Generalized Doubling meets Poincare

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Abstract. We establish a modified segment inequality on metric spaces that satisfy a generalized volume doubling property. This leads to Sobolev and Poincaré inequalities for such spaces. We also give several examples of spaces that satisfy the generalized doubling condition.

1. Introduction

Hajłasz and Koskela showed in [**5**] that a weak Poincare Inequality is sufficient to derive Sobolev Inequalities on any space with a doubling measure. In this paper we demonstrate that by strengthening the metric measure condition we can in fact establish the Poincaré inequality. Namely, instead of insisting on just concentric balls growing at a more or less predictable rate, we define a generalized doubling property. Here, we say that for any compact set and point in our space we can select a set of distance minimizing segments from the set to the point such that the measure of the set of t midpoints of these segments is at least a certain fraction of the measure of the final set. This not only contains the standard doubling condition as a special case, but also implies a slightly modified version of Cheeger-Colding's segment inequality [**3**], and thus a weak $(1,1)$-Poincare Inequality.

Furthermore, we show by example that this condition is easy to check in a number of cases including vector spaces with Minkowski norms, Riemannian manifolds, Finsler manifolds, certain Carnot-Caratheodory spaces, and Gromov-Hausdorff limit spaces of a sequence that satisfies generalized doubling. We also provide examples of easy to define spaces, such as a wedge of spheres, which do not satisfy generalized doubling or, incidentally, Poincare inequality. The fact that our condition holds on such a variety of spaces indicates that it is not a good way of generalizing Ricci curvature. This despite the fact that on Riemannian manifolds our condition is essentially equivalent to lower Ricci curvature bounds. The work in [**9**], [**14**], and [**15**] indicate that a somewhat more intricate condition is needed in order to find the correct framework for Ricci curvature on metric measure spaces.

In [**12**] Semmes discusses a condition similar to our own. He considers the question of existence of a collection of paths with common endpoints each not too much longer than minimal and shows that existence of a measure with special properties on such a collection is equivalent to certain topological information as well as analytic inequalities.

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For more information about analysis on metric spaces we recommend the text [**6**] by Heinonen.

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2. Volume Comparison

Throughout we assume that (X,d,μ) is a metric measure space. The metric space (X,d) is proper in the sense that all closed balls are compact, and in addition all pairs of points can be joined by a segment, i.e., a curve whose length equals the distance between the end points. The measure μ is assumed to be Radon, i.e., it is a regular Borel measure where all compact sets have finite measure. For subsets $A,B\subset X$ let $\Gamma_{A,B}$ be the set of segments $\gamma:[0,1]\to X$ such that $\gamma(0)\in A$ and $\gamma(1)\in B$.

There are several ways of formulating the generalized volume comparison. On a space where $\Gamma_{p,q}$ consists of a single segment for almost all $p,q\in X$ these conditions are all equivalent. In particular they are equivalent for Riemannian manifolds. For $p\in X$ define $E_t:\Gamma_{p,X}\to X$ by $E_t(\gamma)=\gamma(t)$. *Backward comparison* is defined as

$$C\mu\left(E_t\left(E_1^{-1}(A)\right)\right)\geq\mu(A),\text{ for all compact }A\subset X.$$

While *forward comparison* is

$$C\mu(B)\geq\mu\left(E_1\left(E_t^{-1}(B)\right)\right),\text{ for all compact }B\subset X.$$

If we let $B=E_t\left(E_1^{-1}(A)\right)$ then we immediately get that

$$\begin{aligned}\mu\left(E_1\left(E_t^{-1}(B)\right)\right) &= \mu\left(E_1\left(E_t^{-1}\left(E_t\left(E_1^{-1}(A)\right)\right)\right)\right)\\ &\geq \mu\left(E_1\left(E_1^{-1}(A)\right)\right)\\ &= \mu(A).\end{aligned}$$

Thus forward comparison implies backward comparison. In a space where segments are not unique a.e. forward comparison is too strong a condition, while backward comparison could be too weak to tell us much. Thus we make a selection of segments $\Gamma'_{p,X}\subset\Gamma_{p,X}$ such that the restriction $E'_1:\Gamma'_{p,X}\to X$ is a Borel bijection. This uses the Borel measurable selection principle for proper maps (see [**8**] and [**10**]). We then have the two alternate versions

$$C\mu\left(E'_t\left(\left(E'_1\right)^{-1}(A)\right)\right)\geq\mu(A),\text{ for all compact }A\subset X,$$

and

$$C\mu(B)\geq\mu\left(E'_1\left(\left(E'_t\right)^{-1}(B)\right)\right),\text{ for all compact }B\subset X.$$

This selection makes backward comparison a stronger condition while forward comparison becomes weaker. In fact the two conditions are now equivalent. It is again clear that forward comparison gives us backward comparison. To check the reverse we note that

$$E'_t\left(\left(E'_1\right)^{-1}\left(E'_1\left(\left(E'_t\right)^{-1}(B)\right)\right)\right)=E'_t\left(\left(E'_t\right)^{-1}(B)\right)\subset B$$

since E_1' is a bijection. Therefore, if $A = E_1'\left(\left(E_t'\right)^{-1}(B)\right)$ we have

$$\begin{aligned} C\mu(B) &\geq C\mu\left(E_t'\left(\left(E_1'\right)^{-1}(A)\right)\right) \\ &\geq \mu(A) \\ &= \mu\left(E_1'\left(\left(E_t'\right)^{-1}(B)\right)\right) \end{aligned}$$

For a metric measure space (X, d, μ) we say that it satisfies the *generalized volume doubling property with constant* C if for all $p \in X$ there are subsets $\Gamma_{p,X}' \subset \Gamma_{p,X}$ as above such that

$$C\mu\left(E_t'\left(\left(E_1'\right)^{-1}(A)\right)\right) \geq \mu(A), \text{ for all compact } A \subset X, \text{ and } t \geq \frac{1}{2},$$

or equivalently that

$$C\mu(B) \geq \mu\left(E_1'\left(\left(E_t'\right)^{-1}(B)\right)\right), \text{ for all compact } B \subset X, \text{ and } t \geq \frac{1}{2}.$$

If we construct the "contractions"

$$\begin{aligned} F_{p,t} &: X \to X, \\ x &\to E_t'\left(E_1'\right)^{-1}(x) \end{aligned}$$

we see that the two equivalent conditions can be stated as

$$\begin{aligned} C\mu\left(F_{p,t}(A)\right) &\geq \mu(A), \text{ for } t \geq \frac{1}{2}, \\ \left(F_{p,t}\right)_* \mu &\leq C\mu, \text{ for } t \geq \frac{1}{2}. \end{aligned}$$

Note that the contractions have the scaling property:

$$d\left(p, F_{p,t}(x)\right) = t d(p, x),$$

and that the curves

$$\gamma_{p,x}(t) = F_{p,t}(x)$$

are the selected segments from p to $x \in X$.

3. Basic Properties

In this section we establish some simple properties of spaces that satisfy the generalized doubling condition. First note that when $A = B(p, r)$ is a ball centered at p we get the usual volume doubling condition

$$C\mu\left(B\left(p, \frac{r}{2}\right)\right) \geq \mu(B(p, r)).$$

The doubling condition for metric balls centered at p immediately implies that

$$(2t \cdot r)^{\log_2 C} \mu(B(p, t \cdot r)) \geq \mu(B(p, r))$$

and hence that all open sets have nonzero measure. A similar result holds for generalized doubling.

Proposition 1. *Let (X, d, μ) satisfy the generalized doubling condition, then*

$$\left(F_{p,t}\right)_* \mu \leq \left(\frac{t}{2}\right)^{-\log_2 C} \mu.$$

PROOF. The key is to use that

$$F_{p,t_1} \circ F_{p,t_2} = F_{p,t_1 t_2}$$

for all $t_1, t_2 \in [0,1]$. Thus for if $s = t^{\frac{1}{k}} \geq \frac{1}{2}$, then

$$(F_{p,t})_* \mu = (F_{p,s})_*^k \mu \leq C^k \mu$$

So for all $t \in \left[\left(\frac{1}{2}\right)^k, \left(\frac{1}{2}\right)^{k-1} \right)$ we have

$$(F_{p,t})_* \mu \leq C^k \mu = \left(\frac{1}{2}\right)^{-k \log_2 C} \leq \left(\frac{t}{2}\right)^{-\log_2 C} \mu.$$

□

This proof can also be used to show that if

$$(F_{p,t})_* \mu \leq C\mu, \text{ for } t \geq 1 - \varepsilon,$$

then

$$(F_{p,t})_* \mu \leq C^k \mu, \text{ for } t \geq \frac{1}{2},$$

as long as

$$k \geq \frac{-1}{\log_2 (1 - \varepsilon)}.$$

With generalized doubling we can also get an upper bound for the volume of balls.

PROPOSITION 2. *Let (X, d, μ) satisfy the generalized doubling condition. Then*

$$\mu (B (p, \varepsilon)) = O (\varepsilon).$$

PROOF. Consider a set B with $\operatorname{diam} B \leq \varepsilon$ and assume that we can find $p \in X$ such that $d(p, B) \geq R$. Then $F_{p,t}(B) \cap F_{p,s}(B) = \emptyset$ if $|t - s| > \frac{\varepsilon}{R}$. So if we select $\frac{1}{2} \leq t_1 < t_2 < \cdots < t_N \leq 1$ with $t_{i+1} - t_i > \frac{\varepsilon}{R}$, then we get N disjoint sets $F_{p,t_i}(B) \subset B(p, R + \varepsilon)$ and hence

$$\mu (B (p, R + \varepsilon)) \geq \sum_{i=1}^{N} \mu (F_{p,t_i} (B)) \geq C^{-1} N \mu (B).$$

As we can select $N > \frac{R}{2\varepsilon} - 1$ we get that

$$\begin{aligned} \mu (B) &\leq \frac{C\mu (B (p, R + \varepsilon))}{\frac{R}{2\varepsilon} - 1} \\ &\leq \varepsilon \frac{3C\mu (B (p, 2R))}{R} \end{aligned}$$

if $R >> \varepsilon$. □

Finally we show that generalized doubling is preserved under convergence.

PROPOSITION 3. *If $(X_i, d_i, \mu_i) \to (X, d, \mu)$ is a convergent sequence of spaces in the measured Gromov-Hausdorff topology that all satisfy generalized doubling with the same constant C, then the limit space also satisfies generalized doubling with the constant C.*

PROOF. The convergence $(X_i, d_i, \mu_i) \to (X, d, \mu)$ is equivelent to saying that we have Borel measurable maps

$$\begin{aligned} f_i &: X_i \to X, \\ g_i &: X_i \to X, \end{aligned}$$

and a sequence $\varepsilon_i \to 0$ such that

$$\begin{aligned} |d(f_i(x), f_i(y)) - d(x, y)| &\leq \varepsilon_i, \text{ for all } x, y \in X_i \\ |d_i(g_i(x), g_i(y)) - d_i(x_i, y_i)| &\leq \varepsilon_i, \text{ for all } x, y \in X, \end{aligned}$$

$f_i(X_i)$ is ε_i-dense in X, $g_i(X)$ is ε_i-dense in X_i,

$$\begin{aligned} |\mu(A) - (f_i)_* \mu_i(A)| &\leq \varepsilon_i, \text{ for all } A \subset X, \\ |\mu_i(B) - (g_i)_* \mu(B)| &\leq \varepsilon_i, \text{ for all } A \subset X_i. \end{aligned}$$

The selected segments $\gamma_{p,q} : [0,1] \to X$ are obtained as the limits of the (discontinuous) curves $f_i \circ \gamma_{g_i(p),g_i(q)}$ where $\gamma_{g_i(p),g_i(q)} : [0,1] \to X_i$ is the selected segment in X_i. If we let

$$\begin{aligned} F_{p,t} &: X \to X, \\ x &\to \gamma_{p,x}(t) \end{aligned}$$

and similarily

$$\begin{aligned} F^i_{p,t} &: X_i \to X_i, \\ x &\to \gamma_{p,x}(t) \end{aligned}$$

then we note that

$$\begin{aligned} \left(f_i \circ F^i_{g_i(p),t} \circ g_i\right)(x) &= f_i\left(\gamma_{g_i(p),g_i(x)}(t)\right) \\ &\to \gamma_{p,x}(t) \\ &= F_{p,t}(x). \end{aligned}$$

Thus we also have that

$$\left(f_i \circ F^i_{g_i(p),t} \circ g_i\right)_*(\mu) \to (F_{p,t})_* \mu.$$

Since

$$\begin{aligned} |\mu(A) - (f_i)_*(\mu_i)(A)| &\leq \varepsilon_i, \\ C\left(f_i \circ F^i_{g_i(p),t}\right)_*(\mu_i)(A) &\geq (f_i)_*(\mu_i)(A), \\ \left|\left(f_i \circ F^i_{g_i(p),t}\right)_*(\mu_i)(A) - \left(f_i \circ F^i_{g_i(p),t} \circ g_i\right)_*(\mu)(A)\right| &\leq \varepsilon_i. \end{aligned}$$

this finishes the argument. □

Since the generalized doubling property with constant C is invariant under scaling the metric and/or the measure, we see that this property is transferred to the tangent cones of the space.

4. Examples

Here are some basic examples of spaces that satisfy the generalized doubling condition.

EXAMPLE 1. *$X = \mathbb{R}^n$ with the Euclidean metric and $C = 2^n$. Since the space is homogeneous we just need $F_{0,t}(x) = tx$ and that*

$$\det DF_{0,t} = t^n.$$

EXAMPLE 2. *Again $X = \mathbb{R}^n$ with the same measure but a Minkowski norm that comes from letting the unit ball be a convex set containing the origin. Since such a norm is still scaling invariant with respect to positive scalars we can again use $F_{0,t}(x) = tx$. This means that $C = 2^n$ as in the Euclidean case.*

Things are a bit different if we consider the more general volume comparisons were no selection of geodesics is made. Backwards comparison always holds but forwards comparison might not. Take, e.g., the maximum norm on $\mathbb{R}^2$. If $p = (0,0)$ and $q = \left(\frac{1}{2}, 0\right)$ then

$$\{(1,s) : s \in [-1,1]\} \subset E_1\left(E_{\frac{1}{2}}^{-1}(q)\right).$$

So if $B = B(q,\varepsilon)$, then we have that

$$\begin{aligned} \mu(B) &\approx \varepsilon^2, \\ \mu\left(E_1\left(E_{\frac{1}{2}}^{-1}(B)\right)\right) &\approx \varepsilon. \end{aligned}$$

Therefore, the unrestricted forward comparison cannot hold.

EXAMPLE 3. *This leads to an interesting convergence example. Consider $\mathbb{R}^n$ with the p-norm*

$$\|x\| = \left(\left(x^1\right)^p + \cdots + \left(x^n\right)^p\right)^{\frac{1}{p}}$$

when p is an even integer this is a smooth Finsler space. As $p \to \infty$ we see that it converges to $\mathbb{R}^n$ with the maximum norm. In view of the previous example this shows that unrestricted forward comparison is not preserved by measured Gromov-Hausdorff convergence.

EXAMPLE 4. *The Heisenberg group is generalized doubling. Since the space is homogeneous we only need to find the contractions at one point. The space is $\mathbb{R}^3$ with the usual Lebesgue measure, and the metric comes from the "norm"*

$$||(x,y,z)|| = \left(\left(x^2 + y^2\right)^2 + z^2\right)^{\frac{1}{4}}$$

The contraction map is then given by

$$F_{0,t}(x,y,z) = \left(tx, ty, t^2 z\right)$$

This is linear with Jacobian determinant t^4 so we get the generalized doubling property with $C = 2^4$ (see also [**11**] *and* [**7**].*)*

EXAMPLE 5. *Let (M,g) be a Riemannian manifold with* $\mathrm{Ric} \geq (n-1)k$, *then we can use*

$$C = 2\left(\frac{s_k(R)}{s_k\left(\frac{1}{2}R\right)}\right)^{n-1}$$

if $B \subset B(p,R)$. When $k = 0$ this reduces to $C = 2^n$ as in the Euclidean case.

EXAMPLE 6. *Let (M, g) be a compact Riemannian manifold with measure coming from ϕ*dvol, *where ϕ is a positive function. If* $\mathrm{Ric} \geq (n-1)\, k$ *then we can use*

$$C = \frac{\max \phi}{\min \phi} 2 \left(\frac{s_k(R)}{s_k\left(\frac{1}{2}R\right)} \right)^{n-1}.$$

EXAMPLE 7. *If we consider a family of pointed Riemannian manifolds (M_i, g_i, p_i) with the renormalized measures $\mu_i = \frac{1}{\mathrm{vol}(B(p_i,1))} \mathrm{dvol}_{g_i}$ where we assume that $Ric_{g_i} \geq (n-1)\, k$, then we have as in the previous example that these spaces satisfy generalized doubling. This condition will be transfered to a limit space in the measured Gromov-Hausdorff topology (see also* [**2**] *and* [**4**].*)*

EXAMPLE 8. *Let (M, F) be a Finsler space such that* $\mathrm{Ric} \geq (n-1)\, k$ and $|H| \leq h$ *(see* [**13**]*) then we can use*

$$C = 2 \left(e^{\frac{h}{2}R} \frac{s_k(R)}{s_k\left(\frac{1}{2}R\right)} \right)^{n-1}$$

if $B \subset B(p, R)$. In case M is Riemannian or all tangent spaces are isometric to each other we can use $h = 0$. The special case where $k = 0$ and $h = 0$ allows us to use $C = 2^n$ as in the case of Euclidean space with a Minkowski norm.

With the Heisenberg group in mind one can also see that compact regular Carnot-Caratheodory spaces with finite diameter are generalized doubling see also [**1**].

Here is a simple example which is not generalized doubling, but nevertheless a very nice space. A space is Ahlfors d-regular if there are constants $d \geq 1$ and $C > 1$ so that

$$C^{-1} r^d \leq \mu(B(p, r)) \leq C r^d$$

for all $p \in X$ and $r < \mathrm{diam}(M)$. Ahlfors spaces are clearly doubling spaces.

EXAMPLE 9. *A wedge of two spheres has this nice property but is not generalized doubling. $X = S_1 \vee S_2$, where S_1 and S_2 are isometric n-spheres. Let w be the common wedge point and p the antipodal point on S_1. Then consider an ε annulus $B \subset S_2$ around the equator. This set has volume $\approx \varepsilon$. The sets $B_{p,t}$ then mostly look like annuli as well. However when $t \approx \frac{2}{3}$ the set $B_{p,t}$ will be concentrated near the wedge point and volume $\approx \varepsilon^n$. So we have*

$$\begin{aligned} \mu(B) &\approx \varepsilon, \\ \mu\left(B_{p,\frac{2}{3}}\right) &\approx \varepsilon^n. \end{aligned}$$

This shows that there can't be a uniform C in the generalized doubling when $\varepsilon \to 0$.

Similar examples can be constructed where two spaces are glued together along a set which has codimension ≥ 2 in one of the spaces.

5. Segment and Poincaré Inequalities

We claim that generalized doubling implies a modified and slightly stronger segment inequality. The reason that it is stronger lies in the fact that we parametrize our geodesics on $[0, 1]$ rather than by arclength.

LEMMA 1. *Let (X,d,μ) be a metric measure space that satisfies generalized doubling with constant C, then*

$$\int_{A\times B}\mathcal{F}_g(x,y)\,d(\mu\times\mu)\leq\frac{C}{2}(\mu(A)+\mu(B))\int_W gd\mu,$$

where

$$\mathcal{F}_g(x,y)=\inf_{\gamma_{x,y}\in\Gamma_{x,y}}\int_0^1 g\circ\gamma_{x,y}dt,$$

g is a measurable nonnegative function, and W is a set that contains the geodesics in $\Gamma'_{p,B}$ and $\Gamma'_{q,A}$ for all $p\in A$ and $q\in B$.

PROOF. We basically follow the proof in [**3**]. Note that

$$\mathcal{F}_g(p,y)\leq\int_0^1 g\circ\gamma_{p,y}(t)\,dt.$$

We have

$$\begin{aligned}\int_B\int_{\frac{1}{2}}^1 g\circ\gamma_{p,y}(t)\,dtd\mu(y) &= \int_{\frac{1}{2}}^1\int_B g\circ\gamma_{p,y}(t)\,d\mu(y)\,dt\\ &= \int_{\frac{1}{2}}^1\int_B g\circ F_{p,t}d\mu dt\\ &= \int_{\frac{1}{2}}^1\int_{F_{p,t}(B)} gd\left((F_{p,t})_*\mu\right)dt\\ &\leq C\int_{\frac{1}{2}}^1\int_{F_{p,t}(B)} gd\mu dt\\ &\leq C\int_{\frac{1}{2}}^1\left(\int_W gd\mu\right)dt\\ &= \frac{C}{2}\int_W gd\mu.\end{aligned}$$

Integrating this over all $p\in A$ gives

$$\int_A\int_B\int_{\frac{1}{2}}^1 g\circ\gamma_{p,y}dtd\mu(y)\,d\mu(p)\leq\frac{C}{2}\mu(A)\int_W gd\mu.$$

To estimate the other part of the integral

$$\int_A\int_B\int_0^{\frac{1}{2}} g\circ\gamma_{x,y}dtd\mu(y)\,d\mu(x)$$

we can for each $q\in B$ use that

$$\begin{aligned}\int_A\int_0^{\frac{1}{2}} g\circ\gamma_{x,q}dtd\mu &\leq \frac{C}{2}\int_W gd\mu,\\ \int_B\int_A\int_0^{\frac{1}{2}} g\circ\gamma_{x,q}dtd\mu(x)\,d\mu(q) &\leq \frac{C}{2}\mu(B)\int_W gd\mu.\end{aligned}$$

This gives the desired inequality. □

This gives us a weak $(1,1)$-Poincaré inequality:

Lemma 2. *Let (X,d,μ) be a metric measure space that satisfies generalized doubling with constant C, then*

$$\begin{aligned} \frac{1}{\operatorname{vol}\left(B\left(p,\frac{r}{2}\right)\right)}\int_{B\left(p,\frac{r}{2}\right)}\left|f(x)-\bar{f}\right|d\mu(x) &\leq \frac{C^2 r}{\operatorname{vol}(B(p,r))}\int_{B(p,r)} g\,d\mu, \\ \bar{f} &= \frac{1}{\operatorname{vol}\left(B\left(p,\frac{r}{2}\right)\right)}\int_{B\left(p,\frac{r}{2}\right)} f(y)\,d\mu(y) \end{aligned}$$

where g is an upper gradient for f.

Proof. Again we proceed as in [**3**]. Let f be a function on $B=B(p,r)$ with upper gradient g. Then

$$|f(x)-f(y)| \leq d(x,y)\,\mathcal{F}_g(x,y) \leq r\mathcal{F}_g(x,y)$$

for all $x,y\in B\left(p,\frac{r}{2}\right)$ so

$$\int_{B\left(p,\frac{r}{2}\right)\times B\left(p,\frac{r}{2}\right)}|f(x)-f(y)|\,d\mu(x)\,d\mu(y) \leq Cr\operatorname{vol}\left(B\left(p,\frac{r}{2}\right)\right)\int_{B(p,r)} g\,d\mu.$$

Keeping in mind that

$$\begin{aligned} \left|f(x)-\bar{f}\right| &= \frac{1}{\operatorname{vol}\left(B\left(p,\frac{r}{2}\right)\right)}\left|\int_{B\left(p,\frac{r}{2}\right)}(f(x)-f(y))\,d\mu(y)\right| \\ &\leq \frac{1}{\operatorname{vol}\left(B\left(p,\frac{r}{2}\right)\right)}\int_{B\left(p,\frac{r}{2}\right)}|f(x)-f(y)|\,d\mu(y) \end{aligned}$$

we get

$$\int_{B\left(p,\frac{r}{2}\right)}\left|f(x)-\bar{f}\right|d\mu(x) \leq Cr\int_{B(p,r)} g\,d\mu.$$

Since

$$C\operatorname{vol}\left(B\left(p,\frac{r}{2}\right)\right) \geq \operatorname{vol}(B(p,r))$$

we get

$$\begin{aligned} \frac{1}{\operatorname{vol}\left(B\left(p,\frac{r}{2}\right)\right)}\int_{B\left(p,\frac{r}{2}\right)}\left|f(x)-\bar{f}\right|d\mu(x) &\leq \frac{Cr}{\operatorname{vol}\left(B\left(p,\frac{r}{2}\right)\right)}\int_{B(p,r)} g\,d\mu \\ &\leq \frac{C^2 r}{\operatorname{vol}(B(p,r))}\int_{B(p,r)} g\,d\mu. \end{aligned}$$

□

This next corollary follows directly from [**6**, Theorem 4.18 and Theorem 9.19], given the conditions we have imposed on the space and that we have established the weak $(1,1)$-Poincaré inequality

Corollary 1. *Let (X,d,μ) be a compact metric measure space that satisfies generalized doubling with constant C. Then the space is $\log_2 C$ Loewner and satisfies $(q,1)$-Sobolev inequalities for suitable functions with*

$$q \leq \frac{\log_2 C}{(\log_2 C)-1}.$$

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