On the powersets of singular cardinals in HOD

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Abstract

From the assumption that there is a measurable cardinal κ with $o(\kappa) = \kappa^+$, we produce a model in which for all $x \subseteq \aleph_\omega$, HOD_x does not contain the powerset of $\aleph_\omega$. We also prove that this assertion requires large cardinals.

1 Introduction

Shelah [6] proved that for every singular cardinal κ of uncountable cofinality there exists $x \subset \kappa$ such that $\mathcal{P}(\kappa) \subset \mathrm{HOD}_x$. In work by Cummings, Friedman, Magidor, Rinot, and Sinapova [2] it has been shown that the statement is consistently false for a singular κ of cofinality ω, starting from the large cardinal assumption of an inaccessible cardinal λ and an infinite sequence of $< \lambda$-supercompact cardinals below it. In fact they prove the stronger assertion that it is consistent that there is a singular cardinal κ of cofinality ω such that for all $x \subseteq \kappa$, $(\kappa^+)^{\mathrm{HOD}_x} < \kappa^+$.

We introduce a weak homogeneity notion and use it to reduce the large cardinal assumption for the failure of Shelah's theorem at a singular cardinal of cofinality ω. In particular, we prove the following theorem

Theorem 1. *Assuming there is a cardinal κ with $o(\kappa) = \kappa^+$, there is a forcing extension in which κ is singular of cofinality ω and for every subset x of κ, HOD_x does not contain the powerset of κ.*

We also show that κ can be made into $\aleph_\omega$. Our strategy is to use the extender based forcing originally formulated by Gitik and Magidor [3] and revisited by Merimovich [4]. This forcing is typically proper for many models of size κ. It follows that every subset of κ in the extension is captured by the generic extension of such a model. To establish the conclusion, we must construct nontrivial cone isomorphisms which fix the extension of a given model. Thus it makes sense to attempt to reduce the large cardinal assumption to $o(\kappa) = \kappa^+$.

In reducing the large cardinal assumption, we use of a "nice system of ultrafilters" similar to the ones defined in [3] in place of an extender. We use the iteration of Prikry forcing defined in [1] to move from the assumption $o(\kappa) = \kappa^+$ to a particular nice system of ultrafilters. This allows us to define the usual extender-based forcing and argue that it has enough homogeneity to establish the conclusion.

We conjecture the large cardinal in the Theorem is exact, but for now we are only able to prove the following theorem.

Theorem 2. *Let V be transitive model of set theory. If there exists a strong limit singular cardinal κ in V such that for set $x \in \mathcal{P}(\kappa)^V$, $\mathcal{P}(\kappa)^V \subset \mathrm{HOD}_x^V$, then V contains a sharp to a model with a measurable cardinal.*

2 Forcing preliminaries

We work in a model V and assume that there is a measurable cardinal κ with $o(\kappa) = \kappa^+$. Let $\langle U_{\alpha,\tau} \mid \alpha \leq \kappa, \tau < o(\alpha)\rangle$ be a coherent sequence of measures with $o(\kappa) = \kappa^+$ and $o(\alpha) < \alpha^+$ for all $\alpha < \kappa$. We commence by reviewing a forcing method which turns the Mitchell order increasing sequence $\langle U_{\kappa,\tau} \mid \tau < \kappa^+\rangle$ on κ into a Rudin-Keisler increasing sequence $E = \langle E_\tau \mid \tau < \kappa^+\rangle$, which forms a "nice system" in the sense of [3]. This system of measures will allow us to force with a version of the extender based Prikry forcing from [4].

To produce the Rudin-Keisler system of measures from $\vec{U}$, we iterate Prikry/Magidor forcing. Let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa\rangle$ be the nonstationary support iteration of Prikry forcings defined in [1], and let G be $\mathbb{P}_\kappa$-generic. For each $\alpha < \kappa$ with $o(\alpha) > 0$, we let $d_\alpha \subseteq \alpha$ be the club of ordertype $\omega^{o(\alpha)}$ added by G. As in [1], each measure $U_{\kappa,\tau}$ extends to a measure $U^*_{\kappa,\tau}(\emptyset)$ defined in $V[G]$ by $\dot{X} \in U^*_{\kappa,\tau}(\emptyset)$ if and only if there are $p \in G$ and a tree T such that

$$\langle \emptyset, T\rangle^\frown j_\tau(p) \restriction [\kappa+1, j(\kappa)) \Vdash_{j_\tau(\mathbb{P}_\kappa)/G} \kappa \in j_\tau(\dot{X})$$

where j_τ is the ultrapower map by $U_{\kappa,\tau}$.

We recall from [1] that $U^*_{\kappa,\tau}(\emptyset)$ is the unique measure extending $U_{\kappa,\tau}$ and concentrating on the sets $\{\nu < \kappa \mid p \restriction \nu \frown \langle \emptyset, T \restriction \nu\rangle \frown p \setminus \nu + 1 \in G\}$ for $p \in G$ and suitable trees T. By suitable, we mean a particular $\leq^*$-dense open set of trees in the forcing $\mathbb{Q}^\tau_\kappa$ restricted to conditions with empty stem.

We define $E_\tau = U^*_{\kappa,\tau}(\emptyset)$ for each $\tau < \kappa^+$. In V, let $\langle f_\rho \mid \rho < \kappa^+\rangle$ be the canonical sequence of functions from κ to κ. It is straightforward to see that in $V[G]$ if $\tau < \kappa^+$, then the set $X_\tau = \{\alpha < \kappa \mid o(\alpha) = f_\tau(\alpha)\} \in E_\tau$. For

$\tau_1 < \tau_2 < \kappa^+$, we define $\pi_{\tau_2,\tau_1} : X_{\tau_2} \to X_{\tau_1}$ by $\pi_{\tau_2,\tau_1}(\alpha)$ is the least $\beta \in d_\alpha$ such that $o(\beta) = f_{\tau_1}(\beta)$.

Claim 3. *For $\tau_1 < \tau_2 < \kappa^+$ and $Y \subseteq \kappa$, $Y \in E_{\tau_1}$ if and only if $\pi^{-1}_{\tau_2,\tau_1} Y \in E_{\tau_2}$.*

Proof. Fix some $Z \in E_{\tau_2}$ and $Y \in E_{\tau_1}$. It is enough to show that $Z \cap \pi^{-1}_{\tau_2,\tau_1} Y$ is nonempty. Suppose that there is $p \in G$ which forces that this intersection is empty. By strengthening p we can assume that for some trees T_1 and T_2,

$$p \restriction \kappa ^\frown \langle \emptyset, T_1 \rangle ^\frown j_{\tau_1}(p) \setminus \kappa + 1 \Vdash \kappa \in j_{\tau_1}(Y)$$

and

$$p \restriction \kappa ^\frown \langle \emptyset, T_2 \rangle ^\frown j_{\tau_2}(p) \setminus \kappa + 1 \Vdash \kappa \in j_{\tau_2}(Z).$$

For ease of notation we call the first condition above p_{τ_1} and the second condition p_{τ_2}. To understand the relationship between Y and Z we work in the iterated ultrapower. Let M^* be $\mathrm{Ult}(M_{\tau_1}, j_{\tau_1}(U_{\kappa,\tau_2}))$ and $j^* : V \to M^*$ be the iterated ultrapower map. Note that there are embeddings $k_1 : M_{\tau_1} \to M^*$ (the ultrapower map forming M^* inside M_{τ_1}) and $k_2 : M_{\tau_2} \to M^*$ which is just the ultrapower of M_{τ_2} by U_{κ,τ_1}.

It is straightforward to see that $k_2(p_{\tau_2})$ is compatible with $k_1(p_{\tau_1})$. Let q be the natural lower bound. Note that $k_2(\kappa)$ is in the support of q with an empty stem, since κ is in the support of p_{τ_2} with an empty stem. Further, $\kappa \in k_2(T_2)$ is a point of order τ_1 since k_2 is just the ultrapower map by U_{κ,τ_1} inside M_2. It follows that we may add κ to the stem of q on coordinate $k_2(\kappa)$ and restrict $k_2(T_2)$ to force that κ is the first point of order τ_1. We call the resulting condition q^*. By the choice of q^*, it forces that $j^*(\pi^{-1}_{\tau_2,\tau_1} Z) \cap j^*(Y) \neq \emptyset$, but $q^* \leq j^*(p)$. This is a contradiction. □

So our sequence of measures $\langle E_\tau \mid \tau < \kappa^+ \rangle$ is linearly ordered under the Rudin-Keisler ordering as witnessed by the maps π_{τ_2,τ_1} for $\tau_1 < \tau_2 < \kappa^+$. The following claim is standard where we can take the witnessing embedding to be the limit ultrapower of the E_τ for $\tau < \kappa^+$ (see [3]).

Claim 4. *In $V[G]$, there is an elementary embedding $j : V[G] \to M[H]$ satisfying:*

1. *${}^\kappa M[H] \subseteq M[H]$.*
2. *There is an increasing sequence of ordinals $\langle \gamma_\tau \mid \tau < \kappa^+ \rangle$ below $j(\kappa)$ such that $\gamma_0 = \kappa$, $o^M(\gamma_\tau) = \tau$ and $E_\tau = \{X \subseteq \kappa \mid \gamma_\tau \in j(X)\}$.*
3. *For $\tau_1 < \tau_2 < \kappa^+$, $\gamma_{\tau_1} = j(\pi_{\tau_2,\tau_1})(\gamma_{\tau_2})$.*

4. *For all $z \in M[H]$, there are a function $f : \kappa \to V[G]$ and $\tau < \kappa^+$ such that $z = j(f)(\gamma_\tau)$.*

Using this embedding we have the prerequisites to define the forcing from Merimovich's paper [4]. For completeness we include a sketch of the relevant notions. First we define the "Cohen parts".

Definition 5. Let $\mathbb{P}_0$ be the set of functions $f : \alpha + 1 \to [\kappa]^{<\omega}$ where $\kappa \leq \alpha < \kappa^+$ ordered by extension.

We adopt the convention that $[\kappa]^{<\omega}$ consists of finite increasing sequences of ordinals. Next we define the trees of measure one sets which restrict the growth of the Prikry sequences.

Definition 6. For $\tau < \kappa^+$, a set $T \subseteq [\kappa]^{<\omega}$ is an E_τ-tree if it is closed under initial segments and for each $\vec{\nu} \in T$, the collection $\mathrm{Suc}_T(\vec{\nu})$ of immediate successors of $\vec{\nu}$ is in E_τ.

Suppose that $\tau_1 < \tau_2 < \kappa^+$ and F is a function whose domain T is an E_{τ_1}-tree. We define $\pi^{-1}_{\tau_2,\tau_1} T$ be the natural E_{τ_2}-tree obtained by applying $\pi^{-1}_{\tau_2,\tau_1}$ pointwise to the elements of T. Define $\pi^{-1}_{\tau_2,\tau_1}(F)$ to be the function with domain $\pi^{-1}_{\tau_2,\tau_1} T$ which assigns to each $\vec{\nu}$, $F(\pi_{\tau_2,\tau_1} “\vec{\nu})$.

We are now ready to define the main forcing $\mathbb{P}$.

Definition 7. Let $\mathbb{P}$ be the set of pairs (f, F) where $f \in \mathbb{P}_0$ and setting $\mathrm{dom}(f) = \tau + 1$, F is a function from some E_τ-tree T to $[\kappa^+]^{<\kappa}$ satisfying the following properties:

1. For all $\vec{\nu} \in T$, $j\left(\rho \mapsto F(\vec{\nu}^\frown \rho)\right)(\gamma_\tau) = j“\tau$.
2. For all $\vec{\nu} \in T$, and $\rho \in F(\vec{\nu})$, $\rho < \tau$ and $f(\rho)^\frown \pi_{\tau,\rho}“\vec{\nu}$ is an increasing sequence.
3. For all $\vec{\nu} \in T$, $\kappa \in F(\vec{\nu})$.
4. For all $\vec{\nu} \in T$, $|F(\vec{\nu})| \leq \pi_{\tau,0}(\max(\vec{\nu}))$.
5. For all $\vec{\nu}, \vec{\nu}' \in T$ if $\vec{\nu'}$ extends $\vec{\nu}$, then $F(\vec{\nu}) \subseteq F(\vec{\nu}')$.

We note that conditions 2 through 5 occur "on a dense set" in the sense that if we have a pair (f, F) satisfying condition (1), then we can restrict the tree T to obtain conditions 2 through 4 and take unions of F along the branches of T to obtain condition (5). We use the following abbreviations.

For a condition $p = (f, F)$, we write $f^p = f$, $F^p = F$, $T^p = \mathrm{dom}(F)$ and $\mathrm{mc}(p)$ is the unique τ such that $\mathrm{dom}(f) = \tau + 1$.

Next we define the ordering on $\mathbb{P}$ starting with the direct extension order. For $p, q \in \mathbb{P}$, we define $p \leq^* q$ if

1. $f^p \supseteq f^q$,
2. $T^p \subseteq \pi^{-1}_{\mathrm{mc}(p),\mathrm{mc}(q)} T^q$
3. for all $\vec{\nu} \in T^p$, $F^p(\vec{\nu}) \cap \mathrm{mc}(q) = F^q(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}``\vec{\nu})$.
4. for all $\vec{\nu} \in T^p$, and all $\rho \in F^p(\vec{\nu}) \cap \mathrm{mc}(q)$,

$$\pi_{\mathrm{mc}(p),\rho}(\max(\vec{\nu})) = \pi_{\mathrm{mc}(q),\rho}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\max(\vec{\nu}))).$$

Next we define a one step extension. Let $p \in \mathbb{P}$ and $\langle \nu \rangle \in T^p$. We define a condition $p_{\langle \nu \rangle}$ by

1. $f^{p_{\langle \nu \rangle}}(\beta) = f^p(\rho)^\frown \langle \pi_{\mathrm{mc}(p),\beta}(\rho) \rangle$ for $\rho \in F^p(\langle \nu \rangle)$ and $f^{p_{\langle \nu \rangle}}(\rho) = f^p(\rho)$ otherwise.
2. $T^{p_{\langle \nu \rangle}} = \{\vec{\nu} \mid \nu^\frown \vec{\nu} \in T^p\}$ and for $\vec{\nu} \in T^{p_{\langle \nu \rangle}}$, $F^{p_{\langle \nu \rangle}}(\vec{\nu}) = F^p(\nu^\frown \vec{\nu})$.

We can define n-step extensions by iterating one step extensions. We define $p \leq q$ if p can be obtained from q by an alternating sequence of direct extensions and one point extensions. It is straightforward to check that if $p \leq q$, then there is a natural number n such that p is a direct extension of some n-step extension of q.

The proofs in [4] can be repeated to show:

1. $(\mathbb{P}, \leq, \leq^*)$ is a Prikry type forcing.
2. $(\mathbb{P}, \leq^*)$ is κ-closed.
3. For θ large enough, $(\mathbb{P}, \leq)$ is proper for stationarily many $N \prec H_\theta$ of size κ.
4. $(\mathbb{P}, \leq)$ has the κ^{++}-cc.

It follows that $\mathbb{P}$ preserves cardinals.

Remark 8. *The proof of condition (3) above goes extending some $p \in N$ to an $(N, (\mathbb{P}, \leq^*))$-master condition q. The construction of q is a standard argument using the closure of the $\leq^*$ ordering. A careful argument shows that q is also an $(N, \mathbb{P})$-master condition.*

A standard density argument shows that $\mathbb{P}$ adds κ^+ many ω sequences to κ, $\langle t_\tau \mid \tau < \kappa^+ \rangle$, where $t_\tau = \bigcup\{f^p(\tau) \mid p \in G, \tau \in \mathrm{dom}(f^p)\}$.

Finally we need one more fact about the measures E_τ.

Lemma 9. *For all $\rho < \tau < \kappa^+$ and every E_τ-measure one set A, there are distinct $\nu_1, \nu_2 \in A$ such that $\pi_{\tau,\rho}(\nu_1) = \pi_{\tau,\rho}(\nu_2)$.*

This is a straightforward genericity argument in the forcing $\mathbb{P}_\kappa$ used to define the measures. It will be used to generate a particular cone isomorphism in the final argument below.

3 Homogeneity

In this section we prove Theorem 1 by showing that the forcing from the previous section has a certain weak homogeneity property. We reinitialize the notation from the previous section and work in a model V with the forcing $\mathbb{P}$ based on our nice system of ultrafilters.

Definition 10.

1. For a condition $q \in \mathbb{P}$, let $\mathbb{P}/q$ denote the cone of conditions $p \in \mathbb{P}$ extending q.
2. For $q_1, q_2 \in \mathbb{P}$, a cone isomorphism of $\mathbb{P}/q_1, \mathbb{P}/q_2$ is an order preserving bijection $\sigma : \mathbb{P}/q_1 \to \mathbb{P}/q_2$.

Lemma 11. *Let $N \prec H_\theta$ be an elementary substructure of size κ and $q \in \mathbb{P}$ be an $(N, \mathbb{P})$-generic condition. Let λ be an ordinal with $\lambda + 1 \subset N$, and $\dot{y} \in N$ be a name for a subset of λ. Suppose that $q_1, q_2 \in \mathbb{P}$ extend q and $\sigma : \mathbb{P}/q_1 \to \mathbb{P}/q_2$ is a cone isomorphism with the property that for every $r \in N$ if q_1' is below both r and q_1, then $\sigma(q_1')$ is below r. If $G \subset \mathbb{P}$ is a generic filter containing q_1, then $\dot{y}_G = \dot{y}_{\sigma[G]}$.*

Proof. We may assume $\dot{y} \in N$ is a canonical name. Let us show that for every $\rho < \lambda$, $\rho \in \dot{y}_G$ if and only if $\rho \in \dot{y}_{\sigma[G]}$. Let $D_\rho = \{p \in \mathbb{P} \mid p \text{ deciding the statement } ``\check{\rho} \in \dot{y}"\}$. $D_\rho \in N$ since $\lambda \subset N$ and therefore there exists a condition $r \in D_\rho \cap G \cap N$. Let $q_1' \in G$ be a common extension of r and q_1, then $\sigma(q_1') \in \sigma[G]$ extends r as well. It follows that $\rho \in \dot{y}_G$ if and only if $\rho \in \dot{y}_{\sigma[G]}$. Finally, since $\dot{y}_G \subset \lambda \in N$ there exists $p \in N \cap G$ forcing $\dot{y} \subset \check{\lambda}$. Then $p \in N \cap \sigma[G]$ and therefore $\dot{y}_{\sigma[G]} \subset \lambda$. □

Definition 12. Let $\mathrm{HOD}^*(M)$ denote the class of all sets $t \in V$ which are hereditary ordinal definable from parameters $a \in \mathcal{P}(\alpha) \cap M$ for some $\alpha < \sup(M \cap |M|^+)$.

Theorem 13. *Let $N \prec H_\theta$ be an elementary substructure of size κ with $N \cap \kappa^+ \in \kappa^+$. If q is an $(N, \mathbb{P})$-generic condition. Then $q \Vdash \dot{t}_{\rho_N+1} \notin \mathrm{HOD}^*(N[\dot{G}])$.*

Proof. Let $\rho = M \cap \kappa^+$. It is sufficient to show that $t_{\rho+1} \notin \mathrm{HOD}^*(N[G])$ in every generic extension $V[G]$ where $G \subset \mathbb{P}$ contains q. Fix a generic extension $V[G]$ with $q \in G$ and suppose otherwise. Then there exists $a \in \mathcal{P}(\alpha) \cap N[G]$ for some $\alpha < \rho$, and formula $\phi(x, y)$ such that $t_{\rho+1} = \{\beta < \kappa \mid \phi(\beta, a)\}$.

Let $\dot{t}_{\rho+1} = \{\langle \check{\beta}, p\rangle \mid \beta \in f^p(\rho+1)\}$ be the canonical name for $t_{\rho+1}$ and let $\dot{a} \in N$ be a name for a such that for some $\bar{q} \in G$ extending q, $\bar{q} \Vdash \dot{t}_{\rho+1} = \{\beta < \check{\kappa} \mid \phi(\beta, \dot{a})\}$. Denote $f^{\bar{q}}, F^{\bar{q}}$ and $T^{\bar{q}}$ by f, F, T respectively.

We may assume $\rho+1 \in \mathrm{dom}(f)$. Using Lemma 9 there are $\langle \nu_1\rangle, \langle \nu_2\rangle \in T$ satisfying $\pi_{\mathrm{mc}(f),\rho+1}(\nu_1) \neq \pi_{\mathrm{mc}(f),\rho+1}(\nu_2)$, but $\pi_{\mathrm{mc}(f),\rho}(\nu_1) = \pi_{\mathrm{mc}(f),\rho}(\nu_2)$. For each $i \in \{1, 2\}$ let q_i by the direct extension of $\bar{q}_{\langle \nu_i\rangle}$ obtained by reducing the tree to contain only ordinals ν satisfying $\pi_{\mathrm{mc}(f),0}(\nu) > \max(\nu_1, \nu_2)^+$. Note that $\mathrm{dom}(f^{q_1}) = \mathrm{dom}(f^{q_2}) = \mathrm{dom}(f)$. Define a cone isomorphism $\sigma : \mathbb{P}/q_1 \to \mathbb{P}/q_2$ as follows. For a condition $w_1 \in \mathbb{P}/q_1$ let $\sigma(w_1)$ be the condition w_2 defined as follows

- $\mathrm{dom}(f^{w_2}) = \mathrm{dom}(f^{w_1})$,
- for every $\delta \in \mathrm{dom}(f^{w_2}) \setminus \mathrm{dom}(f^{q_1})$, $f^{w_2}(\delta) = f^{w_1}(\delta)$,
- for every $\delta \in \mathrm{dom}(f^{q_1})$, if $f^{w_1}(\delta) = f^{q_1}(\delta)^\frown s$ then $f^{w_2}(\delta) = f^{q_2}(\delta)^\frown s$,
- $F^{w_2} = F^{w_1}$.

It is routine to verify σ is well defined and order preserving map from $\mathbb{P}/q_1$ to $\mathbb{P}/q_2$. It is also easy to see that the function $\tau : \mathbb{P}/q_2 \to \mathbb{P}/q_1$ defined by swapping the roles of q_1 and q_2 in the definition of σ, is also order preserving and inverts σ. Moreover by the choice of ν_1 and ν_2 is straightforward to see that σ satisfies the hypotheses of Lemma 11. So by Lemma 11 applied to $\dot{a}$, $\dot{a}_G = \dot{a}_{\sigma[G]}$.

Finally, it is clear from the definition of σ that $\bar{q}$ is in $\sigma[G]$. Then since $\bar{q} \Vdash \dot{t}_{\rho+1} = \{\beta < \check{\kappa} \mid \phi(\beta, \dot{a})\}$, we have $(\dot{t}_{\rho+1})_{\sigma[G]} = \{\beta < \check{\kappa} \mid \phi(\beta, \dot{a}_{\sigma[G]})\} = \{\beta < \check{\kappa} \mid \phi(\beta, \dot{a}_G)\} = (\dot{t}_{\rho+1})_G$. This is absurd as $\pi_{\mathrm{mc}(f),\rho+1}(\nu_1) \in (\dot{t}_{\rho+1})_G \setminus (\dot{t}_{\rho+1})_{\sigma[G]}$. □

Corollary 14. *Let $G \subset \mathbb{P}$ be a generic filter and $x \in V[G]$ be a subset of κ. Then there exists $\delta < \kappa^+$ such that $t_\delta \notin \mathrm{HOD}_x^{V[G]}$.*

We now give a brief description of how to bring the result down to $\aleph_\omega$. We add collapses between the cardinals in $f(0)$ as well as a constraining function restricting the collapse conditions which are allowed to come next when we add ordinals to $f(0)$.

We start by making a few conventions. For a condition, $p \in \mathbb{P}$ we can associate an E_0 measure one set as follows. For $\vec{\nu} \in T^p$, let $A_{\vec{\nu}} = \pi_{\mathrm{mc}(p),0}``\mathrm{Suc}_T(\vec{\nu})$. We let A^p be the diagonal intersection of $A_{\vec{\nu}}$ for $\vec{\nu} \in T^p$. For a condition $p \in \mathbb{P}$, we let n^p be the size of $f^p(0)$ and we let $f^p(0) = \langle \kappa^p_0, \dots \kappa^p_{n^p-1}\rangle$.

We also note that since $2^\kappa = \kappa^+$, there is (in V) a $\mathrm{Ult}(V, E_0)$-generic filter K for the poset $\mathrm{Coll}(\kappa^{++}, < j(\kappa))$ as computed in $\mathrm{Ult}(V, E_0)$.

We define a forcing $\hat{\mathbb{P}}$ as follows. Conditions are of the form $\langle c, f, F, C\rangle$ where $\langle f, F\rangle$ is a condition $p \in \mathbb{P}$ where ω is the least element of $f^p(0)$ and

1. $c \in \prod_{i<n^p-1} \mathrm{Coll}((\kappa^p_i)^{++}, < \kappa^p_{i+1}) \times \mathrm{Coll}((\kappa^p_{n^p-1})^{++}, < \kappa)$,
2. C is function with domain A^p and for each $\lambda \in A^p$, $C(\lambda) \in \mathrm{Coll}(\lambda^{++}, < \kappa)$,
3. $\min(C) > \max(\mathrm{rng}(c(n^p - 1)))$ and
4. $[C]_{E_0} \in K$.

The definition of the direct extension ordering is straightforward. We can take a direct extension of the part of the condition in $\mathbb{P}$, can strengthen c and can strengthen C pointwise. Suppose that $\langle \nu\rangle \in T$, then we can adjoin $\langle \nu\rangle$ by adjoining it to the part of the condition in $\mathbb{P}$, adjoining $C(\pi_{mc(p),0}(\nu))$ to c and restricting C to the smaller E_0-measure one obtained from adjoining $\langle \nu\rangle$ to the condition in $\mathbb{P}$. A typical extension is just a combination of direct extensions and one point extensions.

All the previous claims remain valid here. In particular, the forcing $\hat{\mathbb{P}}$ is of Prikry-type and for large enough θ is proper for stationarily many elementary substructures of H_θ of size κ. This gives the prerequisites to repeat Theorem 13 in this new context. Lemma 11 can be used without modification, since it is independent of the forcing $\mathbb{P}$. The definition of the cone isomorphism σ is the same, but with the additional requirement that σ fixes both the c and C part of each condition.

4 A lower bound

We prove Theorem 2 to demonstrate that some large cardinal assumption above a measurable cardinal is needed to produce a model as in Theorem

1. This result can be easily generalized to get a sharp to a model with a measurable cardinal κ of a finite Mitchell order. We believe that the result of Theorem 1 is tight in the sense that the large cardinal assumption of $o(\kappa) = \kappa^+$ is needed for the consistency result.

Let us first recall Mitchell's covering Lemma for one measurable cardinal (see [5]).

Theorem 15 (Mitchell)**.** *Assume that $0^\dagger$ does not exist in V then there exists an ordinal definable class $K \subset V$ such that one of the following two statements holds for every set x of ordinals in V.*

1. *There is a set $y \in K$ with $y \supset x$ and $|y| = |x| + \aleph_1$.*
2. *K contains a measure U and there is a set $C \subset \kappa$, which is Prikry generic for U over K and is maximal up to finite initial segments, such that for some $y \in K[C]$, $y \supset x$ and $|y| = |x| + \aleph_1^V$. Moreover, U is unique and C is unique up to finitely many ordinals.*

Proof of Theorem 2. Suppose that κ is a singular cardinal in V so that $\mathcal{P}(\kappa)^V$ is not contained in HOD_X^V for any $X \subset \kappa$. By Shelah's Theorem, we may assume $\mathrm{cf}(\kappa) = \omega$. Let $\langle \kappa_n \mid n < \omega\rangle$ be an increasing and cofinal sequence in κ, and for each $n < \omega$ let $x^n = \langle x_i^n \mid i < \lambda_n\rangle$ be an injective enumeration of $\mathcal{P}(\kappa_n)^V$. Since κ is strong limit, $\lambda_n < \kappa$ for every $n < \omega$. Let $\vec{x} = \langle x^n \mid n < \omega\rangle$, and for every $Y \subset \kappa$ define an ω sequence $\vec{i}(Y) = \langle i_n(Y) \mid n < \omega\rangle$ to be the unique sequence in the product $\prod_n \lambda_n$ so that $Y \cap \kappa_n = x^n_{i_n(Y)}$ for all $n < \omega$. Clearly, Y is definable from $\vec{x}$ and $\vec{i}(Y)$.

Suppose $0^\dagger$ does not exist, then by the covering Lemma and the fact κ is strong limit, there exists a subset $X \subset \kappa$ which codes $\vec{x}$, $\mathcal{P}(\omega_1)^V$, and a maximal Prikry sequence C over K, if there is one in V. Since K is an ordinal definable class, we have that $K[\vec{x}, \mathcal{P}(\omega_1)^V, C] \subset \mathrm{HOD}_X$ (or $K[\vec{x}, \mathcal{P}(\omega_1)^V] \subset \mathrm{HOD}_X$ if C does not exist). Moreover, if $C' \in V$ is another Prikry sequence over K then C' is almost contained in C (i.e., up to finitely many ordinals), and since HOD_X^V contains all finite sequences of ordinals and all subsets of ω in V, C' must also be a member of HOD_X^V.

We derive a contradiction by showing $\mathcal{P}(\kappa)^V \subset \mathrm{HOD}_X^V$. By our choice of $\vec{x}$, which is coded by X, it is sufficient to show $[\kappa]^\omega \subset \mathrm{HOD}_X^V$. To this end, let y be a countable subset of κ in V. By the covering lemma there exists a set $x \supset y$ of size $|x| \leq \aleph_1$ such that either $x \in K$ or $x \in K[C']$ for some Prikry sequence C'. Either way, we get that $x \in \mathrm{HOD}_X^V$. Now, let $\langle \rho_\nu \mid \nu < |x|\rangle$ be an increasing enumeration of x and $a = \{\nu < |x| \mid \rho_\nu \in y\}$. Then y is definable from x and a, but $a \in \mathcal{P}(\omega_1)^V$ and therefore belongs to

HOD_X^V. Hence $y \in \mathrm{HOD}_X^V$ for all $y \in [\kappa]^\omega$, which contradicts our assumption that $0^\dagger$ does not exist. □

References

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