Forcing Magidor Iteration over a Core Model below $0^{\P}$

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Abstract

We study the variety of normal measures which appear on the first measurable cardinal κ, after forcing the Magidor Iteration of Prikry forcings $\mathbb{P}_\kappa$. We prove that by forcing over a core model below $0^{\P}$ then there exists a natural identification between the normal measures on κ in the generic extension and those of the ground model.

1 Introduction

In this work, we study the celebrated Magidor iteration of Prikry forcings. We are primarily interested in the normal measures on the first measurable cardinal κ (i.e., κ complete normal ultrafilters on κ). In [7], Magidor introduced his iterating style of Prikry type forcings and used such an iteration to achieve consistency results comparing the first strongly compact cardinals with the first measurable in one case and with the first super-compact cardinal in the other.

It seems natural to ask weather this iteration can produce some other results. One such issue to be considered is the number of normal measures on the first measurable cardinal κ. This has been extensively studied. Results by Kunen [4] and Stewart Baldwin [2] have been obtained using inner model constructions, while Kunen-Paris [5], Apter, Cummings, Hamkins [1], and Jeffery Scott Leaning [6] obtained results using forcing. The question has been finally settled by Magidor and Friedman [8], which showed how to achieved models with arbitrary number normal measures $\lambda \leq \kappa^{++}$ (with $\kappa^{++} = 2^{2^\kappa}$) on the first measurable cardinal κ from the minimal assumption of a single measurable cardinal. It is still unknown how many normal measures can a strongly compact cardinal have. James Cummings asked whether it is possible to use the Magidor iteration forcing up to a measurable κ over V, in which $o(\kappa)^V = \lambda$ and $\lambda \leq \kappa^{++}$, to achieve an extension in which the first measurable cardinal κ has exactly λ many normal measures. In this paper we address this question giving an affirmative answer when assuming V is a suitable fine structural core model. For every U, a normal measure on κ in V, we associate a normal measure $U^\times$ in the generic extension. The main result of this study is the following Theorem,

Theorem 1.1. *Assuming that $0^{\P}$ does not exist, and V is the core mode, then all the normal measures on κ in a generic extension $V^{\mathbb{P}_\kappa}$ are precisely those of* *the form $U^\times$ where U is a normal measure on κ in V, and they are all distinct.*

In particular if $o(\kappa) = \lambda$ then there are exactly λ many normal measures on κ in a generic extension.

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A road map to this work - In the remainder of this section we survey the necessary preliminary material on the Magidor iteration of Prikry forcing notions up to κ denoted $\mathbb{P}_\kappa$. In section 2 we study the extensions of ground model normal measures in a generic extension. We show that every normal measure on κ in V has a natural extension in $V^{\mathbb{P}_\kappa}$, and study these extensions. The results in this section requires only mild assumptions about V which do not involve inner model theory. In section 3 we incorporate the assumption of the ground model being a core model without overlapping extenders. Given a normal measure W on κ in the generic extension, then by studying the restriction of its corresponding elementary embedding j_W to V we relate W to the normal measures of V. Throughout this work we use forcing conventions by which $p < q$ means that q is *stronger* (more informative) than p. A weakest element of a forcing notion $\mathbb{P}$ will be denoted by $0_\mathbb{P}$. Regarding measurability notations, a *measure* on κ is a κ complete ultrafilter $U \subset \mathcal{P}(\kappa)$ on κ. It is a *normal* if this ultrafilter is closed to diagonal intersections.

1.1 The Magidor Iteration

Let $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \underset{\sim}{Q}_\alpha | \ \alpha < \lambda\rangle$ be the Magidor iteration as defined in [3]. For every $\alpha < \lambda$ either

1. $0_{\mathbb{P}_\alpha}$ forces α is measurable and then $(\underset{\sim}{Q}_\beta, \leq_\beta, \leq^*_\beta)$ is taken to be Prikry forcing [10] by some normal measure $\underset{\sim}{U}^*_\alpha$ on α which does not concentrate on the set of former measurable cardinals below α (i.e. measurable cardinals in V), or

2. $0_{\mathbb{P}_\alpha}$ forces α is not measurable and then $(\underset{\sim}{Q}_\beta, \leq_\beta, \leq^*_\beta)$ is taken to be trivial forcing.

$\mathbb{P}_\alpha$ consists of conditions p of the form $\langle \underset{\sim}{p}_\beta \mid \beta < \alpha\rangle$ where for each $\beta < \alpha$, $0_{\mathbb{P}_\beta}$ forces $\underset{\sim}{p}_\beta \in \underset{\sim}{Q}_\beta$. Whenever α is measurable in $V^{\mathbb{P}_\alpha}$, we denote by $\langle \underset{\sim}{s}_\beta(p), \underset{\sim}{X}_\beta(p)\rangle$ a couple which represents $\underset{\sim}{p}_\beta$ as a condition in the Prikry forcing, i.e., $\underset{\sim}{s}_\beta(p)$ is a name for a finite sequence of ordinals below β, and $\underset{\sim}{X}_\beta(p)$ a name for a (measure one) set in $\underset{\sim}{U}^*_\beta$.

We use standard notation to describe different parts in the elements of $\mathbb{P}_\alpha$. For every condition $p = \langle p_\beta | \ \beta < \kappa\rangle \in \mathbb{P}_\kappa$ and $\alpha < \beta < \kappa$, let $p\restriction\beta = \langle p_\gamma | \ \gamma < \beta\rangle \in \mathbb{P}_\beta$, $p\backslash\beta = \langle p_\gamma | \ \beta \leq \gamma\rangle \in \mathbb{P}_\alpha\backslash\beta$, and $p\restriction_{[\alpha,\beta)} = (p\restriction_\beta)\backslash\alpha$.

Suppose $p = \langle \underset{\sim}{p}_\beta \mid \beta < \alpha\rangle$ and $p' = \langle \underset{\sim}{p}'_\beta \mid \beta < \alpha\rangle$ are conditions of $\mathbb{P}_\alpha$. Then p is an *extension* of p' denoted by $p \geq_{\mathbb{P}_\alpha} p'$, if and only if

1. for every $\beta < \alpha$, $p\restriction\beta \Vdash_{\mathbb{P}_\beta} \underset{\sim}{p}_\beta \geq_\beta \underset{\sim}{p'}_\beta$.
2. There is a finite set $b \subset \alpha$ such that for every $\beta \in \alpha \setminus b$, $p\restriction\beta \Vdash_{\mathbb{P}_\beta} \underset{\sim}{p}_\beta \geq^*_\beta \underset{\sim}{p'}_\beta$.

Also, if $b = \emptyset$ then p is a *direct extension* of p', $p \geq^*_{\mathbb{P}_\alpha} p'$.

So roughly, when extending a condition $p \in \mathbb{P}_\alpha$, we may shrink the measure one sets at each measurable $\beta < \alpha$ but adding elements to the Prikry sequences only in finitely many $\beta' s$.

We are interested only in conditions of $(\mathbb{P}_\alpha, \geq_{\mathbb{P}_\alpha}, \geq^*_{\mathbb{P}_\alpha})$ which are extensions of the zero condition $0_{\mathbb{P}_\alpha} = \langle \underset{\sim}{0}_{Q_\beta} \mid \beta < \alpha \rangle$. Therefore, throughout this work, whenever a generic set $G_\alpha, \subset \mathbb{P}_\alpha$ is taken, we assume it contains $0_{\mathbb{P}_\alpha}$.
We define for every $p = \langle \underset{\sim}{p}_\beta \mid \beta < \alpha \rangle$ the *support* of p, *supp*(p) to be the minimal finite set $b \subset \alpha$ such that for every $\beta \in \alpha \setminus b$, $p\restriction\beta \Vdash_{\mathbb{P}_\beta} \underset{\sim}{p}_\beta \geq^*_\beta 0_{\underset{\sim}{Q}_\beta}$.

Remark 1.2. *Recall that as in any iterated forcing notion, for every $p, q \in \mathbb{P}_\kappa$ such that $p \geq q$, then one can construct a $p-$equivalent condition p', such that for every $\alpha < \kappa$, $0_{\mathbb{P}_\alpha} \Vdash \underset{\sim}{p'}_\alpha \geq \underset{\sim}{q}_\alpha$. In particular, the same is true when replacing $\geq$ with $\geq^*$.*

Proofs for the following can be found in [3]:

Proposition 1.3.

1. *$(\mathbb{P}_\alpha, \geq_{\mathbb{P}_\alpha}, \geq^*_{\mathbb{P}_\alpha})$ is a Prikry type forcing.*
2. *$(\mathbb{P}_\alpha, \geq_{\mathbb{P}_\alpha})$ does not collapse cardinals.*
3. *α is measurable in $V^{\mathbb{P}_\alpha}$ if and only if it is measurable in V.*

Therefore, by forcing this iteration $\mathbb{P}_\kappa$, then for every $\mu < \kappa$, the stage $\mu + 1$ of the iteration $\mathbb{P}_\kappa$ is non-trivial if and only if μ is a measurable cardinal in the ground model V. Denote $\Delta \subset \kappa$,

$$\Delta = \{\mu < \kappa \mid \mu \text{ is measurable}\}.$$

Let us elaborate about the argument (κ is measurable in V) $\longrightarrow$ (κ is measurable in $V^{\mathbb{P}_\kappa}$). Let U be a measure on κ in the ground model V. Denote by $M \cong Ult(V, U)$ the corresponding ultrapower and by $j : V \longrightarrow M$ its elementary embedding. By the elementarity of j we get that *supp*$(j(p)) = j($*supp*$(p))$ for every $p \in \mathbb{P}_\kappa$, and $j($*supp*$(p)) =$ *supp*(p) since *supp*(p) is a finite subset of κ. We conclude that $j(p)\backslash\kappa \geq^* 0_{\mathbb{P}_{j(\kappa)}\backslash\kappa} = \langle 0_{Q_\beta} | \kappa \leq \beta < j(\kappa)\rangle$.

Definition 1.1. *Let $\underset{\sim}{U}^*$ be the $\mathbb{P}_\kappa$-name consists of all pairs $\langle \underset{\sim}{X}, p\rangle$ such that:*

1. *$\underset{\sim}{X}$ is a $\mathbb{P}_\kappa$-name for a subset of κ.*
2. *p is a condition in $\mathbb{P}_\kappa$ for which there is some $q \geq^* j(p)\backslash\kappa$ such that $p^\frown q \Vdash_{\mathbb{P}_{j(\kappa)}} \check{\kappa} \in j(\underset{\sim}{X})$.*

Lemma 1.4. *U^* is a κ-complete ultrafilter extending U.*

Proof. First, let us show that for every name $\underset{\sim}{X}$ for a subset of κ and a condition $p \in \mathbb{P}_\kappa$ then p forces "$\underset{\sim}{X} \in \underset{\sim}{U}^*$" actually implies $\langle \underset{\sim}{X}, p\rangle \in \underset{\sim}{U}^*$. Suppose that $p \Vdash \underset{\sim}{X} \in \underset{\sim}{U}^*$. Let $G_\kappa \subset \mathbb{P}_\kappa$ be a generic set with $p \in G_\kappa$. Turning to $M[G_\kappa]$, we use Prikry condition of $(\mathbb{P}_{j(\kappa)} \backslash \kappa, , \geq, \geq^*)$ to find a direct extension $q \geq^* j(p)\backslash\kappa$ which decides the statement "$\check{\kappa} \in j(\underset{\sim}{X})$". Returning to M, let $\underset{\sim}{q}$ be a uniform name for the described condition q. We get that $p \Vdash_{\mathbb{P}_\kappa} (\underset{\sim}{q} \parallel_{(\mathbb{P}_{j(\kappa)} \backslash \kappa)} \check{\kappa} \in j(\underset{\sim}{X}))$, and claim that $p \Vdash_{\mathbb{P}_\kappa} (\underset{\sim}{q} \Vdash_{\mathbb{P}_{j(\kappa)} \backslash \kappa} \check{\kappa} \in j(\underset{\sim}{X}))$. Suppose otherwise, then there would be some $p^* \geq p$ such that $p^* \Vdash_{\mathbb{P}_\kappa} (\underset{\sim}{q} \Vdash_{\mathbb{P}_{j(\kappa)} \backslash \kappa} \check{\kappa} \notin j(\underset{\sim}{X}))$. So $p^* {}^\frown \underset{\sim}{q} \Vdash_{\mathbb{P}_{j(\kappa)}} \check{\kappa} \notin j(\underset{\sim}{X})$. However, we have $p^* \Vdash \underset{\sim}{X} \in \underset{\sim}{U}^*$ which implies there is a condition p^{**}, compatible with p^*, such that $\langle \underset{\sim}{X}, p^{**}\rangle \in \underset{\sim}{U}^*$. This means we can find a direct extension, $q^{**} \geq^* j(p^{**})\backslash\kappa$ such that $p^{**} {}^\frown q^{**} \Vdash \check{\kappa} \in j(\underset{\sim}{X})$, but this is impossible as $p^* {}^\frown \underset{\sim}{q}$ and $p^{**} {}^\frown q^{**}$ are compatible. We conclude that $p^\frown \underset{\sim}{q} \Vdash \check{\kappa} \in j(\underset{\sim}{X})$.

By the definition of $\underset{\sim}{U}^*$, it is clear that $U \subseteq U^*$, thus U^* is not principle. Let G_κ be a generic subset of $\mathbb{P}_\kappa$. Suppose that $X \subseteq Y \subseteq \kappa$ are sets in $V[G_\kappa]$ and $X \in U^*$. Choose a name $\underset{\sim}{X}$ for X, and a condition $p \in G_\kappa$ which forces "$\underset{\sim}{X} \in \underset{\sim}{U}^*$ and $\underset{\sim}{X} \subset \underset{\sim}{Y}$". Hence we can find a direct extension $q \geq^* j(p)\backslash\kappa$ such that $p^\frown q \Vdash \check{\kappa} \in j(\underset{\sim}{X})$. $p^\frown q \geq j(p)$, and $j(p)$ forces "$j(\underset{\sim}{X}) \subseteq j(\underset{\sim}{Y})$" thus $p^\frown q \Vdash \check{\kappa} \in j(\underset{\sim}{Y})$. Hence, $Y \in U^*$ as well.
Next, let $X_1, X_2 \subseteq \kappa$. Assume that both are in U^*. Take $p \in G_\kappa$ which forces this. Using the previous Lemma, we can find direct extensions, $q_1,\ q_2 \geq^* j(p)\backslash\kappa$ such that

$$p^\frown q_i \Vdash \check{\kappa} \in j(\underset{\sim}{X}_i),\ i = 1, 2.$$

Now both $q_1,\ q_2$ are direct extensions of $j(p)\backslash\kappa$ and therefore they are $\geq^*$ compatible. Let q be a common direct extension of q_1 and q_2. Then $p^\frown q \Vdash \check{\kappa} \in j(\underset{\sim}{X}_1 \cap \underset{\sim}{X}_2)$, thus $X_1 \cap X_2 \in U^*$.

Finally, let as establish $\kappa-$completeness for U^*. Suppose $\langle X_\alpha \mid \alpha < \lambda < \kappa\rangle$ is a partition of κ. We need to show that there is a unique $\tau < \lambda$ such that X_τ belongs to U^*. Pick names $\langle \underset{\sim}{X}_\alpha \mid \alpha < \lambda\rangle$ and a condition $p \in G_\kappa$ which forces "$\langle \underset{\sim}{X}_\alpha \mid \alpha < \lambda\rangle$ is a partition of κ". Using the elementarity of j, we conclude that in M,

$$j(p) \Vdash\ \check{\kappa} \in \biguplus_{\alpha<\lambda} j(\underset{\sim}{X}_\alpha).$$

Since $j(p)\restriction\kappa = p$ it follows that $j(p)\backslash\kappa$ forces the same statement in $M[G_\kappa]$. As $(\mathbb{P}_{j(\kappa)\backslash\kappa}, \leq^*)$ is κ-closed, we are able to find a condition $q \geq^* j(p)\backslash\kappa$ which decides all the statements "$\check{\kappa} \in j(\underset{\sim}{X}_\alpha)$", $\alpha < \lambda$. Hence, there must be a unique $\tau < \lambda$ for which $q \Vdash \check{\kappa} \in j(\underset{\sim}{X}_\tau)$. Back in M, take a suitable name $\underset{\sim}{q}$ for q. Then there are $r \in G_\kappa$ and $\tau < \lambda$ such that $r^\frown \underset{\sim}{q} \Vdash \check{\kappa} \in j(\underset{\sim}{X}_\tau)$, so $X_\tau \in U^*$. □

We introduce a useful operation applied to conditions in $\mathbb{P}_\kappa$.

Definition 1.2.

1. *For every $\alpha < \kappa$ let $p^{-\alpha}$ be the direct extension of p, obtained by throwing out all ordinals $\leq \alpha$ from all the measure one sets of measurable cardinal above α, i.e, for every $\beta \in \Delta$, $p^{-\alpha}\restriction\beta$ forces*

$$X_\beta(p^{-\alpha}) = \begin{cases} X_\beta(p), & \text{if } \beta \leq \alpha \\ X_\beta(p) \setminus (\alpha+1), & \text{if } \beta > \alpha \ . \end{cases}$$

2. *Let p be a condition in $\mathbb{P}_\kappa$ and suppose that $Q = \langle p_{(\alpha)} : \alpha < \kappa\rangle$ is a sequence of condition, all are direct extensions of p. Define a new condition $p_Q \geq^* p$, by taking*

$$X_\alpha(p_Q) = X_\alpha(p_{(\alpha)}) \cap (\triangle_{\beta<\alpha} X_\alpha(p_{(\beta)})), \quad \text{for every } \alpha < \kappa.$$

*We say that p_Q the **diagonalization** of Q above p.*

Note that both operations have uniform definitions (e.g., the construction of $p_\beta^{-\alpha}$ over p_β is independent of the iteration $\mathbb{P}_\beta$), therefore for every $\alpha, \beta < \kappa$ we see that $0_{\mathbb{P}_\beta} \Vdash (p_Q^{-\alpha})_\beta \geq^* (p_{(\alpha)})_\beta$.

It is clear from the definition of p_Q that $p_{(\alpha)}\restriction\alpha$ forces

$$(*) \qquad (p_Q \backslash \alpha)^{-\alpha} \geq^* p_{(\alpha)}^{-\alpha} \backslash \alpha \ \text{ for all } \alpha < \kappa.$$

Assume that for every $\alpha < \kappa$, $p_{(\alpha)}$ extends p only in coordinates $\geq \alpha$, i.e, $p\restriction_\alpha = p_{(\alpha)}\restriction_\alpha$. We have $p_Q\restriction\alpha \ \geq^* \ p_{(\alpha)}\restriction_\alpha$ since $p_Q \geq^* p$, so together with $(*)$ we get that $p_Q{}^{-\alpha} \geq^* p_{(\alpha)}$. Let $M \cong Ult(V, U)$ be the ultrapower of V induced by U, and $j : V \to M$ its elementary embedding. Consider the condition $q \in \mathbb{P}_{j(\kappa)}$ represented by the sequence $Q = \langle p_{(\alpha)} : \alpha < \kappa\rangle$, $q = j(Q)(\kappa)$. Clearly $q \geq^* j(p)$. Since $p_Q{}^{-\alpha} \geq^* p_{(\alpha)}$ for every $\alpha < \kappa$, we see that $j(p_Q)^{-\kappa} \geq^* q$. Given $X \in U^*$ we can apply this construction to some $q \in \mathbb{P}_{j(\kappa)}$ used in definition 1.1 and conclude the following property,

Lemma 1.5. *For every $X \in U^*$ there is some $p \in G_\kappa$ such that*

$$j(p)^{-\kappa} \Vdash \check{\kappa} \in j(\underset{\sim}{X}).$$

2 Prikry function and generic normal measures

We now introduce the notion of generic *Prikry function* d which plays a major role in the characterization of all normal measures in the generic extension. We show that whenever $U \in V$ concentrate in measurable cardinals below κ, then d induces a projection of its extension U^* to a *normal* measure on κ. At the end of this section we list some properties of this generic function which will be used extensively throughout this paper.

Definition 2.1. *For a generic set $G_\kappa \subset \mathbb{P}_\kappa$, let $d : \Delta \to \kappa$ be the induced function sending every (old) measurable $\mu \in \Delta$ to the first element of its Prikry sequence. We call d the Prikry function induced by G_κ.*

Proposition 2.1. *The measure U^* is normal if and only if $\Delta \notin U$. Furthermore, if $\Delta \in U$ then d induces a projection of U^* to a normal measure.*

Remark 2.2. *Trivially, the function $d : \Delta \to \kappa$ is regressive. We claim that d cannot be constant on any infinite set. Indeed, fix $\alpha < \kappa$. Given a condition $p \in \mathbb{P}_\kappa$, let $b = supp(p)$, so b a finite subset of κ. We get that the condition $p^{-\alpha} \geq p$ forces "$\underset{\sim}{d}^{-1}(\{\check{\alpha}\}) \subset \check{b}$". Hence, $|d^{-1}(\{\alpha\})| < \aleph_0$ in the generic extension.*

Proof. For the first part, note that $\Delta \notin U$ implies that $(\mathbb{P}_{j(\kappa)} \backslash \kappa)$ is κ^+ closed as κ is not measurable in M or in $M^{\mathbb{P}_\kappa}$. Therefore, just run the argument used to establish the $\kappa-$completeness of U^*, on a regressive function, i.e., use the κ^+ closure to guess the value $j(\underset{\sim}{f})(\kappa) < \kappa$ whenever f is a regressive function in the generic extension.
Suppose $\Delta \in U$ then $\Delta \in U^*$ as well. As seen in remark 2.2, the Prikry function $d : \Delta \to \kappa$ is regressive but not constant on any infinite set. Let us show d induces a projection of U^* to a normal ultrafilter.
Let G_κ be a generic subset of $\mathbb{P}_\kappa$. Suppose f is a function in $V[G_\kappa]$ with $f(\alpha) < d(\alpha)$ on a measure one set in U^*. Choose conditions $p \in G_\kappa$ and $q \geq^* j(p)\backslash\kappa$ such that $p^\frown q \Vdash j(\underset{\sim}{f})(\check{\kappa}) < j(\underset{\sim}{d})(\check{\kappa})$. κ is measurable in M as $\Delta \in U$. Moreover, proposition 1.3 ensures that it remains measurable in $M[G_\kappa]$. Let U_κ be the $M[G_\kappa]-$measure on κ used for the $\kappa + 1$ stage of the iteration, $\mathbb{P}_{j(\kappa)}$, namely $Q_\kappa = \mathbb{P}(U_\kappa)$. Working in $M[G_\kappa]$, we have $q \Vdash_{\mathbb{P}_{j(\kappa)\backslash\kappa}} j(\underset{\sim}{f})(\check{\kappa}) < j(\underset{\sim}{d})(\check{\kappa}) < \kappa$. Unlike before, $\langle \mathbb{P}_{j(\kappa)\backslash\kappa}, \geq^* \rangle$ is not κ^+-closed, so the proof of the first part above ($\Delta \notin U$) cannot be applied directly in this case. Nevertheless, after forcing at κ, the rest of the iteration $\langle \mathbb{P}_{j(\kappa)\backslash(\kappa+1)}, \geq^* \rangle$ is κ^+-closed. Hence, by applying the argument above, we can find a $\mathbb{P}(U_\kappa)-$name of an ordinal π and a direct extension $t_{>\kappa} \geq^* q\backslash(\kappa + 1)$ such that

$$q_\kappa^\frown t_{>\kappa} \Vdash \pi = j(\underset{\sim}{f})(\check{\kappa}) < j(\underset{\sim}{d})(\check{\kappa}).$$

Now, as both π and $j(\underset{\sim}{d})(\check{\kappa})$ are names in the forcing language of $\mathbb{P}(U_\kappa)$, we conclude that the condition q_κ forces the statement "$\pi < j(\underset{\sim}{d})(\check{\kappa})$" above.

Note that when forcing Prikry forcing $\mathbb{P}(U)$ by a single normal measure U on κ, then for every $\mathbb{P}(U)$ name π of an ordinal and $p \in \mathbb{P}(U)$ such that, $p \Vdash_{\mathbb{P}(U)}$ "π is smaller than the first element of the generic Prikry sequence" then the identity of π can be decided by a direct extension of p. So there exists a direct extension $t_\kappa \geq^*_{\mathbb{P}(U_\kappa)} q_\kappa$ such that $t_\kappa \Vdash_{\mathbb{P}(U_\kappa)} \pi = \check{\beta}$ for some $\beta < \kappa$. Let $t = t_\kappa^\frown t_{>\kappa}$. Then $t \geq^* q$ and $t \Vdash j(\underset{\sim}{f})(\check{\kappa}) = \pi = \check{\beta}$, so there exists some $r \in G_\kappa$ such that $r^\frown t \Vdash j(\underset{\sim}{f})(\check{\kappa}) = \check{\beta}$. As r forces $t, j(r)\backslash\kappa$ are both direct extensions of $0_{j(\mathbb{P}_\kappa)}\backslash\kappa$ it also force they are $\leq^*$compatible. If t' is a name of a common direct extension then $r^\frown t' \geq^* j(r)$ and $r^\frown t \Vdash j(\underset{\sim}{f})(\check{\kappa}) = \check{\beta}$. Hence $f^{-1}(\beta) \in U^*$. □

Notation 2.3. *Denote by $U^\times$ the measure on κ in $V[G_\kappa]$ projected from U^*. By the proposition above $U^\times = U^*$ if $\Delta \notin U$ and $U^\times = d_*(U^*)$ otherwise.*

Let U be a normal measure on κ in V which concentrates on Δ. The considerations above implies that For every $X \subseteq \kappa$ in $V[G_\kappa]$ then $X \in U^\times$ if and only if there are $p \in G_\kappa$ and $q \geq^* j(p)\backslash\kappa$ such that $p^\frown q \Vdash j(\underset{\sim}{d})(\check{\kappa}) \in j(\underset{\sim}{X})$.

Assumption 2.4. *Let us add an additional assumption regarding the construction of $\mathbb{P}_\kappa$. Consider a non trivial stage $\mu \in \Delta$ of the iteration $\mathbb{P}_\kappa$. Previously, the only requirement from the normal measure $\underset{\sim}{U}^*_\mu$, used at stage μ, was that it does not concentrate on $\Delta \cap \mu$. In general this does not determine uniquely the normal measure. In order to avoid unnecessary complication let us add the assumption that in V, every $\mu \in \Delta$ has a unique normal measure $\mathcal{U}(\mu, 0)$ which does not include $\Delta \cap \mu$. Therefore, at stage μ, $\underset{\sim}{Q}_\mu$, is Prikry forcing by the the normal measure $\mathcal{U}(\mu, 0)^*$, extending $\mathcal{U}(\mu, 0)^*$ in $\widetilde{V}^{\mathbb{P}_\mu}$. We note that this assumption holds true in every suitable extender models $L[\mathcal{E}]$ (e.g. as in definition 2.4 of [9]). Under the assumption of $\neg 0^\P$ the core model is of such form. under the assumption of $\neg 0^\P$.*

Lemma 2.5. *Let $G_\kappa \subset \mathbb{P}_\kappa$ be a generic set. Then in the generic extension $V[G_\kappa]$, there is a finite set $b \subset \kappa$ such that every pair of measurable cardinals $\lambda < \mu$ in $\Delta \setminus b$, satisfies $d(\mu) \notin [d(\lambda), \lambda]$. In particular, the elements of the sequence $\langle d(\mu) : \mu \in \Delta \setminus b\rangle$ are pairwise distinct.*

Proof. Fix $\mu \in \Delta$. Suppose $Q_\mu = \mathbb{P}(\underset{\sim}{U}^*_\mu)$ where U_μ is a measure on μ (in V). Denote by $j : V \to M \cong Ult(V, U_\mu)$ the induced elementary embedding. Let B_μ be the $\mathbb{P}_\mu-$name for the set

$$\mu \setminus (\bigcup_{\lambda \in (\Delta \cap \mu)} [d(\lambda), \lambda]).$$

Since we now assume $\Delta \cap \mu \notin U_\mu$, it is easily seen that $j(q)^{-\mu} \Vdash \check{\mu} \in j(B_\mu)$ for every $q \in \mathbb{P}_\mu$, therefore $0_{\mathbb{P}_\nu} \Vdash_{\mathbb{P}_\mu} B_\mu \in \underset{\sim}{U}^*_\mu$.

Given $p \in \mathbb{P}_\kappa$ let $b = supp(p)$, and for every $\mu \in \Delta$ let $A_\mu = X(p_\mu)$ and A'_μ be the name of $B_\mu \cap A_\mu$, then $p\restriction\mu \Vdash_{\mathbb{P}\mu} A'_\mu \in \underset{\sim}{U}^*_\mu$. If $q \geq^* p$ be the direct extension of p obtained by taking A'_μ instead of A_μ in all the coordinates $\mu \in \Delta$. Then q forces that $d(\mu) \notin [d(\lambda), \lambda]$ whenever $\mu > \lambda$ are in Δ and $\mu \notin b$. □

In order to give a better description of the behavior of d, we define the following sets,

Definition 2.2. *Suppose G_κ is a generic subset of $\mathbb{P}_\kappa$. Define:*

1. $\Gamma = d``\Delta$*, the set of first elements of all Prikry sequences in* G_κ.
2. $\Pi = \{\alpha : |d^{-1}(\alpha)| = 1$ *and* $\forall\mu \in \Delta.\ (\mu > d^{-1}(\alpha)) \to (d(\mu) \notin [\alpha, d^{-1}(\alpha)])\ \}$.
3. $\Sigma = \{\alpha < \kappa :$ *if* $\mu > \alpha$ *is measurable then* $d(\mu) \geq \alpha\ \}$.

Clearly, d^{-1} is a well defined function on Π, $d^{-1} : \Pi \to \Delta$. By Lemma 2.5 there is a finite set $b \subset \kappa$ such that $d(\beta) \in \Pi$ for every $\beta > \max(b)$. Hence, $\Gamma \setminus \Pi$ is bounded in κ.

We point out that Σ is a club. Trivially, Σ is closed, thus it is sufficient to show that the complement of Σ is non stationary. Let $g(\alpha) = \min\{\alpha' : \exists\mu > \alpha\ \alpha' = d(\mu)\}$. Clearly, g is regressive on $\kappa \setminus \Sigma$. Hence, if $\kappa \setminus \Sigma$ was stationary then there would be an ordinal $\alpha^* < \kappa$ with $g^{-1}(\{\alpha^*\})$ stationary. However, this is impossible since $g^{-1}(\{\alpha^*\}) \subseteq \max\left(d^{-1}(\{\alpha^*\}\right) < \kappa$ (see definition 2.1).

Definition 2.3. *For every $\alpha < \beta < \kappa$ with $\beta \in \Delta$, let $p^{+(\alpha,\beta)}$ be the condition obtained by adding α to sequence $s_\beta(p)$, i.e, $(p^{+(\alpha,\beta)})_\gamma = p_\gamma$ for every $\gamma \neq \beta$, and*

$$p^{+(\alpha,\beta)}\restriction\beta \Vdash_{\mathbb{P}_\beta} \ s_\beta(p^{+(\alpha\beta)}) = s_\beta(p) \cup \{\alpha\} \ \text{ and } \ X_\beta(p^{+(\alpha,\beta)}) = X_\beta(p)\backslash(\alpha+1)$$

We say that α is first available *at p_β if $p\restriction\beta \Vdash s_\beta(p) = \emptyset$ and $\check{\alpha} \in X_\beta(p)$. In such case we get that $p^{+(\alpha,\beta)} \geq p$ and $p^{+(\alpha,\beta)} \Vdash \underset{\sim}{d}(\check{\beta}) = \check{\alpha}$*

Remark 2.6. *We demonstrate how the sets Σ, Γ, Π indicates on certain closer properties of the generic set G_κ with respect to operations defined in 1.2, 2.3. Fix $p \in G_\kappa$.*
Suppose that α belongs to $\Sigma \setminus \Gamma$. Then neither α nor any $\alpha' < \alpha$ appears as a first element of a Prikry sequence for a measurable cardinal $\beta > \alpha$. Thus, the condition $p^{-\alpha}$ remains in G_κ.
Similarly, if $\alpha \in \Pi \cap \Sigma$ and $\beta = d^{-1}(\alpha)$ then $p^{+(\alpha,\beta)} \in G_\kappa$, and since $\alpha \in \Sigma$ then $p^{+(\alpha,\beta)-\alpha} \in G_\kappa$. Furthermore, the fact $\alpha \in \Pi$ ensures that $p^{+(\alpha,\beta)-\alpha-\beta}$ belongs to G_κ as well.

Let U_1, U_0 be two normal measures on κ in V such that $\Delta \in U_1 \setminus U_0$, and $U_0 \triangleleft U_1$. Denote by $M_1 \cong V^\kappa/U_1$, $M_0 \cong V^\kappa/U_0$ the induced ultrapowers and $j_1 : V \to M_1$, $j_0 : V \to M_0$ their corresponding elementary embeddings. Since $U_0 \in M_1$, we can also take its ultrapower inside M_1. Denote by $j_0^1 : M_1 \to M_{1,0} \cong Ult(M_1, U_0)$ the induced ultrapower of M_1 and its embedding, and by $j_{1,0} = j_0^1 \circ j_1 : V \to M_{1,0}$ the composite elementary embedding.

Recall that the iterated ultrapower $M_{1,0}$ can be perceived in two more different ways: The first is the two step iteration obtained by taking the M_0-ultrapower by the $j_0(\kappa)$ ultrafilter $j_0(U_1)$. The second is by a single ultrapower using the $\kappa-$complete ultrafilter $U_0 \times U_1$ on κ^2. Thus

$$M_0^{j_0(\kappa)}/j_0(U_1) \cong M_{1,0} \cong V^{\kappa^2}/(U_0 \times U_1).$$

Suppose $G_\kappa \subset \mathbb{P}_\kappa$ is a $V-$generic set. We establish straightforward criterions for a subset of κ, $X \in V[G_\kappa]$ to be included in the normal measure $U_1^\times = d^*(U_1^*)$.

Lemma 2.7. *Let u be an condition in $j_1(\mathbb{P}_\kappa)$. If $u\restriction_\kappa \Vdash s(u_\kappa) = \emptyset$ then there exists a direct extension $r \geq^* j_0^1(u)\restriction_{[\kappa, j_0(\kappa))}$ in $M_{1,0}$, such that κ is first available at $t_{j_0(\kappa)}$, where $t = u\restriction_\kappa^\frown r^\frown(j_0^1(u)\backslash j_0(\kappa))$ is a condition in $j_{1,0}(\mathbb{P}_\kappa)$.*

Proof. First, note that κ is measurable in M_1 so it is a non trivial point of the iteration $j_1(\mathbb{P}_\kappa) = \mathbb{P}^{M_1}_{j_1(\kappa)}$. Also, U_0 is the only measure on κ in M_1, which does not include Δ, by the assumption added in 2.4, U_0^* must be the normal measure to be used at stage κ of $\mathbb{P}^{M_1}_{\jmath_1(\kappa)}$. As $\mathbb{P}^{M_1}_{\jmath_1(\kappa)} \restriction \kappa = \mathbb{P}_\kappa$ and $V_{\kappa+1} = (M_1)_{\kappa+1}$ we may apply in M_1 all previous results regarding U_0^* in V. For every $u \in \mathbb{P}^{M_1}_{\jmath_1(\kappa)}$, $u\restriction_\kappa \Vdash_{\mathbb{P}_\kappa} X(u_\kappa) \in \underset{\sim}{U}_0^*$. Therefore there exists a direct extension $r \geq^* j_0^1(u\restriction_\kappa)\backslash\kappa$ such that $u\restriction_\kappa^\frown r \Vdash \check{\kappa} \in j_0^1(X(u_\kappa))$. Note that $j_0^1(X(u_\kappa)) = X_{j_0(\kappa)}(j_0^1(u))$.

Moving to $M_{1,0}$, let t be the direct extension of $j_0^1(u)$ defined by

$$t = u\restriction_\kappa^\frown r^\frown (j_0^1(u)\backslash j_0(\kappa)).$$

Since we assumed $u\restriction_\kappa \Vdash s(u_\kappa) = \emptyset$ we have:

1. $t_{j_0(\kappa)} = j_0^1(u_\kappa)$, thus $j_0(u\restriction\kappa)$ forces $X(t_{j_0(\kappa)}) = j_0^1(X(u_\kappa))$ and
$$j_0(u\restriction_\kappa) \Vdash s(t_{j_0(\kappa)}) = \emptyset.$$
2. $t\restriction_{j_0(\kappa)} = u\restriction_\kappa^\frown r$.

We get that $t\restriction_{j_0(\kappa)} \Vdash \check{\kappa} \in X(t_{j_0(\kappa)})$, hence, κ is first available at $t_{j_0(\kappa)}$ and $t^{+(\kappa, j_0(\kappa))} \geq t$. □

Remark 2.8. *The direct extension r taken inside the iterated ultrapower $M_{1,0}$ can be described as $r = [R]_{U_0 \times U_1}$, where $R : \kappa^2 \longrightarrow V$. Without lost of generality we may assume that for every $\alpha < \beta < \kappa$ $R(\alpha, \beta) \in \mathbb{P}_\kappa \restriction_{[\alpha,\beta)}$.*

We can say more about the structure of possible R: As $\mathbb{P}^V_\kappa = \mathbb{P}^{M_1}_\kappa$ and $u\restriction_\kappa \Vdash_{\mathbb{P}^V_\kappa} X(u_\kappa) \in \underset{\sim}{U}_0^*$ (forces in V), r can be taken from M_0, i.e. $r \geq^* j_0(u\restriction_\kappa)\backslash\kappa$ and

$$u\restriction_\kappa^\frown r \Vdash_{\mathbb{P}_{j_0(\kappa)}} \check{\kappa} \in X_{j_0(\kappa)}(u).$$

Therefore, we can choose a representing function $R : \kappa^2 \longrightarrow V$ for r with a simpler structure. Let $\langle r_\alpha \mid \alpha < \kappa\rangle$ be a sequence representing r in M_0, then the function R defined by $R(\alpha, \beta) = r_\alpha\restriction_\beta$ works.

For every $p \in \mathbb{P}_\kappa$, it is possible to apply the previous Lemma to the condition $u = j_1(p) \in \mathbb{P}_{j_1(\kappa)}$ and conclude

Corollary 2.9. *For every condition $p \in \mathbb{P}_\kappa$ there exists a direct extension $q \geq^* j_{1,0}(p)\backslash\kappa$ in $M_{1,0}$ such that $p^\frown q^{+(\kappa,j_0(\kappa))} \geq p^\frown q$, and $p^\frown q^{+(\kappa,j_0(\kappa))} \Vdash d(j_0(\kappa)) = \kappa$.*

Proposition 2.10. *Let G_κ be a generic subset of $\mathbb{P}_\kappa$. For every $X \subseteq \kappa$ in $V[G_\kappa]$, $X \in U_1^\times$ iff there are conditions $p \in G_\kappa$ and $q \geq^* j_{1,0}(p)\backslash\kappa$ in $M_{1,0}$ such that $p^\frown q^{+(\kappa,j_0(\kappa))} \geq p^\frown q$ and $p^\frown q^{+(\kappa,j_0(\kappa))} \Vdash \check{\kappa} \in j_{1,0}(\underset{\sim}{X})$.*

Proof. Let $U^\times$ be the collection of all sets $X \subseteq \kappa$ in $V[G_\kappa]$ for which there are $p \in G_\kappa$ and $q \geq^* j_{1,0}(p)\backslash\kappa$ such that $p^\frown q^{+(\kappa,j_0(\kappa))} \Vdash \check{\kappa} \in j_{1,0}(\underset{\sim}{X})$. Suppose that the conditions $p_1, p_2 \in \mathbb{P}_\kappa$ are compatible and $q_1, q_2 \in \mathbb{P}_{j_{1,0}(\kappa)}$ are suitable extensions: $q_i \geq^* j_{1,0}(p_i)\backslash\kappa$, $i = 1, 2$. Note that q_1, q_2 are compatible as well, hence the conditions $p_1{}^\frown q_1^{+(\kappa,j_0(\kappa))}$, $p_2{}^\frown q_2^{+(\kappa,j_0(\kappa))}$ are compatible and cannot force contradictive statements. We conclude that for every $X \subseteq \kappa$ in $V[G_\kappa]$, it is impossible to have both X and $\kappa \setminus X$ inside $U^\times$. Hence, in order to establish that $U_1^\times = U^\times$, it is sufficient to show that $U_1^\times \subseteq U^\times$. Let $X \in U_1^\times$. Pick a suitable name $\underset{\sim}{X}$ and conditions $p \in G_\kappa$, $q \geq^* j_1(p)\backslash\kappa$ such that $p^\frown q \Vdash_{\mathbb{P}_{j_1(\kappa)}} j_1(\underset{\sim}{d})(\check{\kappa}) \in j_1(\underset{\sim}{X})$. Applying the elementary embedding j_0^1 we get that

$$j_0^1(p^\frown q) \Vdash_{\mathbb{P}_{j_{1,0}(\kappa)}} j_{1,0}(\underset{\sim}{d})(j_0(\check{\kappa})) \in j_{1,0}(\underset{\sim}{X}).$$

Now, $p \Vdash s(q_\kappa) = \emptyset$ so by Lemma 2.7 (applied to the condition $p^\frown q \in \mathbb{P}_{j_1(\kappa)}$) there is a direct extension $t \geq^* j_0^1(p^\frown q)$ with $t\restriction_\kappa = p$, such that κ is first available at $t_{j_0(\kappa)}$. On the one hand we have $t^{+(\kappa,j_0(\kappa))} \Vdash j_{1,0}(\underset{\sim}{d})(j_0(\check{\kappa})) \in j_{1,0}(\underset{\sim}{X})$ since t forces this statement and $t^{+(\kappa,j_0(\kappa))} \geq t$. But on the other hand $t^{+(\kappa,j_0(\kappa))} \Vdash \check{\kappa} = j_{1,0}(\underset{\sim}{d})\,(j_0(\check{\kappa}))$. It follows thats $t^{+(\kappa,j_0(\kappa))} \Vdash \check{\kappa} \in j_{1,0}(\underset{\sim}{X})$. Finally, let $q' = t\backslash\kappa$, then p forces $q' \geq^* j_0^1(p^\frown q)\backslash\kappa \geq^* j_{1,0}(p)\backslash\kappa$, and $p^\frown q'^{+(\kappa,j_0(\kappa))} \Vdash \check{\kappa} \in j_{1,0}(\underset{\sim}{X})$. □

The following property is a key component in the proof of our main result, Theorem 2.11,

Theorem 2.11. *For every $X \in U_1^\times$ there exists some $A \in U_1$ such that $d``A \cap \Sigma \cap \Pi \subseteq X$.*

We start by proving the following Lemma,

Lemma 2.12. *For every name $\underset{\sim}{X}$ of a subset of κ such that $(\underset{\sim}{X})_{G_\kappa} \in U_1^\times$, there exists some $p \in G_\kappa$ such that*

$$j_{1,0}(p)^{+(\kappa,j_0(\kappa))-\kappa-j_0(\kappa)} \Vdash \check{\kappa} \in j_{1,0}(\underset{\sim}{X}).$$

Note that we do not require that $j_{1,0}(p)^{+(\kappa,j_0(\kappa))-\kappa-j_0(\kappa)}$ is an extension of $j_{1,0}(p)$. This will not be needed in the proof of Theorem 2.11.

Proof. Fix $X \in U_1^\times$. We know there are conditions $p \in G_\kappa$ and $q \geq^* j_1(p)\backslash\kappa$ such that $p^\frown q \Vdash \underset{\sim}{d}(\check{\kappa}) \in j_1(\underset{\sim}{X})$. Moreover, by proposition 1.5 we may assume that $j_1(p)^{-\kappa}$ forces this statement. Applying the elementary embedding j_0^1, we get that $j_{1,0}(p)^{-j_0(\kappa)} \Vdash \underset{\sim}{d}(j_0(\check{\kappa})) \in j_{1,0}(\underset{\sim}{X})$. Also, by lemma 2.7 there is a direct extension $r \geq^* j_{1,0}(p)^{-j_0(\kappa)}\restriction_{[\kappa,j_0(\kappa))}$ in $M_{1,0}$, such that κ is first available at coordinate $j_0(\kappa)$ of the condition $t = p^\frown r^\frown (j_{1,0}(p)\backslash j_0(\kappa))^{-j_0(\kappa)} \in \mathbb{P}_{j_{1,0}(\kappa)}$. Therefore, we have

$$(*) \qquad p^\frown r^\frown (j_{1,0}(p)\backslash j_0(\kappa))^{+(\kappa,j_0(\kappa))-j_0(\kappa)} \Vdash \check{\kappa} \in j_{1,0}(\underset{\sim}{X}).$$

We identify $M_{1,0}$ with the single ultrapower of V by $U_0 \times U_1$, $M_{1,0} \cong Ult(V, U_0 \times U_1)$. Let $R : \kappa^2 \to V$ be a function which represents r in this ultrapower. $(*)$ above indicates there is a measure one set $T \in U_0 \times U_1$ such that for all $(\alpha,\beta) \in T$, the condition $R(\alpha,\beta)$ is a direct extension of $p\restriction_{[\alpha,\beta)}$ with

$$(**) \qquad (p\restriction_\alpha{}^\frown R(\alpha,\beta)^\frown p\backslash\beta)^{+(\alpha,\beta)-\beta} \Vdash \check{\alpha} \in \underset{\sim}{X}.$$

By remark 2.8, we can assume that for some sequence $\langle r_\alpha \mid \alpha < \kappa\rangle$ with $r_\alpha \geq^* p\backslash\alpha$, then $R(\alpha,\beta) = r_\alpha\restriction_\beta$ for all $(\alpha,\beta) \in T$. We may assume that $(r_\alpha)_\beta \geq^* (p\backslash\alpha)_\beta$ is begin forced by $0_{\mathbb{P}_\beta}$ for every $\beta < \alpha$ (otherwise use equivalent conditions as suggested in 1.2).
Consider the sequence of conditions $Q = \langle p_{(\alpha)} | \alpha < \kappa\rangle$ where $p_{(\alpha)} = p\restriction_\alpha{}^\frown r_\alpha$ for each $\alpha < \kappa$. For every $(\alpha,\beta) \in T$ we have $p_{(\alpha)} \geq^* p\restriction_\alpha{}^\frown R(\alpha,\beta)^\frown p\backslash\beta$.

Now, although $p_{(\alpha)}^{+(\alpha,\beta)-\beta}$ need not be an extension of $p_{(\alpha)}$, it is still a direct extension of $(p\restriction_\alpha{}^\frown R(\alpha,\beta)^\frown p\backslash\beta)^{+(\alpha,\beta)-\beta}$, so $(**)$ implies

$$p_{(\alpha)}^{+(\alpha,\beta)-\beta} \Vdash \check{\alpha} \in \underset{\sim}{X}.$$

Let $p_Q \geq^* p$ be the diagonalization of the sequence Q, then $p_Q^{-\alpha} \geq^* p_{(\alpha)}$ for every $\alpha < \kappa$. We get that $p_Q^{+(\alpha,\beta)-\alpha-\beta} \Vdash \check{\alpha} \in \underset{\sim}{X}$ for every $(\alpha,\beta) \in T$, so $j_{1,0}(p_Q)^{+(\kappa,j_0(\kappa))-\kappa-j_0(\kappa)} \Vdash \check{\kappa} \in j_{1,0}(\underset{\sim}{X})$ since $T' \in U_0 \times U_1$. The claim follows by a standard density argument. □

We can now prove the the Theorem .

Proof. (Theorem 2.11) Pick a name $\underset{\sim}{X}$ for X. By the previous lemma, there is a condition $p \in G_\kappa$ such that

$$j_{1,0}(p)^{+(\kappa,j_0(\kappa))-\kappa-j_0(\kappa)} \Vdash \check{\kappa} \in j_{1,0}(\underset{\sim}{X}).$$

Consider the set of all pairs $(\alpha,\beta) \in \kappa^2$ for which $p^{+(\alpha,\beta)-\alpha-\beta} \Vdash \check{\alpha} \in \underset{\sim}{X}$. Denote this set by T. Clearly $T \in U_0 \times U_1$.
Let $\langle U_{0,\beta} | \beta < \kappa\rangle$ be the sequence of normal measures which represents U_0 in the ultrapower M_1. For every $\beta < \kappa$, define

$$T_\beta = \{\alpha < \beta : (\alpha,\beta) \in T\}.$$

The fact that $T \in U_0 \times U_1$ implies there is some $B \in U_1$ such that $T_\beta \in U_{0,\beta}$ for every $\beta \in B$.
For every condition $p \in \mathbb{P}_\kappa$ let p_T be the direct extension of p, obtained by shrinking the one measure sets:

$$X_\beta(p_T) := X_\beta(p) \cap T_\beta, \text{ for all } \beta \in B.$$

Let b be the support of p. Then $p_T \Vdash \underset{\sim}{d}(\beta) \in \underset{\sim}{T}_\beta$ for all $\beta \in (\check{B} \setminus \check{b})$. So by a standard density argument we see that in the generic extension $V[G_\kappa]$ there is a finite set $b \subset \kappa$ such that $d(\beta) \in T_\beta$ for all $\beta \in B \setminus b$.
Let $A = B \setminus b$, then $A \in U_1$. For every $\alpha \in d``A \cap \Sigma \cap \Pi$, let $\beta = d^{-1}(\alpha)$. We get that $\beta \in A$ and $(\alpha,\beta) \in T$, so $p^{+(\alpha,\beta)-\alpha-\beta} \Vdash \check{\alpha} \in \underset{\sim}{X}$. But $p^{+(\alpha,\beta)-\alpha-\beta}$ belongs to G_κ since $p \in G_\kappa$ and $\alpha \in \Sigma \cap \Pi$. Hence, $\alpha \in X$. □

3 Core Model Aspects

In this section we add the assumption there is no inner model with overlapping extenders and consider a generic extension of the core model $\mathcal{K}$.

Let $G_\kappa \subset \mathbb{P}_\kappa$ be a $V-$generic set and suppose W is a measure on κ in $V[G_\kappa]$. Denote by $j_W : V[G_\kappa] \to M_W \cong Ult(V[G_\kappa], W)$ the corresponding elementary embedding. Since V is the core model of $V[G_\kappa]$ then the restriction map $j = j_W \restriction V : V \to M$ is induced by an iterated ultrapower of V. In addition, $M_W = M[G_W]$ where $G_W \subset \mathbb{P}_{j(\kappa)}$ is M generic.

For simplicity, we extend the notations from the previous section. Thus, we write $d : j(\Delta) \to j(\kappa)$ for the extended Prikry function (which sends every measurable cardinal $< j(\kappa)$ in M, to the first element of his Prikry sequence), induced by the generic $G_W \subset \mathbb{P}^M_{j(\kappa)}$. Throughout this section, we use these notations when describing an arbitrary measure on κ in the generic extension $V[G_\kappa]$.

The ground model $V = \mathcal{K} = L[\mathcal{E}]$ has many possible iteration components. The following claim exemplifies that the iterated ultrapower j is much more restricted. This claim is not essential to the rest of this paper, so a reader which is only interested in the main result may skip directly to the next section (3.1).

Lemma 3.1.

1. *Any extender which is iterated infinitely many times in the iteration leading to M must be a normal measure.*

2. *No normal measure is being iterated more than ω many times in the construction of M.*

Proof.

1. Let E be an extender which is iterated ω many times along the construction of M and let $i : V \to M_0$ a finite sub iteration of j such that $E \in M_0$. Let us add notations describing the $\omega-$iteration. Define M_n, E_n, i_n for every $n < \omega$. M_0 is defined, $i_0 = id_{M_0}$, and $E_0 = E$. Given M_n, $i_n : M_0 \to M_n$, then $E_n = i_n E$, $M_{n+1} \cong Ult(M_n, E_n)$, $i_{n,n+1} : M_n \to M_{n+1}$ is the induced ultrapower map, and $i_{n+1} = i_{n,n+1} \circ i_n$. If $i_\omega : M_0 \to M_\omega$ is the direct limit of the directed system $\{M_n, i_{m,n} \mid m < n < \omega\}$. Then $i_\omega \circ i$ is a sub iteration of j.

   Denote the critical point of E by δ. δ is the generator of the normal measure E_δ of E. If E is not a normal measure then E has an independent generator δ' such that $\delta < \delta' < i_1(\delta)$. Then the induced measure $E_{\delta'} = \{X \in \mathcal{P}(\delta)^{M_0} \mid \delta' \in i_1(X)\}$ is not Rudin Kiesler equivalent to E_δ (which is its normal projection, i.e. $E_\delta <_{RK} E_{\delta'}$). Let $\sigma' = \langle \delta'_n \mid n < \omega \rangle$ be the generators of $E_{\delta'}$ accumulated along the $\omega-$iteration by E, that is $\delta'_n = i_n(\delta')$. Denote their limit by δ_ω.

   $\sigma' \in M_W$ since ${}^\kappa M_W \subset M_W$. Therefore, in M_W we can retrieve the $M-$measure $i_\omega(E_{\delta'})$ as $\{X \in \mathcal{P}(\delta_\omega)^M \mid \sigma' \text{ is almost contained in } X\}$.

Since M is the core model of $M_W = M[G_W]$ then $i_\omega(E_{\delta'}) \in M$. Furthermore, σ' satisfies Mathias criterion for a generic Prikry sequence of $i_\omega(E_{\delta'})$. This fact is being stated by M_W, i.e. G_W adds its core model M a generic Prikry sequence to some measure which not Rudin Kiesler equivalent to its normal projection. Hence they same holds true in $V[G_\kappa]$, however this is impossible since the forcing $\mathbb{P}_\kappa$ adds Prikry sequences only to normal measures.

2. Suppose now that E is a normal measure which is iterated at least ω many times in the construction of M. Adopting the notations from the first part of the proof we have $E = E_\delta$. Let $\sigma = \langle \delta_n \mid n < \omega \rangle$ be the generators of E_δ accumulated along the $\omega-$iteration by E_δ, and $\delta_\omega = \bigcup_{n<\omega} \delta_n$. As argued above, the $M-$measure $i_\omega(E_\delta)$ belongs to M_W. Since M is the core model of M_W E_δ appears on the $M-$extender sequence (see [11] Lemma 8.3.4). On the other hand, $i_\omega(E_\delta)$ already appears on the $M_\omega-$extender sequence, and $i_\omega : V \to M_\omega$ is a sub-iteration of $j : V \to M$. This implies $i_\omega(E_\delta)$ is not used in the rest of the iteration, and E is iterated exactly ω many times along j.

□

3.1 Key Generators

Let $G_\kappa \subset \mathbb{P}_\kappa$ be a V-generic set. In 2.2 we defined the sets Γ and Π in $V[G_\kappa]$. We saw that $\Gamma \setminus \Pi$ is bounded in κ, and proved that for every normal measure U on κ in V, then Γ belongs to its corresponding normal measure $U^\times$ in $V[G_\kappa]$.

Let W be any normal measure on κ in $V[G_\kappa]$ such that $\Gamma \in W$. Then $\Pi \in W$ and therefore in $M_W \cong Ult(V[G_\kappa], W)$ we have $\kappa \in j_W(\Pi)$. Using the notations from the beginning of the section to consider M_W as the generic extension $M[G_W]$ of M, it follows that κ is the first element of a Prikry sequence in the generic set $G_W \subset \mathbb{P}_{j(\kappa)}$ and $d^{-1}(\kappa)$ is a unique measurable cardinal in M. As it will turn out (in the end of this section), the ordinal $d^{-1}(\kappa)$ determines a normal measure U_W on κ in V such that $U_W^\times = W$.

Consider $j^{\mathcal{F}in} : V \to M^{\mathcal{F}in}$, a finite sub iteration of j. Let us describe an ultrapower construction of $M^{\mathcal{F}in}, j^{\mathcal{F}in}$. Let $n < \omega$ be the length of the iteration, then there are elementary embeddings : $j_k : V \to M_k$ for every integer $k \leq n$, and extenders E_k for every $k < n$, such that:

1. $M_0 = V$, $j_0 = id_{M_0}$.
2. For every $k < n$, E_k is an extender in M_k, $M_{k+1} \cong Ult(M_k, E_k)$, $j_{k,k+1} : M_k \to M_{k+1}$ is the ultrapower map, and $j_{k+1} = j_{k,k+1} \circ j_k$.
3. $j^{\mathcal{F}in} = j_n$ and $M^{\mathcal{F}in} = M_n$

Definition 3.1. *Under the above notations, consider the critical points $crit(E_k)$ for $k < n$.*

1. *We say $crit(E_k)$ is a normal generator of the iteration if $crit(E_k) \leq j_k(\kappa)$.*

2. *We say $crit(E_k)$ is a key generator if and only if $crit(E_k) = j_k(\kappa)$.*

3. *We denote the largest key generator in $M^{\mathcal{F}in}$ by κ^*.*

Let $\{crit(E_{k_m}) \mid 0 \leq m \leq n^*\}$ the list of key generators. This description $M^{\mathcal{F}in}, j^{\mathcal{F}in}$ as an iterated extender ultrapower is not unique, however the sets of normal generators and key generators are independent of the choice of description. We can assume that our description satisfies that $crit(E_{k_{m_1}}) < crit(E_{k_{m_2}})$ for every $m_1 < m_2 \leq n^*$. Note that $crit(E_0) = \kappa$ since $j^{\mathcal{F}in}$ is a sub iteration of j so it is a key generator. In particular if $n \geq 1$ then there must be a largest key generator $\kappa^* \geq \kappa$.

The first part of the following proof is the only place in this paper in which we make use of the assumption there are no inner models with overlapping extenders.

Lemma 3.2.

1. *The set $\{j^{\mathcal{F}in}(f)(\kappa^*) \mid f \in \kappa^\kappa \cap V\}$ is cofinal in $j^{\mathcal{F}in}(\kappa)$.*

2. *Let $\mu^* < j^{\mathcal{F}in}(\kappa)$ be a measurable cardinal in $M^{\mathcal{F}in}$. If μ^* is bigger than the largest key generator κ^*, then for every condition $p \in \mathbb{P}_\kappa$ there exists a direct extension $p^* \geq^* p$, such that $j^{\mathcal{F}in}(p^*) \Vdash \check{\kappa} \neq \underset{\sim}{d}(\mu^*)$.*

Proof.

1. We use the iteration parts M_k, j_k, E_k, $k \leq n$ described above and prove by induction on $k \leq n$ that if κ^* is the maximal key generator of the sub iteration $j_k : V \to M_k$, then $\{j_k(f)(\kappa^*) \mid f \in \kappa^\kappa \cap V\}$ is cofinal in $j_k(\kappa)$.

   First note that it is sufficient to prove that for all $\delta' < j_k(\kappa)$, if δ' is a generator of the iteration j_k (i.e. a generator of one of the extenders $\langle E_i \mid i < k\rangle$ used to form j_k) then it is bounded by $j_k(f_{\delta'})(\kappa^*)$ for some $f_{\delta'} \in \kappa^\kappa \cap V$. This is because for every ordinal $\gamma < j_k(\kappa)$ there exists $m < \omega$, $h : \kappa^m \to \kappa$, and generators $\delta_0, .., \delta_{m-1} < j_k(\kappa)$ such that $\gamma = j_k(h)(\delta_0, .., \delta_{m-1})$. So by defining $h_\gamma : \kappa \to \kappa$ by $h_\gamma(\nu) = \sup(\{h(\nu_0, .., \nu_{l-m}) \mid \forall i < m, \nu_i < f_{\delta_i}(\nu)\}) + 1$, then $j_k(h_\gamma)(\kappa^*) > \gamma$.

   When $n = 0$ there are no key generators so there is nothing to prove. Suppose this claim holds for some $k < n$ and consider the next ultrapower $j_{k,k+1} : M_k \to M_{k+1} \cong Ult(M_k, E_k)$. Denote $crit(E_k)$ by δ. We split the argument to into four cases:

   (a) If $\delta > j_k(\kappa)$ then an ultrapower by E_k does not change any of the structure below δ. In particular κ^* is still the maximal key generator and the claim holds true.

   (b) Suppose $\kappa^* < \delta < j_k(\kappa)$. Since there are no overlapping extenders in M_k then all generators of E_k are bounded below $j_k(\kappa)$. By our inductive assumption, for every generator δ' of E_k there exists some

$f_{\delta'} \in \kappa^\kappa \cap V$ such that $\delta' = j_k(f_{\delta'})(\kappa^*) > \delta'$. We claim that the same function $f_{\delta'}$ works for $j_{k+1} = j_{k,k+1} \circ j_k$. Indeed $\kappa^* < \delta = crit(E_k)$ hence $j_{k,k+1}(\kappa^*) = \kappa^*$. Hence $j_{k+1}(f_{\delta'})(\kappa^*) = (j_{k,k+1} \circ j_k)(f_{\delta'})(j_{k,k+1}(\kappa^*)) = j_{k,k+1}\left(j_k(f_{\delta'})(\kappa^*)\right) > j_{k,k+1}(\delta') > \delta'$.

(c) Suppose $\delta = j_k(\kappa)$. Then $\kappa^* = \delta$ is the new maximal key generator in M_{k+1}. The generators of $j_{k+1} : V \to M_{k+1}$ are either generators of $j_k : V \to M_k$ or generators of E_k. If δ is a generator of j_k smaller than $j_{k+1}(\kappa)$ then $\delta < j_k(\kappa) = \kappa^*$ so $f_\delta = id_\kappa$ works. As to the generators of E_k, if E_k is a normal measure then this is immediate. Otherwise $o(\delta)^{M_k} > 0$ so there are unbounded many measurable cardinals below δ in M_k, and as $\delta = j_k(\kappa)$, the same is true for κ in V. Define $r : \kappa \to \kappa$ by mapping every $\nu < \kappa$ to the least measurable cardinal above ν. Since there are no overlapping extenders in M_k, then the first measurable cardinal above δ in M_{k+1} is an upper bound to all the generators of E_k. We get that $j_{k+1}(r)(\kappa^*) = j_{k+1}(r)(\delta)$ is greater than all the generators of E_k.

(d) Suppose $\delta < \kappa^*$. Since there is no inner models with overlapping extenders then all E_k's generators are bounded below κ^* i.e. $f_{\delta'} = id_\kappa$ works for every generator δ' of E_k.

2. Pick $m < \omega$, $h : \kappa^m \to \kappa$ in V, and generators $\delta_0, .., \delta_{m-1} < j^{\mathcal{F}in}(\kappa)$ such that $\mu^* = j^{\mathcal{F}in}(h)(\delta_0, .., \delta_{m-1})$. Also for every $i < m$ choose a function $f_{\delta_i} \in \kappa^\kappa \cap V$ such that $\delta_i < j^{\mathcal{F}in}(f_{\delta_i})(\kappa^*)$. Define

$$Z = \{(\theta, \theta^*, \nu_0, .., \nu_{m-1}) \in \kappa^{m+2} \mid \ h(\nu_0, .., \nu_{m-1}) > \theta^* \ , \theta \leq \theta^*, \\ \text{and } \theta \leq \nu_i < f_{\delta_i}(\theta^*) \text{ for every } i < n\}.$$

then $(\kappa, \kappa^*, \delta_0, .., \delta_{m-1}) \in j^{\mathcal{F}in}(Z)$, and $|\{(\theta, \theta^*, \nu_0, .., \nu_{m-1}) \in Z \mid \theta^* = \lambda\}| < \kappa$ for every $\lambda < \kappa$.

Define a closed unbounded sequence in κ, $C = \langle \lambda_i \mid i < \kappa \rangle$. Set $\lambda_0 = 0$. Suppose that $\langle \lambda_j \mid j < i \rangle$ has been defined. When i is a limit ordinal set $\lambda_i = \bigcup_{j<i} \lambda_j$. When $i = i' + 1$, if there Are $\theta, \nu_0, .., \nu_{m-1}$ such that $(\theta, \lambda_{i'}, \nu_0, .., \nu_{m-1}) \in Z$ then define

$$\lambda_i = \bigcup \{h(\nu_0, .., \nu_{m-1}) \mid \exists \theta, \nu_0, .., \nu_{m-1} \ (\theta, \lambda_{i'}, \nu_0, .., \nu_{m-1}) \in Z\} + 1,$$

and $\lambda_i = \lambda_{i'} + 1$ otherwise.

Since C is a closed unbounded in κ and κ^* is a normal generator (i.e. generator of a normal measure) then $\kappa^* \in j^{\mathcal{F}in}(C)$ and therefore $(\kappa, \kappa^*, \delta_0, .., \delta_{m-1}) \in j^{\mathcal{F}in}(Z \restriction C)$ where $Z \restriction C = \{(\theta, \theta^*, \nu_0, .., \nu_{m-1}) \in Z \mid \theta^* \in C\}$.

Define $H : Z \restriction C :\to \kappa$ by $H(\theta, \theta^*, \nu_0, .., \nu_{m-1}) = h(\nu_0, .., \nu_{m-1})$. By the construction of C we get that for every $\nu < \kappa$, if $H^{-1}(\{\nu\})$ is not empty, then there exists a unique $\lambda \in C$, $\lambda < \nu$, such that $\lambda = \theta^*$ for every $(\theta, \theta^*, \nu_0, .., \nu_{m-1}) \in H^{-1}(\{\nu\})$. We then denote λ by $\lambda(\nu)$.

For every $p \in \mathbb{P}_\kappa$ let us construct a direct extension $p^* \geq p$ such that $j^{\mathcal{F}in}(p^*) \Vdash \check{\kappa} \neq \underset{\sim}{d}(\mu^*)$.

For every measurable cardinal $\nu < \kappa$ set (the name of) $X_\nu(p^*)$ to be

$$X_\nu(p^*) \quad = \quad \begin{cases} X_\nu(p) \backslash (\lambda(\nu)+1) & \text{if } H^{-1}(\{\nu\}) \neq \emptyset \\ X_\nu(p) & \text{otherwise} \end{cases}$$

The definitions of $Z{\restriction}C$ and p^* imply that for every $(\theta, \theta^*, \nu_0, .., \nu_{m-1}) \in Z{\restriction}C$ then $p^* \Vdash \check{\theta} \neq \underset{\sim}{d}(h(\nu_0, .., \nu_{m-1}))$. The claim follows as $(\kappa, \kappa^*, \delta_0, .., \delta_{m-1}) \in j^{\mathcal{F}in}(Z{\restriction}C)$ and $\mu^* = j^{\mathcal{F}in}(h)(\delta_0, .., \delta_{m-1})$.

□

Let $M^{\mathcal{F}in}$ be a finite sub-iteration of M. Denote by $i^{\mathcal{F}in} : M^{\mathcal{F}in} \to M$ the embedding of $M^{\mathcal{F}in}$ in M (i.e. $j = i^{\mathcal{F}in} \circ j^{\mathcal{F}in}$). For every $\mu \in M$, we say that μ is represented in $M^{\mathcal{F}in}$ if there is an ordinal $\mu^* \in M^{\mathcal{F}in}$ such that $i^{\mathcal{F}in}(\mu^*) = \mu$. Notice that for every $\mu < j(\kappa)$ in M, the fact that a representative of μ is a key generator of its finite sub-iteration, does not depend on a particular choice of a finite sub-iteration of M (as long as μ is represented there). We expand definition 3.1.

Definition 3.2. *We say that an ordinal $\mu < j(\kappa)$ is a* key generator *of M if and only if for every (some) finite sub-iteration $M^{\mathcal{F}in}$ in which μ is represented by some $\mu^* \in M^{\mathcal{F}in}$, then μ^* is a key-generator of $M^{\mathcal{F}in}$ in the sense of definition 3.1.*

Proposition 3.3. *If $\kappa \in j_W(\Gamma)$ in the ultrapower M_W (namely, $\Gamma \in W$) then $d^{-1}(\kappa) \in M$ is a key generator of M.*

Proof. First, note that κ cannot be a measurable cardinal in M as otherwise we would have to add a Prikry sequence to κ when forcing with $\mathbb{P}^M_{j(\kappa)}$, so $cf(\kappa)^{M[G_W]} = \omega$. This is impossible since $M[G_W] = M_W$ is an ultrapower of the generic extension $V[G_\kappa]$ where κ is measurable. Therefore, there must be a $\kappa-$measure U_0 which is included in the construction of M and does not concentrate on Δ.
Let μ is measurable cardinal in M which is not a key generator of M. Let $M^{\mathcal{F}in}$ be a finite sub-iteration which has representative $\mu^* \in M^{\mathcal{F}in}$ for μ. We can assume that μ^* is bigger then the largest key generator κ^* (since ultrapowers by extenders whose critical points are *above* μ^* are not relevant to μ^*) and that κ is not measurable in $M^{\mathcal{F}in}$ (as we can assume U_0 was included in this finite iteration). By the previous Proposition 3.2, there is a condition $p \in G_\kappa$ such that $j^{\mathcal{F}in}(p) \Vdash \check{\kappa} \neq \underset{\sim}{d}(\mu^*)$. Denote by $i^{\mathcal{F}in} : M^{\mathcal{F}in} \to M$ the embedding of $M^{\mathcal{F}in}$ into the direct limit M. Then $i^{\mathcal{L}im}(\kappa) = \kappa$ since κ is not measurable in $M^{\mathcal{F}in}$. By the elementarity of $i^{\mathcal{L}im}$ we get that $j(p) \Vdash \check{\kappa} \neq \underset{\sim}{d}(\mu)$. But $G_W = j_W(G_\kappa)$, hence $j(p) = j_W(p) \in G_W$ so $\kappa \neq d(\mu)$ in the generic extension $M[G_W] = M_W$. □

3.1.1 One Measure

In this section we establish first results regarding possible normal measures on κ in a generic extension $V^{\mathbb{P}_\kappa}$. These results are based on our assumptions of $\neg 0^{\P}$ and that the V is the core model.

Let U_0 be a measure on κ in the ground model V, which does not concentrate on Δ. In Proposition 2.1 we proved that $U_0^* = U_0^\times$ is a normal ultrafilter on κ in $V^{\mathbb{P}_\kappa}$. In this part of the section we show that U_0^* is the only normal measure on κ in $V^{\mathbb{P}_\kappa}$ which does not include $\Gamma = d``\Delta$. As a corollary we prove that in case κ has a unique normal measure in the ground model $V = \mathcal{K}$, then the same holds in a generic extension $V^{\mathbb{P}_\kappa}$.

Remark 3.4. *In the first part of proof 3.3 we established that whenever $G_\kappa \subset \mathbb{P}_\kappa$ is V generic and W is a measure on κ in $V[G_\kappa]$, then κ is not measurable in $M_W \cong Ult(V[G_\kappa], W)$. Hence $\Delta \notin W$. Since $V = \mathcal{K}$ is the core model of $V[G_\kappa]$ then $W \cap V$ is a normal ultrafilter on κ in V. This must be U_0 since in such suitable extender models there is at most one normal measure on κ which does not include Δ.*

Proposition 3.5. *U_0^* is the only normal measure on κ in $V[G_\kappa]$ which does not concentrate on Γ.*

Proof. Denote by $j_0 : V \to M_0 \cong Ult(V, U_0)$ be the corresponding elementary embedding. By lemma 1.5 for every $X \in U_0^*$ there exists some $p \in G_\kappa$ such that $j_0(p)^{-\kappa} \Vdash \check{\kappa} \in j_0(\underset{\sim}{X})$. It follows that $\Gamma \notin U_0^*$ since $j_0(p)^{-\kappa} \Vdash \underset{\sim}{d}^{-1}(\{\kappa\}) = \emptyset$.

Next, suppose W is a normal measure on κ in $V[G_\kappa]$ with $\Gamma \notin W$, and let us show that $U_0^* = W$. It is sufficient to verify that $U_0^* \subseteq W$. By remark 3.4 we know that $U_0 \subset W$. Suppose $X \in U_0^*$. Choose any $p \in G_\kappa$ such that $j_0(p)^{-\kappa} \Vdash \check{\kappa} \in j_0(\underset{\sim}{X})$ and define the following subset of κ,

$$B = \{\alpha < \kappa \mid p^{-\alpha} \Vdash \check{\alpha} \in \underset{\sim}{X}\}.$$

Clearly, $\kappa \in j_0(B)$, thus $B \in U_0$. In 2.2 we defined Σ, Π, subsets of κ in $V[G_\kappa]$, and proved that Σ is a club, $\Gamma \setminus \Pi$ is bounded in κ, and that for every $\alpha \in \Sigma \backslash \Gamma$ and $p \in G_\kappa$, then $p^{-\alpha} \in G_\kappa$ (see remark 2.6). Therefore if $\alpha \in B \cap (\Sigma \backslash \Gamma)$ then $p^{-\alpha}$ belongs to G_κ and forces "$\check{\alpha} \in \underset{\sim}{X}$". We conclude $(\Sigma \backslash \Gamma) \cap B \subseteq X$. It follows that $X \in W$ since Σ is closed unbounded in κ, $\Gamma \notin W$, and $B \in U_0 \subset W$. □

Corollary 3.6. *Assuming that $0^{\P}$ does not exist, and V is the core model in which κ has a unique normal measure U_0, then in a generic extension $V^{\mathbb{P}_\kappa}$, U_0^* is the only normal measure on κ.*

Proof. Note that since U_0 does not include Δ since it is the only normal measure on κ in V. By Proposition 3.5 we know that U_0^* is the only measure on κ in $V[G_\kappa]$ which does not include Γ. We claim there is no $W \in V[G_\kappa]$ which is normal measure on κ and $\Gamma \in W$. If there was such W, then $d^{-1}(\kappa) \neq \emptyset$. By Proposition 3.3 the ordinal $d^{-1}(\kappa)$ is a measurable cardinal in M which is a key generator. This is impossible because all key generators are obtained by iterating U_0 which not concentrate on the set of measurable cardinals, so all key generators are not measurables in M. □

3.2 Identifying all the Normal Measures on κ

Let W be a normal $\kappa-$complete ultrafilter on κ in $V[G_\kappa]$ such that $\Gamma \in W$. Denote by $j_W : V[G_\kappa] \to M_W$ the corresponding ultrapower and elementary embedding.

$\Gamma \in W$ implies $\Pi \in W$ which in turn, implies that in $M_W = M[G_W]$, $|d^{-1}(\{\kappa\})| = 1$. Denote $\mu = d^{-1}(\kappa)$. By Proposition 3.3, μ is a key generator of the iteration M. Let $j^{\mathcal{F}in} : V \to M^{\mathcal{F}in}$ be a finite sub-iteration of j which has a representative μ^* for μ. We may assume that $\mu^* = \kappa^*$ is the maximal key generator. We focus our attention in the following $\kappa-$complete ultrafilter of V.

Definition 3.3. *Define* $U_W = \{X \subseteq \kappa \mid \mu^* \in k^{\mathcal{F}in}(X)\}$.

Lemma 3.7. *U_W is a normal measure on κ.*

Proof. Let $i : V \to N \cong Ult(V, U_W)$ be the induced ultrapower embedding and let $f : \kappa \to \kappa$ be a function representing κ in this ultrapower. If the statement of the proposition is false then there exists such a function f which is regressive. We will use such f to establish a contradiction by showing that there is a condition $q \in G_\kappa$ such that $j(q) \Vdash \underset{\sim}{d}(\check{\mu}) > \check{\kappa}$. This is absurd as we assumed $(j)``G_\kappa \subseteq G_W$ and $d(\mu) = \kappa$ in $M[\widetilde{G}_W]$. Note that N can be factored from the iterated ultrapower $M^{\mathcal{F}in}$. The map $k : N \to M^{\mathcal{F}in}$ defined by $k([g]_{U_W}) = j^{\mathcal{F}in}(g)(\mu^*)$ is elementary and $j^{\mathcal{F}in} = k \circ i$.

For every $p \in \mathbb{P}_\kappa$ define a direct extension $p^{-f} \geq^* p$ by reducing the (name of) every measure one set $X_\nu(p)$ for all $\nu \in \Delta$ such that

$$p^{-f}\restriction_\nu \Vdash X_\nu(p^{-f}) = X_\nu(p) \setminus (f(\nu) + 1).$$

The definition of p^{-f} implies that there is a finite subset $b \subset \Delta$ such that for every $\nu \in \Delta \setminus b$, $p^{-f} \Vdash \underset{\sim}{d}(\check{\nu}) > f(\check{\nu})$. From this we conclude $i(p^{-f}) \Vdash \underset{\sim}{d}([id]_{U_W}) > \kappa$, and by applying k that $j^{\mathcal{F}in}(p^{-f}) \Vdash \underset{\sim}{d}(\mu^*) > k(\kappa) \geq \kappa$. The result follows by applying a standard density argument and lifting the forcing assertion to M via $i^{\mathcal{F}in}$. $\square$

We are now ready to prove the main result.

Proof. (Theorem 1.1) Suppose κ is measurable in V. As usual, we denote by U_0 the unique normal measure on κ which does not concentrate on Δ. By Proposition 3.5 we know that $U_0^\times = U_0^*$ is the unique normal measure on κ in $V[G_\kappa]$ which does not include Γ. It is therefore sufficient to consider normal measures on κ in $V^{\mathbb{P}_\kappa}$ which include Γ. First note that for every $U \neq U_0$, a normal measure on κ in V, then $\Delta \in U$ and therefore by Proposition 2.1 $\Gamma \in U^\times$. So $U^\times \neq U_0^\times$. Also, as $U \subset U^*$ and d is injective above some bounded set in κ then distinct normal measures U in V induce distinct normal measures $U^\times$ in the generic extension.

Finally, let $W \in V^{\mathbb{P}_\kappa}$ be a normal measure on κ with $\Gamma \in W$. We use the facts and notations established at the start of section 3.2 and Lemma 3.7. We claim that $W = U_W^\times$. It is sufficient to prove $U_W^\times \subset W$ which, by Theorem 2.11, amounts to proving that $d``A \in W$ for every $A \in U_W$ (as Σ is a club and $\Gamma \setminus \Pi$ is bounded below κ). By the definition of U_W, $\mu^* \in j^{\mathcal{F}in}(A)$ for every $A \in U_W$. Applying the direct limit map $i^{\mathcal{F}in}$ we find that $\mu \in j(A)$. The last implies $\kappa \in j_W(d"A)$ in $M_W = M[G_W]$, and hence $d"A \in W$. $\square$

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