The structure of the Mitchell order - II

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Abstract

We address the question regarding the structure of the Mitchell order on normal measures. We show that every well founded order can be realized as the Mitchell order on a measurable cardinal κ, from some large cardinal assumption.

1 Introduction

In this paper we address the question regarding the possible structure of the Mitchell order $\triangleleft$, at a measurable cardinal κ. The Mitchell order was introduced by William Mitchell in [12], who showed it is a well-founded order. The question whether every well-founded order can be realized as the Mitchell order on the set of some specific measurable cardinal, has been open since. By combining various forcing techniques with inner model theory we succeed in constructing models which realize every well-founded order as the Mitchell order on a measurable cardinal.

Given two normal measures U, W, we write $U \triangleleft W$ to denote that $U \in M_W \cong \text{Ult}(V, W)$. For every measurable cardinal κ, let $\triangleleft(\kappa)$ be the restriction of $\triangleleft$ to the set of normal measures on κ, and let $o(\kappa) = \text{rank}(\triangleleft(\kappa))$ be its (well-foundedness) rank.

The research on the possible structure on the Mitchell order $\triangleleft(\kappa)$ is closely related to the question of its possible size, namely, the number of normal measures on κ: The first results by Kunen [8] and by Kunen and Paris [9] showed that this number can take the extremal values of 1 and κ^{++} (in a model of GCH) respectively. Soon after, Mitchell [12] [13] showed

*The paper is a part of the author Ph.D. written in Tel-Aviv University under the supervision of Professor Moti Gitik.

that this size can be any cardinal λ between 1 and κ^{++}, under the large cardinal assumption and in a model of $o(\kappa) = \lambda$. Baldwin [2] showed that for $\lambda < \kappa$ and from stronger large cardinal assumptions, κ can also be the first measurable cardinal. Apter-Cummings-Hamkins [1] proved that there can be κ^+ normal measures on κ from the minimal assumption of a single measurable cardinal; for $\lambda < \kappa^+$, Leaning [10] reduced the large cardinal assumption from $o(\kappa) = \lambda$ to an assumption weaker than $o(\kappa) = 2$. The question of the possible number of normal measures on κ was finally resolved by Friedman and Magidor in [7], where it is shown that κ can carry any number of normal measures $1 \leq \lambda \leq \kappa^{++}$ from the minimal assumption. The Friedman-Magidor method will be extensively used in this work.

Further results were obtained on the possible structure of the Mitchell order: Mitchell [12] and Baldwin [2] showed that from some large cardinal assumptions, every well-order and pre-well-order (respectively) can be isomorphic to $\triangleleft(\kappa)$ at some κ. Cummings [5],[6], and Witzany [18] studied the $\triangleleft$ ordering in various generic extensions, and showed that $\triangleleft(\kappa)$ can have a rich structure. Cummings constructed models where $\triangleleft(\kappa)$ embeds every order from a specific family of orders we call tame. Witzany showed that in a Kunen-Paris extension of a Mitchell model $L[\mathcal{U}]$, with $o^{\mathcal{U}}(\kappa) = \kappa^{++}$, every well-founded order of cardinality $\leq \kappa^+$ embeds into $\triangleleft(\kappa)$.

In this paper, the main idea for realizing well-founded orders as $\triangleleft(\kappa)$ is to force over an extender model $V = L[E]$ with a sufficiently rich $\triangleleft$ structure on a set of extenders at κ. By forcing over V we can collapse the generators of these extenders, giving rise to extensions of these extenders, which are equivalent to ultrafilters on κ. The possible structure of the Mitchell order on arbitrary extenders was previously studied by Steel [17] and Neeman [14] who showed that the well-foundedness of the Mitchell order fails exactly at the level of a rank-to-rank extender.

For the most part, the extenders on κ which will be used do not belong to the main sequence E. Rather, they are of the form $F' = i_\theta(F)$, where $F \in E$ overlaps a measure on a cardinal $\theta > \kappa$, and i_θ is an elementary embedding with $\text{cp}(i_\theta) = \theta$. There is a problem though; the extenders F' may not be $\kappa-$complete. To solve this we incorporate an additional forcing extension by which F' will generically regain its missing sequence of generators[1].

[1]I.e., while $M_{F'} \cong \text{Ult}(V, F')$ may not be closed under $\kappa-$sequences, its embedding $j_{F'} : V \to M_{F'}$ will extend in $V[G]$ to $j' : V[G] \to M_{F'}[G']$ so that ${}^{\kappa}M_{F'}[G'] \subset M_{F'}[G']$.

Forcing the above would translate the $\triangleleft$ on certain extenders F', to $\triangleleft(\kappa)$; however some of the normal measures in the resulting model will be unnecessary and will need to be destroyed. This will be possible since the new normal measures on κ are separated by sets[2] allowing us to remove the undesired normal measures in an additional generic extension, which we refer to as a *final cut*.

The combination of these methods will be used to prove the main result:

Theorem 1.1. Let $V = L[E]$ be a core model. Suppose that κ is a cardinal in V and $(S, <_S)$ is a well-founded order of cardinality $\leq \kappa$, so that

1. there are at least $|S|$ measurable cardinals above κ; let θ be the supremum of the successors of the first $|S|$,
2. there is a $\triangleleft-$increasing sequence of $(\theta+1)-$ strong extenders $\vec{F} = \langle F_\alpha \mid \alpha < \mathrm{rank}(S, <_S)\rangle$

Then there is a generic extension V^* of V in which $\triangleleft(\kappa)^{V^*} \cong (S, <_S)$.

In particular, if E contains a $\triangleleft-$increasing sequence of extenders $\vec{F} = \langle F_\alpha \mid \alpha < \kappa^+\rangle$, so that each F_α overlaps the first κ measurable cardinals above κ, then every well founded order $(S, <_S) \in V$ of cardinality $\leq \kappa$ can be realized as $\triangleleft(\kappa)$ in a generic extension of V. As an immediate corollary of the proof we have that under slightly stronger large cardinal assumptions, including a class of cardinals κ carrying similar overlapping extenders, there is a class forcing extension V^* in which every well founded order $(S, <_S)$ is isomorphic to $\triangleleft(\kappa)$ for some κ.

This paper is the second of a two-parts study on $\triangleleft$. In the first part [4], a wide family of well-founded orders named tame orders, was isolated and it was shown that every tame order of cardinality at most κ can be realized from an assumption weaker than $o(\kappa) = \kappa^+$.
The principal characteristic of tame orders is that they do not embed two specific orders: $R_{2,2}$ on a set of four elements, and $S_{\omega,2}$ on a countable set.

In section 2 we consider a first example in which $V = L[E]$ contains a $\triangleleft-$increasing sequence of extenders in κ, overlapping a single measure on a

[2]Namely, we can associate to each normal measure U on κ a set $X_U \subset \kappa$, which is not contained in any distinct normal measure.

cardinal $\theta > \kappa$. We use some specific $\triangleleft-$configurations in $L[E]$ to produce models which realize $R_{2,2}$ and $S_{\omega,2}$ as $\triangleleft(\kappa)$. We also construct a model in which $o(\kappa) = \omega$ but there is no ω sequence in $\triangleleft(\kappa)$. In Section 3 we extend our overlapping framework to models containing extenders $\vec{F} = \langle F_\alpha \mid \alpha < l(\vec{F})\rangle$ which overlap a sequence of measurable cardinals $\vec{\theta} = \langle \theta_i \mid i < \omega\rangle$. It is in this framework where we first need to deal with non-complete extenders of the form $F_{\alpha,c} = i_c(F_\alpha)$, where i_c result from iterated ultrapowers by measures on the cardinals in $\vec{\theta}$. We introduce a poset designed to force the completeness of (extensions of) $F_{\alpha,c}$, and combine our methods to prove the main theorem. Finally, in Section 4 we list further questions.

For the most part, the notations in this paper continue the conventions in [4]. Note that we use the Jerusalem convention for the forcing order, in which $p \geq q$ means that the condition p is stronger than q. For every $(\kappa, \lambda)-$extender F, and $\gamma < \lambda$, we denote the $\gamma-$th measure in F by $F(\gamma)$.

2 A First Use of Overlapping Extenders

In this section we consider the the Mitchell order in an extender model $V = L[E]$ in the sense of [16], which is minimal with respect to a certain large cardinal property.
An extender F on the coherent sequence E has a critical point $\text{cp}(F)$, and support

$$\nu(F) = \sup(\{\kappa^+\} \cup \{\xi + 1 \mid \xi \text{ is a generator of } F\}).$$

ξ is a generator of F if there are no $a \in [\xi]^{<\omega}$ and $f \in V$ so that $\xi = [f, a]_F$. The index $\alpha = \alpha(F)$ of F on the main sequence E (i.e., $F = E_\alpha$) is given by $\alpha = (\nu(F)^+)^{\text{Ult}(F, L[E\restriction\alpha])}$

Our large cardinal assumptions on $V = L[E]$ include the following requirements:

1. There are measurable cardinals $\kappa < \theta$ where θ is the first measurable above κ.
2. There is a $\triangleleft-$increasing sequence $\vec{F} = \langle F_\alpha \mid \alpha < l(\vec{F})\rangle$, $l(\vec{F}) < \theta$, of $(\kappa, \theta^{++})-$extenders.
3. Each F_α is $(\theta + 2)-$strong, i.e., $V_{\theta+2} \subset \text{Ult}(V, F_\alpha)$.

4. $\vec{F}$ consists of all full $(\kappa, \theta^{++})-$extenders on the main sequence E.

5. There are no stronger extenders on κ in E ($o(\kappa) = \theta^{++} + l(\vec{F})$).

6. There are no extenders $F \in E$ so that $\mathrm{cp}(F) < \kappa$ and $\nu(F) \geq \kappa$.

Remark 2.1. 1. The overlapping assumptions described here are in the realm of almost-linear iterations (i.e., weaker than zero hand-grenade, $0^{¶}$). The existence of the core model for such large cardinal assumptions is established in [15].

2. The unique normal measure on θ, U, is equivalent to an extender on the main sequence E, where $\nu(U) = \theta^{+}$, thus $\alpha(U) < \theta^{++V}$. In particular, the trivial completion of U appears before $F_0 \in \vec{F}$ on the main sequence E. This fact will be used in the proof of Proposition 2.15, where we coiterate certain ultrapowers of $L[E]$ by iterating the least disagreement.

Definition 2.2 (i_n, θ_n, $F_{\alpha,n}$,Θ)**.**

1. For every $\alpha < l(\vec{F})$ let $j_{F_\alpha} : V \to M_{F_\alpha} \cong \mathrm{Ult}(V, F_\alpha)$ be the induced ultrapower embedding. We point out that the fact F_α is $(\theta+2)-$strong guarantees that $U \in M_{F_\alpha}$.

2. For every $n < \omega$, let $i_n : V \to N_n \cong \mathrm{Ult}^n(V, U)$ be the $n-$th ultrapower embedding by U and $\theta_n = i_n(\theta)$. Clearly, θ_n is the first measurable cardinal above κ in N_n.

3. For every $\alpha < l(\vec{F})$ and $n < \omega$, define $F_{\alpha,n} = i_n(F_\alpha)$. $F_{\alpha,n}$ is a (κ, θ^{++V}) extender on κ in N_n since both κ and θ^{++V} are fixed points of i_n.

   $F_{\alpha,n}$ is clearly a $\kappa-$complete extender on κ in N_n. Since V and N_n have the same subsets of κ it follows that $F_{\alpha,n}$ is an extender of V as well.

4. For every $\nu < \kappa$, let $\Theta(\nu)$ be the first measurable cardinal above ν for every $\nu < \kappa$. Since $\Theta(\nu)$ is not a limit of measurable cardinals it carries a unique normal measure denoted by $U_{\Theta(\nu)}$.

Lemma 2.3. For every $\alpha < l(\vec{F})$ and $n < \omega$, $F_{\alpha,n}$ is the $(\kappa, \theta^{++})-$extender derived in V from the composition $i_n^{M_{F_\alpha}} \circ j_{F_\alpha} : V \to M_{\alpha,n}$ where $M_{\alpha,n} \cong \mathrm{Ult}^{(n)}(M_{F_\alpha}, U)$.

Proof. For simplicity of notations, we prove the Lemma for $n = 1$. The proof for arbitrary $n < \omega$ is similar. Here, $i_1^{M_{F_\alpha}} : M_{F_\alpha} \to N_{\alpha,1}$ is the $1-$st ultrapower embedding of M_{F_α} by U. Let $j_{\alpha,1} = i_1^{M_{F_\alpha}} \circ j_{F_\alpha} : V \to M_{\alpha,1}$ denote this composition. We need to verify that for every $\gamma < \theta^{++}$ and $X \subset \kappa$, $X \in F_{\alpha,1}(\gamma)$ if and only if $\gamma \in j_{\alpha,1}(X)$, where $F_{\alpha,1}(\gamma)$ is the $\gamma-$th measure in $F_{\alpha,1}$. As an element in $N_1 \cong \mathrm{Ult}(V, U)$, γ is represented as $\gamma = i_1(g)(\theta)$ where $g : \theta \to \theta^{++}$ is a function in V. Note that $F_{\alpha,1} = i_1(F_\alpha) \in N_1$ and $X = i_1(X) \in N_1$, so $X \in F_{\alpha,1}(\gamma)$ if and only if $i_1(X) \in i_1(F_\alpha)(\gamma) = i_1(F_\alpha)(i_1(g)(\theta))$. By Lós Theorem, the last is equivalent to $\{\nu < \theta \mid X \in F_\alpha(g(\nu))\} \in U$, thus

$$\gamma \in j_{\alpha,1}(X) \iff \{\nu < \theta \mid X \in F_\alpha(g(\nu))\} \in U.$$

Consider again $j_{\alpha,1} = i_1^{M_{F_\alpha}} \circ j_{F_\alpha}$. Note that every $f : \theta \to \theta^{++}$ in V can be easily coded as a subset of $\theta \times \theta^+ \times \theta^+$ and therefore belongs to $V_{\theta+2} \subset M_{F_\alpha}$. It follows that $\gamma = i_1^{M_{F_\alpha}}(g)(\theta)$ is represented by the same function $g \in N_{\alpha,1} \cong \mathrm{Ult}(M_{F_\alpha}, U)$, and hence

$$\gamma \in i_{\alpha,1}(X) \iff i_1^{M_{F_\alpha}}(g)(\theta) \in i_1^{M_{F_\alpha}}(j_\alpha(F))(i_1^{M_{F_\alpha}}(X)) \iff \{\nu < \theta \mid g(\nu) \in j_{F_\alpha}(X)\} \in U.$$

The claim follows as $g(\nu) \in j_{F_\alpha}(X)$ if and only if $X \in F_\alpha(g(\nu))$. □

The fact $l(\vec{F}) < \theta$ implies that the generators of the extenders derived from each $j_{\alpha,n}, M_{\alpha,n}$ belong to $[\kappa, \theta^{++})$[3]. We conclude the following

Corollary 2.4. For every $\alpha < l(\vec{F})$ and $n < \omega$ we have

1. The ultrapower of V by $F_{\alpha,n}$ is given by

$$j_{\alpha,n} = i_n^{M_{F_\alpha}} \circ j_{F_\alpha} : V \to M_{\alpha,n} \cong \mathrm{Ult}(V, F_{\alpha,n})$$

2. $F_{\alpha,n}$ is a $\kappa-$complete extender in V, i.e., $M_{\alpha,n}$ is closed under $\kappa-$sequences from V.

3. $j_{\alpha,n}(\Theta)(\kappa) = \theta_n$ and $U_{j_{\alpha,n}(\Theta)(\kappa)} = i_n(U)$.

[3]I.e., for every $x \in M_{\alpha,n}$ there is $\gamma \in [\kappa, \theta^{++})$ and $f : \kappa \to V$ so that $x = j_{\alpha,n}(f)(\gamma)$

2.1 Collapsing and Coding

Let $\nu \leq \kappa$ and suppose that V^* is a set generic extension of $V = \mathcal{K}(V^*)$ which preserves all stationary subsets of ν^+. For every $A \subset \nu^+$ in V^*, we define a coding poset $\mathrm{Code}(\nu^+, A)$ to be forced over V^*.

Definition 2.5 ($\mathrm{Code}(\nu^+, A)$)**.** Let $\vec{S} = \langle S_i \mid i < \nu^+ \rangle$ be the $<_{\mathcal{K}(V^*)}$ minimal $\diamondsuit_{\nu^+}$ sequence in $V = \mathcal{K}(V^*)$. This sequence is definable from $H(\nu^+)^V$. Let $\langle T_i \mid i < \nu^+ \rangle$ be a sequence of pairwise disjoint stationary subsets of $\mathrm{Cof}(\nu) \cap \nu^+$, defined by $T_i = \{\mu \in \mathrm{Cof}(\nu) \cap \nu^+ \mid S_\mu = \{i\}\}$.
Conditions $c \in \mathrm{Code}(\nu^+, A)$ are closed bounded subsets $c \subset \nu^+$ which satisfy the following properties for every $i < \nu$:

- $c \cap T_{4i} = \emptyset$ if $i \in A$, and
- $c \cap T_{4i+1} = \emptyset$ if $i \notin A$.

For every $c, d \in \mathrm{Code}(\nu^+, A)$, $d \geq c$ if

- d is an end extension of c,
- $(d \setminus c) \cap T_{4i+2} = \emptyset$ if $i \in c$, and
- $(d \setminus c) \cap T_{4i+3} = \emptyset$ if $i < \max(c)$, $i \notin c$.

The forcing $\mathrm{Code}(\nu^+, A)$ is $< \nu^+-$distributive. Let $C \subset \nu^+$ be a $\mathrm{Code}(\nu^+, A)-$generic club over V^*. It is clear from the definition of $\mathrm{Code}(\nu^+, A)$ that for every $i < \nu^+$, C is almost disjoint from T_{4i} if $i \in A$; from T_{4i+1} if $i \notin A$; from T_{4i+2} if $i \in C$; from T_{4i+3} if $i \notin C$. The sets T_j, $j < \nu^+$, which are not required to be almost disjoint from C, remain stationary (see [7]). Therefore, for every $i < \nu^+$, exactly one of T_{4i},T_{4i+1} is not stationary in V^*, and exactly one of T_{4i+2},T_{4i+3} is not stationary in V^*.

Let $g : \nu^+ \to \tau$ be a surjection from ν^+ onto some ordinal τ, and let $A_g \subset \nu^+$ be a canonical encoding of g defined by $\langle \mu_0, \mu_1 \rangle \in A_g$ if and only if $g(\nu_0) \leq g(\nu_1)$. Here $\langle \cdot, \cdot \rangle$ is the Godel pairing function $\nu^+ \times \nu^+$. Note that g can be easily reconstructed from A_g. We define $\mathrm{Code}(\nu^+, g)$ to be $\mathrm{Code}(\nu^+, A_g)$.

Definition 2.6 (Collapsing and Coding Iteration $\mathcal{P}$)**.** Define an iteration $\mathcal{P} = \mathcal{P}_{\kappa+1} = \langle \mathcal{P}_\nu, \dot{Q}_\nu \mid \nu \leq \kappa \rangle$ over $V = L[E]$. We use the Friedman-Magidor non-stationary support, i.e., every $p \in \mathcal{P}_\nu$ belongs to the inverse limit of the posets $\langle \mathcal{P}_\mu \mid \mu < \nu \rangle$ with the restriction that if ν is inaccessible, then the set of $\mu < \nu$ such that p_μ is nontrivial is a nonstationary subset of ν. For every $\nu \leq \kappa$ if ν is not an inaccessible limit of measurable cardinals then $\Vdash_{\mathcal{P}_\nu} \dot{Q}_\nu = \emptyset$. Otherwise, $\Vdash_{\mathcal{P}_\nu} \dot{Q}^1_\nu = \mathrm{Coll}(\nu^+, \Theta(\nu)^{++}) * \mathrm{Code}(\nu^+, \dot{g}_\nu)$ where

1. $\mathrm{Coll}(\nu^+, \Theta(\nu)^{++})$ is the Levy collapsing forcing, introducing a surjection $g_\nu : \Theta(\nu)^{++} \twoheadrightarrow \nu^+$, and

2. $\mathrm{Code}(\nu^+, \dot{g}_\nu)$ is the coding of the generic collapsing function.

   Remark 2.7. $\dot{Q}_\nu$ is $< \nu-$closed and $< \nu^+-$distributively closed for every $\nu < \kappa$. The uniqueness of the $\mathrm{Code}(\nu^+)$ generic at every nontrivial stage $\nu \leq \kappa$ implies that for every $V-$generic filter $G \subset \mathcal{P}$, G is the unique $\mathcal{P}$ generic over V in $V[G]$.

1. The iteration $\mathcal{P}$ is a variant of the Friedman-Magidor iteration ([7]) where the collapsing posets $\mathrm{Coll}(\nu^+, \Theta(\nu)^{++})$ are replaced with generalized Sacks forcings.

In order to force with $\mathrm{Code}(\nu^+, \dot{g}_\nu)$ over a $\mathcal{P} \restriction \nu * \mathrm{Coll}(\nu^+, \Theta(\nu)^{++})-$generic extension of V, it is necessary that the extension preserves the stationary subsets of ν^+ in V, which are used in $\mathrm{Code}(\nu^+, \dot{g}_\nu)$. This is indeed the case. The proof is similar to the one given by Friedman-Magidor.

The next results follow from the arguments by Friedman and Magidor[4]:

Lemma 2.8. 1. For every inaccessible limit of measurable cardinals $\nu \leq \kappa$, the poset $\mathcal{P} \restriction \nu * \mathrm{Coll}(\nu^+, \Theta(\nu)^{++})$ preserves stationary subsets of ν^+.

2. The iteration $\mathcal{P}$ is $\sigma-$closed and does not collapse inaccessible limits of measurable cardinals in V.

3. For every function $\phi : \kappa \to \kappa$ in a generic extension of V by $\mathcal{P}$, there exists some $f : \kappa \to [\kappa]^{<\kappa}$ in V, so that $|f(\nu)| \leq \Theta^{++}(\nu)$, and $\phi(\nu) \in f(\nu)$ for every $\nu < \kappa$.

We proceed to study the normal measures on κ in a $\mathcal{P}$ generic extension. Let $G \subset \mathcal{P}$ be generic over V.

[4] I.e., Lemma 5 and Lemma 14 in [7]

Lemma 2.9. For every $\alpha < l(\vec{F})$ and $n < \omega$, there is a unique $M_{\alpha,n}-$generic filter $G_{\alpha,n} \subset j_{\alpha,n}(\mathcal{P})$ so that $j_{\alpha,n}``G \subset G_{\alpha,n}$.

Proof. We have that $j_{\alpha,n}(\mathcal{P}) \restriction (\kappa+1) = \mathcal{P}$. The coding generics and the fact $H(\kappa^+)^{M_{\alpha,n}} = H(\kappa^+)^V$ imply that $G \in V[G]$ is the only possible $M_{\alpha,n}-$generic filter for $\mathcal{P}$. We prove that $\{j_{\alpha,n}(q') \setminus (\kappa+1) \mid q' \in G\}$ generates a $M_{\alpha,n}[G]-$generic filter for $j_{\alpha,n}(\mathcal{P}) \setminus (\kappa+1)$. The proof consists of two parts: In the first part (**1**) we restrict ourselves to conditions $q \in G \restriction \kappa \subset \mathcal{P}_\kappa$ and $j_{\alpha,n}(q) \in j_{\alpha,n}(\mathcal{P}_\kappa) = j_{\alpha,n}(\mathcal{P}) \restriction j_{\alpha,n}(\kappa)$ (i.e., we do not address the last forcing step $j_{\alpha,n}(\dot{Q}_\kappa)$). We show that for every $\mathcal{P}-$name of a dense open set $D \subset j_{\alpha,n}(\mathcal{P}_\kappa) \setminus (\kappa+1)$ there is $q \in G \restriction \kappa$ such that $j_{\alpha,n}(q) \restriction (\kappa+1) \Vdash j_{\alpha,n}(q) \setminus (\kappa+1) \in D$. Then, in the second part (**2**), we build on the results of the first part and deal with dense open sets $E \subset j_{\alpha,n}(\mathcal{P}) \setminus (\kappa+1)$.

1. Let D be a $\mathcal{P}-$name for a dense open set of $j_{\alpha,n}(\mathcal{P}_\kappa) \setminus (\kappa+1)$. There are $\gamma < \theta^{++}$ and $f : \kappa \to V$ so that $D = j_{\alpha,n}(f)(\gamma)$. We may assume that for every $\mu < \kappa$, if $\kappa(\mu) \leq \mu$ is the largest cardinal up to μ which is a limit of measurable cardinals, then $f(\mu)$ is a $\mathcal{P} \restriction (\kappa(\mu)+1)$ name for a dense open set of $\mathcal{P}_\kappa \setminus (\kappa(\mu)+1)$

Let p be a condition in $\mathcal{P}_\kappa$. Working in V, we simultaneously define three sequences:

1. $\langle p_i \mid i \leq \kappa \rangle$, an increasing sequence of conditions above p.
2. $\langle C_i \mid i \leq \kappa \rangle$, a sequence of closed unbounded subset of κ.
3. $\vec{\nu} = \langle \nu_i \mid i < \kappa \rangle$, a continuous increasing sequence of ordinals below κ.

We take p_0 to be an extension of p which belongs to $f(0)$, $C_0 \subset \kappa$ to be a closed unbounded set, disjoint from $\text{supp}(p_0)$, and $\nu_0 = \min(\{\nu \in C_0 \mid \nu \text{ is inaccessible and limit of measurable cardinals}\})$. The forcing $\mathcal{P}_\kappa \setminus (\nu_0 + 1)$ is $\Theta(\nu_0)^{+3}-$closed and therefore the intersection $\bigcap\{f(\mu) \mid \mu \leq \Theta(\nu_0)^{++}\}$ is dense in $\mathcal{P}_\kappa \setminus (\nu_0+1)$. Let p_1 be an extension of p_0 so that

1. $p_1 \restriction \nu_0 + 1 = p_0 \restriction \nu_0 + 1$, and
2. $p_1 \restriction \nu_0 + 1 \Vdash p_1 \setminus \nu_0 + 1 \in \bigcap\{f(\mu) \mid \mu \leq \Theta(\nu_0)^{++}\}$.

We take $C_1 \subset C_0$ to be a closed unbounded set so that $C_1 \cap \text{supp}(p_1) = \emptyset$ and $C_1 \cap (\nu_0 + 1) = C_0 \cap (\nu_0 + 1)$. We then choose $\nu_1 = \min(\{\nu \in C_1 \setminus (\nu_0 + 1) \mid \nu \text{ is inaccessible and limit of measurable cardinals}\})$.
Suppose that the three sequences have been constructed up to some $i^* \leq \kappa$, and satisfy the following properties:

1. $\langle p_j \mid j < i^* \rangle$ is an increasing sequence of conditions and $\langle \nu_j \mid j < i^* \rangle$ is a continuous increasing sequence of ordinals below κ,

2. $p_{j_1} \restriction \nu_{j_1} + 1 = p_{j_2} \restriction \nu_{j_1} + 1$ and $C_{j_2} \subset C_{j_1}$ for every $j_1 < j_2 < i^*$,

3. $p_{j+1} \restriction \nu_j + 1 \Vdash p_{j+1} \setminus \nu_j + 1 \in \bigcap \{f(\mu) \mid \mu \leq \Theta(\nu_j)^{++}\}$ whenever $j + 1 < i^*$,

4. $C_j \cap \text{supp}(p_j) = \emptyset$ for every $j < i^*$,

5. $\{\nu_j \mid j < i^*\} \subset \bigcap_{i<i^*} C_i$ if $i^* < \kappa$.

If $i^* = i+1$ is a successor ordinal then we define $p_{i+1}, C_{i+1}, \nu_{i+1}$ from p_i, C_i, ν_i the same way p_1, C_1, ν_1 were defined from p_0, C_0, ν_0.
Suppose $i^* = \delta$ is a limit ordinal. Let $\nu_\delta = \cup_{i<\delta} \nu_i$ and define C_δ to be $\bigcap_{i<\delta} C_i$ if $\delta < \kappa$, and $\triangle^{\vec{\nu}} C_i = \{\alpha < \kappa \mid \forall i < \kappa. (\nu_i < \alpha) \to (\alpha \in C_i)\}$ if $\delta = \kappa$. For every $j \leq \delta$ we have that $\{\nu_i \mid i < \delta\} \subset C_j$ and therefore $\nu_\delta \in C_j$.

Let us define p_δ. We first construct $p_\delta \restriction \nu_\delta$: For every $i < \nu_\delta$ let $(p_\delta)_i = (p_j)_i$ where $j < \delta$ is such that $i < \nu_j$. It follows that $p_\delta \restriction \nu_j + 1 = p_j \restriction \nu_j + 1$ for every $j < \delta$. Let us verify that $p_\delta \restriction \nu_\delta$ has nonstationary support. For every inaccessible cardinal $\gamma < \nu_\delta$ we have that $p_\delta \restriction \gamma = p_\gamma \restriction \gamma$ hence $\text{supp}(p_\delta \restriction \nu_\delta) \cap \gamma$ is nonstationary in γ. If ν_δ is inaccessible (i.e., $\nu_\delta = \delta$) then $\text{supp}(p_\delta \restriction \nu_\delta) \subset \nu_\delta$ is nonstationary in ν_δ as it is disjoint $\{\nu_i \mid i < \delta\}$.
If $\delta = \kappa$ then $\nu_\delta = \kappa$, so $p_\delta = p_\delta \restriction \nu_\delta$ and we are done. Suppose that $\delta < \kappa$. Set $(p_\delta)_{\nu_\delta} = 0_{\dot{Q}_{\nu_\delta}}$ and let $p_\delta \setminus (\nu_\delta + 1)$ be a common extension of $\{p_i \setminus \nu_\delta + 1 \mid i < \delta\}$ so that $\text{supp}(p_\delta \setminus \nu_\delta + 1) = \bigcup_{i<\delta} \text{supp}(p_i \setminus \nu_\delta + 1)$. This is clearly possible as $\mathcal{P}_\delta \setminus (\nu_\delta + 1)$ is δ^+-closed, and it is not difficult to see that p_δ, C_δ, and ν_δ satisfy the inductive assumptions.
This concludes the construction of the sequence $\langle p_i \mid i \leq \kappa \rangle$. Let $q = p_\kappa$. For every $i < \kappa$ we have that $q \restriction \nu_i + 1 \Vdash q \setminus (\nu_i + 1) \in \bigcap \{f(\mu) \mid \mu \leq \Theta(\nu_i)^{++}\}$. Let $j_{n,\alpha}(\{\nu_i \mid i < \kappa\}) = \{\nu_i \mid i < j_{n,\alpha}(\kappa)\}$. Since $\{\nu_i \mid i < \kappa\}$ is a closed unbounded set in κ we conclude that $\nu_\kappa = \kappa$ and that

$$j_{\alpha,n}(q) \restriction (\kappa + 1) \Vdash j_{\alpha,n}(q) \setminus (\kappa + 1) \in \bigcap \{j_{\alpha,n}(f)(\mu) \mid \mu \leq \theta_n^{++}\}.$$

The last intersection includes $D = j_{\alpha,n}(f)(\gamma)$.

2. Suppose now that E is dense open set of $j_{\alpha,n}(\mathcal{P}) \setminus (\kappa+1)$. There are $e : \kappa \to V$ and $\gamma < \theta^{++}$ such that $E = j_{\alpha,n}(e)(\gamma)$. We may assume that for every $\mu < \kappa$, $e(\mu)$ is a $\mathcal{P}_{\kappa(\mu)+1}$ name of a dense open set of $\mathcal{P} \setminus (\kappa(\mu)+1)$. Let $\dot{E}(G(j_{\alpha,n}(\mathcal{P}_\kappa))$ be a $j_{\alpha,n}(\mathcal{P}_\kappa)$ for the set of all conditions $e \in j_{\alpha,n}(\dot{Q}_\kappa)$ for which there is some t in the generic filter $G(j_{\alpha,n}(\mathcal{P}_\kappa))$ such that $t^\frown \dot{e} \in E$. $\dot{E}(G(j_{\alpha,n}(\mathcal{P}_\kappa))$ is dense open in $j_{\alpha,n}(\dot{Q}_\kappa)$. Similarly, for every $\mu < \kappa$ $e(\mu)(G(\mathcal{P}_\kappa))$ is a name for a dense open set of $\dot{Q}_\kappa$. Fix a condition $p' = \langle p'_\alpha \mid \alpha \leq \kappa\rangle \in \mathcal{P}$. The fact $\dot{Q}_\kappa$ is $< \kappa^+-$distributive implies there is a $\mathcal{P}_\kappa$ name q'_κ so that $p' \restriction \kappa \Vdash p'_\kappa \leq q'_\kappa \in \bigcap_{\mu<\kappa} e(\mu)(G(\mathcal{P}_\kappa))$. Let $p = p' \restriction \kappa$. We have that $j_{\alpha,n}(p) \Vdash j_{\alpha,n}(q'_\kappa) \in \dot{E}(G(j_{\alpha,n}(\mathcal{P}_\kappa))$. Let $D \subset j_{\alpha,n}(\mathcal{P}_\kappa) \setminus (\kappa+1)$ be a $\mathcal{P}-$name for the set of all $t' \in j_{\alpha,n}(\mathcal{P}_\kappa) \setminus (\kappa+1)$ for which $t'^\frown j_{\alpha,n}(q'_\kappa) \in E$. D is a dense open set of $j_{\alpha,n}(\mathcal{P}_\kappa) \setminus (\kappa+1)$ so by the construction in the first part (1), there is a condition $q \geq p$ in $\mathcal{P}_\kappa$ such that $j_{\alpha,n}(q) \restriction (\kappa+1) \Vdash j_{\alpha,n}(q) \setminus (\kappa+1) \in D$. Taking $q' = q^\frown q'_\kappa$, we conclude that $q' \geq p'$ and $j_{\alpha,n}(q') \restriction (\kappa+1) \Vdash j_{\alpha,n}(q') \setminus (\kappa+1) \in E$. □

Definition 2.10. Define the following in $V[G]$:

1. Let $G_{\alpha,n} \subset j_{\alpha,n}(\mathcal{P})$ denote the unique generic filter over $M_{\alpha,n}$ with $j_{\alpha,n}$“$G \subset G_{\alpha,n}$, whose existence was proved in Lemma 2.9.
2. Let $i_{\alpha,n} : V[G] \to M_{\alpha,n}[G_{\alpha,n}]$ be the unique extension of $j_{\alpha,n}$ to an embedding in $V[G]$. $i_{\alpha,n}$ is defined by $i_{\alpha,n}(\dot{x}_G) = j_{\alpha,n}(\dot{x})_{G_{\alpha,n}}$.
3. Let $U_{\alpha,n}$ be the normal measure defined by

$$U_{\alpha,n} = \{X \subset \kappa \mid \kappa \in i_{\alpha,n}(X)\}.$$

Remark 2.11. The proof of Lemma 2.9 implies that for every $X = \dot{X}_G \subset \kappa$ in $V[G]$, $X \in U_{\alpha,n}$ if and only if there exists a condition $p \in G$ so that in $M_{\alpha,n}[G]$,

$$j_{\alpha,n}(p) \setminus (\kappa+1) \Vdash_{j_{\alpha,n}(\mathcal{P}) \setminus (\kappa+1)} \check{\kappa} \in j_{\alpha,n}(\dot{X}).$$

Equivalently, $X \in U_{\alpha,n}$ if and only if there is $p \in G$ so that

$$p^\frown (j_{\alpha,n}(p) \setminus \kappa+1) \Vdash_{j_{\alpha,n}(\mathcal{P})} \check{\kappa} \in \dot{X}.$$

Let $\vec{\rho} = \langle \rho_\zeta \mid \zeta < \kappa^+ \rangle$ be a sequence of canonical functions at κ, so that each $\rho_\zeta \in {}^\kappa\kappa$ has Galvin-Hajnal degree ζ. We have that

1. for every $\zeta_0 < \zeta_1 < \kappa^+$, the set $\{\nu < \kappa \mid \rho_{\zeta_0}(\nu) \geq \rho_{\zeta_1}(\nu)\}$ is bounded in κ,
2. for every elementary embedding $j : V \to M$ in V, then $\zeta = j(\rho_\zeta)(\kappa)$ for all $\zeta < \kappa^+$.

Let $\vec{g} = \langle g_\nu \mid \nu < \kappa \rangle$ be the generic sequence of collapsing functions induced from G, i.e., $g_\nu : \nu^+ \twoheadrightarrow \Theta(\nu)^{++}$ for every $\nu \leq \kappa$ which is an inaccessible limit of measurable cardinals.

We can use each $g_\nu : \nu^+ \twoheadrightarrow \Theta(\nu)^{++}$ to construct a surjection $h_\nu : \nu^+ \twoheadrightarrow \Theta^{++}(\nu)$, which is canonically defined from g_ν[5]. For every $\delta < \theta^{++}$ let ζ_δ be the unique $\zeta < \kappa^+$ so that $\delta = h_\kappa(\zeta_\delta)$.

Definition 2.12. Let $\delta < \theta^{++}$. Define a function $\psi_\delta : \kappa \to \kappa$ by $\psi_\delta(\nu) = h_\nu(\rho_\zeta(\nu))$ for every $\nu < \kappa$.

It follows that for every $\delta_0, \delta_1 < \theta^{++}$, the set $\{\nu < \kappa \mid \psi_{\delta_0}(\nu) \neq \psi_{\delta_1}(\nu)\}$ is bounded in κ. Also, for every $\alpha < l(\vec{F})$ and $n < \omega$, it is easy to check that $i_{\alpha,n}(\psi_\delta)(\kappa) = \delta$ for every $\delta < \theta$.

Lemma 2.13. $i_{\alpha,n} : V[G] \to M_{\alpha,n}[G_{\alpha,n}]$ coincides with the ultrapower embedding of $V[G]$ by $U_{n,\alpha}$.

Proof. Since $U_{\alpha,n}$ is derived from $i_{\alpha,n}$, it is sufficient to verify that every $x \in M_{\alpha,n}[G_{\alpha,n}]$ can be represented as $x = i_{\alpha,n}(\phi)(\kappa)$, for some $\phi : \kappa \to V[G]$. Every $x \in M_{\alpha,n}[G_{\alpha,n}]$ is of the form $x = (\dot{x})_{G_{\alpha,n}}$, where $\dot{x} = j_{\alpha,n}(f)(\delta) = i_{\alpha,n}(f)(\delta)$, for some $f : \kappa \to V$ in V and $\delta < \lambda$. We may assume that $f(\nu)$ is a $\mathcal{P}$ name, for every $\nu < \kappa$. Define $\phi : \kappa \to V[G]$ by $\phi(\nu) = f(\psi_\delta(\nu))_G$. It follows that $x = i_{\alpha,n}(\phi)(\kappa)$ as $\delta = i_{\alpha,n}(\psi_\delta)(\kappa)$ and $G_{\alpha,n} = i_{\alpha,n}(G)$. □

[5]For example, restrict the domain of g_ν to ordinals $\mu < \nu^+$ so that $\mu = \min(g_\nu^{-1}(\{\tau\}))$ for some $\tau < \Theta^{++}(\nu)$, and then collapsing the restricted domain to achieve the bijection h_ν.

2.2 The Mitchell Order in $V[G]$

We show that $U_{\alpha,n}$, $\alpha < l(\vec{F})$, $n < \omega$ are the only normal measures on κ in $V[G]$, and that $U_{\alpha',n'} \triangleleft U_{\alpha,n}$ if and only if $\alpha' < \alpha$ and $n' \geq n$. To prove these results, we use Schindler's description ([15]) of ultrapower restrictions.

Proposition 2.14. Let W be a normal measure on κ in $V[G]$. There exists some $\alpha < l(\vec{F})$ and $n < \omega$, such that $W = U_{\alpha,n}$.

Proof. Let $j_W : V[G] \to M_W \cong \mathrm{Ult}(V[G], W)$ be the induced ultrapower embedding and $j = j_W \restriction V : V \to M$ be its restriction to $V = \mathcal{K}(V[G])$. According to Schindler ([15]) there is an iteration tree T on V and a cofinal branch b so that $\pi^T_{0,b} = j_W \restriction V : V \to M$ results from the normal iteration of T along b. Furthermore $M_W = M[G_W]$ where $G_W = j_W(G) \subset j(\mathcal{P})$ is M-generic. In 2.12 we defined a sequence of functions $\langle \psi_\delta \mid \delta < \lambda \rangle$ so that so that $\{\nu < \kappa \mid \psi_{\delta_0}(\nu) = \psi_{\delta_1}(\nu)\}$ is bounded in κ for every distinct $\delta_0, \delta_1 < \theta$. It follows that $j_W(\psi_{\delta_0})(\kappa) \neq j_W(\psi_{\delta_1})(\kappa)$ for every $\delta_0 \neq \delta_1$, thus $j_W(\kappa) \geq \theta^{++}$. Furthermore, as $\psi_\delta(\nu) < \Theta^{++}(\nu)$ for every $\delta < \theta^{++}$ and $\nu < \kappa$, it follows that $j_W(\psi_\delta)(\kappa) < j_W(\Theta^{++})(\kappa) = j(\Theta^{++})(\kappa)$ thus $\theta^{++} \leq j(\Theta^{++})(\kappa)$. As $M_W = M[G_W]$ and $G_W \subset j(\mathcal{P})$ is generic, $(\theta^{++})^V$ is collapsed to κ^+ in M_W. With this in mind, let us consider the iteration tree T and the cofinal branch b inducing $\pi^T_{0,b} = j$. We first claim that the iteration of b does not use the same extender more the finitely many times. Otherwise there would be an ordinal $\delta \leq \pi_0(\kappa)$ which is the the limit of the critical points of an $\omega-$subiteration by the same measure. In particular $\mathrm{Cof}^{V[G]}(\delta) = \omega$. Since δ is the image of the critical points it must be inaccessible in M. Furthermore, its cofinality in $M[G_W]$ is ω because $M[G_W] = M_W$ is closed under ω sequence in $V[G]$. It follows that the $M-$ generic set $G_W \subset j(\mathcal{P})$ introduces a cofinal $\omega-$sequence in δ. However this is impossible as $j(\mathcal{P})$ is a $\sigma-$closed forcing.
Since $\mathrm{cp}(\pi^T_{0,b}) = \mathrm{cp}(j_W) = \kappa$, $\pi^T_{0,b}$ factors into $\pi^T_{0,b} = \pi^T_{1,b} \circ j_F$, where F is an extender on κ in V and $\mathrm{cp}(\pi^T_{1,b}) > \kappa$. By the elementarity of $j_F : V \to M_F$, $j_F(\kappa)$ is an inaccessible limit of measurable cardinals in $M_F \cong \mathrm{Ult}(V, F)$ and there are no extenders $F' \in M_F$ for which $\mathrm{cp}(F') < j_F(\kappa)$ and $\alpha(F') \geq j_F(\kappa)$. Since the rest of the iteration along b can apply each extender finitely many times, $j_F(\kappa)$ must be fixed point of $\pi^T_{1,b}$, and an inaccessible and limit of measurable cardinals in M. By Lemma 2.8, $j_F(\kappa)$ is not collapsed in $M_W = M[G_W]$. It follows that $j_F(\kappa) \geq \theta^{++}$ so $\nu(F) \geq \theta^{++}$. Since the normal measure U on θ is the only measure which overlaps extenders on κ, it follows that F belongs to $N_\eta \cong \mathrm{Ult}^\eta(V, U)$ for some ordinal η. We claim that $\eta < \omega$.

Otherwise, if θ_ω is the $\omega-$th image of $\theta = \mathrm{cp}(U)$ the θ_ω is an inaccessible cardinal in both N_η and $M_F \cong \mathrm{Ult}(V, F)$. This is impossible as $\mathrm{Cof}^{V[G]}(\theta_\omega) = \omega$ but $j(\mathcal{P})$ $\sigma-$closed. We conclude that $\eta = n$ for some finite $n < \omega$, and as $\nu(F) \geq \theta^{++}$ we get that $F = F_{\alpha,n}$ for some $\alpha < l(\vec{F})$. Finally, we verify that $W = U_{\alpha,n}$. We can rewrite he restriction $j_W \restriction V = j$ as $j = \pi_1 \circ j_{\alpha,n}$ where $\mathrm{cp}(\pi_1) > \kappa$. We clearly have that $j(\mathcal{P}) \restriction (\kappa+1) = \mathcal{P}$. The coding posets in $\mathcal{P}$ and the fact $H(\kappa^+)^{M_W} = H(\kappa^+)^V$ imply that $G = G_W \restriction (\kappa+1)$. Since we also have that $j``G \subset G_W$ we conclude that $p^\frown(j(p) \setminus \kappa+1) \in G_W$ for every $p \in G$. Suppose that $X = \dot{X}_G \in U_{\alpha,n}$. According to Remark 2.11, there is some $p \in G$ so that $p^\frown(j_{\alpha,n}(p) \setminus \kappa+1) \Vdash \check{\kappa} \in j_{\alpha,n}(\dot{X})$. By applying π_1 we conclude that $p^\frown(j(p) \setminus \kappa+1) \Vdash \check{\kappa} \in j(\dot{X})$, thus $X \in W$. □

Proposition 2.15. For every $\alpha, \alpha' < l(\vec{F})$ and $n, n' < \omega$, $U_{\alpha',n'} \triangleleft U_{\alpha,n}$ if and only if $\alpha' < \alpha$ and $n' \geq n$.

Proof. Suppose first that $n' \geq n$ and $\alpha' < \alpha$. It is clear that $F_{\alpha',n'} \in M_{\alpha,n}$ and that $\mathcal{P}^V = \mathcal{P}^{M_{\alpha,n}}$. The construction of $U_{\alpha',n'}$ requires $F_{\alpha',n'}$,$V_{\kappa+1}$, and G; all belong to $M_{\alpha,n}[G_{\alpha,n}] = \mathrm{Ult}(V[G], U_{\alpha,n})$, thus $U_{\alpha',n'} \triangleleft U_{\alpha,n}$.

Suppose now that $U_{\alpha',n'} \triangleleft U_{\alpha,n}$. To simplify our notations, let us denote $W_{\alpha',n'}$ by W' and $W_{\alpha,n}$ by W. Accordingly, let $j_{W'} : V[G] \to M' \cong \mathrm{Ult}(V[G], W')$ and $j_W : V[G] \to M \cong \mathrm{Ult}(V[G], W)$. Note that $\mathcal{K}(M) = M_{\alpha,n}$ and $\mathcal{K}(M') = M_{\alpha',n'}$are both extender models. Let us denote $\mathcal{K}(M)$ by $L[E^M]$ and $\mathcal{K}(M')$ by $L[E^{M'}]$. Since $W' \triangleleft W$, we can form an ultrapower by W' in M. Let $i' : M \to N' \simeq \mathrm{Ult}(M, W')$, and denote $\mathcal{K}(N')$ by $L[E^{N'}]$. According to Schindler ([15]), $\mathcal{K}(N')$ results from a normal iteration of $K(M)$. Namely, there is an iteration tree T of $K(M)$, and a cofinal branch b, so that $j_{W'} \restriction K(M) = \pi^{\mathcal{K}(M)}_{0,b} : \mathcal{K}(M) \to \mathcal{K}(N')$. Moreover, the proof of Theorem 2.1 in [15] implies that iteration tree T is the tree which results from a comparison between $\mathcal{K}(M)$ and $\mathcal{K}(N')$. Thus, when applying a comparison process to $L[E^M]$ and $L[E^{N'}]$ we get that $L[E^{N'}]$ does not move, and the iteration tree T on $L[E^M]$, is determined by comparing the extender sequence of the iterands of $L[E^M]$ with $E^{N'}$.
Now, M is the ultapower of $V[G]$ by a normal measure on κ, so $M \cap H(\kappa^+) = V[G] \cap H(\kappa^+)$. Therefore, when taking the ultrapower of both models by W', we get that $i' \restriction \kappa^+ = j_{W'} \restriction \kappa^+$ and that $M' \cap V[G]_{i'(\kappa)} = N' \cap V[G]_{i'(\kappa)}$. In particular, $E^{N'} \restriction i'(\kappa) = E^{M'} \restriction i'(\kappa)$.
It follows that when coiterating $\mathcal{K}(M) = L[E^M]$ with $\mathcal{K}(M') = L[E^{M'}]$,

the $L[E^{M'}]-$side does not move below $j_{W'}(\kappa)$, namely $E^{M'} \restriction j_{W'}(\kappa)$ is fixed throughout the comparison.

We now conclude that $\alpha' < \alpha$ and $n' \geq n$ by coiterating $\mathcal{K}(M) = M_{\alpha,n}$ with $\mathcal{K}(M') = M_{\alpha',n'}$. Let $m = \min(n, n')$. If $n \neq n'$ then the first point of distinction between the E^M and $E^{M'}$ is at the index of the measure $U^m = i_m(U)$, which is $\alpha = (\theta_m^{++})^{\mathrm{Ult}^{m+1}(V,U)} < \theta^{++} < j_{W'}(\kappa)$. We must have that $n' \geq n$, as otherwise the first steps in the coiteration would consists of ultrapowers by $i_{n'}(U) \in E^{M'} \restriction j_{W'}(\kappa)$ on the $L[E^{M'}]$ side. This contradicts the assumption that $E^{M'} \restriction j_{W'}(\kappa)$ remains fixed. Let $k = n' - n \geq 0$. It follows that the first k steps of the coiteration coincide with the $k-$ultrapower of $L[E^M]$ by $i_n(U)$. Let us denote the resulting ultrapower of $L[E^M]$ by $L[E^{M,k}]$. It is clear that the extender sequences $E^{M'}$ and $E^{M,k}$ agree on indices up to θ^{++}. Therefore, the first possible difference between $E^{M'}$ and $E^{M,k}$ would be at an extender F so that $\nu(F) = \theta^{++}$. These are the extenders with support θ^{++} in the models $M_{\alpha,n'}$ and $M_{\alpha',n'}$. We claim that $\alpha' \neq \alpha$. Otherwise the agreement between $E^{M'}$ and $E^{M,k}$ would go above all extenders F with critical point $\mathrm{cp}(F) = \kappa$. But this would imply that the whole iteration T of $\mathcal{K}(M)$ is above $\kappa+1$ (Note that the critical points in the first k steps of the iteration are above κ) and so $\mathrm{cp}(\pi_{0,b}^{\mathcal{K}(M)}) > \kappa$. This is absurd as $\pi_{0,b}^{\mathcal{K}(M)} = i' \restriction \mathcal{K}(M)$ where $\mathrm{cp}(i') = \kappa$. We conclude that $\alpha' \neq \alpha$. Finally, if $\alpha' > \alpha$ then the first disagreement between $E^{M'}$ and $E^{M,k}$ would be at the extender $F_{\alpha',n'} \in E^{M'} \restriction j_{W'}(\kappa)$. This would contradict the fact that $E^{M'} \restriction j_{W'}(\kappa)$ remains fixed in the comparison. It follows that $\alpha' < \alpha$. $\square$

2.3 A Final Cut

We apply a final cut forcing over $V[G]$. For every $X \subset \kappa$ in $V[G]$, let $\mathcal{P}^X$ be the final cut forcing by X, defined in [4] (see Section 7).

Lemma 2.16. Suppose that X is a subset of κ in $V[G]$ and let $G^X \subset \mathcal{P}^X$ be generic over $V[G]$. For every normal measure U on κ in $V[G]$, if $X \notin U$ then U has a unique extension U^X in $V[G * G^X]$. Furthermore, these are the only normal measures on κ in $V[G * G^X]$.

Proof. Suppose $U \in V[G]$ is a normal measure on κ such that $X \notin U$ and let $j : V[G] \to M[G_U] \cong \mathrm{Ult}(V[G], U)$ be its ultrapower embedding. It is clear that $j(\mathcal{P}^X) \restriction \kappa = \mathcal{P}^X$, and stage κ of $j(\mathcal{P}^X)$ is trivial as $\kappa \notin j(X)$. The Friedman-Magidor iteration style implies that $j``G^X$ determines a unique

generic filter $H^X \subset j(\mathcal{P}^X) \setminus \kappa$ over $M[G^X]$. Setting $G^* = G^X * H^X$, we get that $G^* \subset j(\mathcal{P}^X)$ is the unique generic filter over M for which $j``G^X \subset G^*$. It follows that $j^* : V[G*G^X] \to M[G_U]$ is the unique extension of $j : V[G] \to M$ in $V[G*G^X]$ and that $U^X = \{Y \subset \kappa \mid \kappa \in j^*(Y)\}$ is the only normal measures extending U in $V[G * G^X]$.
Let W be a normal measure on κ in $V[G*G^X]$ and $j_W : V[G*G^X] \to M_W \cong \text{Ult}(V[G * G^X], W)$. The embedding $j = j_W \restriction V : V \to M$ results from a normal iteration of V given by a cofinal branch b in an iteration tree T. Furthermore, $M_W = M[G_W * G^X_W]$ where $G_W * G^X_W \subset j(\mathcal{P} * \mathcal{P}^X)$ is generic over M. It is easy to verify that the arguments Proposition 2.14 applies here as well. Note that the replacement of $\mathcal{P}$ with $\mathcal{P} * \mathcal{P}^X$ does not affect the argument, as $\mathcal{P} * \mathcal{P}^X$ (like $\mathcal{P}$) does not add new $\omega-$sequences and does not collapse inaccessible limits of measurable cardinals. It follows that there are $\alpha < l(\vec{F})$ and $n < \omega$ so that $U_{\alpha,n} \subset W$.
We claim that $X \notin W$. Otherwise $\kappa \in j(X)$, so stage κ in $j(\mathcal{P}^X)$ is $\text{Code}^*(\kappa)$. Let $C \in M[G_W * G^X_W]$ be the $\text{Code}^*(\kappa)$ generic club determined from G^X_W. We have that $C \cap X^\kappa_0 = \emptyset$ and that C is closed unbounded in $V[G * G^X]$ (as $M[G_W * G^X_W] = M_W$ is closed under $\kappa-$sequences). Thus X^κ_0 is nonstationary in $V[G * G^X]$, but this is impossible since $\mathcal{P}^X$ preserves all stationary subset of κ^+. We conclude that $W \cap V[G] = U_{\alpha,n}$ and that $X \notin U_{\alpha,n}$. Let us verify $U^X_{\alpha,n} \subseteq W$. Let $Y = (\dot{Y})_{G^X}$ be a set in $U^X_{\alpha,n}$. Since $j_{U_{\alpha,n}}``G^X$ generates a $j_{U_{\alpha,n}}(\mathcal{P}^X)$ generic filter over $M_{U_{\alpha,n}} \cong \text{Ult}(V[G], U_{\alpha,n})$, it follows that there is a condition $p \in G^X$ so that $j_{U_{\alpha,n}}(p) \Vdash \check{\kappa} \in j_{U_{\alpha,n}}(\dot{Y})$. Let $Y' = \{\nu < \kappa \mid p \Vdash \check{\nu} \in \dot{Y}\}$. It follows that $\kappa \in j_{U_{\alpha,n}}(Y')$. We conclude that $Y' \in U_{\alpha,n}$ and thus $Y' \in W$. But $Y' \subset Y$ because $p \in G^X$, so $Y \in W$. It follows that $U^X_{\alpha,n} = W$. □

A simple inspection of the proof of Proposition 2.15 shows that the Mitchell order on the set of normal measures $U^X_{\alpha,n}$ in $V[G * G^X]$ inherits the $\triangleleft$ structure from $V[G]$.

Corollary 2.17. $U^X_{(\alpha',n')} \triangleleft U^X_{(\alpha,n)}$ if and only if $\alpha' < \alpha$ and $n' \geq n$.

2.4 Applications

We conclude this section with several applications, showing how to realize some new non-tame orders as $\triangleleft(\kappa)$ in generic extensions of the form $V[G][G^X]$.

In part I ([4]), we introduced a class of well founded order called tame orders, and proved that every tame order $(S, <_S)$ of size at most κ, can be consistently realized as $\triangleleft(\kappa)$. An order $(S, <_S)$ is tame if and only if it does not contain two specific orders:

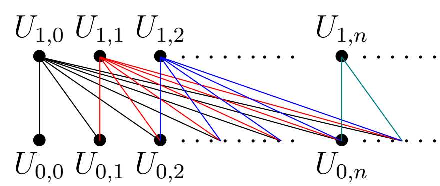
1. $(R_{2,2}, <_{R_{2,2}})$ is an order on a set of four elements $R_{2,2} = \{x_0, x_1, y_0, y_1\}$, defined by $<_{R_{2,2}} = \{(x_0, y_0), (x_1, y_1)\}$.

2. $(S_{\omega,2}, <_{S_{\omega,2}})$ is an order on a disjoint union of two countable sets $S_{\omega,2} = \{x_n\}_{n<\omega} \uplus \{y_n\}_{n<\omega}$, defined by $<_{S_{\omega,2}} = \{(x_m, y_n) \mid m \geq n\}$.

Therefore $R_{2,2}$ and $S_{\omega,2}$ are the principal examples of orders which cannot be realized by the methods of Part I. Let us show that $R_{2,2}$ and $S_{\omega,2}$ can be realized in our new settings.

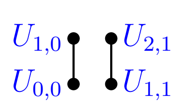
First application - Realizing $S_{\omega,2}$

Suppose that in $l(\vec{F}) = 2$ in $V = L[E]$, i.e., $\vec{F} = \langle F_0, F_1 \rangle$. Propositions 2.14 and 2.15 imply that the normal measures in κ in $V[G]$ are given by $\{U_{\delta,n} \mid \delta < 2, n < \omega\}$, where $U_{\delta',m} \triangleleft U_{\delta,n}$ if and only if $\delta' = 0$, $\delta = 1$ and $m \geq n$. Therefore $\triangleleft(\kappa)^{V[G]} \cong <_{S_{\omega,2}}$.



Second application - Realizing $R_{2,2}$

Suppose that $l(\vec{F}) = 3$ in $V = L[E]$, i.e. $\vec{F} = \langle F_0, F_1, F_2 \rangle$. The measures on κ in $V[G]$ are given by in $\{U_{\delta,n} \mid \delta < 3, n < \omega\}$. Let $S = \{U_{0,0}, U_{1,0}, U_{1,1}, U_{2,1}\}$. The restriction of $\triangleleft(\kappa)$ to S includes the relations $U_{0,0} \triangleleft U_{1,0}$ and $U_{1,1} \triangleleft U_{2,1}$, therefore $\triangleleft(\kappa)^{V[G]} \restriction S \simeq R_{2,2}$. Since there are only $\aleph_0$ many normal measures on κ in $V[G]$, then the normal measures in $\{U_{\delta,n} \mid \delta < 3, n < \omega\}$ are separated by sets, and there is some $X \subset \kappa$ so that $X \notin U_{\delta,n}$ if and only if $U_{\delta,n} \in S$. By forcing with $\mathcal{P}^X$ over $V[G]$ we get a model $V[G][G^X]$ in which $\triangleleft(\kappa)^{V[G][G^X]} \cong R_{2,2}$.



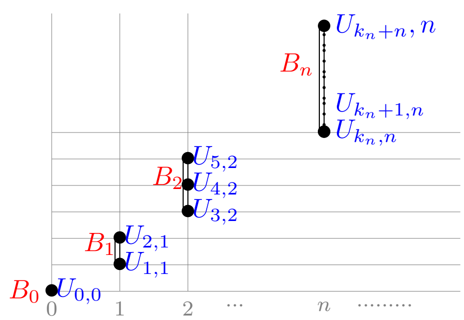
Third application - $o(\kappa) = \omega$ but no $\omega-$increasing sequence in $\triangleleft$ Suppoes that $l(\vec{F}) = \omega$, i.e., $\vec{F} = \langle F_i \mid i < \omega \rangle$. The normal measures on κ in a $\mathcal{P}-$generic extension $V[G]$, are of the form $U_{i,n}$ where $i < \omega$ and $n < \omega$. We define blocks of normal measures in $V[G]$:

$$B_n = \{U_{i,n} \mid k_n \leq i \leq k_n + n\}, \quad k_n = \frac{n(n+1)}{2}$$

We get that for every $n < \omega$:

1. $|B_n| = n + 1$, and
2. The last $i-$index in a measure $U_{i,n} \in B_n$ is equal to the minimal $i-$index of a measure in B_{n+1}.

Proposition 2.15 implies that for every $k < \omega$, B_k is linearly ordered by $\triangleleft(\kappa)$, so $\triangleleft(\kappa)^{V[G]} \restriction B_k$ is $(k+1)-$increasing sequence in $\triangleleft$. Moreover two measures from different blocks are $\triangleleft$ incomparable. The normal measures in $V[G]$ are separated by sets, so there is a set $X \subset \kappa$ which separates the measures in $S = \bigcup_{k<\omega} B_k$ from the rest of the normal measures on κ. Let $G^X \subset \mathcal{P}^X$ be generic over $V[G]$. It follows that $\triangleleft(\kappa)^{V[G][G^X]}$ is isomorphic to a disjoint union of linear orders on $(k+1)-$elements, for every $k < \omega$. In particular $o^{V[G][G^X]}(\kappa) = \omega$, but there is no $\omega-$increasing sequence in $\triangleleft(\kappa)$.



3 The Main Theorem

This section is devoted to proving the main Theorem (1.1). In subsection 3.1, we first introduce a family of orders $(R^*_{\rho,\lambda}, <_{R^*_{\rho,\lambda}})$ $\rho, \lambda \in \text{On}$, and show that every well-founded order $(S, <_S)$ embeds into $R^*_{\rho,\lambda}$ for $\rho = \text{rank}(S, <_S)$ and

$\lambda = |S|$. We then proceed to describe the revised ground model assumptions of $V = L[E]$ and introduce extenders $F_{\alpha,c}$ which are used to realize $R^*_{\rho,\lambda}$ using $\triangleleft$. The forcing we apply to V has two main goals: To generate extensions of $F_{\alpha,c}$ (in a generic extension) which are $\kappa-$complete, and to collapse their generators to κ^+, thus introducing equivalent normal measures on κ.
The generic extension V^2 of V is obtained by a poset which includes three components: $\mathcal{P}^0$. $\mathcal{P}^1$, and $\mathcal{P}^2$.

1. $\mathcal{P}^0$ is the Friedman-Magidor forcing, splitting the measures and extenders on κ in V into λ $\triangleleft-$equivalent extensions. Much like the use of the Friedman-Magidor forcing in Part I ([4]), the purpose of $\mathcal{P}^0$ is to allow simultaneously dealing with many different extenders in a single forcing. In subsection 3.2 we describe certain extensions of the extenders $F_{\alpha,c} \in V$ in a $\mathcal{P}^0$ generic extension V^0, and introduce the key iterations and embeddings used in the subsequent extensions.

2. $\mathcal{P}^1$ is a Magidor iteration of one-point Priky forcings. The purpose of this poset is to introduce extensions of the extenders $F_{\alpha,c}$ which are $\kappa-$complete (i.e., extension of the $V-$iterated ultrapower by $F_{\alpha,c}$ which are closed under $\kappa-$sequences). In subsection 3.3 we describe a $\mathcal{P}^1$ generic extension V^1. We then introduce and further investigate a collection of κ-complete extenders in V^1.

3. $\mathcal{P}^2$ is a collapse and coding iteration, similar to the collapsing and coding poset $\mathcal{P}$ introduced in the previous section. In subsection 3.4 we describe the normal measures on κ in $V^2 = (V^1)^{\mathcal{P}^2}$ and $\triangleleft(\kappa)$ in V^2, which embeds $R^*_{\rho,\lambda}$.

Finally, in subsection 3.5 we apply a final cut extension to form a model in which $\triangleleft(\kappa) \cong (S, <_S)$.

3.1 The Orders $(R^*_{\rho,\lambda}, <_{R^*_{\rho,\lambda}})$

For every ordinal λ, and $c, c' : \lambda \to \mathrm{On}$, we write $c \geq c'$ when $c(i) \geq c'(i)$ for every $i < \lambda$.

Definition 3.1 $(R^*_{\rho,\lambda}, <_{R^*_{\rho,\lambda}})$**.**
For ordinals ρ, λ let $R^*_{\alpha,\lambda} = \rho \times {}^{\lambda}2$. Let $<_{R^*_{\rho,\lambda}}$ be a order relation on $R^*_{\rho,\lambda}$ defined by $(\rho_0, c_0) <_{R^*_{\alpha,\lambda}} (\rho_1, c_1)$ if and only if $\rho_0 < \rho_1$ and $c_0 \geq c_1$.

Lemma 3.2. Let $(S, <_S)$ be a well-founded order so that $|S| \leq \lambda$, and $\operatorname{rank}(S, <_S) = \rho$, then $(S, <_S)$ can be embedded in $(R^*_{\rho,\lambda}, <_{R^*_{\rho,\lambda}})$

Proof. Suppose that $(S, <_S)$ is a well founded order. For every $x \in S$ let $\bar{u}(x) = \{y \in S \mid x <_S y \text{ or } y = x\}$, and define a function $c'_x : S \to 2$ to be the characteristic function of $\bar{u}(x)$ in S, i.e., for every $y \in S$,

$$c'_x(y) = \begin{cases} 1 & \text{if } y \in \bar{u}(x) \\ 0 & \text{otherwise.} \end{cases}$$

Let x, y be distinct elements in S. Note that if $x <_S y$ then $\bar{u}(y) \subset \bar{u}(x)$ hence $c'_x \geq c'_y$, and if $x \not<_S y$ the $y \in \bar{u}(y) \setminus \bar{u}(x)$ so $c'_x \not\geq c'_y$.
Let $\sigma : \lambda \to S$ be a bijection and define for every $x \in S$, $c_x = c'_x \circ \sigma : \lambda \to 2$.
It follows that for every $x \neq y$ in S, $c_x \geq c_y$ if and only if $x <_S y$.
We conclude that the function $\pi : S \to R^*_{\rho,\lambda}$ defined by $\pi(x) = \langle \operatorname{rank}_S(x), c_x \rangle$, is an embedding of $(S, <_S)$ into $(R^*_{\rho,\lambda}, <_{R^*_{\rho,\lambda}})$. □

We conclude that the order $(S, <_S)$ is isomorphic to a restriction of $(R^*_{\rho,\lambda}, <_{R^*_{\rho,\lambda}})$ to a subset of the domain.

Let us assume from this point on that $S \subset R^*_{\rho,\lambda}$ and $<_S = <_{R^*_{\rho,\lambda}} \restriction S$.

Definition 3.3 ($S' = \langle x_i \mid i < \lambda \rangle$)**.**
Define $S' \subset \mathcal{P}(\lambda)$,

$$S' = \{x \subset \lambda \mid \text{ there is } (\alpha, c) \in S \text{ such that } x = c^{-1}(\{1\})\}.$$

We fix a surjective enumeration $\langle x_i \mid i < \lambda \rangle$ of S'.

Suppose that $V = L[E]$ is a core model which satisfy the following requirements:

1. There are λ measurable cardinals above κ. Let $\vec{\theta} = \langle \theta_i \mid i < \lambda \rangle$ be an increasing enumeration of the first measurable cardinals above κ and let $\theta = \bigcup_{i<\lambda} \theta_i^+$.

2. There is a $\triangleleft$−increasing sequence $\vec{F} = \langle F_\alpha \mid \alpha < l(\vec{F}) \rangle$, $l(\vec{F}) < \theta_0$, of (κ, θ^+)−extenders.

3. Each F_α is $(\theta + 1)$−strong.

4. $\vec{F}$ consists of all full $(\kappa, \theta^+)-$extenders in E.

5. There are no stronger extenders on κ in E $(o(\kappa) = \theta^+ + l(\vec{F}))$.

6. There are no extenders $F \in E$ so that $\text{cp}(F) < \kappa$ and $\nu(F) \geq \kappa$.

For every $i < \lambda$, let U_{θ_i} be the unique normal measure on θ_i in V.

Definition 3.4 (i_c, N_c, $F_{\alpha,c}$, $M_{\alpha,c}$, $j_{\alpha,c}$)**.**

1. For any $c : \lambda \to 2$ let $i_c : V \to N_c$ be the elementary embedding formed by a linear iteration of the measures U_{θ_i} for which $c(i) = 1$. We refer to this iteration as the $c-$derived iteration.
   Therefore every set $x \in N_c$ is of the form $x = i_c(f)(\theta_{i_0}, \ldots, \theta_{i_{n-1}})$, where $\{i_0, \ldots, i_n\}$ is a finite subset of $c^{-1}(\{1\})$ and $f : \prod_{k<n} \theta_{i_k} \to V$ is a function in V.

2. For every $i < \lambda$, let $\theta_i^{c(i)} = i_c(\theta_i)$. Therefore $\theta_i^{c(i)}$ is the $i-$th measurable cardinal above κ in N_c.

3. For every $\alpha < l(\vec{F})$ let $F_{\alpha,c} = i_c(F_\alpha)$. $F_{\alpha,c}$ is a $(\kappa, \theta^+)-$extender on κ in V.

4. Note that $\langle U_{\theta_i} \mid i < \lambda \rangle$ belongs to M_α. Let $i_{\alpha,c} : M_{F_\alpha} \to M_{\alpha,c}$ be the embedding which results from the $c-$derived iteration of M_α.

5. Let $j_{F_{\alpha,c}} : V \to M_{F_{\alpha,c}} \cong \text{Ult}(V, F_{\alpha,c})$ denote the iterated ultrapower of V by $F_{\alpha,c}$.

Lemma 3.5. For each $\alpha < l(\vec{F})$ and $c : \lambda \to 2$, $F_{\alpha,c}$ is the $(\kappa, \theta^+)-$extender derived from the embedding $i_{\alpha,c} \circ j_\alpha : V \to M_{\alpha,c}$. Therefore $M_{F_{\alpha,c}} = M_{\alpha,c}$ and $j_{F_{\alpha,c}} = i_{\alpha,c} \circ j_\alpha$.

Proof. We need to verify that for every $\gamma < \theta^+$ and $Y \subset \kappa$, $Y \in F_{\alpha,c}(\gamma)$ if and only if $\gamma \in i_{\alpha,c} \circ j_\alpha(Y)$.
We represent γ as an element of the $c-$derived iteration if M_α. Let $i_0, \ldots, i_{n-1} < \lambda$ be a finite sequence of indices, and let $f : \prod_{k<n} \theta_{i_k} \to \theta^+$, so that $\gamma = i_{\alpha,c}(\theta_{i_0}, \ldots, \theta_{i_{n-1}})$.
Since $V_{\theta+1} \subset M_\alpha$ we get that $i_c \restriction V_{\theta+1} = i_c^{M_\alpha} \restriction V_{\theta+1}$. In particular $\gamma = i_c(\theta_{i_0}, \ldots, \theta_{i_{n-1}})$. As $F_{\alpha,c} = i_c(F_\alpha)$ we have that $Y \in F_{\alpha,c}(\gamma)$ if and only if

$$\{\vec{\nu} \in \prod_{k<n} \theta_{i_k} \mid Y \in F_\alpha(f(\vec{\nu}))\} \in \prod_{i<n} U_{\theta_{i_k}}. \qquad (1)$$

Note that for every $\vec{\nu} \in \prod_{k<n} \theta_{i_k}$, $Y \in F_\alpha(f(\vec{\nu}))$ if and only if $f(\vec{\nu}) \in j_\alpha(Y)$. Therefore 1 is equivalent to

$$\{\vec{\nu} \in \prod_{k<n} \theta_{i_k} \mid f(\vec{\nu}) \in j_\alpha(Y)\} \in \prod_{k<n} U_{\theta_{i_k}}. \tag{2}$$

The claim follows since 2 can be seen as a statement of M_α which is equivalent to $i_c^{M_F}(f(\theta_{i_0}, \ldots, \theta_{i_{n-1}}) \in i_c \circ j_{F_\alpha}(Y)$. □

3.2 The Poset $\mathcal{P}^0$

Definition 3.6 (Ω' and $\mathcal{P}^0$)**.**
Let Ω' denote the set of $\nu < \kappa$ which are inaccessible limits of measurable cardinals.

The poset $\mathcal{P}^0 = \mathcal{P}^0_{\kappa+1} = \langle \mathcal{P}^0_\nu, \mathcal{Q}^0_\nu \mid \nu \leq \kappa \rangle$ is a Friedman-Magidor poset ([7]) for splitting the $(\kappa, \theta^+)-$extenders in V into $\lambda \lhd-$equivalent extensions. $\mathcal{P}^0$ is a nonstationary support iteration so that $\dot{Q}^0_\nu$ is not trivial for $\nu \in \Omega' \cup \{\kappa\}$, and $\Vdash_{\mathcal{P}^0_\nu} \dot{Q}^0_\nu = \text{Sacks}_\lambda(\nu) * \text{Code}(\nu)$, where $\text{Sacks}_\lambda(\nu)$ is a Sacks forcing with $\rho_\lambda(\nu)-$splittings, and $\text{Code}(\nu)$ codes the generic Sacks function $s_\nu : \nu \to \lambda(\nu)$ and itself. Here

$$\lambda(\nu) = \begin{cases} \lambda & \text{if } \lambda < \kappa \\ \nu & \text{if } \lambda = \kappa \end{cases}$$

Let $G^0 \subset \mathcal{P}^0$ be a generic filter. We denote $V[G^0]$ by V^0. According to the analysis of Friedman and Magidor, for every $\alpha < l(\vec{F})$, the ultrapower embedding $j_\alpha : V \to M_\alpha \cong \text{Ult}(V, \alpha)$ has exactly $\lambda-$different extensions of the form $j^0_{\alpha,i} : V[G^0] \to M_\alpha[G^0_{\alpha,i}]$, satisfying the following properties:

1. $j^0_{\alpha,i} \restriction V = j_\alpha$,
2. $j^0_{\alpha,i}(G^0) = G^0_{\alpha,i}$,
3. $s^{G^0_{\alpha,i}}_{j_\alpha(\kappa)}(\kappa) = i$, where $s^{G^0_{\alpha,i}}_{j_\alpha(\kappa)} : j_\alpha(\kappa) \to j_\alpha(\lambda)$ is the $G^0_{\alpha,i}-$generically derived Sacks function.
4. $V^0_{\theta+1} \subset M_\alpha[G^0_{\alpha,i}]$.

Definition 3.7 ($M^0_{\alpha,x}$, $j^0_{\alpha,x}$, $F^0_{\alpha,x}$, Ω'_x)**.**
Let $x \in S'$ and suppose that $\mathbf{x} = \mathbf{x_i}$ in the enumeration of S' introduced in Definition 3.3.

1. Let us denote $M_\alpha[G^0_{\alpha,i}]$ by $M^0_{\alpha,x}$, and $j^0_{\alpha,i} : V[G^0] \to M_\alpha[G^0_{\alpha,i}]$ by $j^0_{\alpha,x} : V[G^0] \to M^0_{\alpha,x}$.

2. Let $F^0_{\alpha,x}$ denote the $(\kappa, \theta^+)-$extender in $V[G^0]$ derived from $j^0_{\alpha,x} : V^0 \to M^0_{\alpha,x}$.

3. Let $\Omega'_x = \{\nu \in \Omega' \mid s_\nu = s_\kappa \restriction \nu \text{ and } s_\kappa(\nu) = i\}$.

It easily follows that for every $x, y \in S'$, $\Omega'_x \in F^0_{\alpha,y}(\kappa)$ if and only if $x = y$. As $|\mathcal{P}^0| < \theta_0$ we have that for every $i < \lambda$, the normal measure U_{θ_i} in V generates a unique normal measure on θ_i in $V^0 = V[G^0]$. Let us denote this extension by $U^0_{\theta_i}$. Note that $U^0_{\theta_i} \in M^0_{\alpha,x}$ for every $\alpha < \vec{F}$ and $x \in S'$.

Definition 3.8 ($i^0_{\alpha,x,c}$, $N^0_{\alpha,x,c}$, $F^0_{\alpha,x,c}$)**.**

1. Suppose that $\alpha < l(\vec{F})$, $x \in S'$, and $c : \lambda \to \omega$. We define a linear iteration
$$\langle N^0_{\alpha,x,c,j}, i^0_{\alpha,x,c,j,j'} \mid j < j' \leq \lambda\rangle \tag{3}$$
associated with α, x, and c:

   - $N^0_{\alpha,x,c,0} = M^0_{\alpha,x}$.
   - for every $j < \lambda$,
   $$i^0_{\alpha,x,c,j,j+1} : N^0_{\alpha,x,c,j} \to N^0_{\alpha,x,c,j+1} \cong \mathrm{Ult}^{c(j)}(N^0_{\alpha,x,c,j}, U^0_{\theta_j})$$
   is the $c(j)-$th iterated ultrapower embedding of $N^0_{\alpha,x,c,j}$ by $U^0_{\theta_j} = i^0_{\alpha,x,c,0,j}(U^0_{\theta_j})$.
   - for every limit ordinal $j' \leq \lambda$, $N^0_{\alpha,x,c,j'}$ is the direct limit of the iteration up to j', and $i^0_{\alpha,x,c,j,j'}$, $j < j'$, are the limit embeddings.
   - We denote $N^0_{\alpha,x,c,\lambda}$ by $N^0_{\alpha,x,c}$, and $i^0_{\alpha,x,c,0,\lambda}$ by $i^0_{\alpha,x,c} : M^0_{\alpha,x} \to N^0_{\alpha,x,c}$.

   We refer to this iteration as the $c-$derived iteration of $M^0_{\alpha,x}$, and to $i^0_{\alpha,x,c} : M^0_{\alpha,x} \to N^0_{\alpha,x,c}$ as the $c-$derived embedding of $M^0_{\alpha,x}$.

2. Let $i^0_c : V^0 \to N^0_c$ be similarly defined $c-$derived iteration of V^0.

3. Let $F^0_{\alpha,c,x} = i^0_c(F_{\alpha,x})$. $F^0_{\alpha,c,x}$ is a $(\kappa,\theta^+)-$extender on κ in V^0 as $\mathrm{cp}(i^0_c) > \kappa$ and $\theta^+ = i^0_c(\theta^+)$. Moreover, the fact $V^0_{\theta+1} \subset M^0_{\alpha,x}$ implies that $i^0_c \restriction \theta^+ = i^0_{\alpha,x,c} \restriction \theta^+$.

The proof of Lemma 3.5 can be easily modified to show the following

Corollary 3.9. For every $\alpha < l(\vec{F})$, $x \in S'$, and $c : \lambda \to \omega$, $F^0_{\alpha,x,c}$ is the $(\kappa,\theta^+)-$extender derived from $i^0_{\alpha,x,c} \circ j^0_{\alpha,x} : V^0 \to M^0_{\alpha,x,c}$, and therefore $M^0_{\alpha,x,c} \cong \mathrm{Ult}(V^0, F^0_{\alpha,x,c})$.

For every $\alpha < l(\vec{F})$, $c : \lambda \to \omega$, and $x, y \in S'$, we saw that $\Omega'_x \in F^0_{\alpha,y}(\kappa)$ if and only if $x = y$. Since $\kappa < \mathrm{cp}(i^0_c)$ it follows that the same is true for Ω'_x and $F^0_{\alpha,y,c}(\kappa)$.

3.3 The poset $\mathcal{P}^1$

Definition 3.10 ($x-$suitable functions, Θ_j, Θ)**.**

1. Let $c_x : \lambda \to 2$ be the characteristic function of $x \subset \lambda$. We say that function $c : \lambda \to \omega$ is $x-$*suitable* if and only if $c(j) = c_x(j)$ for all but finitely many $j < \lambda$.

2. For every $j < \lambda$ and $\nu < \kappa$ let $\Theta_j(\nu)$ to be the $j-$th measurable cardinal above ν, if $j < \nu$, and 0 otherwise.
   Define $\Theta : \kappa \to \kappa$ by $\Theta(\nu) = \bigcup_{j<\lambda} \Theta_j(\nu)^+$.

Definition 3.11 ($\mathcal{P}^1$)**.**
The poset $\mathcal{P}^1 = \langle \mathcal{P}^1_\mu, \dot{\mathcal{Q}}^1_\mu \mid \mu < \kappa \rangle$ is a Magidor iteration of Prikry type forcing, where For each $\mu < \kappa$ the forcing $\mathcal{Q}^1_\mu$ is nontrivial if and only if there are $x \in S'$, $\nu \in \Omega'_x$, and $i \in x \cap \nu$ such that $\mu = \Theta_i(\nu)$. Note that ν is the unique such cardinal and $\mu = \Theta_i(\nu)$ is not a limit of measurable cardinals, thus $\Theta_i(\nu)$ carries a unique normal measure $U_{\Theta_i(\nu)}$ in V. Furthermore, as $\mathcal{P}^0$ factors into $\mathcal{P}^0_{\nu+1} * \mathcal{P}^0 \setminus (\nu+1)$ where $|\mathcal{P}^0_{\nu+1}| < \Theta_i(\nu)$ and $\mathcal{P}^0 \setminus (\nu+1)$ is $(2^{\Theta_i(\nu)})^+-$distributive, we get that $U_{\Theta_i(\nu)}$ has a unique extension $U^0_{\Theta_i(\nu)}$ in V^0. Similarly, the fact μ is not a limit of measurable cardinals implies that $|\mathcal{P}^1_\mu| < \mu$. By an argument of Levy and Solovay [11], $U_{\Theta_i(\nu)}$ has a unique extension $U^1_{\Theta_i(\nu)}$ in a $\mathcal{P}^1_\nu$ generic extension of V^0.
Define $\mathcal{Q}^1_\mu = Q(U^1_{\Theta_i(\mu)})$ where $Q(U^1_{\Theta_i(\mu)})$ is the one-point Prikry forcing by $U^1_{\Theta_n(\mu)}$, and introduces a single Prikry point $d(\mu) < \mu$.

Definition 3.12 ($\mathcal{F}$,$Z_{\alpha,x,c}(j)$, local Prikry functions)**.**
Let $\alpha < l(\vec{F})$, $x \in S'$, and $c : \lambda \to \omega$.

1. Let $\mathcal{F}$ denote the set of functions $f : \kappa \to \mathcal{P}(\kappa)$ in V^0, such that $f(\mu) \in U^0_\mu$ for every nontrivial iteration stage $\mu < \kappa$. Note that $|\mathcal{F}| = \kappa^+$ and $j^0_{\alpha,x}(f)(\theta_j) \in U^0_{\theta_j}$ for every $f \in \mathcal{F}$ and $j < \lambda$. Furthermore, as $V^0_{\theta+1} \subset M^0_{\alpha,x}$ it follows that $\{j^0_{\alpha,x}(f)(\theta_j) \mid f \in \mathcal{F}\} \in M^0_{\alpha,x}$ and that $|\{j^0_{\alpha,x}(f)(\theta_j) \mid f \in \mathcal{F}\}|^{M^0_{\alpha,x}} = |\mathcal{F}| = \kappa^+$ for every $j < \lambda$. Thus $\bigcap\{j^0_{\alpha,x}(f)(\theta_j) \mid f \in \mathcal{F}\} \in U^0_{\theta_j}$.

2. For every $j < \lambda$, let $Z_{\alpha,x}(j) = \bigcap\{j^0_{\alpha,x}(f)(\theta_j) \mid f \in \mathcal{F}\}$.

3. For every $c : \lambda \to \omega$ let $Z_{\alpha,x,c}(j) = i^0_{\alpha,x,c}(Z_{\alpha,x}(j))$.

4. Suppose that $c : \lambda \to \omega$ is $x-$suitable. We say that a function $\delta \in \prod_{j\in x} \theta_j^{c(j)}$ is a *local Prikry function* with respect to α, x, c when
   - $\delta(j) \in Z_{\alpha,x,c}(j)$ for every $j \in x$.
   - $\delta(j) = \theta_j$ for all but finitely many $j \in x$.

Note that for every $j < \lambda$, $Z_{\alpha,x,c}(j) \in U^0_{\theta_j^{c(j)}} = i^0_c(U^0_{\theta_j})$ is not empty. Furthermore, if $c(j) > 0$ then the $c-$derived iteration includes an ultrapower by $U^0_{\theta_j}$ whose critical point is θ_j, thus $\theta_j \in i^0_{\alpha,x,c}(Z_{\alpha,x}(j)) = Z_{\alpha,x,c}(j)$.

Let $G^1 \subset \mathcal{P}^1$ be a V^0-generic filter and denote $V^0[G^1] = V[G^0 * G^1]$ by V^1.

Definition 3.13 ($i^0_{\alpha,x,c\restriction\sigma}$, $N^0_{\alpha,x,c\restriction\sigma}$, $k^0_{\alpha,x,c\restriction\sigma}$, $(\alpha, c, \delta, G^1)-$compatible conditions)**.**
Let $\alpha < l(\vec{F})$ and $x \subset \lambda$, and suppose that $c : \lambda \to \omega$ is a $x-$suitable funciton.

1. For every $\sigma \in \mathcal{P}_\omega(\lambda)$ let $i^0_{\alpha,x,c\restriction\sigma} : M^0_{\alpha,x} \to N^0_{\alpha,x,c\restriction\sigma}$ be the embedding obtained from restricting the iteration of $i^0_{\alpha,x,c}$ to the ultrapowers $\mathrm{Ult}^{c(j)}(*, U^0_{\theta_j})$ for $j \in \sigma$.

2. It follows that $N^0_{\alpha,x,c}$ is the direct limit of $\{N^0_{\alpha,x,c\restriction\sigma} \in \mathcal{P}_\omega(\lambda)\}$ with obvious embeddings. For every $\sigma \in \mathcal{P}_\omega(\lambda)$ let $k^0_{\alpha,x,c\restriction\sigma} : N^0_{\alpha,x,c\restriction\sigma} \to N^0_{\alpha,x,c}$ be the resulting limit embedding.

3. Let $\mu < \kappa$ so that μ is a nontrivial stage in $\mathcal{P}^1$. Suppose τ is a $\mathcal{P}^1_\mu$ name for an ordinal less than μ. For every $p \in \mathcal{P}^1$ let $p^{+(\tau,\mu)}$ be the condition obtained by replacing p_μ with τ. Therefore $p^{+(\tau,\mu)} \Vdash \dot{d}(\check{\mu}) = \tau$. Note that $p^{+(\tau,\mu)} \geq p$ whenever $p \restriction \mu \Vdash \tau \in p_\mu$.

4. Suppose δ is a local Prikry function with respect to α, x, c. For every $p \in \mathcal{P}^1$ and $q \in j^0_{\alpha,x,c}(\mathcal{P}^1)$, we say that q is $(\alpha, c, \delta, p)-$compatible if there are $\sigma \in \mathcal{P}_\omega(x)(x = \mathrm{dom}(\delta))$ and $q' \in i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}(\mathcal{P}^1)$ so that

   (a) $q' \geq^* \left(i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}(p)\right)^{+\langle(\delta(j),\theta_j^{c(j)})|j\in\sigma\rangle}$.
   (b) $q' \restriction \kappa = p$.
   (c) $q = k^0_{\alpha,x,c\restriction\sigma}(q')$.

   We refer to $\sigma \subset \lambda$ as the *ultrapower support* of q, and denote it by $\sigma(q)$. It is clearly unique.

5. We say that $q \in j^0_{\alpha,x,c}(\mathcal{P}^1)$ is $(\alpha, c, \delta, G^1)-$compatible if it is $(\alpha, c, \delta, p)-$compatible for some $p \in G^1$.

Definition 3.14 ($F^1_{\alpha,c,\delta}$)**.**
Let $\alpha < l(\vec{F})$, $x \in S'$, and suppose that c is a $x-$suitable function and δ is a local Prikry function with respect to α, x, c. Define a $(\kappa, \theta^+)-$extender $F^1_{\alpha,c,\delta}$ in V^1 as follows: For every $\gamma < \theta^+$ and $Y = (\dot{Y})_{G^1} \subset \kappa$, $Y \in F^1_{\alpha,c,\delta}(\gamma)$ if and only if there is a $(\alpha, c, \delta, G^1)-$compatible condition $q \in j^0_{\alpha,x,c}(\mathcal{P}^1)$ so that $q \Vdash \check{\gamma} \in j^0_{\alpha,x,c}(\dot{Y})$.

Clearly, $F^1_{\alpha,c,\delta}$ extends $F^0_{\alpha,x,c}$.

The following notations will be useful in the analysis of $F^1_{\alpha,c,\delta}$.

Definition 3.15 ($\vec{\theta}^c_j$, $\vec{\theta}^c_\sigma$)**.**

1. For every $j < \lambda$ and $c : \lambda \to \lambda$, let $\vec{\theta}^c_j = \langle \theta^0_j, \dots, \theta_j^{c(j)-1} \rangle$ denote the sequence of critical points of the $c(j)-$the iterated ultrapower by $U^0_{\theta_j}$.

2. Suppose that $\sigma = \{j_0, \dots, j_{m-1}\}$ is a finite subset of λ, let $\vec{\theta}^c_\sigma = \vec{\theta}^c_{j_0} {}^\frown \dots {}^\frown \vec{\theta}^c_{j_{m-1}}$.

Note that $\vec{\theta}^c_\sigma$ enumerates the generators of $i^0_{\alpha,x,c\restriction\sigma}$.

Lemma 3.16. $F^1_{\alpha,c,\delta}(\gamma)$ is a $\kappa-$complete ultrafilter for every $\gamma < \theta^+$.

Proof. Let p be a condition in G^1 and suppose that $\dot{f}$ is a $\mathcal{P}^1$ name for a function from κ to some $\beta < \kappa$. Note that $\dot{f}$ belongs to $N^0_{\alpha,x,c\restriction\sigma}$ for every relevant α, x, c, σ.
Choose $\sigma = \{j_0, \ldots, j_{m-1}\} \in \mathcal{P}_\omega(\lambda)$ and $\gamma' \in N^0_{\alpha,x,c\restriction\sigma}$ so that $\gamma = k^0_{\alpha,x,c\restriction\sigma}(\gamma')$.
Let $t = \left(i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}(p)\right)^{+\langle(\delta(j),\theta_j^{c(j)})|j\in\sigma\rangle}$. Since $\mathcal{P}^1$ satisfies the Prikry condition and the direct extension order in $\left(i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}(\mathcal{P}^1)\right)\backslash\kappa$ is κ^+-closed,there are $p' \geq p$ in G^1 and

$$t' \geq^* \left(i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}(p')\right)^{+\langle(\delta(j),\theta_j^{c(j)})|j\in\sigma\rangle} \setminus \kappa$$

so that the condition $q' = p'^\frown t'$ decides the statement $\check{\gamma}' \in i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}(\dot{f}(\check{\nu}))$, for every $\nu < \beta$. Let ν^* be the unique $\nu < \beta$ so that $p'^\frown t' \Vdash \check{\gamma}' \in i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}(\dot{f}(\check{\nu}))$. We conclude that $q = k^0_{\alpha,x,c\restriction\sigma}(q')$ is $(\alpha, c, \delta, G^1)-$compatible and forces that $\check{\gamma} \in j^0_{\alpha,c,x}(\dot{X}_{\nu^*})$, thus $X_{\nu^*} \in F^1_{\alpha,c,\delta}(\gamma)$. □

A similar argument shows that $F^1_{\alpha,c,\delta}(\kappa)$ is a normal measure on κ in V^1.

Remark 3.17. Suppose that $c : \lambda \to \omega$ is $x-$suitable and δ is a local Prikry function with respect to α, x, c. It follows there is a finite set $\sigma^* \subset \lambda$ so that for every $j \in \lambda \setminus \sigma^*$, $(c(j) = 0)$ if $j \notin x$, and $(c(j) = 1) \wedge (\delta(j) = \theta_j = \theta^0_j)$ if $j \in x$. We can also assume that $c(j) > 1$ for every $j \in \sigma^*$, therefore the generators of the $c-$derived iteration are the ordinals in $\vec{\theta}^c_{\sigma^*}{}^\frown\vec{\theta}_{x\setminus\sigma^*}$, where $\vec{\theta}_{x\setminus\sigma^*} = \langle\theta_j \mid j \in x \setminus \sigma^*\rangle$ and $\langle\theta_j \mid j \in x \setminus \sigma^*\rangle = \langle\delta(j) \mid j \in x \setminus \sigma^*\rangle$.
Working in V^1, let $\vec{d}_x : \Omega'_x \to \kappa^{<\kappa}$ denote the local Prikry function, where for every $\nu \in \Omega'_x$, $\vec{d}_x(\nu) = \langle d(\Theta_j(\nu)) \mid j \in x\cap\nu\rangle$. Since q is $(\alpha, c, \delta, G^1)-$compatible we get that for every finite $\sigma \subset x \setminus \sigma^*$, if $\sigma \subset \sigma(q)$ then $q \Vdash j^0_{\alpha,x,c}(\dot{\vec{d}}_x)(\check{\kappa}) \restriction \sigma = \langle\theta_j \mid j \in \sigma\rangle = \vec{\theta}^c_\sigma$.

Lemma 3.18. Suppose that $c : \lambda \to \omega$ is $x-$suitable and δ is a local Prikry function with respect to α, x, c, then $F^1_{\alpha,c,\delta}$ is a $\kappa-$complete extender in V^1.

Proof. It is sufficient to prove that $\mathrm{Ult}(V^1, F^1_{\alpha,c,\delta})$ is closed under $\kappa-$sequence of ordinals below θ^+. Let $\langle\gamma_i \mid i < \kappa\rangle$ be a sequence in V^1 of ordinals below θ^+. We use the notations and observations given in the previous remark (3.17).

For every $i < \kappa$ there is a finite set $\sigma_i \subset x \setminus \sigma^*$ and a function $f_i \in M^0_{\alpha,x}$ so that $\gamma_i = i^0_{\alpha,c,x}(f_i)(\vec{\theta}^c_{\sigma^*} {}^\frown \vec{\theta}^c_{\sigma_i})$.
Since $\vec{\theta}^c_{\sigma^*}$ is a finite sequence of ordinals below θ^+ we have that $\vec{\theta}^c_{\sigma^*} = j^0_{\alpha,x,c}(F^{\sigma^*})(\gamma^*)$ for some $\gamma^* < \theta^+$ and a function $F^{\sigma^*} \in V^0$.
Let $\vec{f}$ denote $\langle f_i \mid i < \kappa\rangle$ and $\vec{\sigma} = \langle \sigma_i \mid i < \kappa\rangle$. The model $M^0_{\alpha,x} \cong \mathrm{Ult}(V^0, F^0_{\alpha,x})$ is closed under $\kappa-$sequence so $\vec{f}, \vec{\sigma} \in M^0_{\alpha,x}$. Note that $j^0_{\alpha,x,c}(\vec{\sigma}) \restriction \kappa = \vec{\sigma}$ and that $i^0_{\alpha,x,c}(\vec{f}) = \langle i^0_{\alpha,c,x}(f_i) \mid i < \kappa\rangle$. Since $i^0_{\alpha,x,c}(\vec{f}) \in M^0_{\alpha,x,c}$, we may assume that there is a function $F^{\vec{f}} \in V^0$ so that $j^0_{\alpha,x,c}(F^{\vec{f}})(\gamma^*) = i^0_{\alpha,x,c}(\vec{f})$.

Let $\Omega^*_x \subset \kappa$ denote the set of $\mu < \kappa$ so that $\max(\Omega'_x \cap \mu + 1)$ exists, and denote it by $\kappa(\mu)$. Note that $\Omega^*_x \in F^0_{\alpha,x,c}(\gamma)$ for every $\kappa \le \gamma < \theta^+$, and that $j^0_{\alpha,x,c}([\mu \mapsto \kappa(\mu)])(\gamma) = \kappa$.
Let $\dot{h}$ be a $\mathcal{P}^1$ name for the function $h \in V^1$, so that $h : \Omega^*_x \to \kappa^{<\kappa}$ and for $\mu \in \Omega^*_x$ we have that

- $h(\mu)$ is a function, $\mathrm{dom}(h(\mu)) = \kappa(\mu)$,
- for every $i < \kappa(\mu)$, $h(\mu)(i) = \vec{d_x}(\kappa(\mu)) \restriction (\sigma_i \cap \kappa(\mu)) = \langle d\,(\Theta_j(\kappa(\mu))) \mid j \in \sigma_i \cap \kappa(\mu)\rangle$.

Note that this definition makes sense as $\sigma_i \subset x$ and $\kappa(\mu) \in \Omega'_x$.
We get that for every $\gamma < \theta^+$, $j^0_{\alpha,x,c}(\dot{h})(\gamma)$ is a $j^0_{\alpha,x,c}(\mathcal{P}^1)$ name of a function with domain κ, and for every $i < \kappa$,

$$j^0_{\alpha,x,c}(\dot{h})(\gamma)(i) = \langle j^0_{\alpha,x,c}(\dot{d})(\theta^c_j) \mid j \in \sigma_i\rangle.$$

If q is a $(\alpha, c, \delta, G^1)-$compatible condition and $\sigma_i \subset \sigma(q)$, then

$$q \Vdash j^0_{\alpha,x,c}(\dot{h})(\gamma)(i) = \langle \theta_j \mid j \in \sigma_i\rangle = \vec{\theta}^c_{\sigma_i}.$$

Finally, let $\dot{g}$ be the $\mathcal{P}^1$ name of the function $g \in V^1$ so that for every $\mu \in \Omega^*_x$,

- $g(\mu)$ is a function, $\mathrm{dom}(g(\mu)) = \kappa(\mu)$,
- for every $i < \kappa(\mu)$, $g(\mu)(i) = \left(F^{\vec{f}}(\mu)_i\right)(F^{\sigma^*}(\mu) {}^\frown h(\mu))$.

Therefore if q is a $(\alpha, c, \delta, G^1)-$compatible and $\sigma_i \subset \sigma(q)$, then

$$q \Vdash j^0_{\alpha,x,c}(\dot{g})(\gamma^*)(i) = (f_i)\,(\vec{\theta}^c_{\sigma^*} {}^\frown \vec{\theta}^c_{\sigma_i}) = \gamma_i.$$

We conclude that in V^1, for every $i < \kappa$,

$$\{(\mu_0, \mu_1) \in \kappa^2 \mid \mu_0 = g(\mu_1)\} \in F^1_{\alpha,c,\delta}(\gamma_i) \times \mathcal{F}^1_{\alpha,c,\delta}(\gamma^*),$$

hence $\langle \gamma_i \mid i < \kappa \rangle$ is represented by $[g, \gamma^*]_{F^1_{\alpha,c,\delta}} \in \mathrm{Ult}(V^1, F^1_{\alpha,c,\delta})$. □

Definition 3.19 ($j^1_{\alpha,c,\delta}$, $M^1_{\alpha,c,\delta}$).
Let $\alpha < l(\vec{F})$ and $x \in S'$, and suppose that $c : \lambda \to \omega$ is $x-$suitable and δ is a local Prikry function with respect to α, x, c.
Let $j^1_{\alpha,c,\delta} : V^1 \to M^1_{\alpha,c,\delta} \cong \mathrm{Ult}(V^1, F^1_{\alpha,c,\delta})$ be the ultrapower embedding of V^1 by $F^1_{\alpha,c,\delta}$, and $G^1_{\alpha,c,\delta} = j^1_{\alpha,c,\delta}(G^1)$.

Lemma 3.18 implies that $M^1_{\alpha,c,\delta}$ is closed under $\kappa-$sequences in V^1. We also see that $G^1_{\alpha,c,\delta} \subset j^0_{\alpha,x,c}(\mathcal{P}^1)$ is generic over $M^0_{\alpha,x,c}$ and that $j^1_{\alpha,c,\delta} \restriction V^0 = j^0_{\alpha,x,c}$.

3.3.1 The restriction $j^1_{\alpha,c,\delta} \restriction V$

We proceed to describe the restrictions of the embeddings $j^1_{\alpha,c,\delta}$ to V^0 and V. We define an iterated ultrapower of V^0, generating an embedding $\pi^0_{\alpha,c,\delta} : V^0 \to Z^0_{\alpha,c,\delta}$ which coincides with the restriction of $j^1_{\alpha,c,\delta} \restriction V^0$. The technical arguments justifying $\pi^0_{\alpha,c,\delta} = j^1_{\alpha,c,\delta} \restriction^0$ will be similar to those given in [4] and are therefore omitted here.
Let $T^0_{\alpha,c,\delta} = \langle Z^0_i, \sigma^0_{i,j} \mid i < j < \zeta \rangle$ be a normal iteration defined by $Z^0_0 = V^0$ $\sigma^0_{0,1} : Z^0_0 \to Z^0_1$ is the ultrapower embedding of V^0 by $F^0_{\alpha,x,c}$ ($x = \mathrm{dom}(\delta)$).
Let $\vec{\mu} = \langle \mu_i \mid i < \zeta \rangle$ be an increasing enumeration of all the ordinals μ in $\sigma^0_{0,1}(\Omega^*)$, i.e., which are of the form $\mu = \sigma^0_{0,1}(\Theta_i)(\nu)$ where $\nu \in j^0_{\alpha,x,c}(\Omega'_y) \setminus \theta^+$ for some $y \in S'$ and $i \in \sigma^0_{0,1}(y) \cap \nu$. Therefore $\vec{\mu}$ enumerates all the nontrivial forcing stages in the iteration $\sigma^0_{0,1}(\mathcal{P}^1) \setminus (\kappa + 1)$.
For every $i < \zeta$ let $U^0_{\mu_i}$ be the unique normal measure on μ_i in $Z^0_1 = N^0_{\alpha,x,c}$.
For every $0 < i < \zeta$, let $\sigma^0_{i,i+1} : Z^0_i \to Z^0_{i+1} \cong \mathrm{Ult}(Z^0_i, \sigma^0_{0,i}(U^0_{\mu_i}))$. For every limit $i^* \leq \zeta$, $Z^0_{i^*}$ is the direct limit of the iteration $T_{F,c} \restriction i^* = \langle Z^0_i, \sigma^0_{i,j} \mid i < j < i^* \rangle$. Then $\pi^0_{\alpha,c,\delta} = \sigma^0_{0,\zeta}$ and $Z^0_{\alpha,c,\delta} = Z^0_\zeta$.

Corollary 3.20. The restriction of $j^1_{\alpha,c,\delta} : V^1 \to M^1_{\alpha,c,\delta}$ to V^0 is $\pi^0_{\alpha,c,\delta} : V^0 \to Z^0_{\alpha,c,\delta}$.

Let $G^1_{\alpha,c,\delta} = j^1_{\alpha,c,\delta}(G^1)$. Then $G^1_{\alpha,c,\delta} \subset \pi^0_{\alpha,c,\delta}(\mathcal{P}^1)$ is generic over $Z^0_{\alpha,c,\delta}$.

It is clear that the description of $j^1_{\alpha,c,\delta}$ to V results from the restriction of the V^0 iteration $T^0_{\alpha,c,\delta} = \langle Z^0_i, \sigma^0_{i,j} \mid i < j < \zeta\rangle$ to V, where the embedding $\sigma^0_{0,1} = j^0_{\alpha,x,c}$ is replaced with $\sigma_{0,1} = j_{\alpha,c}$, and for every $i < \zeta$, $\sigma^0_{i,i+1} : Z^0_i \to Z^0_{i+1} \cong \mathrm{Ult}(Z_i, \sigma^0_{0,i}(U^0_{\mu_i}))$ is replaced with $\sigma_{i,i+1} : Z_i \to Z_{i+1} \cong \mathrm{Ult}(Z_i, \sigma_{0,i}(U_{\mu_i}))$. We denote the limit model by $Z_{\alpha,c}$. Thus $Z_{\alpha,c} = \mathcal{K}(Z^0_{\alpha,c,\delta}) = \mathcal{K}(M^1_{\alpha,c,\delta})$.

3.3.2 Further results concerning $\mathcal{P}^1$

We conclude this subsection with a few results concerning the poset $\mathcal{P}^1$ and extenders $F^1_{\alpha,c,\delta}$, which will be used in the study of the normal measures on κ in the next generic extension V^2.

Lemma 3.21. For every $\gamma < \theta^+$ and $X = \dot{X}_{G^1} \in F^1_{\alpha,c,\delta}(\gamma)$ there is $t \in G^1$ and a (α, c, δ, p) compatible condition $q \in j^0_{\alpha,x,c}(\mathcal{P}^1)$ so that

- $t \Vdash \check{\gamma} \in j^0_{\alpha,x,c}(\dot{X})$,
- $t \setminus \theta^+ = j^0_{\alpha,x,c}(p) \setminus \theta^+$

Proof. Suppose that $q \in j^0_{\alpha,x,c}(\mathcal{P}^1)$ be a $(\alpha, c, \delta, p)-$compatible condition so that $q \Vdash \check{\gamma} \in j^0_{\alpha,x,c}(\dot{X})$. Let $\sigma = \sigma(q)$ be the support of q and $q' \in N^0_{\alpha,x,c\restriction\sigma}(\mathcal{P}^1)$ so that $q' \geq^* \left(i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}(p)\right)^{+\langle(\delta(j),\theta_j^{c(j)})|j\in\sigma\rangle}$ and $q = k^0_{\alpha,x,c\restriction\sigma}(q')$. Let us denote $i^0_{\alpha,x,c\restriction\sigma} \circ j^0_{\alpha,x}$ by $j^0_{\alpha,x,c\restriction\sigma}$.
Note that θ^+ is a fixed point of $k^0_{\alpha,x,c\restriction\sigma}$. Let $r = q' \setminus \theta^+$. r is a $j^0_{\alpha,x,c\restriction\sigma}(\mathcal{P}^1) \restriction \theta^+$ name for a condition in $j^0_{\alpha,x,c\restriction\sigma}(\mathcal{P}^1) \setminus \theta^+$, and $q' \restriction \theta^+ \Vdash r' \geq^* j^0_{\alpha,x,c\restriction\sigma}(p) \setminus \theta^+$. Take $R\kappa \to \kappa$ and $\gamma < \theta^+$ so that $r = j^0_{\alpha,x,c\restriction\sigma}(R)(\gamma)$. We may assume that for every $\nu \in \Omega'$ and $\mu \in [\nu, \Theta(\nu)^+)$, $R(\mu) \geq^* p \setminus \Theta(\nu)^+$. For every $\mu \in \Omega^* \setminus \mathrm{supp}(p)$ and $\nu = \max(\Omega' \cap \mu)$ let p^*_μ be a $\mathcal{P}^1_\mu$ name of $\bigcap\{R(\mu')_\mu \mid \mu' \in \Omega^* \cap \nu\}$. The fact $|\Omega^* \cap \nu| \leq \nu < \mu$ implies that p^*_μ is a $\mathcal{P}^1_\mu$ name for a set in $Q(U^1_\mu)$. We conclude that $p^* \geq^* p$, and that for every $\nu \in \Omega'$ and $\mu' \in [\nu, \Theta(\nu)^+)$, $p^* \restriction \Theta(\nu)^+ \Vdash p^* \setminus \Theta(\nu)^+ \geq^* R(\mu')$. Let $t' = q' \restriction \theta^{+\frown} \left(j^0_{\alpha,x,c\restriction\sigma}(p^*) \setminus \theta^+\right)$ and $t = k^0_{\alpha,x,c\restriction\sigma}(t')$. Then $t \in j^0_{\alpha,x,c}(\mathcal{P}^1)$ is $(\alpha, c, \delta, p^*)-$compatible extension of q. The Lemma follows from a standard density argument. $\square$

Lemma 3.22. Suppose that $\mathcal{P} = \mathcal{P}_0 * Q_\mu * \mathcal{P}_1$ is an iteration of Prikry forcings, so that

1. $|\mathcal{P}_0| < \mu$,

2. $Q_\mu = Q(U_\mu)$ as a one point Prikry forcing,

3. $\leq^*_{\mathcal{P}_1}$ is μ^+-closed.

Let $p \in \mathcal{P}$ such that $p_{Q_\mu} \in U_\mu$, and let $\dot\eta$ be a $\mathcal{P}-$name of an ordinal so that $p \Vdash \dot\eta < \check\mu$. There are $p^* \geq^* p$ and a function $\phi : \mu \to [\mu]^{|\mathcal{P}_0|}$ so that

1. $p^* \restriction (\mathcal{P}_0 * Q_\mu) = p \restriction (\mathcal{P}_0 * Q_\mu)$ and

2. $p^* \Vdash \dot\eta \in \check\phi(\dot d(\check\mu))$,

where $\dot{d(\mu)}$ is the $Q_\mu-$name for the generic Prikry point.

Proof. Since $\leq^*_{\mathcal{P}_1}$ is μ^+-closed and $p \Vdash \dot\eta < \mu$, there is an is a $p^* \geq^* p$ with $p^* \restriction (\mathcal{P}_0 * Q_\mu) = p \restriction (\mathcal{P}_0 * Q_\mu)$, so that p^* reduces $\dot\eta$ to a $\mathcal{P}_0 * Q_\mu$ name, namely, $p^* \Vdash \dot\eta = \tau$ where τ is a $\mathcal{P}_0 * Q_\mu$ name for an ordinal below μ.
For every ordinal ν let $A_\nu \subset \mathcal{P}_0$ be a maximal antichain of conditions which decide $\check\nu \in p_{Q_\mu}$. Note that if $p_0 \in A_\nu$ forces $\check\nu \in p_{Q_\mu}$, the forcing $\mathcal{P}_0 * Q_\mu$ above $(p_0 * p_{Q_\mu})^{+(\nu,\mu)}$ is equivalent to the forcing $\mathcal{P}_0$ above p_0. For each $p_0 \in A_\delta$ which forces $\check\nu \in p_{Q_\mu}$, let A_{ν,p_0} be a maximal antichain above p_0, which consists of conditions which decides the value of σ as an ordinal below μ. Let $\phi(\nu) \subset \mu$ be the collection of all possible values of σ forced by conditions A_{ν,p_0} for suitable $p_0 \in A_\nu$. It is clear that $|\phi(\nu)| \leq |\mathcal{P}_0|$ and that $p \restriction (\mathcal{P}_0 * Q_\mu) \Vdash$ if $\dot d(\check\mu) = \check\nu$ then $\tau \in \check{\phi(\nu)}$. ☐

By applying the previous Lemma consecutively $\omega-$times, we conclude the following result:

Corollary 3.23. Let $\vec\mu = \langle \mu_n \mid n < \omega \rangle$ be a sequence of non-trivial $\mathcal{P}^1$ forcing stages, and $\vec v = \langle \dot{\eta_n} \mid n < \omega \rangle$ be a $\mathcal{P}^1-$name of an ordinal sequence. Suppose $p \in \mathcal{P}^1$ forces that $\vec v \in \prod_{n<\omega} \check{\Theta_{\mu_n}}$, then there is $p^* \geq^* p$, and a sequence of functions $\vec\psi = \langle \psi_n \mid n < \omega \rangle$ in V^0 so that

1. $\psi_n : \mu_n \to [\mu_n]^{|\mathcal{P}^1_{\mu_n}|}$ for every $n < \omega$.

2. $p^* \Vdash \forall n < \omega .\, \dot{\eta_n} \in \check{\psi_n}\left(\dot d(\check{\mu_n})\right)$.

3.4 The Poset $\mathcal{P}^2$

Definition 3.24 ($\mathcal{P}^2$).
$\mathcal{P}^2 = \langle \mathcal{P}^2_\nu, \mathcal{Q}^2_\nu \mid \nu \le \kappa \rangle$ is a collapsing and coding iteration with a Friedman-Magidor (nonstationary) support. The nontrivial stages of the iteration are at $\nu \in \Omega' \cup \{\kappa\}$, and $\mathcal{Q}^2_\nu = \mathrm{Coll}(\nu^+, \Theta(\nu)^+) * \mathrm{Code}(\dot{g}_\nu)$ where

- $\mathrm{Coll}(\nu^+, \theta(\nu)^+)$ is a Levy collapsing, which introduces a surjection $g_\nu : \nu^+ \to \Theta(\nu)^+$.
- $\mathrm{Code}(\dot{g}_\nu)$ codes g_ν and itself using a closed unbounded set in ν^+, as in Definition 2.5.

Let $G^2 \subset \mathcal{P}^2$ be a generic filter of the forcing $\mathcal{P}^2$ over V^1. We denote $V^1[G^2] = V[G^1 * G^2]$ by V^2.
The arguments in Section 2 (addressing the extension of the embeddings $j_{F,n}$ in V to a $V^{\mathcal{P}}$ embeddings) guarantee that the following can be defined:

Definition 3.25 ($U^2_{\alpha,c,\delta}$,$j^2_{\alpha,c,\delta}$, $M^2_{\alpha,c,\delta}$).
Every $j^1_{\alpha,c,\delta} : V^1 \to M^1_{\alpha,c,\delta}$ has a unique extension to a $V^1[G^2]$ embedding, denoted by $j^2_{\alpha,c,\delta} : V^1[G^2] \to M^2_{\alpha,c,\delta}$, so that

1. $M^2_{\alpha,c,\delta} = M^1_{\alpha,c,\delta}[G^2_{\alpha,c,\delta}]$ where $G^2_{\alpha,c,\delta} = j^2_{\alpha,c,\delta}(G^2) \subset j^1_{\alpha,c,\delta}(\mathcal{P}^2)$ is generic over $M^1_{\alpha,c,\delta}$.
2. The set $\{p^\frown(j^1_{\alpha,c,\delta}(p) \setminus \kappa+1) \mid p \in G^2\}$ meets every dense open set in $j^1_{\alpha,x,\delta}(\mathcal{P}^2)$ and generates $G^2_{\alpha,c,\delta}$.
3. The set $U^2_{\alpha,c,\delta} = \{X \subset \kappa \mid \kappa \in j^2_{\alpha,c,\delta}(X)\}$ is a normal measure on κ in $V^1[G^2]$.
4. $j^2_{\alpha,c,\delta} : V^1[G^2] \to M^2_{\alpha,c,\delta}$ coincides with the ultrapower embedding and model of $V^1[G^2]$ by $U^2_{\alpha,c,\delta}$.

Theorem 3.26. Suppose that W is a normal measure on κ in V^2. Then $W = U^2_{\alpha,c,\delta}$ for some $\alpha < l(\vec{F})$, $x \in S'$, a function $c : \lambda \to \omega$ which is $x-$suitable, and a local Prikry function δ with respect to α, x, c.

Let $j_W : V^2 \to M_W \cong \mathrm{Ult}(V^2, W)$. $j_W \restriction V : V \to M$ results from a normal iteration $\pi^T_{0,b}$ generated by an iteration tree T of V and a branch b. Moreover $M_W = M[G^0 * G^1_W * G^2_W]$ where $G^0_W * G^1_W * G^2_W = j_W(G^0 * G^1 * G^2) \subset$

$j(\mathcal{P}^0 * \mathcal{P}^1 * \mathcal{P}^2)$ is generic over M.

The description of a collapsing and coding generic extension $V^{\mathcal{P}}$ in Section 2 applies to $\mathcal{P}^2$. We get that in V^2, there is a sequence of functions $\langle \psi_\delta \mid \delta < \theta^+ \rangle \subset {}^\kappa\kappa$ so that for every $\delta_0 \neq \delta_1$, $\{\nu < \Omega' \mid \psi_{\delta_0}(\nu) \neq \psi_{\delta_1}(\nu)\}$ is bounded in κ. Since $\Omega' \in W$ we get $j_W(\kappa) \geq \theta^+$.
The forcing $\mathcal{P}^0 * \mathcal{P}^1 * \mathcal{P}^2$ does not add cofinal $\omega-$sequences to inaccessible cardinals in V. Therefore $G^0_W * G^1_W * G^2_W$ do not add such sequences to inaccessible cardinals in M. Since M_W is closed under $\omega-$sequences in V^2, it follows that there are no ordinals μ of countable cofinality in V^2, which are inaccessible in M. Therefore, the iteration T cannot use the same measure/extender more than finitely many times. Since $\mathrm{cp}(j) = \kappa$ and $j(\kappa) \geq \theta^+$, we conclude that $j = k \circ j_{F'}$, where F' is an extender on κ in V with $\nu(F') = \theta^+$ and $\mathrm{cp}(k) > \theta^+$.
Since $\{U_{\theta_j} \mid j < \lambda\}$ are the only normal measures which overlaps extenders on κ in V, and each can be applied finitely many times, we conclude that there are $\alpha < l(\vec{F})$ and $c : \lambda \to \omega$ such that $F' = i_c(F_\alpha) = F_{\alpha,c}$.
Let $s^{G^0_W}_{j(\kappa)} : j(\kappa) \to j(\lambda)$ be the G^0_W-induced generic Sacks function. Let $i = s^{G^0_W}_{j(\kappa)}(\kappa) < \min(\kappa, \lambda)$, and let $x \in S'$ so that $x = x_i$. It follows that $\Omega'_x \in W$, and that the the $j_W(\mathcal{P}^1)$ non-trivial forcing stages between κ and θ^+, are at the points $\theta^{c(j)}_j$, $j \in x$. Let $\delta \in \prod_{j \in x} \theta^{c(j)}_j$ be the function defined by $\delta(j) = j_W(d)(\theta^{c(j)}_j)$.

The fact that $\Omega'_x \in W$ implies that $j^0 = j_W \restriction V^0$ is of the form $j^0 = k^0 \circ j^0_{\alpha,x,c}$, where $j^0_{\alpha,x,c}$ is the ultrapower embedding of V^0 by $F^0_{\alpha,x,c} = i^0_x(F_{\alpha,x})$ and $\mathrm{cp}(k^0) > \kappa$.
Recall that $j^0_{\alpha,c,x} = i^0_{\alpha,x,c} \circ j^0_{\alpha,x}$ where $j^0_{\alpha,x}$ is the ultrapower embedding of V^0 by $F^0_{\alpha,x}$ and $i^0_{\alpha,x,c}$ is the $c-$derived embedding of $M^0_{\alpha,x} \cong \mathrm{Ult}(V, F^0_{\alpha,x})$.

Claim 3.27. *c is a $x-$suitable function and δ is a local Prikry function with respect to α, x, c.*

We prove the claim for the more difficult case when x is infinite. The proof for the finite case x is simpler. We separate the claim into a series of subclaims:

subclaim 3.27.1. $c(j) \geq 1$ for all but finitely many $j \in x$.

Suppose otherwise, and fix a countable set $y \subset x$ so that $c(j) = 0$ for all $j \in y$, i.e., $\theta^{c(j)}_j = \theta_j$. We have that $i^0_{\alpha,x,c}(\theta_j) = \theta_j$ for every $j \in y$, so

$i^0_{\alpha,x,c}$ maps each θ_j cofinally to itself. In particular, there is $\gamma_j < \theta_j$ so that $i^0_{\alpha,x,c}(\gamma_j) > \delta(j)$. Let $\vec{\gamma} = \langle \gamma_j \mid j \in y \rangle$. $\vec{\gamma} \in V[G^1 * G^2]$. Since G^2 does not add new $\omega-$sequences and $\mathcal{P}^0 * \mathcal{P}^1$ satisfies $\kappa^{++}.c.c$, it follows that in V^0 there is a sequence $\vec{\gamma}^* = \langle \gamma^*_n \mid j \in y \rangle \in \prod_{j \in y} \theta_j^{c(j)}$ so that $\gamma_j \geq \delta(j)$ for every $j \in y$. In particular $i^0_{\alpha,x,c}(\vec{\gamma^*}) \in N^0_{\alpha,x,c} \cap V^0_{\theta+1} = M^0_{\alpha,x,c} \cap V^0_{\theta+1} = M[G^0_W] \cap V^0_{\theta+1}$. We therefore define in $M[G^0_W]$ a set $D = \{p \in j_W(\mathcal{P}^1) \mid i^0_{\alpha,x,c}(\gamma^*_j) \cap p_{\theta_j^{c(j)}} = \emptyset$ for some $j \in y\}$. D is dense in $j_W(\mathcal{P}^1)$ since y is infinite, but it is clear we cannot have $G^1_W \cap D \neq \emptyset$.

subclaim 3.27.2. $\delta(j) \leq \theta_j^{c(j)-1}$ for all but finitely many $j \in x$.

Suppose otherwise, then there is a countable set $y \subset x$ so that $c(j) \geq 1$ and $\delta(j) > \theta_j^{c(j)-1}$ for all $j \in y$. For every $j \in y$ let $\sigma_j \in \mathcal{P}_\omega(\lambda)$ and a function f_j so that $\delta(j) = i^0_{\alpha,x,c}(f_j)(\vec{\theta}^c_{\sigma_j})$. We may assume that $j \in \sigma_j$ is the maximal ordinal in σ_j. We separate the last generator $\theta_j^{c(j)-1}$ from $\vec{\theta}^c_{\sigma_j}$ and write $\vec{\theta}^c_{\sigma_j} = \vec{\theta}^* \frown \langle \theta_j^{c(n)-1} \rangle$. If follows that $\delta(j) = i_c(f_j)(\vec{\theta}^* \frown \theta_j^{c(j)-1})$ where $f_j \in V^0$ is a function, $f_j : \prod_{j' \in y} [\theta_{j'}]^{c(j')} \to \theta_j$. Let $k = \sum_{j' \in \sigma_j \setminus \{j\}} c(j') + (c(j) - 1)$, $k < \omega$, and define $g_j : \theta_j \to \theta_j$ in V^0 by

$$g_j(\mu) = \sup(\{f_j(\vec{\nu}^* \frown \langle \mu \rangle \mid \vec{\nu}^* \in \mu^k\})$$

It is clear that $\delta(j) \leq i^0_{\alpha,x,c}(g_j)(\theta_j^{c(j)-1}) < \theta_j^{c(j)}$. Let $E_j \subset \theta_j$ be the set of closure points of g_j, then E_j is closed unbounded in θ_j and $\delta(j) \notin i_c(E_j)$ as we assumed $\theta_j^{c(j)-1} < \delta(j)$.

We point out that although $E_j \in V^0$ for each $j \in y$, the sequence $\vec{E} = \langle E_j \mid j \in y \rangle$ may not be in V^0. Instead, note that $\vec{E} \in V^1$ as $\mathcal{P}^2$ does not introduce new ω sequence, and that $\mathcal{P}^0 * \mathcal{P}^1$ satisfies $\kappa^{++}.c.c.$ It follows that there is a sequence $\vec{E}^* = \langle E^*_j \mid j \in y \rangle \in V^0$ so that $E^*_j \subset E_j$ for each $j \in y$. We get that $i^0_{\alpha,x,c}(\vec{E}^*) = \langle i^0_{\alpha,x,c}(E^*_j) \mid j \in y \rangle$ belongs to $M[G^0_W]$, and we can therefore define in $M[G^0_W]$ the set $D = \{p \in j_W(\mathcal{P}^1) \mid p_{\theta_j} \subset i^0_{\alpha,x,c}(E^*_j)$ for some $j \in y\}$. D is dense in $j_W(\mathcal{P}^1)$ since y is infinite, but we cannot have that $D \cap G^1_W \neq \emptyset$.

subclaim 3.27.3. $\delta_j \geq \theta_j^{c(j)-1}$ for all but finitely many $j \in x$.

Suppose otherwise and let $y \subset x$ be a countable set so that $c(j) \geq 1$ and $\delta(j) < \theta_j^{c(j)-1}$ for every $j \in y$. Since $M_W \cong \mathrm{Ult}(V^2, W)$ is closed under κ sequences, it follows that the $\langle \theta_j^{c(j)-1} \mid j \in y \rangle$ belongs to $M_W = M[G^0 *$

$G^1_W * G^0_W]$. Furthermore, $\langle \theta_j^{c(j)-1} \mid j \in y\rangle \in M[G^0 * G^1]$ because $\mathcal{P}^2$ does not add new $\omega-$sequence. Let $\vec{v}$ be a $j_W(\mathcal{P}^0 * \mathcal{P}^1)$ name for this sequence. By Corollary 3.23 there is a sequence of functions $\vec{\phi} = \langle \phi_j \mid j \in y$ in $M[G^0_W]$so that for every $j \in y$, $\mathrm{dom}(\phi_j) = \theta_j^{c(j)}$, $|\phi_j(\delta(j)))| < \delta(j)$, and $\theta_j \in \phi_j(\delta(j))$.
As an element of $M[G^0_W] \cap V^0_{\theta+1} = N^0_{\alpha,x,c} \cap V^0_{\theta+1}$, we see that $\vec{\phi} = i^0_{\alpha,x,c}(F)(\vec{\theta^c_\sigma})$ for some finite $\sigma \subset \lambda$. Fix an ordinal $j \in y \setminus \sigma$. Let $c' : \lambda \to \omega$ defined by $c'(j) = c(j) - 1$ and $c'(j') = c(j')$ for every $j' \neq j$. We can factor $i^0_{\alpha,x,c} : V^0 \to N^0_{\alpha,x,c}$ into $i^0_{\alpha,x,c} = k' \circ i^0_{\alpha,x,c'}$ where $k' : N^0_{\alpha,x,c'} \to N^0_{\alpha,x,c} \cong \mathrm{Ult}(N_{c'}, i^0_{\alpha,x,c'}(U_{\theta_j}))$ is the ultrapower embedding of $N^0_{\alpha,x,c'}$ by $i^0_{\alpha,x,c'}(U_{\theta_j})$. So $\mathrm{cp}(k') = \theta_j^{c(j)-1}$ and $k'(\theta_j^{c(j)-1}) = \theta_j^{c(j)}$.
Let $\vec{\phi'} = i^0_{\alpha,x,c'}(F)(\vec{\theta^c_\sigma})$. It is clear that $\vec{\phi} = k'(\vec{\phi'})$. $\vec{\phi'} = \langle \phi'_j \mid j < y\rangle$ where $\phi_j = k'(\phi'_j)$ for every $j \in y$. Moreover, since we assumed that $\delta(j) < \theta_j^{c(j)-1} = \mathrm{cp}(k')$, we get that $\phi_j(\delta(j)) = k'(\phi'_j(\delta(j)))$.
We conclude that $\phi'_j(\delta(j)) \subset \theta_j^{c(j)-1}$ and $|\phi'_j(\delta(j))| < \delta(j)$, but this implies $\theta_j^{c(j)-1} \notin k'(\phi'_j(\delta(j))) = \phi_j(\delta(j))$, contradicting the above.

subclaim 3.27.4. $c(j) = 1$ for all but finitely many $j \in x$.

Otherwise, there would be a countable set $y \subset x$ so that $c(j) > 1$ and $\delta(j) = \theta_j^{c(j)-1}$ for all $j \in y$. Let $\vec{\phi} = \langle \phi_j \mid j \in y\rangle$ be a sequence in M, so that $\theta^0_j \in \phi_j(\theta_j^{c(j)-1})$, where $\phi_j(\theta_j^{c(j)-1}) \in [\theta_j^{c(j)}]^{\leq \theta_{j-1}^{cj-1}}$. Let $\sigma \subset \lambda$ be a finite set so that $\vec{\phi} = N^0_{\alpha,x,c\restriction\sigma}$, and fix an ordinal $j \in y \setminus \sigma$. We know there exists a function $f_j : \theta_j \to [\theta_j]^{\leq \theta_{j-1}}$ so that $\phi_j = i_c(f_j)(\vec{\theta^c_\sigma})$, and we can write

$$\theta^0_j \in \left(i^0_{\alpha,x,c\restriction\sigma}(f_j)(\vec{\theta^c_\sigma})\right)(\theta_j^{c(j)-1}) \tag{4}$$

Let $i' : V^0 \to M' \cong \mathrm{Ult}(V^0, U^0_{\theta_j})$ be the ultrapower embedding of V by $U^0_{\theta_j}$. Note the $f_j \in M'$, so we can see 4 as a statement of M', where θ^1_j is the $j-$th measurable cardinal above κ, $U^0_{\theta^1_j} = i'(U^0_{\theta_j})$ is the unique normal measure on θ^1_j. Also from the perspective of M', $\theta_j^{c(j)-1} \geq \theta^1_j$ is the image of θ^1_j under the $c(j) - 2 \geq 0$ iterated ultrapower embedding by $U^0_{\theta^1_j}$. Back in V we get that

$$\{\mu < \theta_j \mid \mu \in \left(i^0_{\alpha,x,c\restriction\sigma}(f_j)(\vec{\theta^c_\sigma})\right)(\theta_j^{c(j)-2})\} \in U^0_{\theta_j}.$$

However, this is impossible as $\left(i^0_{\alpha,x,c\restriction\sigma}(f_j)(\vec{\theta^c_\sigma})\right)(\theta_j^{c(j)-2})$ has cardinality at most $\theta_{j-1}^{c(j-1)} < \theta_j$.

subclaim 3.27.5. $c(j) = 0$ for all but finitely many $j \in \lambda \setminus x$.

Suppose otherwise. Let $y \subset \lambda \setminus x$ be a countable set, so that $c(j) > 0$ for all $j \in y$. Let $\vec{\theta} \restriction y = \langle \theta_j \mid j \in y \rangle$. $\vec{\theta} \restriction y$ belongs to $M[G^0_W * G^1_W]$ because $M_W = M[G^0_W * G^1_W * G^2_W]$ is closed under ω sequences in V^2, and G^2_W does not add new ω sequences of ordinals.

Let us assume that $x \setminus j \neq \emptyset$ for every $j \in y$. The case $x \setminus j = \emptyset$ is treated similarly. For every $j \in y$ let $j^* = \min(x \setminus j)$. We may assume that $c(j^*) = 1$ and that $\delta(j^*) = \theta_{j^*}$.

By Corollary 3.23 there is a sequence of functions $\vec{\psi} = \langle \psi_j \mid j \in y \rangle$ in $M[G^0_W]$, with $\psi_j : \theta^{c(j^*)}_{j^*} \to [\theta^{c(j^*)}_{j^*}]^{|\mathcal{P}^1 \restriction \theta^0_{j^*}|}$ so that $\theta_j \in \psi_j(\delta(j^*)) = \psi_j(\theta_{j^*})$ for every $j \in y$. Since the non-trivial $j_W(\mathcal{P}^1)$ iteration stages between κ and θ^+ are $\theta^{c(j^*)}_{j^*}$, $j^* \in x$, it follows that $|\mathcal{P}^1 \restriction \theta^0_{j^*}| < \theta_j$ for every $j \in y$.

Since $\vec{\psi} \in M[G^0_W] \cap V^0_{\theta+1} = N^0_{\alpha,c,x} \cap V^0_{\theta+1}$, there is a finite set $\sigma \subset \lambda$ and $\vec{\psi'} \in N^0\alpha, x, c \restriction \sigma$ so that $\vec{\psi} = k^0_{\alpha,x,c\restriction\sigma}(\vec{\psi'})$. Let $\vec{\psi'} = \langle \psi'_j \mid j \in y \rangle$. Pick some $j \in y \setminus \sigma$. We may assume that $j^* \in \sigma$, thus $k^0_{\alpha,x,\sigma}(\theta_{j^*}) = \theta_{j^*}$. It follows that $\theta_j \in k^0_{\alpha,x,c\restriction\sigma}\left(\psi'_j(\theta_{j^*})\right)$. This is impossible as θ_j is a critical point of the embedding $k^0_{\alpha,x,c\restriction\sigma}$ and $|\psi'_j(\theta_{j^*})| < \theta_j$.

□[Claim 3.27]

Since c is $x-$suitable and δ is a local Prikry function with respect to α, x, c, it follows that the normal measure $U^2_{\alpha,c,\delta}$ exists in V^2.

Claim 3.28. If $p \in G^1$ and $q \in j^0_{\alpha,x,c}(\mathcal{P}^1)$ is a $(\alpha, c, \delta, p)-$compatible condition so that $q \setminus \theta^+ = j^0_{\alpha,x,c}(p) \setminus \theta^+$, then $k^0(q) \in G^1_W$.

For every $p \in G^1$ we have that $j_W(p) \in G^1_W$. Note that if $q \in j^0_{\alpha,x,c}(\mathcal{P}^1)$ is (α, c, δ, p) compatible then $q \restriction \kappa \in G^1 = G^1_W \restriction \kappa$. We have that for every $j \in x$, q has an extension in $j^0_{\alpha,x,c}(\mathcal{P}^1)$ which forces that $\left(j^0_{\alpha,x,c}(\mathcal{P}^1)(\dot{d})\right)(\theta^{c(j)}_j) = \check{\delta(j)}$ and by the definition of local Prikry function δ, we have $\delta(j) = j_W(d)(\theta^{c(j)}_j)$. It follows that $q \restriction \theta^+ \in G^1_W \restriction \theta^+$, and as $\text{cp}(k^0) > \theta^+$, we get that

$k^0(q) \restriction \theta^+ \in G^1_W \restriction \theta^+$. Finally, the assumption that $q \setminus \theta^+ = j^0_{\alpha,x,c}(p) \setminus \theta^+$ implies that $k^0(q) \setminus \theta^+ = j^0(p) \setminus \theta^+ = j_W(p) \setminus \theta^+$. Hence $k^0(q) \in G^1_W$.

□[Claim 3.28]

Let us show that $U_{\alpha,c,\delta} \subset W$. Suppose that $X = \dot{X} \in U^2_{\alpha,c,\delta}$. According to Definition 3.25 there is a condition $p^2 \in G^2$ so that

$$p^2{}^\frown(j^1_{\alpha,c,\delta}(p^2) \setminus \kappa+1) \Vdash \check{\kappa} \in j^1_{\alpha,c,\delta}(\dot{X}). \tag{5}$$

The definition of $j^1_{\alpha,c,\delta}$ implies there are $p^1 \in G^1$ and a $(\alpha, c, \delta, p^1)-$compatible condition $q^1 \in j^0_{\alpha,x,c}(\mathcal{P}^1)$ so that q^1 forces a $j^0_{\alpha,x,c}(\mathcal{P}^1)-$statement equivalent to 5 (where $j^1_{\alpha,c,\delta}$ is replaced with $j^0_{\alpha,x,c}$ and $p^2, \check{\kappa}, \dot{X}$ are replaced with their $\mathcal{P}^1-$names). Furthermore by Lemma 3.16 we may assume that $q^1 \setminus \theta^+ = j^0_{\alpha,x,c}(p^1) \setminus \theta^+$.
It follows that $k^0(q^1)$ forces a $j_W(\mathcal{P}^1)-$statement equivalent to

$$p^2{}^\frown(j_W(p^2) \setminus \kappa+1) \Vdash \check{\kappa} \in j_W(\dot{X}), \tag{6}$$

and Claim 3.28 (showing $k^0(q^1) \in G^1_W$) guarantees 6 holds. Finally, the coding posets in $\mathcal{P}^2$ and the fact $j``G^2 \subset G^2_W$ guarantee that $p^2{}^\frown(j(p^2)\setminus\kappa+1) \in G^2_W$, hence $X \in W$.

□[Theorem 3.26]

Proposition 3.29. For every $U^2_{\alpha,c,\delta}$, $U^2_{\alpha',c',\delta'}$ in $V^2 = V[G^0 * G^1 * G^2]$, $U^2_{\alpha',c',\delta'} \lhd U^2_{\alpha,c,\delta}$ if and only if $\alpha' < \alpha$ and $c' \geq c$.

Proof. Suppose that $\alpha' < \alpha$ and $c' \leq c$, and let us verify that $U^2_{\alpha',c',\delta'} \in M^2_{\alpha,c,\delta} \cong \mathrm{Ult}(V^2, U^2_{\alpha,c,\delta})$.
Let $x = \mathrm{dom}(\delta) \subset \lambda$. Then $x \in S'$ and c is a $x-$suitable function. We described the embedding $j = j^2_{\alpha,c\delta} \restriction V = j^1_{\alpha,c,\delta} \restriction V$ as an iterated ultrapower of V, $j : V \to Z_{\alpha,c}$. We know that

1. $j = k \circ j_{\alpha,c}$ where $j_{\alpha,c} : V \to M_{\alpha,c} \cong \mathrm{Ult}(V, F_{\alpha,c})$ and $\mathrm{cp}(k) > \theta^+$. Therefore $M_{\alpha,c}$ and $Z_{\alpha,c}$ share the same $(\kappa, \theta^+)-$extenders.

2. $Z_{\alpha,c} = \mathcal{K}(M^2_{\alpha,c,\delta})$ and $M^2_{\alpha,c,\delta} = Z_{\alpha,c}[G^0_{\alpha,x,c} * G^1_{\alpha,c,\delta} * G^2_{\alpha,c,\delta}]$, where $G^0_{\alpha,x,c} * G^1_{\alpha,c,\delta} * G^2_{\alpha,c,\delta}$ is $j(\mathcal{P}^0 * \mathcal{P}^1 * \mathcal{P}^2)$ generic over $Z_{\alpha,c}$, satisfying $G^0 = G^0_{\alpha,x,c} \restriction \kappa+1$, $G^1 = G^1_{\alpha,c,\delta} \restriction \kappa$, and $G^2 = G^2_{\alpha,c,\delta} \restriction \kappa+1$. Hence $G^0, G^1, G^2 \in M^2_{\alpha,c,\delta}$.

3. $j_{\alpha,c} = i_{\alpha,c} \circ j_\alpha$ where $j_\alpha : V \to M_\alpha \cong \mathrm{Ult}(V, F_\alpha)$ and $i_{\alpha,c}$ is the $c-$derived (iterated ultrapower) embedding of M_α. The fact that $\vec{F}$ is $\triangleleft-$increasing implies that $\vec{F} \restriction \alpha = \langle F_\beta \mid \beta < \alpha \rangle \in M_\alpha$, which in turn, implies that $i_c(\vec{F} \restriction \alpha) = \langle F_{\beta,c} \mid \beta < \alpha \rangle$ belongs to $M_{\alpha,c}$, and thus also to $Z_{\alpha,c,} = \mathcal{K}(M^2_{\alpha,c,\delta})$. In particular $F_{\alpha',c} \in M^2_{\alpha,c,\delta}$, but $c' \geq c$ so it is clear that $F_{\alpha',c'} \in M^2_{\alpha,c,\delta}$ as well.

4. $\delta' \in M^2_{\alpha,c,\delta}$ since $M^2_{\alpha,c,\delta}$ is closed under $\kappa-$sequences and $\mathrm{dom}(\delta') \subset \lambda \leq \kappa$.

Since the definition of $U^2_{\alpha',c',\delta'}$ is based on $F_{\alpha',c'}, G^0, G^1, G^2$, and δ', we conclude that $U^2_{\alpha',c',\delta'} \in M^2_{\alpha,c,\delta}$.

Suppose next that $U^2_{\alpha',c',\delta'} \triangleleft U^2_{\alpha,c,\delta}$. Let $Z_{\alpha,c} = L[E_{\alpha,c}] = \mathcal{K}(M^2_{\alpha,c,\delta})$ and $Z_{\alpha',c'} = L[E_{\alpha',c'}] = \mathcal{K}(M^2_{\alpha',c',\delta'})$ be the iterated ultrapowers described in the paragraph proceeding Corollary 3.23. The proof of Proposition 2.15 shows that the fact $U^2_{\alpha',c',\delta'} \triangleleft U^2_{\alpha,c,\delta}$ implies that the $Z_{\alpha',c'}-$side of the coiteration with $Z_{\alpha,c}$ does not involve ultrapowers on by extenders indexed below θ^+. Moreover, the coiteration must use an ultrapower by a full extender on κ, on the $Z_{\alpha,c}$.
From the description of $Z_{\alpha',c'}$ and $Z_{\alpha,c}$ we see that the first possible difference between $Z_{\alpha',c'}$ and $Z_{\alpha,c}$ is in the normal measures on the first λ measurable cardinals above κ, which are determined by c' and c respectively. Here the coiteration involves an ultrapower on the $Z_{\alpha',c'}-$side whenever there is $j \in \lambda$ such that $c'(j) < c(j)$, therefore $c' \geq c$. The iteration on the $Z_{\alpha,c}-$side will be the $(c - c')$ derived iteration whose critical points are all above κ. Let us denote the resulting iterand on the $Z_{\alpha,c}-$side by $L[E']$. The next possible disagreement between E' and $E_{\alpha',c'}$ is at the full (κ, θ^+)-extenders. The $(\kappa, \theta^+)-$extenders on E' and $E_{\alpha',c'}$ are $\langle F_{\beta,c'} \mid \beta < \alpha \rangle$ and $\langle F_{\beta,c'} \mid \beta < \alpha' \rangle$ respectively. These are the last extenders with critical point κ on both sequences. As we know that the coiteration must include an ultrapower on the $Z_{\alpha,c}$ side with critical point κ, we must have that $\alpha' < \alpha$. □

3.5 Separation by sets and a Final Cut - $\mathcal{P}^X$

Proposition 3.30. The normal measures on κ in V^2 are separated by sets.

Proof. Following Definition 2.12, let $\langle \psi_\tau \mid \tau < \theta^+ \rangle \subset {}^\kappa\kappa$ be a sequence of representing functions, defined from a sequence of canonical functions

$\langle \rho_\zeta \mid \zeta < \kappa^+ \rangle$ and the G^2-derived collapsing functions $\langle g_\nu : \kappa^+ \twoheadrightarrow \Theta(\nu)^+ \mid \nu \in \Omega' \cup \{\kappa\} \rangle$. We get that for every $\gamma < \theta^+$ and $U^2_{\alpha,c,\delta} \in V^2$, $j^2_{\alpha,c,\delta}(\psi_\gamma)(\kappa) = \gamma$. For every ordinal ν, let $o'(\nu) < \Theta(\nu)^+$ be the length of the maximal sequence $\vec{F}_\nu \subset E^{\mathcal{K}}$ of $(\nu, \Theta(\nu)^+)$ full extenders (note $\mathcal{K}(V^2) = V$ so $E^{\mathcal{K}} = E$). Note that $o'(\kappa)^{M_\alpha} = \alpha$ for every $\alpha < l(\vec{F})$.

Suppose that $\alpha < \theta^+$, $\delta : \lambda \to \theta$, and that $c : \lambda \to \omega$ is $x = \mathrm{dom}(\delta)$ suitable. Define $X_{\alpha,c,\delta} \subset \kappa$ to be the set of $\nu \in \Omega'_x$ satisfying

1. $o'(\nu) = \psi_\alpha(\nu)$,
2. $\Theta_j(\nu) = \psi_{\theta_j^{c(j)}}(\nu)$ for every $j < \lambda \cap \nu$, and
3. $d(\Theta_j(\nu)) = \psi_{\delta(j)}(\nu)$ for every $j \in x \cap \nu$.

For every $U^2_{\alpha',c',\delta'} \in V^2$ it is straightforward to verify $X_{\alpha,c,\delta} \in U^2_{\alpha',c',\delta'}$ if and only if $\alpha = \alpha'$, $c = c'$, and $\delta = \delta'$. $\square$

Suppose that $X \subset \kappa$ and let $\mathcal{P}^X$ be the final cut iteration by X, introduced in [4], Section 7. It is easy to see that the arguments in the proof of Lemma 7.2 and Corollary 7.3 apply to final cut extensions of V^2, therefore if $V^2[G^X]$ is a $\mathcal{P}^X$ extension of V^2 then

1. The normal measure on κ in $V^2[G^X]$ are of the form U^X, where U is a normal measure in V^2 with $X \notin U$, and U^X is its unique normal extension in $V^2[G^X]$.
2. For every $U^X, W^X \in V^2[G^X]$, extending $U, W \in V^2$ respectively, $U^X \triangleleft W^X$ if and only if $U \triangleleft W$.

We apply a final cut extension to obtain a model in which $\triangleleft(\kappa) \cong <_{R^*_{\rho,\lambda}} \restriction S$. By the definition of S' (Definition 3.3) we know that for every element $(\alpha, c) \in S$, the set $x = c^{-1}(\{1\})$ belongs to S' and $c = c_x$ is the characteristic function of x. Let $\delta_c \in \prod_{j \in x} \theta^1_j$ be the local Prikry function, mapping each $j \in x$ to $\delta_c(j) = \theta_j < \theta^1_j = \theta^{c(j)}_j$. δ_c is clearly a local Prikry function with respect to α, x, c. Thus U^2_{α,c,δ_c} exists, and if (α', c') is an additional element in S then $U^2_{\alpha',c',\delta_{c'}} \triangleleft U^2_{\alpha,c,\delta}$ if and only if $(\alpha', c') <_{R^*_{\rho,\lambda}} (\alpha, c)$.
As $|S| \leq \kappa$, the final cut Lemma (Lemma 7.4 in [4]) implies there is a set $X \subset \kappa$ in V^2 so that $\triangleleft(\kappa)^{V^2[G^X]} \cong \triangleleft(\kappa)^{V^2} \restriction \{U^2_{\alpha,c,\delta_c} \mid (\alpha, c) \in S\} \cong <_{R^*_{\rho,\lambda}} \restriction$

S. $\square$[Theorem 1.1]

Note that small forcings of cardinality $< \kappa$ do not interrupt the construction at κ. It is therefore possible to apply the main construction on different cardinals to obtain a global $\triangleleft$ behavior.

Corollary 3.31. Let $V = L[E]$ is a core model. Suppose that for every cardinal λ in V there is proper class of measurable cardinals κ which carry a $\triangleleft-$increasing sequence of (κ, θ^+) extenders $\vec{F} = \langle F_\alpha \mid \alpha < \lambda \rangle$, where

- $\theta = \sup_{i<\lambda} \theta_i^+$ so that θ_i is the $i-$th measurable cardinal above κ, and
- each F_α is $(\theta + 1)$ strong.

Then there is a class generic extension V^* of V, so that every well founded order $(S, <_S)$ in V^* is realized as $\triangleleft(\kappa)$ at some measurable cardinal κ.

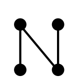
4 Open Questions

We conclude the two-part study, presented here and in [4] with a few questions:

Question 1. Is it possible to realize every well-founded order of size κ^+ as $\triangleleft(\kappa)$?

Question 2. What is the consistency strength of $\triangleleft(\kappa) \cong R_{2,2}$, and of $\triangleleft(\kappa) \cong S_{\omega,2}$? Are overlapping extenders necessary?

Question 3. What is the consistency strength of $\triangleleft(\kappa) \cong S_{2,2}$?



Note that $\text{rank}(S_{2,2}) = 2$ but $\text{Trank}(S_{2,2}) = 3$, therefore the method in Part I can realize $S_{2,2}$ as $\triangleleft(\kappa)$ from the assumption of $o(\kappa) \geq 3$.

Acknowledgements -

The author would like to express his gratitude to his supervisor Professor Moti Gitik, for many fruitful conversations, valuable guidance and encouragement.

The author is grateful to the referee for making many valuable suggestions and comments which greatly improved both the content and structure of this paper.

The author would also like to thank Brent Cody and Dror Speiser for their valuable remarks and suggestions.

References

[1] Arthur Apter, James Cummings, and Joel Hamkins, *Large cardinals with few measures*, Proceedings of the American Mathematical Society, vol. 135 (7) (2007), 2291-2300.

[2] Stewart Baldwin, *The* $\triangleleft$*-Ordering on Normal ultrafilters*, The Journal of Symbolic Logic, **50** (1985) ,936-952.

[3] Omer Ben-Neria, *Forcing Magidor iteration over a core model below* $0^{\P}$, to appear in the Archive for Mathematical Logic.

[4] Omer Ben-Neria, *The structure of the Mitchell order - I*, to appear.

[5] James Cummings, *Possible behaviours for the Mitchell ordering*, Annals of Pure and Applied Logic, Volume 65 (2) (1993), 107-123.

[6] James Cummings, *Possible Behaviours for the Mitchell Ordering II*, Journal of Symbolic Logic, Volume 59 (4) (1994), 1196-1209.

[7] Sy-David Friedman and Menachem Magidor, *The number of normal measures*, The Journal of Symbolic Logic, **74** (2009) ,1069-1080.

[8] Kenneth Kunen, *Some application of iterated ultrapowers in set theory*, Annals of Mathematical Logic **1** (1970), 179-227.

[9] Kenneth Kunen and Jeffrey Paris, *Boolean extensions and measurable cardinals*, Annals of Mathematical Logic **2** (1970/71), 359-377.

[10] Jeffery Leaning and Omer Ben-Neria, *Disassociated indiscernibles*, Mathematical Logic Quarterly **60** (2014) , 389-402.

[11] Azriel Levy and Robert Solovay, *Measurable cardinals and the continuum hypothesis*, Israel J. Math., **5** (1967), 234-248.

[12] William Mitchell, *Sets Constructed from Sequences of Measures: Revisited*, The Journal of Symbolic Logic, Volume 39 (1) (1974), 57-66.

[13] William Mitchell, *Sets Constructed from Sequences of Measures: Revisited*, The Journal of Symbolic Logic, Volume 48 (3) (1983), 600-609.

[14] Itay Neeman, *The Mitchell order below rank-to-rank*, The Journal of Symbolic Logic, Volume 69 (4) (2004), 1143-1162.

[15] Ralf Schindler, *Iterates of the Core Model*, The Journal of Symbolic Logic, Volume 71 (1) (2006), 241 - 251.

[16] John Steel, *An Outline of Inner Model Theory* , Handbook of set theory (Foreman, Kanamori editors), Volume 3, 1601-1690.

[17] John Steel, *The Well-Foundedness of the Mitchell Order*, The Journal of Symbolic Logic, Volume 58 (3) (1993), 931-940.

[18] Jiri Witzany, *Any Behaviour of the Mitchell Ordering of Normal Measures is Possible*, Proceedings of the American Mathematical Society, Volume 124 (1) (1996), 291-297.