259A Fall 2024: "Introduction to C*- and W*-algebras"

Instructor: **Sorin Popa**. Meetings: **MWF 3-4pm, room TBA**

This is an introductory course in algebras of operators on Hilbert space, which are, roughly speaking, noncommutative versions of the algebras $C(X)$ of continuous complex functions on compact spaces X (the C*-*algebras*) and respectively of the bounded measurable functions $L^\infty(X, \mu)$ on a measure space (X, μ) (the W*-*algebras*).

Algebras of Operators allow extending topics in Analysis, Measure Theory, Ergodic Theory, Dynamics, Geometry and Topology to a "noncommutative setting". They are the natural framework for modern areas of mathematics such as quantum information theory, quantum computing, quantum groups.

The prerequisite for this class are the Real Analysis 245 and Functional Analysis 255 classes. Here is the material I would like to cover:

1. Revisiting the geometry of a Hilbert space $\mathcal{H}$. You have done this in 245, 255 classes, but I will nevertheless go through definitions and basic results, without proofs.

2. The space $\mathcal{B}(\mathcal{H})$ of all linear bounded operators on the Hilbert space $\mathcal{H}$: (a) the adjoint (or *) operation; (b) the operator norm, the *wo* and *so*-topologies; (c) special classes of operators (self-adjoint, positive, unitaries, compact, etc);

3. Definition of C*-algebras and W*-algebras as subalgebras of $\mathcal{B}(\mathcal{H})$; (a) first examples; (b) the commutant of a self-adjoint set in $\mathcal{B}(\mathcal{H})$ is a W*-algebra; (c) von Neumann's bicommutant theorem.

4. Abstract C*-algebras (in case you have done $(a) - (f)$ hereafter in 255B then I'll go over it without proofs): (a) definition and examples; (b) characterization of commutative C*-algebras as $C(X)$ and continuous functional calculus; (c) ideals and morphisms; (d) positivity; (e) positive functionals; (f) GNS construction and representation of an abstract C*-algebra as algebra of operators on Hilbert space.

5. More examples of C*-algebras: group C*-algebras (reduced and universal), crossed product constructions, inductive limits, UHF-algebras, AF-algebras.

6. Some basic facts on tensor products of C*-algebras.

7. Completely positive maps: definition, conditional expectations and Tomyiama's theorem, Arveson's extension theorem, Stinespring dilation theorem.

8. Back to W*-algebras: Borel functional calculus and polar decomposition in $\mathcal{B}(\mathcal{H})$ and in W*-algebras; Kaplansky density theorem; geometry of projections (quantum logic); finite and properly infinite W*-algebras; classification by type; W*-algebras with trace and II$_1$ factors.

9. Examples of II$_1$ factors: (a) group factors; (b) group measure space factors;

All registered students will get an A, but will have to make a presentation on Mondays 4-6pm, towards the end of the quarter.

Useful texts: J. Dixmier *C*-algebras and their representations*; S. Sakai *C*-algebras and W*-algebras*; G. Pedersen *C*-algebras and their automorphism groups.*
My notes on II1 factors: https://www.math.ucla.edu/popa/Books/IIunV15.pdf
Notes from my 259A Fall 2020:
https://drive.google.com/drive/folders/1RGsIvFo2-ZfSRXecovKqiyUvY9drjVtj