WEAK PREDICTION PRINCIPLES

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Abstract. We prove the consistency of the failure of the weak diamond Φ_λ at strongly inaccessible cardinals. On the other hand we show that the very weak diamond Ψ_λ is equivalent to the statement $2^{<\lambda} < 2^\lambda$ and hence holds at every strongly inaccessible cardinal.

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0. Introduction

The prediction principle $\diamondsuit_\lambda$ (diamond on λ) was discovered by Jensen, [6], who proved that it holds over any regular cardinal λ in the constructible universe. This principle says that there exists a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ of sets, $A_\alpha \subseteq \alpha$ for every $\alpha < \lambda$, such that for every $A \subseteq \lambda$ the set $\{\alpha < \lambda : A \cap \alpha = A_\alpha\}$ is stationary.

Jensen introduced the diamond in 1972, and the main focus was the case of $\lambda = \aleph_1$. It is immediate that $\diamondsuit_{\aleph_1} \Rightarrow 2^{\aleph_0} = \aleph_1$, but consistent that $2^{\aleph_0} = \aleph_1$ along with $\neg\diamondsuit_{\aleph_1}$. Motivated by algebraic constructions, Devlin and Shelah [2] introduced a weak form of the diamond principle which follows from the continuum hypothesis:

Definition 0.1 (The Devlin-Shelah weak diamond)**.** Let λ be a regular uncountable cardinal.
The weak diamond on λ (denoted by Φ_λ) is the following principle:
For every function $c : {}^{<\lambda}2 \to 2$ there exists a function $g \in {}^{\lambda}2$ such that $\{\alpha \in \lambda : c(f \upharpoonright \alpha) = g(\alpha)\}$ is a stationary subset of λ whenever $f \in {}^{\lambda}2$.

The idea is that we replace the prediction of the initial segments of a set (or a function) by predicting only their color. The function c is a coloring, and the function g is the weak diamond function which gives stationarily many guesses for the c-color of the initial segments of every function f. It is easy to see that the real diamond implies the weak diamond.

Concerning cardinal arithmetic, $\Phi_{\aleph_1}$ follows indeed from the continuum hypothesis. Moreover, the weak continuum hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ implies $\Phi_{\aleph_1}$ as proved in [2]. On the other hand, $\Phi_{\aleph_1}$ implies $2^{\aleph_0} < 2^{\aleph_1}$ (as noted by Uri Abraham) so the two assertions are equivalent.

The diamond and the weak diamond are prediction principles, while $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} = \aleph_1$ belong to cardinal arithmetic. The above statements show that a simple connection may exist. Actually, if $\lambda > \aleph_1$ and $\lambda = \kappa^+$ then the situation becomes even simpler:

Proposition 0.2 (Diamonds and cardinal arithmetic)**.** *Assume that* $\lambda = \kappa^+ > \aleph_1$. *Then* $\diamondsuit_\lambda \Leftrightarrow 2^\kappa = \kappa^+$ *and* $\Phi_\lambda \Leftrightarrow 2^\kappa < 2^{\kappa^+}$.

For the assertion concerning the diamond principle see [10]. The assertion for the weak diamond is a straightforward generalization of [2], the easy direction appears explicitly in [3] Proposition 1.2 and the substantial direction can be extracted from [2] upon replacing $\aleph_1$ by κ^+. The negation of $\diamondsuit_{\aleph_1}$ with the affirmation of $2^{\aleph_0} = \aleph_1$ has been recognized as a peculiarity of the first uncountable cardinal.

However, the situation is totally different if $\lambda = \mathrm{cf}(\lambda)$ is a limit cardinal. One direction is still easy and has a general nature. If $\diamondsuit_\lambda$ then $2^{<\lambda} = \lambda$ and if Φ_λ then $2^{<\lambda} < 2^\lambda$ (see Claim 1.1 below). Notice that these formulations coincide with the above description of the successor case $\lambda = \kappa^+$. But can we prove the opposite implication?

If $\lambda = \mathrm{cf}(\lambda)$ is a limit cardinal then λ is a large cardinal (in the philosophical sense; its existence cannot be established in ZFC). Now if λ is large enough then $\diamondsuit_\lambda$ holds. Measurability suffices, and actually much less. If λ is ineffable (a property which may live happily with V = L) or even subtle then $\diamondsuit_\lambda$ and hence also Φ_λ [7].

A striking (and unpublished) result of Woodin shows that small large cardinals are different than successor cardinals. Woodin proved the consistency of a strongly inaccessible cardinal λ for which $\diamondsuit_\lambda$ fails. Moreover, the construction can be strengthened to strongly Mahlo. In the current paper we prove the same assertion, upon replacing the diamond by the weak diamond. For both the diamond and the weak diamond we do not know what happens if λ is weakly compact.

We conclude that the cardinal arithmetic assumption $2^{<\lambda} < 2^\lambda$ is strictly weaker than the prediction principle Φ_λ. It is still tempting to look for a prediction principle which is characterized by $2^{<\lambda} < 2^\lambda$. We define the following:

Definition 0.3 (The very weak diamond)**.** Let λ be an uncountable cardinal.
The very weak diamond on λ (denoted by Ψ_λ) is the following principle:
For every function $c : {}^{<\lambda}2 \to 2$ there exists a function $g \in {}^\lambda 2$ such that $\{\alpha \in \lambda : c(f \restriction \alpha) = g(\alpha)\}$ is an unbounded subset of λ whenever $f \in {}^\lambda 2$.

As we shall see, $\Psi_\lambda \Leftrightarrow 2^{<\lambda} < 2^\lambda$ whenever $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. Let us mention Galvin's property which says that every collection $\{C_\alpha : \alpha < \lambda\}$ of club subsets of κ^+ has a sub-collection of size κ^+ whose intersection is a club. This principle follows from $2^{<\lambda} < 2^\lambda$ but consistent with $2^{<\lambda} = 2^\lambda$ (see [3]). Bringing all these principles together we have a systematic hierarchy of weak prediction principles, each of which implied by the stronger one but strictly weaker from the next stage.

The basic tool for proving $\neg\Phi_\lambda$ over small large cardinals is Radin forcing, [8]. Since there are several ways to introduce this forcing notion we indicate that our approach is taken from [4], and in particular we use the Jerusalem notation i.e. $p \leq_{\mathbb{R}} q$ means that q is stronger than p.

1. Very weak diamond and cardinal arithmetics

We commence with the easy direction about the connection between Ψ_λ and cardinal arithmetic.

We commence with the easy direction about the connection between Φ_λ and cardinal arithmetic. Actually, the claim below applies to Ψ_λ as well. A parallel assertion can be proved easily for $\Diamond_\lambda$.

Theorem 1.1 (The basic claim)**.** *Let λ be an uncountable cardinal. If Ψ_λ holds then $2^{<\lambda} < 2^\lambda$.*

Proof.
Assume that $2^{<\lambda} = 2^\lambda$. We will show that Ψ_λ fails. Let $b : 2^{<\lambda} \to 2^\lambda$ be a surjection, such that for every $\delta < \alpha < \lambda$ and every $t \in {}^\alpha 2$, if $\varepsilon \in [\delta, \alpha) \Rightarrow t(\varepsilon) = 0$, then $b(t) = b(t \upharpoonright \delta)$. We are trying to describe a coloring $F : {}^{<\lambda}2 \to 2$, which exemplifies the failure of the very weak diamond.

Assume $\alpha < \lambda$ and $\eta \in {}^\alpha 2$. Set $F(\eta) = [b(\eta)](\alpha)$. Let $g \in {}^\lambda 2$ be a function. We will show that g does not predict F. Let $h \in {}^\lambda 2$ be the opposite function, i.e. $h(\alpha) = 1 - g(\alpha)$ for every $\alpha < \lambda$. Let $t \in {}^{<\lambda}2$ be any mapping for which $b(t) = h$. Let $\delta < \lambda$ be such that $t \in {}^\delta 2$. We define $f \in {}^\lambda 2$ as an extension of t as follows. If $\alpha < \delta$ then $f(\alpha) = t(\alpha)$ and if $\alpha \geq \delta$ then $f(\alpha) = 0$. Observe that $b(f \upharpoonright \alpha) = b(f \upharpoonright \delta)$ for every $\alpha \in (\delta, \lambda)$. Consequently,

$$F(f \upharpoonright \alpha) = [b(f \upharpoonright \alpha)](\alpha) = [b(f \upharpoonright \delta)](\alpha) = [b(t)](\alpha) = h(\alpha) \neq g(\alpha),$$

so g fails to predict $F(f \upharpoonright \alpha)$ on an end-segment of λ, as wanted. $\square_{1.1}$

It follows from the above claim that the weak diamond and the very weak diamond are equivalent in the successor case. We comment that the same holds for the common diamond over successor cardinals, if one replaces the requirement of stationary set of guesses by an unbounded set. That is, if $\Diamond'_\lambda$ says that every $A \subseteq \lambda$ is guessed by an unbounded set of A_α's then $\Diamond_\lambda \Leftrightarrow \Diamond'_\lambda$ whenever $\lambda = \kappa^+$. For the following corollary we recall that a cardinal λ is weakly inaccessible iff λ is a regular limit cardinal.

Corollary 1.2 (Φ_λ and Ψ_λ)**.** *If $\lambda = \kappa^+$ then $\Phi_\lambda \Leftrightarrow \Psi_\lambda$. The same holds true if λ is weakly inaccessible and $2^{<\lambda} = 2^\kappa$ for some $\kappa < \lambda$.*

Proof.
The implication $\Phi_\lambda \Rightarrow \Psi_\lambda$ results from the definition for both cases mentioned in the above statement. For the opposite direction, if Ψ_λ then $2^{<\lambda} < 2^\lambda$ by the above theorem, and hence Φ_λ follows from Proposition 0.2 in the successor case and from Claim 1.3 in the weakly inaccessible case. $\square_{1.2}$

Back to the general case, Theorem 1.1 gives one direction by showing that Φ_λ implies $2^{<\lambda} < 2^\lambda$. As we shall see in Theorem 2.5 below, the other direction cannot be proved. Indeed, every strongly inaccessible cardinal satisfies $2^{<\lambda} < 2^\lambda$, but the negation of Φ_λ can be forced over some strongly inaccessible cardinal. However, we can prove that $2^{<\lambda} < 2^\lambda$ is equivalent to the very weak diamond Ψ_λ. We shall use the fact that if λ is weakly inaccessible and $2^{<\lambda} = 2^\kappa < 2^\lambda$ for some $\kappa < \lambda$ then Φ_λ. This fact appears already in [2], without explicit proof. Since it plays a key-role in the theorem below, we spell out the proof:

Claim 1.3 (Weak diamond out of nowhere)**.** *If λ is weakly inaccessible, $2^{<\lambda} < 2^\lambda$ and $2^\kappa = 2^{<\lambda}$ for some $\kappa < \lambda$, then Φ_λ.*

Proof.
Assume towards contradiction that $\neg(\Phi_\lambda)$, and choose a coloring $F : {}^{<\lambda}2 \to 2$ which exemplifies it. Denote the club filter over λ by $\mathcal{D}_\lambda$.

Let $\mathcal{S}$ be the collection of all the sequences s, of length $1 + \beta + \beta$ such that:

- $s(0), \beta < \lambda$.
- $s(1 + i), s(1 + \beta + i) \in {}^{s(0)}2$ for all $i < \beta$.

We will denote $\alpha^s = s(0)$, $g^s_\nu = s(1 + \nu)$, $f^s_\nu = s(1 + \beta + \nu)$. We will omit the superscript s where it is clear from the context.

Observe that $|\mathcal{S}| = 2^{<\lambda}$, which equals 2^κ by the assumption of the theorem. Let h be a one-to-one mapping between $\mathcal{S}$ and ${}^\kappa 2$.

Assume $d \in {}^\lambda 2$ is any function. We will define, by induction on $n < \omega$, functions $g_{\kappa\cdot n+\delta}$, $f_{\kappa\cdot n+\delta}$, $\delta < \kappa$, and clubs C_n so that each f_η codes g_η using the failure of Φ_λ. We will show, eventually, that this coding process produces an unique code from each function from ${}^\lambda 2$ and thus enables us to obtain a surjection from ${}^{<\lambda}2$ onto ${}^\lambda 2$.

For $n = 0$ let $g_0 = d, f_0 = f$ for some f such that the set $\{\alpha \mid F(f \restriction \alpha) = g(\alpha)\}$ contains a club, C. We set $C_0 = C$. Let g_δ, f_δ be arbitrary for $0 < \delta < \kappa$.

Let $n > 0$. Let us assume that f_ν, g_ν where defined for every $\nu < \kappa \cdot n$ and let us assume that C_{n-1} is defined. We want to define the functions $f_{\kappa\cdot n+\delta}, g_{\kappa\cdot n+\delta}$ and the club C_n.

For each $\alpha < \lambda$ let $\beta_{\alpha,\eta}$ be the first member of C_{n-1} greater than α. We define simultaneously the sequence of functions $\langle g_{\kappa\cdot n+\delta} \mid \delta < \kappa\rangle$. For this end, we have to determine the value of all these functions for every $\alpha < \lambda$.

For $\alpha < \lambda$, let s^n_α be the following member of $\mathcal{S}$:

- $s^n_\alpha(0) = \beta_{\alpha,\eta}$,
- $s^n_\alpha(1 + \nu) = g_\nu \restriction \beta_{\alpha,\eta}$ for every $\nu < \kappa \cdot n$ and
- $s^n_\alpha(1 + \beta_{\alpha,\eta} + \nu) = f_\nu \restriction \beta_{\alpha,\eta}$ for every $\nu < \kappa \cdot n$.

Recall that $h(s^n_\alpha) \in {}^\kappa 2$. Let us define $g_{\kappa\cdot n+\delta}(\alpha) = h(s^n_\alpha)(\delta)$.

Having the functions $g_{\kappa\cdot n+\delta}$ at hand for every $\delta < \kappa$, we choose for each one of them a function $f_{\kappa\cdot n+\delta} \in {}^\lambda 2$ such that

$$A_{\kappa\cdot n+\delta} = \{\alpha < \lambda : F(f_{\kappa\cdot n+\delta} \restriction \alpha) = g_{\kappa\cdot n+\delta}(\alpha)\} \in \mathcal{D}_\lambda.$$

Finally, we choose a club C_n of λ such that

$$C_n \subseteq \bigcap_{\delta<\kappa} A_{\kappa\cdot n+\delta} \cap C_{n-1}.$$

Let $\gamma = \min \bigcap_{n<\omega} C_n$. Let us define the *code* of d, $\mathcal{C}^d$, to be the following sequence of length $\kappa \cdot \omega \cdot 2$:

(1) $\mathcal{C}^d(0) = \gamma$.
(2) $\mathcal{C}^d(1+\nu) = g_\nu \restriction \gamma$ for $\nu < \kappa \cdot \omega$.
(3) $\mathcal{C}^d(\kappa \cdot \omega + \nu) = f_\nu \restriction \gamma$ for $\nu < \kappa \cdot \omega$.

$\mathcal{C}^d \in \lambda \times {}^{\kappa\cdot\omega\cdot 2}({}^{<\lambda}2)$, so there are $2^{<\lambda}$ possible values for the code $\mathcal{C}^d$. Let us show that one can reconstruct d from $\mathcal{C}^d$ and thus conclude that $2^{<\lambda} = 2^\lambda$.

Indeed, let $\mathcal{C}^d$ be the code of some function d. Let g_ν, f_ν and C_n be the functions and the clubs which were generated in the course of the construction of $\mathcal{C}^d$. Let d' be another function and let g'_ν, f'_ν, C'_n be the functions and the clubs which were generated in the course of constructing $\mathcal{C}^{d'}$. Let us assume that $\mathcal{C}^d = \mathcal{C}^{d'}$ and show that $d = d'$.

Let us show, by induction on $\alpha \in \bigcap C_n$, that $g_\nu \restriction \alpha = g'_\nu \restriction \alpha$, $f_\nu \restriction \alpha = f'_\nu \restriction \alpha$ and $\bigcap_{n<\omega} C_n \cap (\alpha+1) = \bigcap_{n<\omega} C'_n \cap (\alpha+1)$. This is enough, as $g_0 = d$, $g'_0 = d'$.

For simplicity of notations, let $D = \bigcap_{n<\omega} C_n$, $D' = \bigcap_{n<\omega} C'_n$.

For $\alpha = \min D = \min D'$, the inductive assumption holds since $\mathcal{C}^d = \mathcal{C}^{d'}$.

Let us assume that the claim is true for every $\beta < \alpha$ in D and let us show its validity for α. If α is an accumulation point of D then $\alpha \in D$, since D is a club. Similarly, $\alpha \in D'$. The rest of the inductive assumption holds trivially.

Let α be nonaccumulation point, above the minimal point of D, and let $\gamma = \max D \cap \alpha$. Since the clubs C_n are decreasing, $\sup \beta_{n,\gamma} = \alpha$.

Since $\gamma \in C_n \cap C'_n$ for all n and since $f_\nu \restriction \gamma = f'_\mu \restriction \gamma$ for all $\nu < \kappa \cdot \omega$,

$$F(f_\nu \restriction \gamma) = g_\nu(\gamma) = F(f'_\nu \restriction \gamma) = g'_\nu(\gamma)$$

But $g_{\kappa\cdot n+\delta}(\gamma) = h(s^n_\gamma)(\delta)$ and since h is one to one, we conclude that $\beta_{n,\gamma} = \beta'_{n,\gamma}$, $g_\nu \restriction \beta_{n,\gamma} = g'_\nu \restriction \beta_{n,\gamma}$ and $f_\nu \restriction \beta_{n,\gamma} = f'_\nu \restriction \beta_{n,\gamma}$, for all $\mu < \kappa \cdot n$. This is true for all $n < \omega$, and thus we conclude that the induction assumption holds for α. $\square_{1.3}$

Remark 1.4. Another way to phrase the idea in the above proof is by noticing that if $2^{<\lambda} = 2^\kappa$ then $\neg\Phi_\lambda$ codes a one-to-one mapping from 2^λ into $2^{<\lambda}$, and hence $2^{<\lambda} = 2^\lambda$.

$\square_{1.4}$

Now we can prove the following:

Theorem 1.5 (Very weak diamond and cardinal arithmetic)**.** *For every regular uncountable cardinal λ we have $2^{<\lambda} < 2^\lambda$ iff Ψ_λ.*

Proof.
The successor cardinal case follows from Theorem 1.1 and the results of [2], so we may assume that λ is a limit cardinal. If Ψ_λ holds then $2^{<\lambda} < 2^\lambda$ holds by Theorem 1.1. For the opposite direction, assume that $2^{<\lambda} < 2^\lambda$. If $2^{<\lambda} = 2^\kappa = 2^\lambda$ for some $\kappa < \lambda$ then Φ_λ holds by Claim 1.3 and hence also

Ψ_λ. So assume that this is not the situation (as always happens in the case of a strongly inaccessible cardinal).

We claim that Φ_α holds for unbounded set of α's below λ. For proving this assertion, define $C = \{\kappa < \lambda : \forall\gamma < \kappa, 2^\gamma < 2^\kappa\}$. By the assumptions on λ, C is a club subset of λ. Enumerate the uncountable elements of C by $\{\alpha_\varepsilon : \varepsilon < \lambda\}$.

We claim that $\Phi_{\alpha_\varepsilon}$ holds whenever ε is a successor ordinal. So let $\varepsilon = \zeta+1$. If α_ε is a successor cardinal then $\Phi_{\alpha_\varepsilon}$ follows from [2]. If not, then $2^\gamma = 2^\zeta$ for every sufficiently large $\gamma \in [\alpha_\zeta, \alpha_\varepsilon)$. In which case, α_ε is regular by the Bukovsky-Hechler Theorem (see Corollary 5.17 in [5] and the historical notes in p. 61 there) and $\Phi_{\alpha_\varepsilon}$ follows from Claim 1.3 with α_ζ here standing for κ there.

We claim now that Ψ_λ holds. For this, let $c : {}^{<\lambda}2 \to 2$ be a coloring. For every $\zeta < \lambda$ let $\varepsilon = \zeta + 1$ and let c_ε be the restriction $c \restriction 2^{<\alpha_\varepsilon}$. Choose a function g_ε which exemplifies $\Phi_{\alpha_\varepsilon}$ with respect to c_ε for every $\varepsilon = \zeta+1 < \lambda$. Define $h : \lambda \to \lambda$ as follows. For every $\beta < \lambda$ let $h(\beta)$ be the first ordinal $\varepsilon < \lambda$ so that $\alpha_\varepsilon \leq \beta < \alpha_{\varepsilon+1}$.

We define $g \in {}^\lambda 2$ as follows. Given $\beta < \lambda$, if $h(\beta) = \varepsilon$ is a limit ordinal then $g(\beta) = 0$. If $h(\beta) = \varepsilon$ is a sucessor ordinal then $g(\beta) = g_\varepsilon(\beta)$. Let us show that g exemplifies Ψ_λ.

Assume $f \in {}^\lambda 2$. For every successor oridnal $\varepsilon = \zeta+1 < \lambda$ let $f_\varepsilon = f \restriction \alpha_\varepsilon$. By $\Phi_{\alpha_\varepsilon}$ we can choose an ordinal $\beta_\varepsilon \in [\alpha_\zeta, \alpha_\varepsilon)$ for which $g_\varepsilon(\beta_\varepsilon) = c_\varepsilon(f_\varepsilon \restriction \beta_\varepsilon)$. By the above definitions it folows that $g(\beta_\varepsilon) = c(f \restriction \beta_\varepsilon)$. Since we have unboundedly many β_ε of this form, we are done. $\square_{1.5}$

2. Failure of weak diamond at strongly inaccessible cardinal

In this section we demonstrate the fact that Ψ_λ is strictly weaker than Φ_λ.

Our next goal is to demonstrate the fact that Ψ_λ is strictly weaker than Φ_λ. We shall use Radin forcing $R(\vec{U})$. For the most part, our presentation and arguments follow Gitik's handbook chapter, [4].

Definition 2.1 (Measure sequences)**.** Let κ be a cardinal and $\vec{U} = \langle\kappa\rangle^\frown\langle U_\alpha \mid \alpha < \ell(\vec{U})\rangle$ be a sequence such that each U_α is a measure on V_κ (i.e., a κ complete normal ultrafilter on V_κ). For each $\beta < \ell(\vec{U})$, let $\vec{U} \restriction \beta$ denote the initial segment $\langle\kappa\rangle^\frown\langle U_\alpha \mid \alpha < \beta\rangle$. In particular, $\vec{U} \restriction 0 = \langle\kappa\rangle$.

We say $\vec{U}$ is a **measure sequence** on κ if either $\vec{U} = \langle\kappa\rangle$ or $\ell(\vec{U}) > 0$ and there exists an elementary embedding $j : V \to M$, whose critical point is κ and ${}^\kappa M \subset M$, such that for each $\beta < \ell(\vec{U})$, $\vec{U} \restriction \beta \in M$ and $U_\beta = \{X \subset V_\kappa \mid \vec{U} \restriction \beta \in j(X)\}$.

Note that the length of a measure sequence $\vec{U}$, as sequence of sets, is $1 + \ell(U)$. If $\ell(U) > 0$, let $\cap\vec{U}$ denote the filter $\bigcap_{\alpha<\ell(\vec{U})} U_\alpha$, and $\mathcal{MS}(\kappa)$ denote the set of measure sequences $\vec{\mu}$ on measurable cardinals below κ. If κ is clear from the context, we omit the subscript.

Let $\vec{\mu}$ be a sequence of the form $\langle\nu\rangle^\frown\langle u_i \mid i < \ell(\vec{\mu})\rangle$, where each u_i is a measure on V_ν. We denote ν by $\kappa(\vec{\mu})$, and $\cap\{u_i \mid i < \ell(\vec{\mu})\}$ by $\cap\vec{\mu}$. If $\ell(\vec{\mu}) = 0$, $\cap\vec{\mu}$ is undefined.

The following lemma is well known:

Lemma 2.2. *Let $\vec{U}$ be a measure sequence on κ with nonzero length. Let $\langle A_{\vec{\nu}} \mid \nu \in \mathcal{MS}(\kappa)\rangle$ be a sequence of sets such that $A_{\vec{\nu}} \in \cap\vec{U}$. Then the **diagonal intersection** defined by:*

$$A^\star = \triangle_{\vec{\nu}} A_{\vec{\nu}} = \{\vec{\mu} \in \mathcal{MS}(\kappa) \mid \forall \vec{\nu} \in \mathcal{MS}(\kappa) \cap V_{\kappa(\vec{\mu})}, \vec{\mu} \in A_{\vec{\nu}}\}$$

belongs to $\cap\vec{U}$.

Clearly, if one take a diagonal intersection over smaller set than $\mathcal{MS}(\kappa)$, the resulting set is only larger and thus belongs to $\cap\vec{U}$.

Let us fix a measurable cardinal κ and let $\vec{U}$ be a measure sequence on κ of nonzero length. We proceed to define the Radin forcing $R(\vec{U})$. Define first a sequence of sets $A^n \subset \mathcal{MS}(\kappa)$, $n < \omega$. Let $A^0 = \mathcal{MS}(\kappa)$, and for every $n < \omega$, $A^{n+1} = \{\vec{\mu} \in A^n \mid A^n \cap V_{\kappa(\vec{\mu})} \in \cap\vec{\mu}\}$. Define

$$\bar{A} = \bigcap_{n<\omega} A^n.$$

It is routine to verify $\bar{A} \in \cap\vec{U}$.

Definition 2.3 (Radin forcing)**.** The Radin partial ordered set, $R(\vec{U})$, consists all finite sequences $p = \langle d_i \mid i \leq k\rangle$ satisfying the following conditions.

- (ℵ) $\vec{d} = \langle d_i \mid i < k\rangle$ is a finite sequence. For every $i \leq k$, d_i is either of the form $\langle\kappa_i\rangle$ where $\kappa_i < \kappa$ is an ordinal, or of the form $d_i = \langle\vec{\mu_i}, a_i\rangle$ where $\vec{\mu_i}$ is a measure sequence on a measurable cardinal $\kappa_i = \kappa(\vec{\mu_i}) \leq \kappa$ and $a_i \in \cap\vec{\mu_i}$. For each $i \leq k$ we denote κ_i by $\kappa(d_i)$ and a_i by $a(d_i)$.
- (ℶ) $\langle\kappa_i \mid i < k\rangle$ is increasing.
- (ℷ) $d_k = \langle\vec{U}, A\rangle$, where $A \in \bigcap\vec{U}$ is a subset of $\bar{A}$.

Given a condition $p = \langle d_i \mid i \leq k\rangle$ as above, we will frequently separate $\langle\vec{U}, A\rangle$ from the other components and write $p = \vec{d}^\frown\langle\vec{U}, A\rangle$ where $\vec{d} = \langle d_i \mid i < k\rangle$. We call d the **stem** of p and denote it by stem$p$. We A condition $p^* = \langle d^*_i \mid i \leq k^*\rangle$ is a **direct extension** of $p = \langle d_i \mid i \leq k\rangle$ if $k^* = k$ and $a(d^*_i) \subset a(d_i)$ wherever $a(d_i)$ exists. A condition p' is a **one-point extension** of p if there exists $j \leq k$ and a measure sequence $\vec{\nu} \in a(d_j)$ with $\kappa(\vec{\nu}) > \kappa(d_{j-1})$, and p' is either $\langle d_i \mid i < j\rangle^\frown\langle\vec{\nu}\rangle^\frown\langle d_i \mid i \geq j\rangle$ if $\vec{\nu} = \langle\alpha\rangle$ is an ordinal, or $\langle d_i \mid i < j\rangle^\frown\langle\vec{\nu}, a(d_j) \cap V_{\kappa(\vec{\nu})}\rangle^\frown\langle d_i \mid i \geq j\rangle$ if $\vec{\nu}$ is a nontrivial measure sequence. We refer to p' as the one-point extension of p by $\vec{\nu}$, and further denote it by $p^\frown\langle\vec{\nu}\rangle$.

Following the standard notions, the symbol "$^\frown$" is overloaded in the context of Radin forcing. When we use the concatenation symbol, $^\frown$, with a member of the Radin forcing as the left argument, its meaning is the weakest condition in the Radin forcing which is stronger than the left argument and contains the right argument in its sequence. The interpretation differs from the standard one which is appending the right argument to the left sequence.

A condition q extends p if it is obtained from p by a finite sequence of one-point extensions and direct extensions.

Let $\vec{U} = \langle\kappa\rangle^\frown\langle U_\tau \mid \tau < \kappa^+\rangle$ be a measure sequence of length κ^+, derived from an elementary embedding $j : V \to M$ as above. The following results are established in [4].

Facts 2.4. (1) *$R(\vec{U})$ satisfies κ^+.c.c.*

(2) *$R(\vec{U})$ is a Prikry type forcing notion. Namely, for every condition $p \in R(\vec{U})$ and a statement σ of the forcing language, p has a direct extension p^* which decides σ.*
(3) *$R(\vec{U})$ preserves all cardinals.*
(4) *κ remains regular and strong limit in a $R(\vec{U})$ generic extension.*
(5) *Suppose $G \subset R(\vec{U})$ is a generic filter. Let $MS_G \subset \mathcal{MS}$ be the set of all $\vec{u} \in \mathcal{MS}$ for which there exists some $p \in G$ of the form $p = \vec{d}^\frown\langle\vec{U}, A\rangle$ such that $\vec{d} = \langle u_0, a_0\rangle, \ldots, \langle u_{k-1}, a_{k-1}\rangle$ and $\vec{u} = u_i$ for some $i < k$. The set $C_G = \{\kappa(\vec{u}) \mid \vec{u} \in MS_G\}$ is a closed unbounded subset of κ, called the generic Radin club associated with G.*

Theorem 2.5 (Strong inaccessibility and $\neg\Phi_\kappa$)**.** *Assuming the existence of a measure sequence $\vec{U} = \langle\kappa\rangle^\frown\langle U_\alpha \mid \alpha < \kappa^+\rangle$ derived from an embedding $j : V \to M$, such that $\vec{U} \in M$ and $M \models 2^\kappa = 2^{\kappa^+}$, it is consistent that there is an inaccessible cardinal κ such that $\neg\Phi_\kappa$.*

The large cardinal assumption of the Theorem is known to be consistent relative to a certain hypermeasurablity assumption. For example, the existence of a strong cardinal κ suffices to obtain a model V satisfying $2^\kappa = 2^{\kappa^+} = \kappa^{++}$ with an embedding $j : V \to M$ from which a measure sequence $\langle\kappa\rangle^\frown\langle U_\alpha \mid \alpha < \kappa^+\rangle \in M$ can be derived.

Proof.
Suppose $\vec{U} = \langle\kappa\rangle^\frown\langle U_\tau \mid \tau < \kappa^+\rangle$ is a measure sequence of a measurable cardinal κ derived from an elementary embedding $j : V \to M$ satisfying $M \models 2^\kappa = 2^{\kappa^+}$. Therefore, $U_\tau = \{X \subset V_\kappa \mid \vec{U} \restriction \tau \in j(X)\}$ for all $\tau < \kappa^+$. Clearly,

$$M \models 2^\kappa = 2^{\kappa^+} = |([V_\kappa]^{<\omega} \times \mathcal{P}(V_\kappa))^{\kappa\times\kappa^+}|.$$

In V, let $H : \kappa \to V_\kappa$ be a partial function with

$$\mathrm{dom}(H) = \{\alpha < \kappa \mid 2^\alpha = 2^{\alpha^+}\},$$

such that for every $\alpha \in \mathrm{dom}(H)$, $H(\alpha) : 2^\alpha \leftrightarrow ([V_\alpha]^{<\omega} \times \mathcal{P}(V_\alpha))^{\alpha\times\alpha^+}$ is a bijection. Let $H(\kappa) = j(H)(\kappa)$ (by slightly abuse of notations). By elementarity, $H(\kappa)$ is a bijection between $(2^\kappa)^M$ and $\left(([V_\kappa]^{<\omega} \times \mathcal{P}(V_\kappa))^{\kappa\times\kappa^+}\right)^M$.

Note that both the domain and co-domain of $H(\kappa)$, are evaluated in M. It is likely to assume $H(\kappa)$ is not surjective on $\left(([V_\kappa]^{<\omega} \times \mathcal{P}(V_\kappa))^{\kappa \times \kappa^+}\right)^V$, as we do not assume that M agrees with V on $\mathcal{P}(\kappa^+)$.

For notational simplicity, we denote $H(\alpha)$ by H_α, for each $\alpha \leq \kappa$.

Let $R(\vec{U})$ be the Radin forcing associated with $\vec{U}$. Choose a generic set $G \subseteq R(\vec{U})$. We claim there is no weak diamond on κ in $V[G]$. To show this, it will be convenient to identify conditions $p = \vec{d}^\frown \langle \vec{U}, A\rangle \in R(\vec{U})$ with pairs $\langle \vec{d}, A\rangle \in V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa)$. We say that $\langle \vec{d}, A\rangle \in V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa)$ is a simple representation of p. Since $R(\vec{U})$ satisfies κ^+*.c.c* we can represent antichains in $R(\vec{U})$ using elements in $(V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa))^\kappa$. We use the phrase simple representation in this context as well. Note that those antichains can have cardinality strictly less than κ (and even empty). We code those antichains as well by appending the list of simple representations of the elements in the antichains with the emptyset that does not represent any condition in $R(\vec{U})$.

We first work in V. Let $\underset{\sim}{g}$ be an $R(\vec{U})$ name for a function from κ to 2 and $p = \vec{d}^\frown \langle \vec{U}, A\rangle$ a condition.

Without loss of generality (by taking a direct extension) we may assume that either $\mathrm{len}(\mathrm{stem}(p)) = 0$ or that for every $\vec{\mu} \in A$, $\kappa(\vec{\mu}) > \kappa_{\mathrm{len}(\mathrm{stem}(p))-1}$.

Fix for a moment $\vec{\mu} \in A$ and consider the one-point extension

$$p^\frown \langle \vec{\mu}\rangle = \vec{d}^\frown \langle \vec{\mu}, A \cap V_{\kappa(\vec{\mu})}\rangle^\frown \langle \vec{U}, A\rangle.$$

The forcing $R(\vec{U})/(p^\frown \langle \vec{\mu}\rangle)$ factors into a product

$$R(\vec{\mu}) \times R(\vec{U})/(\langle \vec{U}, A \setminus V_{\kappa(\vec{\mu})+1}\rangle).$$

Let $\langle q_\xi \mid \xi < 2^{\kappa(\vec{\mu})}\rangle$ be an enumeration of the conditions in $R(\vec{\mu})$. Let us define by induction sets $A^{\vec{\mu}}_\xi \in \bigcap \vec{U}$ such that $q_\xi{}^\frown \langle \vec{U}, A^{\vec{\mu}}_\xi\rangle$ decides whether $\underset{\sim}{g}(\check{\kappa}(\vec{\mu})$ is 0, 1 or if q_ξ does not force its value in the generic extension by the right component in the decomposition of $R(\vec{U})/p$. Since $\cap\vec{U}$ is κ-complete and since $2^{\kappa(\vec{\mu})} < \kappa$, $A_{\vec{\mu}} = \bigcap_{\xi < 2^{\kappa(\vec{\mu}}} A^{\mu}_\xi \in \cap\vec{U}$.

Clearly, for densely many conditions $q \in R(\mu)$, $\langle \vec{U}, A_{\vec{\mu}}\rangle$ forces that they decide the value of $\underset{\sim}{g}(\check{\kappa}(\vec{\mu})$.

As $\vec{\mu}$ was arbitrary, by the considirations above, for every $\vec{\mu} \in A$, the condition $p^\frown \vec{\mu}$ has a direct extension of the form $\vec{d}^\frown \langle \vec{\mu}, A \cap V_{\kappa(\vec{\mu})}\rangle^\frown \langle \vec{U}, A_{\vec{\mu}}\rangle$ forcing $\underset{\sim}{g}(\check{\kappa}(\vec{\mu})) = \sigma(\vec{\mu})$, where $\sigma(\vec{\mu})$ is a $R(\vec{\mu})$ name for an ordinal in $\{0, 1\}$. Let

$$A^* = \Delta_{\vec{\mu} \in A} A_{\vec{\mu}} = \{\vec{\nu} \in V_\kappa \mid \vec{\nu} \in A_{\vec{\mu}} \text{ if } \vec{\mu} \in V_{\kappa(\vec{\nu})}\}$$

and $p^* = \vec{d}^\frown \langle \vec{U}, A^*\rangle$. Thus, p^* is a direct extension of p and $p^\frown \langle \vec{\mu}\rangle \Vdash \underset{\sim}{g}(\check{\kappa}(\vec{\mu})) = \sigma(\vec{\mu})$ for all $\vec{\mu} \in A^*$.

Let us fix a well order in M of $V_{\kappa+\omega}$, $\trianglelefteq$. In M, we define a function $h : \kappa^+ \to (V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa))^\kappa$.

For every $\tau < \kappa^+$, let $h(\tau) \in (V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa))^\kappa$ be the $\trianglelefteq$-least simple representation of a maximal antichain $A_\tau \subset R(\vec{U} \restriction \tau)$ of conditions $q \in R(\vec{U} \restriction \tau)$ which force $j(\sigma)(\vec{U} \restriction \tau) = \check{0}$. We point out the definition of h relies on the assumption $\vec{U} \in M$, which implies M contains a enumeration of the posets $\langle R(\vec{U} \restriction \tau) \mid \tau < \kappa^+\rangle$. With this enumeration in M, we can identify h with an element in $\left(V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa)^{\kappa\times\kappa^+}\right)^M$. Let $f = H_\kappa^{-1}(h) : \kappa \to 2$. Back to V, define a function $F' : 2^{<\kappa} \to V_\kappa$ as follows. For every $\alpha \in \mathrm{dom} H$, $w \in 2^\alpha$, $F'(w)$ is a function whose domain is the collection of all measure sequences $\vec{\nu}$ with $\kappa(\vec{\nu}) = \alpha$. Recall that for every $\alpha < \kappa$ and $w \in 2^\alpha$, $H_\alpha(w) \in (V_\alpha^{<\omega} \times \mathcal{P}(V_\alpha))^{\alpha\times\alpha^+}$.

Let $F'(w)(\vec{\nu})$ be the set of all $q \in R(\vec{\nu})$ such that q is simply represented by an element of $H_\alpha(w)(\mathrm{len}(\vec{\nu}))$, where $\mathrm{len}(\vec{\nu})$ is the length of the sequence. By our choice of $f = H_\kappa^{-1}(h)$ we see that the set of all $\vec{\nu} \in V_\kappa$ such that $F'(f \restriction \kappa(\vec{\nu}))(\vec{\nu})$ is an antichain of conditions $q \in R(\vec{\nu})$ such that $q \Vdash \sigma(\vec{\nu}) = \check{0}$, and is maximal relative to this restriction, is in $\bigcap \vec{U}$.

Finally, we define $F : 2^{<\kappa} \to 2$ in $V[G]$. For every $\vec{\nu} \in MS_G$, let $G(\vec{\nu})$ denote the $R(\vec{\nu})$ generic filter induced by G. For each $w \in 2^{\kappa(\vec{\nu})}$ we set

$$F(w) = \begin{cases} 0 & \text{if } F'(w)(\vec{\nu}) \cap G(\vec{\nu}) \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Define $F(w) = 0$ for every other $w \in 2^\alpha$.

We may assume G contains a condition $p^* = \vec{d}^\frown\langle \vec{U}, A^*\rangle$ as above. Let $X = \{\kappa(\vec{\nu}) \mid \vec{\nu} \in A^* \cap B\}$. $C_G \subset^* X$ since $X \in \bigcap \vec{U}$. The claim follows as $F(f \restriction \alpha) = g(\alpha)$ for every $\alpha \in X \cap C_G$.

$\square_{2.5}$

3. Weak diamond at weakly inaccessible cardinals

To round out the picture, we are left with the case of weakly but not strongly inaccessible cardinal. We distinguish three cases. If $2^{<\lambda} = 2^\lambda$ then we already know that $\neg\Phi_\lambda$ and if $2^{<\lambda} = 2^\kappa < 2^\lambda$ for some $\kappa < \lambda$ then Φ_λ. The remaining case is when the sequence $\langle 2^\theta : \theta < \lambda\rangle$ is not eventually constant. In this case $2^{<\lambda} < 2^\lambda$, and we do not know if the weak diamond holds (see [9], Question 1.28) though we have seen that the very weak diamond holds. It is possible to force Φ_λ in such cases:

Theorem 3.1. *It is consistent that λ is weakly inaccessible, $\langle 2^\theta : \theta < \lambda\rangle$ is not eventually constant, λ is not strongly inaccessible and Φ_λ holds.*

Proof.
We begin with a strongly inaccessible cardinal λ in the ground model, aiming to blow up 2^θ for every regular uncountable $\theta \leq \lambda$. Let $\mathbb{P}$ be the Easton support product of $\mathrm{Add}(\theta, \lambda^{+\theta+1})$ for every regular uncountable $\theta \leq \lambda$. Notice that $\mathbb{P}$ neither collapses cardinals, nor changes cofinalities.

The forcing $\mathbb{P}$ is $\aleph_1$-complete, as a product of $\aleph_1$-complete forcing notions with Easton support. Let us show that $\mathbb{P}$ is λ^+-cc. Every condition in $\mathbb{P}$ can be represented as a partial function from $\lambda^{+\lambda+1}$ to 2 with support s such that for every regular uncountable cardinal $\theta \leq \lambda$, $|s \cap [\lambda^{+\theta}, \lambda^{+\theta+1})| < \theta$, the collection $t = \{\theta \leq \lambda \mid s \cap [\lambda^{\theta}, \lambda^{+\theta+1})\}$ consists only of regular cardinals and it is an Easton set, namely $|t \cap \alpha| < \alpha$ for every inaccessible cardinal α. Note that, in particular, $|s| < \lambda$. From this point we will always assume that our conditions are represented in this way.

Let $\{p_i \mid i < \lambda^+\}$ be a collection of conditions in $\mathbb{P}$. Let s_i be the support of p_i (represented as a partial function on $\lambda^{+\lambda+1}$). Then by the Δ-system lemma, there is a subcollection $I \subseteq \lambda^+$, and a set $r \subseteq \lambda^{+\lambda+1}$, such that for every i, j in I, if $i \neq j$ then $s_i \cap s_j = r$. Let us narrow down I further to a set J for which there is some constant function $q_\star$ such that for every i, $p_i \restriction r = q_\star$. This is possible since $2^{<\lambda} = \lambda < \lambda^+$ and $|r| < \lambda$. Thus, every pair of conditions p_i, p_j, such that $i, j \in J$ are compatible.

By Easton Theorem, $\langle 2^\theta : \theta < \lambda\rangle = \langle \lambda^{+\theta+1} : \theta < \lambda\rangle$ in the generic extension, so this sequence is not eventually constant. Although λ is not strongly inaccessible any more, it is still weakly inaccessible. It remains to show that Φ_λ holds in $V^{\mathbb{P}}$.

Let $\underset{\sim}{c}$ be a name of a coloring function from ${}^{<\lambda}2$ into 2 in the generic extension. Observe that $2^{<\lambda} < 2^\lambda = \lambda^{+\lambda+1}$ in $V^{\mathbb{P}}$. Let us pick, for every member of ${}^{<\lambda}2$, $\underset{\sim}{t}$, a maximal antichain of conditions in $\mathbb{P}$ that decides the value of $\underset{\sim}{c}(\underset{\sim}{t})$. Let B be the union of the supports of all the conditions that appear in one of those antichains. Without loss of generality, $B \supseteq \lambda^{+\lambda}$.

For a set C, let us denote by $\mathbb{P} \restriction C$ the set of conditions $p \in \mathbb{P}$ such that $\operatorname{supp} p \subseteq C$. Clearly, $\mathbb{P} \cong \mathbb{P} \restriction B \times \mathbb{P} \restriction (\lambda^{+\lambda+1} \setminus B)$ and $\underset{\sim}{c} \in V^{\mathbb{P}\restriction B}$. Denote $\mathbb{P} \restriction B$ by $\mathbb{Q}$ and $\mathbb{P} \restriction (\lambda^{+\lambda+1} \setminus B)$ by $\mathbb{R}$. Since $\mathbb{Q}$ is $\aleph_1$-complete, in the generic extension by $\mathbb{Q}$, $\mathbb{R}$ is still $\aleph_1$-complete. Let g be a new λ-Cohen set, added by $\mathbb{R}$.

Working in $V^{\mathbb{Q}}$, Let $\underset{\sim}{f}$ be an $\mathbb{R}$-name of a function from λ into 2, and let $\underset{\sim}{D}$ be an $\mathbb{R}$-name of a club subset of λ. For every condition $r \in \mathbb{R}$ we can define by induction on ω a sequence $\langle r_n : n \in \omega\rangle$ of conditions in $\mathbb{R}$ such that:

- (a) $r_0 = r$ and $r_n \leq r_{n+1}$.
- (b) $r_{n+1} \Vdash \underset{\sim}{f} \restriction \alpha_n = \check{g}_n$ for some $g_n \in V^{\mathbb{Q}}$.
- (c) $\operatorname{dom}(r_{n+1}) = \alpha_{n+1}$.
- (d) $r_{n+1} \Vdash \exists \check{\beta}_n, \check{\beta}_n \in \underset{\sim}{D}$ and $\check{\alpha}_n < \check{\beta}_n < \check{\alpha}_{n+1}$.

Let p be $\bigcup_{n\in\omega} r_n$. By the $\aleph_1$-completeness of $\mathbb{R}$, p is a condition in $\mathbb{R}$. Denote $\bigcup_{n\in\omega} \alpha_n$ by α. Since $r_{n+1} \leq p$ for every $n \in \omega$ we see that $p \Vdash \underset{\sim}{f} \restriction \alpha = \bigcup_{n\in\omega} g_n$. Likewise, $p \Vdash \check{\alpha} \in \underset{\sim}{D}$ since $\underset{\sim}{D}$ is forced to be closed and $\alpha = \bigcup_{n\in\omega} \beta_n$.

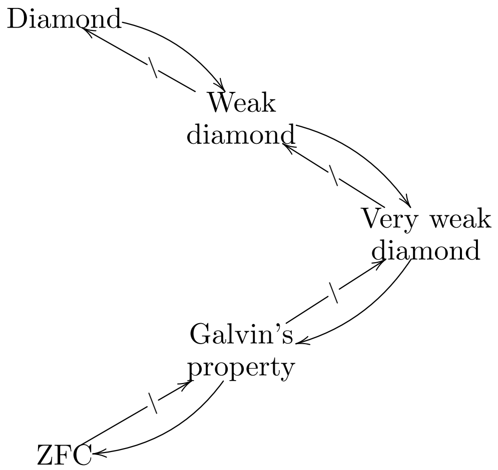
Our goal is to show that the fixed g chosen above can serve as a weak diamond function from λ into 2. For this, we shall prove that the condition

p can be extended to force $g(\alpha) = \underset{\sim}{c}(\underset{\sim}{f} \restriction \check{\alpha})$. As $\underset{\sim}{c} \in V^{\mathbb{Q}}$, the value of $\underset{\sim}{c}(\underset{\sim}{f} \restriction \check{\alpha})$ is determined by the condition p. However, functions from λ into 2 are not determined in a bounded stage. In particular, we can extend p to a condition q which forces $g(\alpha) = \underset{\sim}{c}(\underset{\sim}{f} \restriction \check{\alpha})$. It follows that for every $\underset{\sim}{f} : \check{\lambda} \to 2$ and every club $\underset{\sim}{D}$ there exists an ordinal $\check{\alpha} \in \underset{\sim}{D}$ for which $\Vdash_{\mathbb{P}} g(\alpha) = c(f \restriction \alpha)$, so we are done.

$\square_{3.1}$

4. Conclusions and Open Problems

The following diagram summarizes the relationship between the various prediction principles considered in this paper:



The downward positive implications $\diamondsuit_\lambda \Rightarrow \Phi_\lambda \Rightarrow \Psi_\lambda$ are trivial. The fact that Ψ_λ implies Galvin's property appears in [3], as well as the negative direction upwards (i.e. Galvin's property does not imply Ψ_λ). The consistency of Ψ_λ with $\neg\Phi_\lambda$ can be exemplified by a strongly inaccessible cardinal for which $\neg\Phi_\lambda$ is forced. The consistency of Φ_λ with $\neg\diamondsuit_\lambda$ can be forced by simple cardinal arithmetic considerations. Finally, it is shown in [1] that Galvin's property is not a theorem of ZFC, as its negation can be forced.

We mention the question of Shelah from [9], which seems to be the last open case:

Question 4.1. Assume λ is weakly inaccessible and $\langle 2^\theta : \theta < \lambda \rangle$ is not eventually constant. Is it consistent that $\neg\Phi_\lambda$ holds?

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