a common finite group (cf. [**Su, J1**]). We will prove in another paper that this class contains some other products with amalgamation, such as $\mathbb{F}_2 \oplus \mathbb{Z} = \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$. Note that not all free products with amalgamation of amenable groups belongs to $\mathbb{G}(R)$. Indeed, by 5.8 we have that $G = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$ does not belong to $\mathbb{G}(R)$.

6. Cohomology of free Bernoulli shifts.

We prove in this section that the same results as in 5.5-5.8 hold true for actions of infinite property T groups G (or of the group $G = SL(2, \mathbb{Z}) \rtimes \mathbb{Z}^2$) on the free group factor $L(\mathbb{F}_G) \simeq L(\mathbb{F}_\infty)$, by free Bernoulli shifts.

Thus, we consider a special case of the construction in 2.3, which for simplicity we describe in an alternative way: We lable the free generators of the group $\mathbb{F}_\infty$ by $\{a_g \mid g \in G\}$ and re-denote this group by $\mathbb{F}_G$. We denote by σ the (left) action of the group G on $L(\mathbb{F}_G)$ determined by:

$$\sigma_g(a_h) = a_{g^{-1}h}, \forall g, h \in G,$$

and call it the action of G on $L(\mathbb{F}_G)$ by (left) free shifts.

6.1 Theorem. *With $G, L(\mathbb{F}_G), \sigma$, as above, we have:*

1°. *Any weak 1-cocycle w for σ is weakly trivial and any 1-cocycle is equivalent to a character.*

2°. *For each $t > 0$, let σ^t be a choice of a cocycle action on $(L(\mathbb{F}_G))^t (\simeq L(\mathbb{F}_G)$, by [**V3, Ra1**]), obtained by reducing, or amplifying the action σ, as in 3.1. If $0 < t < 1$ then σ^t cannot be perturbed by inner automorphisms to a genuine action.*

3°. *With the notation in* 2°, *the cocycle actions* $\{\sigma^t\}_{t>0}$, *all regarded as acting on* $L(\mathbb{F}_G) \simeq (L(\mathbb{F}_G))^t$, *are mutually non-outer conjugate. In particular, by taking* t *to be integers, it follows that the sequence of genuine actions* $\{\sigma \otimes id_n\}_n$ *on* $L(\mathbb{F}_G)$ *are not outer conjugate.*

Proof. For simplicity, denote $N = L(\mathbb{F}_\infty)$. To prove the statement it is suficient to show that if w is a generalized weak 1-cocycle with support p for the action σ then $p = 1$ and there exists $v \in \mathcal{U}(N)$ such that $vw_g\sigma_g(v)^* \in \mathbb{C}, \forall g \in G$. Indeed, because then the arguments in the proof of $5.5 - 5.8$ apply identically to get the above $1^\circ, 2^\circ$.

The proof of the vanishing of the generalized weak 1-cocycles is basically the same as the proof of 4.1, except that the role of the tensor product $N \otimes N$ will this time be played by the free product $N * N$. Due to this, it is convenient to keep the notation $\{a_g\}_g$ for the generators of $N * \mathbb{C}$ and denote these same generators of $\mathbb{C} * N$ by $\{b_g\}_g$.

We split the proof in a number of "steps".

Step 1. We first show that there exists a continuous action $\alpha : \mathbb{R} \to \mathrm{Aut}(N * N)$ and a period 2 automorphism $\beta \in \mathrm{Aut}(N * N)$ such that

(a). $\alpha_1(N * \mathbb{C}) = N_0$ where N_0 is free with respect to $N = N * \mathbb{C}$ and N, N_0 generate $N * N$.

(b). $N * \mathbb{C} \subset (N * N)^\beta$.

(c). $\beta\alpha_t\beta = \alpha_{-t}, \forall t$.

(d). α and β commute with the free shift $\theta = \sigma * \sigma$.

This construction plays the role of Lemma 2.5.1 in the sequel.

To construct these automorphisms, let $h_g \in \mathbb{C} * N$ be self-adjoint elements with spectrum in $[0, 2]$ such that $b_g = \exp(\pi i h_g), \forall g$. We then put $\alpha_t(a_g) = \exp(\pi i t h_g) a_g$ and $\alpha_t(b_g) = b_g, \forall g \in G, t \in \mathbb{R}$. By ([**V1**]) $\exp(\pi i t h_g) a_g$ and b_g are mutually free and they clearly generate the same von Neumann algebra as a_g, b_g. Thus, α_t defines an automorphism of N, $\forall t$. Similarily, $b_g a_g$ is free with respect to b_g and they jointly generate the same algebra as a_g, b_g do. Thus, (a) is satisfied.

Moreover, by the definition we clearly have $\alpha_t\alpha_s = \alpha_{t+s}, \forall t, s \in \mathbb{R}$. Thus, α is a continuous action and it is immediate to see that it commutes with $\theta = \sigma * \sigma$.

Also, define $\beta(a_g) = a_g$ and $\beta(b_g) = b_g^*, \forall g \in G$. This clearly defines a period 2 automorphism of $N * N$ that commutes with σ.

Moreover, $\alpha_1(N * \mathbb{C}) = N_0$, with N, N_0 generating $N * N$ and $N * \mathbb{C} \subset (N * N)^\beta$. Further on, we have

$$\begin{aligned}\beta(\alpha_t(\beta(a_g))) &= \beta(\alpha_t(a_g)) \\ &= \beta(\exp(\pi i t h_g) a_g) = \exp(\pi i t h_g)^* a_g \\ &= \exp(-\pi i t h_g) a_g = \alpha_{-t}(a_g).\end{aligned}$$

Similarily, we get

$$\beta(\alpha_t(\beta(b_g))) = b_g = \alpha_{-t}(b_g).$$

This shows that all the required conditions are satisfied.

Step 2. Denote by $\tilde{\sigma}^w$ the representation of G on $L^2(Np, \tau)\overline{\otimes}L^2(pN, \tau)$, as in 3.4. Apply 3.4.3° for $\mathcal{N} = N$ and $H_1 = \{1\}$ to deduce that if $\tilde{\sigma}^w$ contains the trivial representation then $p = 1$ and there exists a unitary element $v \in N$ such that $vw_g\sigma_g(v)^* \in \mathbb{C}, \forall g \in G$. Thus, we are left to prove the Theorem 6.1 when $\tilde{\sigma}^w$ doesn't contain the trivial representation. More precisely, we want to show that this assumption leads to a contradiction.

Step 3. Denote $\sigma' = \mathrm{Ad}(w) \circ \sigma$ and $B_0 = (pNp)^{\sigma'}$. Consider the action of $G \oplus \mathbb{R}$ implemented by the commuting actions $\theta = \sigma * \sigma$ and α on $N * N$. Then define $p_t \in N * N$ the same way as in Lemma 4.2 and $u(t)$ the same way as in Lemma 4.3. Thus, $u(t)$ is an isometry in $N * N$ such that if we put $l(t) = u(t)u(t)^*, r(t) = u(t)^*u(t)$ then conditions 1°-3° of Lemma 4.3 are satisfied, where B_0 is identified with $B_0 * \mathbb{C}$.

Step 4. The function $t \mapsto p_t$ follows continuous, with respect to the norm-2 topology, in exactly the same way as in Lemma 4.4, by using the property T of the group G.

Step 5. Lemma 4.8 holds in this "free product" context as well: Indeed, we just replace $\tilde{N}$ by $N * N$ and $N \otimes \mathbb{C}$ by $N * \mathbb{C}$ throughout the proof of that lemma to deduce that if there exists a diffuse von Neumann subalgebra $B^0 \subset N * \mathbb{C}$ and a partial isometry $v_0 \in N * N$ satisfying $v_0^*v_0 = 1_{B^0}$ and $v_0B^0v_0^* \subset \alpha_{1/2^n}(N * \mathbb{C})$ for some $n \geq 1$ then there exists a partial isometry $v_n \in N * N$ and a diffuse algebra $B_n \subset N * \mathbb{C}$ such that $\tau(v_nv_n^*) = \tau(v_0v_0^*)$, $v_n^*v_n = 1_{B_n}$ and $v_nB_nv_n^* \subset N_0$, where $N_0 = \alpha_1(N * \mathbb{C})$ as denoted in *Step 1*.

Since $N = N * \mathbb{C}$ and N_0 are mutually free and they generate $N * N$, by (4.3 in [**Po6**]) it follows that $v_n = 0$, thus $v_0 = 0$ and $B^0 = 0$.

Step 6. One applies the results in *Steps 4, 5* to $B_0, u(1/2^n)$ the same way as in the proof of Lemma 4.5, to first conclude that $B_0l(1/2^n)$ is atomic, then pursue with the same argument as in that proof to conclude that all B_0 follows atomic, and that on each minimal central projection of B_0 $\mathrm{Ad}(u(1/2^n))$ implements $\alpha_{1/2^n}$, for n sufficiently large.

Step 7. We conclude, by using exactly the same construction as in the proof of Lemma 4.6, that there exists a partial isometry $u \in N * N$ such that $uu^* = 1_{B_0}$ and

$$w_gU_gu = u\alpha_1(w_g)U_g, \forall g \in G,$$

where U_g are the canonical unitaries in $(N * N) \rtimes G$ implementing the action θ of G on $N * N$.

Step 8. To end the proof of Theorem 6.1 we will prove that the relation in *Step 7* entails $u = 0$, giving us the desired contradiction. To show this we prove that $\tau(uX) = 0, \forall X \in N * N$.

We use the fact that $N * N$ is generated by $N = N * \mathbb{C}$ and $N_0 = \alpha_1(N * \mathbb{C})$, with N, N_0 mutually free. We denote by $b'_g = \alpha_1(a_g)$. Thus, it is suficient to prove that $\tau(uX) = 0$ for X of the form $X_1Y_1X_2Y_2...$ or of the form $Y_0X_1Y_1...$, where X_i are words in a_g and Y_j are words in b'_g.

By the relation in *Step 7* we have:

$$\tau(uX) = \tau(puXp) = \tau(U'_g(uX)U'^*_g)$$

$$= \tau(w_gU_g(uX)U^*_gw^*_g) = \tau(u\alpha_1(w_g)U_gXU^*_gw^*_g)$$

$$= \tau(u\alpha_1(w_g)\theta_g(X)w^*_g)$$

Thus, if X is a word of length at least 3 in X_i, Y_i, with all "letters" X_i, Y_i of trace 0, then the "middle" letters have only θ_g acting on them, being un-altered by the left multiplication by $w_g \in N * \mathbb{C}$ and by the right multiplication by $\alpha_1(w_g) \in N_0$. Since u can be approximated arbitrarily well by an element in the algebra generated by $\{a_h, b'_h \mid g, h \in S\}$, for some finite subset $S \subset G$, it follows that we have

$$\lim_{g\to\infty} \tau(u\alpha_1(w_g)\theta_g(X)w^*_g) = 0$$

for any such X. From the above equations, this implies $\tau(uX) = 0$. A similar argument shows that if X is of the form X_1Y_1 with $\tau(X_1) = \tau(Y_1) = 0$ then $\tau(uX) = 0$.

Thus, we are left to check the equality $\tau(uX) = 0$ for $X = Y_0X_1$. But this brings us to a situation equivalent to the hypothesis of Lemma 4.7. By using *Step 2* and Lemma 4.7 in the same way as in the *Proof of* 4.1.1°, it follows that $\tau(uX) = 0$ in this case as well. Q.E.D.

6.2. Remark. Note that Theorem 6.1 provides a different class of cocycle actions by property T groups G on $L(\mathbb{F}_\infty)$ that cannot be perturbed to genuine actions, than the cocycle action of Connes and Jones ([**CJ**]; see the Appendix 2 in this paper).

Indeed, each of our cocycle actions σ^t on the factor $(L(\mathbb{F}_\infty))^t$ (which is isomorphic to $L(\mathbb{F}_\infty), \forall t$, by [**V3, Ra1**]), obtained by reducing a free Bernoulli shift σ by a projection of trace t, has the property that its t^{-1} amplification can be perturbed to an action. While none of the amplifications of the cocycle action in ([**CJ**]) can be perturbed to an action. Indeed, the argument is the same: $(L(\mathbb{F}_n))^s$ still has

Haagerup's compact approximation property ([**H**]), $\forall s > 0$, so $L_\mu(G)$ cannot be embedded in it, $\forall \mu$ a scalar 2-cocycle of G.

6.3. Remarks. 1°. Related to 6.1, to the examples in ([**CJ**]) and to the problem 5.7.3, the following question is in place:

Find the class $\mathbb{G}$ *of all groups for which any cocycle action on any type* II_1 *factor can be perturbed to a genuine action.*

Note that, like for cocycle actions on the hyperfinite type II_1 factor, the class $\mathbb{G}$ contains all amenable groups ([**Po1**]) and, more generally, all the free products of amenable groups with amalgamation over finite subgroups (this follows trivialy by applying [**Su, J1**] and [**Po1**]). We do not know of any other example of group in this class.

2°. The Connes-Jones cocycles seem to be "universally bad", so the class of groups G (with some specified presentation $\mathbb{F}_n \to G \to 1$) for which such cocycles vanish (or do not vanish) is an interesting test example to study for the Problem 6.3.1° above. The obvious obstruction for vanishing cohomology to look at in this case would be the non-embeddability of the group algebra $L(G)$ into $L(\mathbb{F}_n)$, like in the argument in ([**CJ**]). So the following problem (see [**Ge**]), seems to be very interesting :

What is the class of von Neumann algebras that can be embedded in the free group factors $L(\mathbb{F}_n)$ *? In particular, what are the groups* G *for which* $L(G)$ *can be embedded in* $L(\mathbb{F}_n)$ *?.*

3°. Along these lines, Liming Ge has speculated that $L(\mathbb{F}_n)$ does not contain non-injective von Neumann subalgebras N_0 with $N_0' \cap L(F_n)$ diffuse (without atoms). Equivalently, if N_0 is not injective and $\mathcal{Z}(N_0) \simeq L^\infty([0,1])$ then N_0 cannot be embedded in $L(\mathbb{F}_n), n \geq 2$. If true, Ge's conjecture would imply that a group G_0 is non-amenable if and only if $L(G_0 \oplus \mathbb{Z})$ cannot be embedded into $L(\mathbb{F}_n)$. Related to this, Shlyakhtenko has asked whether the factor $(L(\mathbb{F}_\infty)\overline{\otimes}L^\infty([0,1],\lambda)) * L(\mathbb{F}_\infty)$ is isomorphic to $L(\mathbb{F}_\infty)$. An affirmative answer to Ge's conjecture would imply these factors are not isomorphic.

In fact, it even seems possible that no *thin* type II_1 factors ([**Po2**]) other than the hyperfinite ones can be embedded into $L(\mathbb{F}_n)$. In particular, this would imply that no non-injective factors with the property Γ of Murray and von Neumann ([**MvN**]) can be embedded into $L(\mathbb{F}_n)$ (such factors being thin by [**GePo**]). In this respect, let us point out the following:

6.4. Proposition. *Assume the type* II_1 *factor* N *(with separable predual) contains a non-injective von Neumann subalgebra* N_0 *such that* $N_0' \cap N^\omega$ *is a diffuse von Neumann algebra, where* N^ω *is an ultrapower algebra of* N*. Then there exists a non-injective von Neumann subalgebra* $N_1 \subset N$ *such that* $N_1' \cap N$ *is diffuse.*

Moreover, if N_0 is a factor then N_1 can be taken to be a factor. In particular, if N contains a non-hyperfinite subfactor with the property Γ then it contains a non-hyperfinite subfactor with the property Γ with diffuse relative commutant.

Proof. Since N_0 in not injective, there exists a finite set $\{x_1, x_2, \dots, x_n\} \subset N_0$ such that the von Neumann algebra it generates is non-injective. By Connes' characterisation of injectivity with hypertraces it follows that there exists $\varepsilon > 0$ such that if $\{x'_1, \dots, x'_n\} \subset N$ are so that $\|x'_i - x_i\|_2 \leq \varepsilon$ then $\{x'_i\}_i$ generates a non-injective von Neumann subalgebra in N.

Since $N'_0 \cap N^\omega$ is diffuse, it follows by induction that there exists a sequence of mutually commuting, τ-independent two dimensional abelian *-subalgebras $A_n \subset N$, with minimal projections of trace 1/2, such that if we denote by $B_n = A_1 \vee A_2 \vee ...A_n$ then

$$\|E_{A'_{n+1} \cap N}(x_i) - x_i\|_2 < \varepsilon/2^{n+1}, \forall i.$$

But then we also have

$$\|E_{B'_n \cap N}(x_i) - E_{B'_n \cap N}(E_{A'_{n+1} \cap N}(x_i))\|_2 < \varepsilon/2^{n+1}, \forall 1 \leq i \leq n,$$

Since $E_{B'_n \cap N} \circ E_{A'_{n+1} \cap N} = E_{B'_{n+1} \cap N}$, if we denote by $A = \vee_n B_n$ and take into account that $E_{A' \cap N} = \lim_n E_{B'_n \cap N}$ (see e.g., [**Po**]), then by triangle inequalities we get

$$\|x_i - E_{A' \cap N}(x_i)\|_2 \leq \varepsilon, \forall i.$$

Thus, if we take N_1 to be the von Neumann algebra generated by

$$x'_i = E_{A' \cap N}(x_i),\ 1 \leq i \leq n,$$

then N_1 satisfies the required conditions.

To get N_1 to be a factor when N_0 is a factor, we take at the same time with the elemnts x_i some additonal finite sets of unitaries, at each step of the induction argument, such that averaging of words of fixed length of elements previously chosen are close to scalars. Q.E.D.

Remarks 6.5. 1°. There is a striking analogy between the symmetry structure of the free group algebras and of the hyperfinite type II_1 factor, with the free group factor $L(\mathbb{F}_\infty)$ seemingly having an edge....

Thus, all discrete groups act freely on the factor $L(\mathbb{F}_\infty)$, by free Bernoulli shifts. This should be compared with the fact that all groups act freely on the hyperfinite factor, by Bernoulli shifts. Moreover, the first examples of embeddings of arbitrary finite groups G in some $\text{Aut}N/\text{Int}N$, with a given arbitrary scalar 3-cocycle, were constructed in ([**Su**]) on a "free-like" factor. Soon after, such examples were constructed in ([**J1**]) for arbitrary amenable groups G, on the hyperfinite type II_1 factor N. In fact all "amenable symmetries" are now known to be present on the hyperfinite II_1 factor by ([**Po8, 9**]).

All "finite symmetries" were shown to act on the free group factors $L(\mathbb{F}_n), \forall 2 \leq n \leq \infty$, in ([**Ra2**]). In ([**ShU**]) it has been shown that there exist prime actions of any quantum group $SU_q(n)$ on the factor $L(\mathbb{F}_\infty)$. More generally, it was recently proved in ([**ShU**]) that any symmetry "acts" on the free group factor $L(\mathbb{F}_\infty)$. In fact, by ([**PoSh**]) it seems that any kind of symmetry-phenomenon (the word being taken in a very broad sense) that can be realized on some type II_1 factor, can be realized on $L(\mathbb{F}_\infty)$ as well. Theorem 6.1 above, which parallels the "hyperfinite" results 5.5-5.8, seems to confirm this fact. Moreover, it seems that $L(\mathbb{F}_\infty)$ is the unique factor having such a universality property with respect to symmetries.

2°. Related to the comments above, we have already mentioned in Remark 5.10.3° Choda's result showing that there exist actions of $SL(n, \mathbb{Z}), n \geq 3$ on the hyperfinite factor such that the associated cross product algebra has the property T (in the sense of ([**CJ**]). Yet we do not know to construct an action of a property T group on $L(\mathbb{F}_\infty)$ that would give rise to a property T factor, via cross-product construction. Same for $L(\mathbb{F}_n), 2 \leq n < \infty$.

In fact, while the free group algebras with finitely many generators apriori seem to have lesser symmetries, it seems more "feasable" to construct such rigid actions of property T groups on $L(\mathbb{F}_n)$ than on $L(\mathbb{F}_\infty)$. On the other hand, there is a lack of examples of actions of arbitrary groups G on $L(\mathbb{F}_n)$ that would behave as the free Bernoulli shifts on $L(\mathbb{F}_\infty)$ (for instance, actions of G on $L(\mathbb{F}_n)$ that would implement a multiple of the left regular representation of G on $\ell^2(\mathbb{F}_n)$, as in [**J2**], while not normalizing any non-trivial hyperfinite subalgebra of $L(\mathbb{F}_n)$, as in [**Po3**]).

3°. Related to the proof of Theorem 6.1 above and to (Lemma 4.3.2 in [**Po4**]), the following question on the group of automorphisms of the free group factors seems natural to ask:

Is $\mathrm{Aut}(L(\mathbb{F}_n)), 2 \leq n \leq \infty$, *path-wise connected?*

One should note that $\mathrm{Aut}R$ does have this property, i.e., it is path-wise connected. Indeed, this follows trivially by the classification of single automorphisms in ([**C2, 3**]), which reduces the problem to some model automorphisms and ultimately to aperiodic such automorphisms, as follows:

Any automorphism $\theta \in \mathrm{Aut}R$ is conjugate to an inner perturbation of a model automorphism ρ_0. But $\mathrm{Ad}(u) \circ \rho_0$ is path-wise connected to ρ_0 and $\rho_0 \sim \rho_0 \otimes id_R$. Taking the "model" aperiodic automorphism to be an irrational rotation ρ_1, it follows that ρ_1 is path-wise connected to id_R. Thus, ρ_0 follows path-wise connected to $\rho_0 \otimes \rho_1$. But the latter is aperiodic, so it is connected to ρ_1, thus to id_R.

APPENDIX.

A.1. Definition of property T. We recall here a well known equivalent formulation of the property T for groups (see e.g., [**DKi**] or [**dHV**]; see also Lemma 4.1.5 in [**Po4**] for a general such statement).

A.1.1. Lemma. *A group G has the property T of Kazhdan if and only if there exist $g_1, g_2, ..., g_n \in G$, $\varepsilon_0 > 0$ and $K > 0$ such that for any representation π of G on a Hilbert space $\mathcal{H}$ and any vector $\xi \in \mathcal{H}$ satisfying*

$$\|\pi(g_i)\xi - \xi\| \leq \varepsilon_0 \|\xi\|, \forall i,$$

one has

$$\sup_{g \in G} \|\pi(g)\xi - \xi\| \leq K \max_{1 \leq i \leq n} \|\pi(g_i)\xi - \xi\|$$

A.2. Group algebra extensions. An important class of cocycle actions can be obtained from inclusions of group von Neumann algebras, as follows. Let $1 \to H \hookrightarrow K \to G \to 1$ be an exact sequence of discrete groups and note that one has:

A.2.1. Lemma. *Let $H \subset K$ be an inclusion of discrete groups. Then $L(H)' \cap L(K) = \mathbb{C}$ if and only if K has infinite conjugacy classes relative to H, i.e., $\{hgh^{-1} \mid h \in H\}$ is an infinite set $\forall g \in G$, $g \neq e$.*

Proof. The proof is identical to the classical case for single groups ([**MvN**], [**Di**]). Q.E.D.

For each $g \in G$ let $k(g) \in K$ be a lifting of g in K and denote by $\sigma_g \in \text{Aut}\, L(H)$ the automorphism $\sigma_g(x) = u_{k(g)} x u^*_{k(g)}$, $x \in L(H)$. Also for each $g_1, g_2 \in G$ denote by v_{g_1,g_2} the unitary element $u_h \in L(H) \subset L(K)$, where $h = k(g_1)k(g_2)k(g_1g_2)^{-1} \in H$. Then (σ, v) is clearly a properly outer cocycle action and $L(H) \subset L(K)$ can be viewed as the cocycle cross product $N \subset N \rtimes_{\sigma,v} G$, where $N = L(H)$ (see [**NT**]).

Along these lines, a particularly interesting example is the *Connes-Jones cocycle* (cf. [**CJ**]): Let G be an arbitrary countable discrete group with generators $g_1, g_2, \cdots, g_n$, where $1 \leq n \leq \infty$, but with $G \neq \mathbb{F}_n$. Let $\mathbb{F}_n \to G \to 1$ be the corresponding presentation of $(G; g_1, \ldots, g_n)$. Let $H = \ker(\mathbb{F}_n \to G)$ and note that $H \simeq \mathbb{F}_{nm}$, where $m = |G|$. Then $\mathbb{F}_n$ has infinite conjugacy class relative to its normal subgroup H. Thus $(L(H) \subset L(\mathbb{F}_n)) = (L(H) \subset L(H) \rtimes_{\sigma,v} G)$ for an appropriate properly outer cocycle action (σ, v) of G on $L(H)$, with $H \simeq \mathbb{F}_\infty$ whenever G is an infinite group with generators $g_1, \ldots, g_n$ not all mutually free.

By ([**CJ**]), the Connes-Jones cocycle is non-vanishing when G is an infinite group with the property T. The proof is based on the observation that $L(G)$ cannot be embedded in $L(\mathbb{F}_n)$, due to Haagerup's compact approximation property for $L(\mathbb{F}_n)$

([**H**]). The same argument actually works for the group $G = SL(2,\mathbb{Z}) \rtimes \mathbb{Z}^2$, which doesn't have the property T. For the proof, one uses the rigidity of the inclusion $\mathbb{Z}^2 \subset G$ to conclude that if one takes the approximation of the identity on $L(\mathbb{F}_n)$ by compact, unital, trace preserving maps then the identity on $L(\mathbb{Z}^2)$ follows close to a compact operator, uniformly on all the unitaries in $L(\mathbb{Z}^2)$, a contradiction (see [**Po4**] for a complete argument along these lines).

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MATH.DEPT., UCLA, LA, CA 90095-155505
E-mail address: popa@math.ucla.edu