HOMOGENEOUS CHANGES IN COFINALITIES WITH APPLICATIONS TO HOD

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Abstract. We present a new technique for changing the cofinality of large cardinals using homogeneous forcing. As an application we show that many singular cardinals in V can be measurable in HOD. We also answer a related question of Cummings, Friedman and Golshani by producing a model in which every regular uncountable cardinal θ in V is θ^+-supercompact in HOD.

1. Introduction

Canonical inner models and covering lemmas play an important role in set theory. The first examples are Gödel's constructible universe L and Jensen's covering lemma. Jensen's covering lemma states that either $0^\#$ exists or every uncountable set of ordinals in V is contained in a set of ordinals in L of the same size. In a very general setting, we can think of a definable inner model M (canonical or not) as a subclass of the hereditarily ordinal definable sets HOD and a covering lemma for M as measuring "how close" V is to M. For example with Jensen's covering lemma, if $0^\#$ exists, then every V cardinal is inaccessible in L. So in this case V is very far from L. On the other hand if $0^\sharp$ does not exist, then the second part of the theorem asserts that V is close to L. Covering lemmas for L and other canonical inner models are essential to proving lower bounds on consistency strength.

We are interested in the extent to which V can be far from HOD, since if V is far from HOD, then it is also far from any definable inner model. We measure this by proving consistency results where many cardinals in V are large cardinals in HOD. The relevant covering lemma is due to Woodin [12]:

Theorem 1.1 (HOD Dichotomy). *If δ is an extendible cardinal then one of the following holds.*

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(1) *Every singular cardinal $\alpha > \delta$ (in V) is singular in* HOD *and* $(\alpha^+)^{\mathrm{HOD}} = \alpha^+$.
(2) *Every regular cardinal above δ is ω-strongly measurable in* HOD.

Moreover he has conjectured that

Conjecture 1.2 (HOD conjecture)**.** *There is a proper class of regular uncountable cardinals α which are not ω-strongly measurable in* HOD.

ω-strong measurability is a very strong form of measurability which entails that for some stationary set $S \in \mathrm{HOD}$, $S \subset \alpha \cap \mathrm{Cof}(\omega)$, the club filter restricted to S forms an ultrafilter on $\mathcal{P}(S) \cap \mathrm{HOD}$. The main results of this paper show that V can be quite far from HOD, but they are still very far from making progress on the HOD conjecture.

A recent theorem in this area is due to Cummings, Friedman and Golshani [2] who proved that from large cardinals it is consistent that for every cardinal γ, $(\gamma^+)^{\mathrm{HOD}} < \gamma^+$. So in this model V is far from HOD in the sense that it does not compute any successor cardinal correctly. The second main theorem (Theorem 1.4) of the paper is an improvement of this result.

An unexplored aspect of this area is whether cardinals which are *singular* in V can be inaccessible (or larger) in HOD. In this direction there are an obvious limitation and an obvious example. The limitation is that certain singular cardinals in V must be singular in HOD. For example $\aleph_\omega$ is singular in HOD, since $\{\aleph_n \mid n < \omega\}$ is in HOD. Pushing this further, we see that any definable class club in HOD contains a singular cardinal in V which is also singular in HOD. This remark will show that our first main theorem (Theorem 1.3) cannot be improved in a certain way.

The obvious example of a cardinal which is singular in V, but inaccessible in HOD is a singular cardinal that is the result of Prikry forcing [11]. Suppose that a cardinal κ is measurable and $V = \mathrm{HOD}$. If G is Prikry generic over V then in $V[G]$ κ is singular of cofinality ω but remains inaccessible (even measurable) in $\mathrm{HOD}^{V[G]}$. The key point of the argument that κ remains inaccessible in HOD is that Prikry forcing is homogeneous.

There are clear obstacles to similar results for uncountable cofinalities. In particular the relevant Magidor forcing [8] is not homogeneous and adds an ordinal definable club. We overcome this obstacle and produce a homogeneous forcing which changes the cofinality of some large cardinal to some prescribed regular cardinal. The forcing is based on Gitik's technique [4] for iterating Prikry/Magidor forcings. A careful analysis of the Gitik iteration suggests that it is not homogeneous. In particular the extended measures which are used to construct the forcing trees are highly sensitive to automorphisms since their construction is based on a choice of master sequences. To overcome this we define an iteration of Prikry forcing with *nonstationary* support. Using this technique we prove:

Theorem 1.3. *Suppose that κ is Mahlo and for every $\tau < \kappa$ the set $\Delta_\tau = \{\alpha < \kappa \mid o(\alpha) = \tau\}$ is stationary in κ. There is a forcing extension in which κ remains inaccessible and there is a club $C \subseteq \kappa$ of cardinals which are singular in V and measurable in* HOD.

As mentioned in Remark 4.7 following the proof of Theorem 1.3, the large cardinal property of the ordinals $\alpha \in C$ can be strengthened to any large cardinal property. Theorem 1.3 improves a result of Gitik in [5] where it is shown that from the same large cardinal assumptions there is a cardinal preserving generic extension $V[G]$ which contains a closed unbounded set $C \subset \kappa$ consisting of regular cardinals in the ground model V. Note that by our remarks about ordinal definable class clubs, this result cannot be improved to make C ordinal definable.

We also present an improvement of the Cummings-Friedman-Golshani result:

Theorem 1.4. *Suppose that $\kappa < \theta$ are cardinals where κ is 2^θ-supercompact where $\theta > \kappa$ is the least cardinal which is θ^+ supercompact. There is a generic extension W in which the rank initial segment W_κ of W is a model of* ZFC *and every regular uncountable cardinal θ in W_κ is θ^+ supercompact in* HOD^{W_κ}.

The large property in Theorem 1.4 can be replaced with any local large cardinal property such as huge cardinal. In February 2015, Gitik presented a result where all regular cardinals in V are measurable in HOD. The result appears in [7] and can be used to obtain a result similar to Theorem 1.4. The model constructed in [7] is based on the supercompact extender based Radin forcing which was introduced in [9]. The method used in our paper is not based on an extender based Prikry type forcing. It is obtained by comparing the Radin forcings by certain measures before and after collapsing many supercompact cardinals to become the successors typical points on the Radin generic closed unbounded set.

The paper is organized as follows. In Section 2, we outline a method for iterating Prikry type forcings with nonstationary support. In Section 3, we apply the method from the previous section to the model for

Theorem 1.3. In Section 4, we prove that our iteration of Prikry forcings is homogeneous and finish the proof of Theorem 1.3. In Section 5, we give the proof of Theorem 1.4.

2. NonStationary Support Iteration of Prikry type forcings

In this section we give the basic setup for our iteration of Prikry forcing. We define an iteration of Prikry forcings $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \lambda \rangle$ where each $p \in \mathbb{P}_\alpha$ is a partial function with domain $s \subseteq \alpha$ where s satisfies that for every inaccessible $\beta \leq \alpha$, $s \cap \beta$ is nonstationary in β. For $p \in \mathbb{P}_\alpha$, we write $s(p)$ for the domain of p.

For every $\beta < \alpha$ we require that $0_{\mathbb{P}_\beta} \Vdash (\mathbb{Q}_\beta, \leq_\beta, \leq^*_\beta)$ is a Prikry type forcing notion with $|\mathbb{Q}_\beta| \leq 2^\beta$, and $\leq^*_\beta$ is $< \beta$-directed closed. As usual, we require that for every $\beta \in s(p)$, $p \restriction \beta \in \mathbb{P}_\beta$ and that $p \restriction \beta \Vdash p(\beta) \in \mathbb{Q}_\beta$.

If $p, q \in \mathbb{P}_\alpha$ then $p \geq q$ (p extends q) if $s(q) \subset s(p)$ and for every $\gamma \in s(q)$, $p \restriction \gamma \Vdash p(\gamma) \geq_\gamma q(\gamma)$ and there is a finite set $b \subset s(q)$ so that for every $\gamma \in s(q) \setminus b$, $p \restriction \gamma \Vdash p(\gamma) \geq^*_\gamma q(\gamma)$. If $b = \emptyset$ then we say that p is a direct extension of q, denoted by $p \geq^* q$.

Lastly for $\alpha < \beta < \lambda$, we let $\mathbb{P}_\beta \setminus \alpha$ be the natural $\mathbb{P}_\alpha$-name for the quotient forcing needed to pass to the extension by $\mathbb{P}_\beta$.

Remark 2.1. *If $\dot{p}$ is a name for an element of $\mathbb{P}_\beta \setminus \alpha$, then we can find a direct extension of $\dot{p}$ which is in $\mathbb{P}_\beta \setminus \alpha$ whose domain is in V. Let s be the union of possible values for $s(\dot{p}) \cap (\alpha, \beta)$. It is easy to check that $s \cap \gamma$ is nonstationary for every inaccessible γ. For each $\gamma \in s$, let $p^*(\gamma)$ be a name for an element of $\dot{\mathbb{Q}}_\gamma$ which is $\dot{p}(\gamma)$ if γ is forced into $s(\dot{p})$ and $0_{\dot{\mathbb{Q}}_\gamma}$ otherwise. It follows that p^* is the desired direct extension.*

Under GCH each $\mathbb{P}_\alpha$ for $\alpha \leq \lambda$ has cardinality α^+ and so preserves cardinals above α^+. Next we establish the Prikry condition for $\mathbb{P}$, which is key in showing that cardinals up to λ are preserved by $\mathbb{P}$.

Lemma 2.2. $(\mathbb{P}, \leq, \leq^*)$ *is a Prikry type forcing notion.*

Proof. We prove by induction that $\mathbb{P}_\alpha$ has the Prikry property for all $\alpha \leq \lambda$. For the base case there is nothing to prove. The successor step is standard for iterations of Prikry type forcing. So we are left with the limit step. Suppose that $\delta \leq \lambda$ is a limit ordinal and that for all $\alpha < \delta$, $\mathbb{P}_\alpha$ has the Prikry property. Note that it is immediate from the induction hypothesis that for any $\nu' < \nu < \delta$ the quotient $\mathbb{P}_\nu \setminus \nu'$ has the Prikry property in the extension by $\mathbb{P}_{\nu'}$. We will make use of this below.

We distinguish between two main cases; the regular (inaccessible) case, and the singular case. When considering conditions $p \in \mathbb{P}_\delta$, there is an additional restriction on the support of p when δ is inaccessible. If δ is singular, then there is no additional restriction and $\mathbb{P}_\delta$ is the inverse limit of $\mathbb{P}_\alpha$ for $\alpha < \delta$. We proceed by describing in detail the argument in the inaccessible (more involved) case, and explain at the end how to modify the construction to also deal with the singular case.

Suppose that δ is an inaccessible cardinal. Let $p \in \mathbb{P}_\delta$ and let σ be a statement in the forcing language for $\mathbb{P}_\delta$. We prove that there is a direct extension of p deciding σ. The argument goes in two steps. In the first step we construct three sequences, a $\leq^*$-increasing sequence of conditions p_i in $\mathbb{P}_\delta$, a sequence of ordinals ν_i and sequence of clubs C_i with the following properties.

(1) The sequence ν_i is increasing and continuous.
(2) Each C_i is club in δ.
(3) For all i, $s(p_i) \cap C_i = \emptyset$.
(4) For all $i < j$, $p_i \restriction \nu_i + 1 = p_j \restriction \nu_i + 1$.

We set $p_0 = p$ and $\nu_0 = 0$ and take C_0 to be a club disjoint from the support of p. For the successor step assume that p_i, ν_i and C_i have been defined for some i. We let $\nu_{i+1} = \min(C_i \setminus \nu_i + 1)$. We construct p_{i+1} by specifying its restrictions. Let $p_{i+1} \restriction [\nu_{i+1}, \delta)$ be a name for a direct extension of $p_i \restriction [\nu_{i+1}, \delta)$ which decides σ if one exists in the extension by $\mathbb{P}_{\nu_{i+1}}$. If there is no such direct extension we can assume that $p_{i+1} \restriction [\nu_{i+1}, \delta)$ evaluates to be a direct extension of $p_i \restriction [\nu_{i+1}, \delta)$ where every coordinate added to the support is trivial. This uses Remark 2.1. To complete the successor step, we let $p_{i+1} \restriction \nu_{i+1} = p_i \restriction \nu_{i+1}$ and take $C_{i+1} \subset C_i$ to be club in δ which is disjoint from the support of p_{i+1}.

Let $j \leq \delta$ be a limit ordinal. Let $\nu_j = \sup_{i<j} \nu_i$ and C_j be the club of limit points of $\triangle_{i<j} C_i$. We define p_j by specifying its restrictions. Note that for all $i < j$, $\nu_j \notin s(p_i)$. We set $p_j \restriction \nu_j + 1 = \bigcup_{i<j} p_i \restriction \nu_i + 1$. By our construction we have that $p_j \restriction \nu_j + 1$ is a direct extension of $p_i \restriction \nu_j + 1$ for all $i < j$. Further, $p_j \restriction \nu_j + 1$ forces that $\langle p_i \restriction [\nu_j + 1, \delta) \mid i < j \rangle$ is an increasing sequence in the direct extension ordering on $\mathbb{P}_\delta \restriction [\nu_j + 1, \delta)$. Hence the sequence is forced to have an upperbound, since the relevant poset is forced to be $|\nu_j|^+$-closed. We let $p_j \restriction [\nu_j + 1, \delta)$ be such an upperbound. This completes the limit step. Since our construction works with when $\nu_j = \delta$, we have a condition $q = p_\delta$ which is a direct extension of each p_i.

The main feature of this construction and its resulting condition q is that for all $q^* \geq q$ if q^* decides σ and η is a limit point of the club $C = \langle \nu_i \mid i < \delta \rangle$ which is above the finite set of nondirect extensions in $s(q)$, then $q^* \restriction \eta \cup q \setminus \eta$ decides σ. In fact there is some $\nu_i < \eta$ such that $q^* \restriction \nu_{i+1} \cup q \setminus \nu_{i+1}$.

Let C' be the club of limit points of C. Note that $C' \cap s(q) = \emptyset$. Let ν'_i for $i < \delta$ be an increasing continuous enumeration of C'. For the second step, we construct a $\leq^*$-increasing sequence of conditions p'_i. We set $p'_0 = q$.

Suppose that we have constructed p'_i for some $i < \delta$. We specify p'_{i+1} by its restrictions. Let $p'_{i+1} \restriction [\nu'_{i+1}, \delta) = p'_i \restriction [\nu'_{i+1}, \delta)$ and $p'_{i+1} \restriction \nu'_i + 1 = p'_i \restriction \nu'_i + 1$. Let $p'_{i+1} \restriction (\nu'_i, \nu'_{i+1})$ be a name for a direct extension of $p'_i \restriction (\nu'_i, \nu'_{i+1})$ which decides the statement "$q \restriction [\nu'_{i+1}, \delta)$ decides σ" and if it decides positively also decides whether it forces σ or $\neg\sigma$. Again we use Remark 2.1 to obtain a condition with domain in V. At the limit steps we can take upperbounds using the same method as the first step of the construction. Let $q' = p'_\delta$.

We claim that there is a direct extension of q' which decides φ. We let $q^* \geq q'$ decide σ where the size of the set $b \subseteq s(q')$ with a nondirect extension is minimal. We can assume without loss of generality it forces σ. We assume for a contradiction that $b \neq \emptyset$. We let i be largest such that $[\nu'_i, \nu'_{i+1}) \cap b$ is nonempty. Note that $\nu'_i \notin b$ since ν'_i does not belong to the support of q'. By the first step of the construction $q^* \restriction \nu'_{i+1}$ forces that $p_{i+1} \restriction [\nu'_{i+1}, \delta)$ (and hence $q' \restriction [\nu'_{i+1}, \delta)$) decides σ. By the second step of the construction, we have that $q^* \restriction \nu'_i + 1$ forces that $p'_{i+1} \restriction (\nu'_i, \nu'_{i+1})$ (hence $q' \restriction (\nu'_i, \nu'_{i+1})$) forces that $q' \restriction [\nu'_{i+1}, \delta)$ forces σ . Translating this to a statement about $\mathbb{P}_\delta$, we have that $q^* \restriction \nu'_i + 1 \frown q' \restriction (\nu'_i, \delta)$ forces σ. This contradicts the minimality of the size of b in the choice of q^*.

Finally, let us explain how to adjust the construction for the case when δ is singular. Since δ is singular, $\mathbb{P}_\delta$ is the inverse limit of $\langle \mathbb{P}_\gamma \mid \gamma < \delta \rangle$. The situation is similar to the Magidor iteration of Prikry type forcing notions[1]. Suppose $\mathrm{cf}(\delta) = \rho < \delta$ and let $\langle \nu_i \mid i < \rho \rangle$ be a continuous increasing enumeration of a club C in δ such that $\nu_0 > \rho$. It follows that for each limit $j < \rho$, $\nu_j > j$ is singular as well. We may also assume $C \subset \mathrm{s}(p)$. We modify the first part of the construction above to define a $\leq^*$-increasing sequence $\langle p_i \mid i \leq \rho \rangle$ above $p_0 = p$ so that $p_i \restriction \nu_i + 1 = p_j \restriction \nu_i + 1$ for all $i < j < \rho$. The definition of p_i for both the successor and limit steps is identical to the above, and the only potential issue in the current construction is that ν_j can be a member of $\mathrm{s}(p_i)$ for all $i < j$. Nevertheless, this is not a problem since the direct extension order of $\mathbb{Q}_{\nu_j}$ is $< \nu_j$-closed and $j < \nu_j$. In the second part

[1]See [6] for survey of the Magidor iteration and its applications.

of the construction, we can take $\nu'_i = \nu_i$ for each $i < \rho$. The definition of the conditions p'_i is similar to above, except when constructing p'_{i+1} from p'_i, we take a direct extension of p'_i in the interval $[\nu'_i, \nu'_{i+1})$ as opposed to (ν'_i, ν'_{i+1}) before. Finally, a straightforward modification in the last part of the argument above shows that q' has a direct extension which decides σ. □

An argument similar to the one in the first step of the previous proof can be used to show the following.

Lemma 2.3. *Let δ be a limit ordinal and e be a function on δ such that for all $\eta \in \mathrm{dom}(e)$, $e(\eta)$ is a $\mathbb{P}_{\eta+1}$-name for a dense subset of $\mathbb{P}_\delta \setminus \eta+1$ with the $\leq^*$ ordering. For all $p \in \mathbb{P}_\delta$ and $\nu < \delta$, there are $p^* \geq^* p$ with $p^* \restriction \nu = p \restriction \nu$ and a club C in δ such that for all η in C, $p^* \restriction \eta+1$ forces that $p^* \restriction [\eta+1, \delta) \in e(\eta)$.*

Lemma 2.4. *Suppose that δ is a limit ordinal and $D \subset \mathbb{P}_\delta$ is a dense open set. For all $p \in \mathbb{P}_\delta$ and $\nu < \delta$ there exists a direct extension $p^* \geq^* p$ so that $p^* \restriction \nu = p \restriction \nu$ and for every $q \in D$ extending p^* if $\gamma \in \mathrm{s}(p^*)$ is the maximal nondirect extension coordinate then $q \restriction \gamma+1 \frown p^* \setminus \gamma+1 \in D$.*

The second lemma is an immediate consequence of the first one as we can set $e(\eta)$ to be the canonical $\mathbb{P}_{\eta+1}$-name for the set of $\hat{p} \in \mathbb{P}_\delta \setminus \eta+1$ such that for some $\bar{p} \in \dot{G}_{\eta+1}$, $\bar{p} \frown \hat{p} \in D$ or for all $\bar{p} \in \dot{G}_{\eta+1}$ and $\hat{p}' \geq^* \hat{p}$, $\bar{p} \frown \hat{p}' \notin D$.

Remark 2.5. *Lemma 2.4 (applied with $\nu+1$ in place of ν) gives a $\mathbb{P}_{\nu+1}$-name for a dense open subset of $\mathbb{P}_\delta \setminus \nu+1$. This set is of the correct form to be used as $e(\nu)$ as in Lemma 2.3.*

Corollary 2.6. *Assume GCH. If each nontrivial $\dot{\mathbb{Q}}_\beta$ is forced to preserve cardinals, then $\mathbb{P}$ preserves cardinals.*

Proof. By induction on cardinals ν, we check that $\mathbb{P}$ does not collapse ν. Using the inductive assumption, GCH and the Prikry Lemma, it is straightforward to verify that $\mathbb{P}$ preserves ν for all ν which are not successor of an inaccessible cardinal $\leq \lambda$.

Suppose that $\nu = \delta^+$ for some inaccessible $\delta \leq \lambda$. Factor $\mathbb{P}$ into $\mathbb{P}_\delta * \mathbb{P} \setminus \delta$. Then $\mathbb{P} \setminus \delta = \mathbb{Q}_\delta * \mathbb{P} \setminus (\delta+1)$, where $\mathbb{Q}_\delta$ is a Prikry type forcing at stage δ preserving cardinals. Note that $\mathbb{P} \setminus (\delta+1)$ satisfies the Prikry Lemma and its direct extension order is $(2^\delta)^+$-closed. It follows that $\mathbb{P} \setminus \delta$ does not collapse δ^+.

It remains to show that $\mathbb{P}_\delta$ does not collapse δ^+. Let $\dot{f}$ be a $\mathbb{P}_\delta$-name for a function from δ to δ^+. Let $p \in \mathbb{P}_\delta$. By the remarks preceding the Corollary, we may apply Lemma 2.3 with respect to dense sets deciding the values of the names $\dot{\tau} = \dot{f}(\check{\alpha})$ for $\alpha < \delta$, to form a direct extension $p^* \geq^* p$ and a function $f^* : \delta \to [\delta^+]^\delta$ so that for every $\alpha < \delta$, $p^* \Vdash \dot{f}(\check{\alpha}) \in f^*(\alpha)$. Hence p^* forces that $\mathrm{rng}(\dot{f})$ is bounded in δ^+. □

3. Changing Cofinalities

With the basic setup in place, we show how to use it to change cofinalities. Let κ be Mahlo and let $\mathcal{U} = \langle U_{\alpha,\tau} \mid \alpha < \kappa, \tau < o^{\mathcal{U}}(\alpha)\rangle$ be a coherent sequence of normal measures. We assume that for each $\tau < \kappa$ the set of α with $o(\alpha) = \tau$ is stationary in κ and define an iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa\rangle$ of Prikry forcing notions with nonstationary support using the previous section.

By induction on α, we define $\mathbb{P}_\alpha$ and $\mathbb{Q}_\alpha$ so that the following holds:

(1) $\mathbb{Q}_\alpha$ is nontrivial if and only if $o(\alpha) > 0$.
(2) $(\mathbb{Q}_\alpha, \leq_\alpha, \leq^*_\alpha)$ is a Prikry type forcing notion and $\leq^*_\alpha$ is $< \alpha$-closed.
(3) A nontrivial $\mathbb{Q}_\alpha$ generic filter assigns a club $c_\alpha \subset \alpha$ of order-type $\omega^{o(\alpha)}$ which is Prikry/Magidor generic with respect to the measures $\langle U_{\alpha,\tau} \mid \tau < o(\alpha)\rangle$.

Suppose that $\alpha \leq \kappa$ and that $\mathbb{P}_\alpha$ has been defined. Let $G \subset \mathbb{P}_\alpha$ be a generic filter. Working in $V[G]$, we fix an ordinal $\rho < \alpha$ and say that a finite increasing sequence of ordinals $t = \langle \alpha_0, \ldots, \alpha_{k-1}\rangle \in [\alpha]^{<\omega}$ is ρ-coherent if for each $i < k$ the following hold:

(1) $o(\alpha_i) < \rho$.
(2) If $i < k-1$ and $o(\alpha_{i+1}) > o(\alpha_i)$ then $c_{\alpha_{i+1}}$ end extends c_{α_i}.
(3) If $i < k-1$ and $o(\alpha_{i+1}) \leq o(\alpha_i)$ then $\min(c_{\alpha_{i+1}}) > \alpha_i$.

We define $c_t = \bigcup_{i<k}(c_{\alpha_i} \cup \{\alpha_i\})$, and $\alpha_t = \alpha_{k-1} = \sup(c_t)$.

Suppose that t is a ρ-coherent sequence and $\rho' < \rho$. Let $n_{\rho'} \leq k$ be the minimal $n \leq k = |t|$ so that $o(\alpha_i) < \rho'$ for each $n \leq i < k$, and let $t \restriction \rho' = \langle \alpha_n, \ldots, \alpha_{k-1}\rangle$. Note that $t \restriction \rho'$ is ρ'-coherent and may be empty. We say that t is coherent if it is ρ-coherent for some ρ and that two coherent sequences t, t' are equivalent if $c_t = c_{t'}$. Note that $t \restriction \rho$ and $t' \restriction \rho$ are equivalent if t and t' are. A coherent sequence $t = \langle \alpha_0, \ldots, \alpha_{k-1}\rangle$ is maximal if the sequence $\langle o(\alpha_i) \mid i \leq k\rangle$ is weakly decreasing. It is easy to see that every coherent sequence is equivalent to a maximal coherent sequence.

To determine $\mathbb{Q}_\alpha$, we define posets $\mathbb{Q}^\tau_\alpha$ for $\tau \leq o(\alpha)$ by induction on τ and set $\mathbb{Q}_\alpha = \mathbb{Q}^{o(\alpha)}_\alpha$. For each $\tau \leq o(\alpha)$, forcing with $\mathbb{Q}^\tau_\alpha$ over $V[G]$ adds a generic club $c^\tau_\alpha \subset \alpha$ which is Prikry/Magidor generic for the sequence $\langle U_{\alpha,\gamma} \mid \gamma < \tau\rangle$.

The poset $\mathbb{Q}^\tau_\alpha$ will be defined using measures $U_{\alpha,\rho}(t)$ where $\rho < \tau$ and t is τ-coherent. The definition of these measures depends on the posets $\mathbb{Q}^\rho_\alpha$ for $\rho < \tau$. We postpone the definition of the measures until after the definition $\mathbb{Q}^\tau_\alpha$ and note for now that each $\rho < \tau$ and two τ-coherent sequences t, t', if t and t' are equivalent, then $U_{\alpha,\rho}(t) = U_{\alpha,\rho}(t')$, and this measure concentrates on the set of $\beta < \alpha$ such that $o(\beta) = \rho$, and $c_{t\restriction\rho}$ is an initial segment of the G-induced generic club c_β. Further, it will be clear from our definition of the measures that $U_{\alpha,\rho}(t) = U_{\alpha,\rho}(t \restriction \rho)$.

Remark 3.1. *Let t be a $\mathbb{P}_\gamma$-name for a coherent sequence. To define a $\mathbb{P}_\gamma$ extension of $U_{\gamma,\tau}$ of the form $U_{\gamma,\tau}(t)$ we need to be able to treat t as a $\mathbb{P}_\alpha$ name for many ordinals $\alpha < \gamma$. Since $\mathbb{P}_\gamma$ does not satisfy γ.c.c in general, t need not be $\mathbb{P}_\alpha$-name for any $\alpha < \gamma$. However, the fact $\mathbb{P}_\gamma$ satisfies γ^+-cc and that each V-ultrapower by $U_{\gamma,\tau}$, $\tau < o(\gamma)$, is closed under γ-sequences, allows us to represent t in each ultrapower. For each $\tau < o(\gamma)$ let $t_\tau : \gamma \to V^{\mathbb{P}_\gamma}$ be a function which represents t in the ultrapower by $U_{\gamma,\tau}$. We may assume $t_\tau(\alpha)$ is a $\mathbb{P}_\alpha$-name for a coherent sequence for every $\alpha < \gamma$. For notational simplicity we shall abuse this notation and write t instead of $t_{o(\alpha)}(\alpha)$ whenever a reflection of t to a cardinal α is needed.*

We proceed with our inductive definition of the posets $\mathbb{Q}^\tau_\alpha$ and the measures $U_{\alpha,\tau}(t)$. If $\tau = 0$ then $\mathbb{Q}^0_\alpha$ is the trivial forcing. Suppose that $0 < \tau \leq o(\alpha)$ and $\mathbb{Q}^\rho_\alpha, U_{\alpha,\rho}(t)$ have been defined for every $\rho < \tau$ and τ-coherent t, so that $U_{\alpha,\rho}(t) = U_{\alpha,\rho}(t \restriction \rho)$ for each $\rho < \tau$.

Conditions $q \in \mathbb{Q}^\tau_\alpha$ are of the form $\langle t, T\rangle$ where t is ρ-coherent for some $\rho < \tau$ and $T \subset [\alpha]^{<\omega}$ is a tree which satisfies the following conditions.

1. (1) $\emptyset \in T$ is the stem of T.
2. (2) For each $s \in T$, $t \frown s$ is τ-coherent and $\mathrm{Succ}_T(s) \in \bigcap\{U_{\alpha,\rho}(t \frown s) \mid \rho < \tau\}$, where $\mathrm{Succ}_T(s) = \{\beta \mid s \frown \beta \in T\}$.

We say that two conditions $q = \langle t, T\rangle$ and $q' = \langle t^*, T^*\rangle$ are *equivalent* if t and t^* are equivalent and $T = T^*$. We say that a condition $q^* = \langle t^*, T^*\rangle$ is a *direct extension* of $q = \langle t, T\rangle$, denoted by $q^* \geq^* q$, if $t = t^*$ and $T^* \subset T$. A condition q' is called a *one-point extension* of q if $q' = \langle t^\frown\langle\beta\rangle, T_{\langle\beta\rangle} \setminus V_{\beta+1}\rangle$ where $\beta \in \mathrm{Succ}_T(\emptyset)$ and $T_{\langle\beta\rangle} = \langle \vec{z} \mid \langle\langle\beta\rangle^\frown\vec{z} \in T\rangle$. For the ordering we set $q' \geq q$ if q' is equivalent to a condition obtained from q by a finite combination of one-point extensions and direct extensions. By work from [4], $\mathbb{Q}^\tau_\alpha$ has the Prikry Property.

It remains to define the ultrafilters $U_{\alpha,\rho}(t)$ for every τ-coherent sequence t and $\rho < \tau$. Let $j = j_{\alpha,\rho} : V \to M = M_{\alpha,\rho} \cong \mathrm{Ult}(V, U_{\alpha,\rho})$ be the ultrapower embedding. Note that $j_{\alpha,\rho}(\mathbb{P}_\alpha) \restriction \alpha = \mathbb{P}_\alpha$ and G is also

M-generic. Define $U_{\alpha,\rho}(t)$ as follows. Let $X \subset \alpha$ and $\dot{X}$, a $\mathbb{P}_\alpha$-name for X. $X \in U_{\alpha,\rho}(t)$ if and only if there is some $p \in G \subset \mathbb{P}_\alpha$ and a tree T so that

$$p^\frown \langle t \restriction \rho, T\rangle^\frown j_{\alpha,\rho}(p) \setminus (\alpha+1) \Vdash \alpha \in j_{\alpha,\rho}(\dot{X})$$

It is easy to see that $U_{\alpha,\rho}(t) = U_{\alpha,\rho}(t \restriction \rho)$, and that the measure contains the set of all $\beta < \alpha$ so that c_β end extends $c_{t\restriction\rho}$.

Lemma 3.2. *For every $\leq^*$ dense open set $E \subset j(\mathbb{P}_\alpha) \setminus (\alpha+1)$ there is $p \in G$ so that $j(p) \restriction (\alpha+1) \Vdash_{j(\mathbb{P}_\alpha)\restriction\alpha+1} j(p) \setminus \alpha+1 \in E$.*

Proof. Let $e : \alpha \to V$ be such that $E = j(e)(\alpha)$. We may assume that for every inaccessible $\nu < \alpha$, $e(\nu)$ is a $\mathbb{P}_{\nu+1}$-name for a dense open subset of $\mathbb{P}_\alpha \setminus (\nu+1)$. For each $p \in \mathbb{P}$, we can apply Lemma 2.3 with e as defined and $\nu = 0$ to obtain a direct extension p^* of p. It follows that the set of $p \in \mathbb{P}$ satisfying the conclusion of Lemma 2.3 with respect to e is dense. So let $p \in G$ be such a condition. It follows that $j(p) \restriction \alpha+1$ forces that $j(p) \setminus \alpha+1 \in j(e)(\alpha) = E$ as required. □

Proposition 3.3. *For each τ-coherent sequence t and $\rho < \tau$, $U_{\alpha,\rho}(t)$ is an α-complete ultrafilter on α in $V[G]$.*

Proof. It is routine to verify $U_{\alpha,\rho}(t)$ is a filter. To show it is an α-complete ultrafilter, it is sufficient to verify that for every $\delta < \alpha$, if $\vec{X} = \langle X_i \mid i < \check{\delta}\rangle$ is $\mathbb{P}_\alpha$-name for a partition of α, then there exists $i^* < \delta$ so that $(X_{i^*})_G \in U_{\alpha,\rho}(t)$. Let E be a name for a subset of $j_{\alpha,\rho}(\mathbb{P}_\alpha) \setminus (\alpha+1)$ consisting of all $r \in j_{\alpha,\rho}(\mathbb{P}_\alpha) \setminus (\alpha+1)$ so that for some $i < \delta$, $r \Vdash \dot{\alpha} \in j_{\alpha,\rho}(X_i)$. E is $\leq^*$ dense open since the direct extension order of $j_{\alpha,\rho}(\mathbb{P}_\alpha) \setminus (\alpha+1)$ is α-closed and $j_{\alpha,\rho}(\mathbb{P}_\alpha)$ satisfies the Prikry condition. Therefore there is $p \in G$ so that $j_{\alpha,\rho}(p) \restriction (\alpha+1) \Vdash j_{\alpha,\rho}(p) \setminus (\alpha+1) \in E$. Working in $M[G]$, this means that there is $p \in G$ so that $(j_{\alpha,\rho}(p))(\alpha) \Vdash j_{\alpha,\rho}(p) \setminus (\alpha+1) \in E$. $(j_{\alpha,\rho}(p))(\alpha) = 0_{\mathbb{Q}_\alpha}$ as $\mathrm{s}(p) \subset \alpha$ is nonstationary and by the induction hypothesis, $\mathbb{Q}^\rho_\alpha$ satisfies the Prikry condition. Therefore, there is a tree T and $i^* < \delta$ so that, in $M[G]$, $\langle t \restriction \rho, T\rangle \Vdash_{\mathbb{Q}^\rho_\alpha} j_{\alpha,\rho}(p) \setminus (\alpha+1) \Vdash \check{\alpha} \in j_{\alpha,\rho}(X_{i^*})$. Back in M there is some $q \in G$, $q \geq p$ forcing the last assertion. It follows that

$$q^\frown \langle t \restriction \rho, T\rangle^\frown j_{\alpha,\rho}(q) \setminus (\alpha+1) \Vdash \alpha \in j_{\alpha,\rho}(X_{i^*})$$

Hence $(X_{i^*})_G \in U_{\alpha,\rho}(t)$. □

Let $G \subset \mathbb{P}_\alpha$ be a generic filter and $t \in [\alpha]^{<\omega}$ is a coherent sequence in $V[G]$. For a tree $T \subset [\alpha]^{<\omega}$ and $\nu < \alpha$ let $T \restriction \nu$ denote $T \cap [\nu]^{<\omega}$. For $p \in G$ and T such that $\langle t, T\rangle \in \mathbb{Q}^\rho_\alpha$, let $A_t(p, T, G) = \{\nu < \kappa \mid p \restriction \nu \frown \langle t \restriction o(\nu), T \restriction \nu\rangle \frown p \setminus (\nu+1) \in G\}$. The following is an immediate consequence of the definition of the measures $U_{\alpha,\rho}(t)$.

Corollary 3.4. *For each every $\rho < o(\alpha)$, $U_{\alpha,\rho}(t)$ is generated by the collection $\{Y \cap A_t(p,T,G) \mid Y \in U_{\alpha,\rho}, p \in G, \langle t,T\rangle \in \mathbb{Q}^{\rho}_{\alpha}\}$.*

Remark 3.5. *We note that for every $\leq^*$ dense set of trees $E \subset \{T \subset [\alpha]^{<\omega} \mid \langle t,T\rangle \in \mathbb{Q}^{\rho}_{\alpha}\}$ then $U_{\alpha,\rho}(t)$ is generated by the slightly smaller collection $\{Y \cap A_t(p,S,G) \mid Y \in U_{\alpha,\rho}, p \in G, S \in E\}$.*

We end the section by showing that our iteration preserves stationary sets.

Lemma 3.6. *Let τ be a $\mathbb{P}_\delta$ name for an ordinal. For every condition $p \in \mathbb{P}_\delta$ there exists a direct extension $p' \geq^* p$ and a set of ordinals $x \in V$ with $|x| < \delta$ such that $p' \Vdash \tau \in \check{x}$.*

Proof. Let $D \subset \mathbb{P}_\delta$ be the dense open set of conditions $q \in \mathbb{P}_\delta$ for which there exists x_q in V with $|x_q| < \delta$ such that $q \Vdash \tau \in \check{x}_q$. Let $p^* \geq^* p$ be as in Lemma 2.4 with respect to D. We construct a direct extension $p' \geq^* p^*$ with $\mathrm{s}(p') = \mathrm{s}(p^*)$. Let $\gamma \in \mathrm{s}(p^*)$ and suppose we have constructed $p' \restriction \gamma$. Consider a generic filter $G_\gamma \subset \mathbb{P}_\gamma$. Working in $V[G_\gamma]$ we denote the condition $(p^*_\gamma)_{G_\gamma} \in \mathbb{Q}_\gamma$ by $\langle t,T\rangle$. Let Z_γ be the set of conditions $z \in \mathbb{Q}_\gamma$ such that $z \frown p^* \setminus \gamma+1 \Vdash \tau \in x(z)$ for some $x(z) \in V[G_\gamma]$ with $|x(z)| < \delta$.

Let $A \subset \mathrm{Succ}_T(\emptyset)$, be the set of all $\nu < \gamma$ such that $\langle t \frown \langle\nu\rangle, T_{\langle\nu\rangle}\rangle$ has a direct extension z_ν which belongs to Z_γ. If $A \in U_{\gamma,\rho}(t)$ for some $\rho < o(\gamma)$ then $\langle t,T\rangle$ has a direct extension $p^1_\gamma = \langle t,T^1\rangle$ such that the set $\{z_\nu \mid \nu \in A\}$ is predense above p^1_γ, and thus $p^1_\gamma \frown p^* \setminus \gamma+1 \Vdash \tau \in x^1$ where $x^1 = \bigcup_{\nu\in A} x(z_\nu)$. If A is not a member of $U_{\gamma,\rho}(t)$ for all $\rho < o(\tau)$ then $\langle t,T\rangle$ has a direct extension $\langle t,T^1\rangle$ such that for every $\nu \in \mathrm{Succ}_{T^1}(\emptyset)$, $\langle t \frown \langle\nu\rangle, T^1_{\langle\nu\rangle}\rangle$ does not have a direct extension in Z_γ. Continuing in this fashion, by applying a similar level-by-level construction, it is possible to construct direct extensions $p^n_\gamma = \langle t,T^n\rangle$, $n < \omega$, such that for each n, either $p^n_\gamma \in Z_\gamma$ or for every $s \in T_n$ of length $|s| \leq n$, the condition $\langle t \frown s, T^n_s\rangle$ does not have a direct extension in Z_γ. We finally set $T' = \bigcap_n T^n$. It follows that either $\langle t,T'\rangle$ belongs to Z_γ or has no extension in Z_γ. Back in V, let p'_γ be a $\mathbb{P}_\gamma$ name for the condition $\langle t,T'\rangle \in \mathbb{Q}_\gamma$. This concludes the construction of p'. Let $n < \omega$ be minimal such that there exists $q \geq p'$ and $x(q) \in V$ with $|x(q)| < \delta$, and $q \Vdash \tau \in \check{x(q)}$, and $|b(p',q)| = n$ where $b(p',q) = \{\gamma \in s(p') \mid q_\gamma \text{ is not a direct extension of } p'_\gamma\}$. It is sufficient to show $n = 0$. Suppose otherwise, let $q \geq p'$ be an extension as above with $|b(p',q)| = n$, and $\gamma = \max(b(p',q))$. Since $p' \geq^* p^*$, $q \restriction \gamma+1 \frown p^* \setminus \gamma+1 \Vdash \tau \in \check{x}$ for some $x \in V$ with $|\dot{x}| < \delta$. Consequently, $q \restriction \gamma \Vdash p'_\gamma \in Z_\gamma$. Hence, there is a $\mathbb{P}_\gamma$ name $\dot{x}'$ for a subset of δ of size less than δ such that $q \restriction \gamma \frown p' \setminus \gamma \Vdash \tau \in \dot{x}'$. Since $|\mathbb{P}_\gamma| < \delta$, there exists a set $x \in V$ with $|x| < \delta$ such that $q \restriction \gamma \Vdash \dot{x}' \subset x$. It follows that $q \restriction \gamma \frown p' \setminus \gamma \Vdash \tau \in \check{x}$, contradicting the minimality of $n > 0$. □

Corollary 3.7. *Let $\dot{C}$ be a $\mathbb{P}_\delta$ name for a club subset of δ. It is forced by $\mathbb{P}_\delta$ that there is a club subset D of δ in V such that $D \subseteq \dot{C}$.*

For each $\alpha < \delta$, let τ_α be a $\mathbb{P}_\delta$ name for the least ordinal in $\dot{C}$ above α. Combine the previous lemma and Lemma 2.3 to obtain a condition p^* such that for all $\alpha < \delta$ there is a set x_α of size less than δ such that p^* forces that $\tau_\alpha \in x_\alpha$. Let D be the set of closure points of the function α maps to $\sup(x_\alpha)$. Clearly p^* forces that $D \subseteq \dot{C}$.

It is immediate that $\mathbb{P}_\delta$ preserves stationary subsets of δ.

4. Homogeneity

The purpose of this section is to show that $\mathbb{P}_\delta$ is weakly homogeneous for every $\delta \leq \kappa$. Recall that a poset $\mathbb{Q}$ is weakly homogeneous if for every $q_1, q_2 \in \mathbb{Q}$, there is an automorphism π of $\mathbb{Q}$ such that $\pi(q_1)$ is compatible with q_2.

We start by showing that $\mathbb{Q}_\alpha$ is weakly homogeneous when $o(\alpha)$ is a successor ordinal. In this case $\mathbb{Q}_\alpha$ is isomorphic to a *tree Prikry forcing* as defined in [6] and this tree Prikry forcing is weakly homogeneous. Let $\mathbb{Q} = \mathbb{Q}(U)$ be the tree Prikry forcing via an α-complete ultrafilter on a measurable cardinal α. Recall that conditions in $\mathbb{Q}$ are of the form $\langle t, T\rangle$ where t is a finite increasing sequence of ordinals in α and $T \subset [\alpha \setminus \max(t)+1]^{<\omega}$ consists of finite increasing sequences of ordinals, $stem(T) = \emptyset$ and $\mathrm{Succ}_T(s) \in U$ for every $s \in T$. The general order and direct extension order are defined in the natural way.

Lemma 4.1. $\mathbb{Q}$ *is weakly homogeneous.*

Proof. Let $q = \langle t, T\rangle$ and $q' = \langle t', T'\rangle$ be two conditions in $\mathbb{Q}$. If $t = t'$ then q, q' are compatible. Suppose otherwise. Let $D \subset \mathbb{Q}$ be the $\leq^*$ dense open set of all conditions $\langle s, S\rangle$ so that $S \subset [\alpha \setminus (\max(t \cup t')+1)]^{<\omega}$. Define an automorphism π of $\mathbb{Q}$ with domain D as follows. For every $p = \langle s, S\rangle \in D$,

$$\pi(p) = \begin{cases} p & \text{if none of } t, t' \text{ is an initial segment of } s \\ \langle t' \frown r, S\rangle & \text{if } s = t \frown r \text{ for some finite sequence } r \\ \langle t \frown r, S\rangle & \text{if } s = t' \frown r \text{ for some } r. \end{cases}$$

It is straightforward to verify that this is an automorphism. □

Lemma 4.2. *For every V-measurable $\alpha < \kappa$ and $\mu < o(\alpha)$, $\mathbb{Q}_\alpha^{\mu+1}$ is weakly homogeneous.*

Proof. Let $\mathbb{Q} = \mathbb{Q}_\alpha^{\mu+1}$. By the previous lemma, it is sufficient to show that $\mathbb{Q}$ is isomorphic to the Prikry forcing $\mathbb{Q}(U)$ where $U = U_{\alpha,\mu}(\emptyset)$. Let Δ_μ denote the set of all β with $o(\beta) = \mu$ and $D \subset \mathbb{Q}$ be the dense set of conditions $q = \langle t, T\rangle$ where t is a finite sequence of cardinals $\beta \in \Delta_\mu$. Note that the set D is dense since every $\mu+1$-coherent u can be extended to such a t by adding a point of order μ on top and passing to an equivalent sequence which removes points of order less than μ.

For every $q = \langle t, T\rangle \in D$ let $\bar{q} = \langle t, \bar{T}\rangle$ where $\bar{T} = T \cap [\Delta_\mu]^{<\omega}$. It is straightforward to verify that the map $q \mapsto \bar{q}$ is an isomorphism between D and the Prikry forcing $\mathbb{Q}(U_{\alpha,\mu}(\emptyset))$. □

If $o(\alpha)$ is limit then $\mathbb{Q}_\alpha$ is not isomorphic to Prikry forcing by a single measure. However, an argument similar to the proof of the previous Lemma shows that for every $\mu < o(\alpha)$ the forcing $\mathbb{Q}_\alpha = \mathbb{Q}_\alpha^{o(\alpha)}$ is isomorphic to a similar poset $\mathbb{Q}_\alpha^{>\mu}$ which is solely based on measures $U_{\alpha,\tau}(t)$ for $\mu < \tau < o(\alpha)$. For this, note that $\mathbb{Q}_\alpha$ has a dense subset which consists of conditions $q = \langle t, T\rangle \in \mathbb{Q}_\alpha$ so that the coherent sequence t is equivalent to a sequence $\bar{t}$ which consists only of $\beta < \alpha$ with $o(\beta) > \mu$. Let $\Delta_{>\mu} = \{\beta < \alpha \mid o(\beta) > \mu\}$ and $\bar{T} = T \cap [\Delta_{>\mu}]^{<\omega}$ then the map $\langle t, T\rangle \mapsto \langle \bar{t}, \bar{T}\rangle$ introduces the desired isomorphism.
It is easy to see that $\mathbb{Q}_\alpha$ is not weakly homogeneous when $o(\alpha)$ is a limit ordinal. The reason is that for every condition $\langle t, T\rangle \in \mathbb{Q}_\alpha$ there is always a measure one set of ordinals $\nu \in \mathrm{Succ}_T(\emptyset)$ so that c_t is an initial segment of c_ν. To deal with this issue and establish weakly homogeneity, we first need to deal with the iteration $\mathbb{P}_\alpha$ and modify c_ν for measure one many $\nu < \alpha$. This is done by construction an automorphism π of $\mathbb{P}_\alpha$. Conversely, if we choose to apply an automorphism π on $\mathbb{P}_\alpha$ which swaps t, t' on many $\nu < \alpha$ then must also swap t, t' appropriately in conditions $q \in \mathbb{Q}_\alpha$. Therefore, in this sense, the action of an automorphism π of $\mathbb{P}_{\alpha+1}$ on $\mathbb{Q}_\alpha$ is essentially determined by its action on $\mathbb{P}_\alpha$.

We note that the overall effect of an isomorphism π of $\mathbb{P}_\alpha$ on conditions $\langle t, T\rangle \in \mathbb{Q}_\alpha$ is sensitive to the choice of the iteration support for $\mathbb{P}_\alpha$. For example, similar maps π can be defined on the iteration $\mathbb{P}_\alpha$ constructed in [4]. These maps have an additional undesirable effect on the measures corresponding to the splitting levels of T. The problem is that the definition of the measures $U_{\alpha,\tau}(t)$ on α in [4] is also based on a choice of master sequences which is not preserved under automorphisms of $\mathbb{P}_\alpha$. The key feature of the nonstationary support iteration $\mathbb{P}$ is that by Corollary 3.4, the measures $U_{\alpha,\tau}(t)$ are completely determined by α, τ, and t.

We describe the main guidelines we will follow throughout the inductive construction of the automorphisms π of $\mathbb{P}$ defined below.

(1) The definition of each automorphism π is be based on a sequence of dense open sets $\langle D_\alpha \mid \alpha < \kappa\rangle$, where each D_α is forced by $0_{\mathbb{P}_\alpha}$ to be a $\leq^*$ dense open subset of $\mathbb{Q}_\alpha$. π will be then defined on a dense set $D \subset \mathbb{P}_\kappa$ of all conditions p satisfying $p \restriction \alpha \Vdash p(\alpha) \in D_\alpha$ for every $\alpha \in s(p)$.

(2) π will be constructed by induction on its restrictions to $\mathbb{P}_\gamma$, for each $\gamma \leq \kappa$. That is, the restriction of π to $D \cap \mathbb{P}_\gamma$, denoted by $\pi \restriction \gamma$, will introduce an automorphism of the complete boolean algebra of $\mathbb{P}_\gamma$. At successor steps $\gamma+1$ we shall define a $\mathbb{P}_\gamma$-name for a function π_γ with domain D_γ and set $\pi \restriction (\gamma+1) = \pi \restriction \gamma \frown \pi_\gamma$.

(3) Suppose that $\pi \restriction \gamma$ is an automorphism of $\mathbb{P}_\gamma$. To show $\pi \restriction (\gamma + 1)$ introduces an automorphism of $\mathbb{P}_{\gamma+1}$, it is sufficient to verify that for every generic filter $G_\gamma \subset \mathbb{P}_\gamma$, π_γ defines an order preserving dense injection between the posets $\mathbb{Q}_\gamma(G_\gamma)$ and $\mathbb{Q}_\gamma(G'_\gamma)$, which are the posets $\mathbb{Q}_\gamma$ defined relative to the generic filters G_γ and $G'_\gamma = (\pi \restriction \gamma)[G_\gamma]$ respectively. Let $\vec{c} = \langle c_\alpha \mid \alpha < \gamma\rangle$ and $\vec{c}' = \langle c'_\alpha \mid \alpha < \gamma\rangle$ be the Prikry/Magidor generic sequences introduced by G_γ and G'_γ respectively. For a finite sequence $t = \langle \nu_0, \ldots, \nu_k\rangle$, we say t is coherent with respect to G_γ (respectively G'_γ) if $c_{\nu_0}, \ldots, c_{\nu_k}$ (respectively $c'_{\nu_0}, \ldots, c'_{\nu_k}$) are coherent in the obvious sense.

(4) To guarantee the continuity of the inductive construction, we will require the following conditions hold for every $\langle u, S\rangle \in D_\gamma \subset \mathbb{Q}_\gamma(G_\gamma)$.
   (a) π_γ will always act trivially on the tree part, namely, $\pi_\gamma(\langle u, S\rangle) = \langle u', S\rangle$ for some coherent G'_γ sequence u'.
   (b) For every $s \in S$, the sequences $u \frown s$ and $u' \frown s$ are coherent with respect to G_γ and G'_γ.
   (c) If $p \in \mathbb{P}_\gamma$ and $p \Vdash \pi_\gamma(\langle u, S\rangle) = \langle u', S\rangle$ then for every $\tau < o(\gamma)$ there exists $Z \in U_{\gamma,\tau}$ so that for every $\nu \in Z$ there is a $\mathbb{P}_\nu$-name for a tree T so that $p \restriction \nu \Vdash \pi_\nu(\langle u, T\rangle) = \langle u', T\rangle$.
   (d) For every $s \in S$, $\pi_\gamma(\langle u \frown s, S_s\rangle) = \langle u' \frown s, S_s\rangle$.

We claim conditions (a)-(d) imply π_γ introduces an isomorphism between $\mathbb{Q}_\gamma(G_\gamma)$ and $\mathbb{Q}_\gamma(G'_\gamma)$. We need to check that $\langle u', S\rangle = \pi_\gamma(\langle u, S\rangle) \in \mathbb{Q}_\gamma(G'_\gamma)$ for each $\langle u, S\rangle$, and that π_γ is order preserving. The latter is an immediate consequence of conditions (a) and (d) above. Therefore, let us check that $\langle u', S\rangle$ is a valid condition in $\mathbb{Q}_\gamma(G'_\gamma)$. By (b), all finite sequences from some extension of $\langle u', S\rangle$ are coherent with respect to G'_γ. Therefore, showing $\langle u', S\rangle \in \mathbb{Q}_\gamma(G'_\gamma)$ amounts to verifying that the splitting sets $\mathrm{Succ}_S(s)$ for $s \in S$ are measure one sets with respect to the measures $U_{\gamma,\tau}(s)$, $\tau < o(\gamma)$, which are defined relative to the $\mathbb{P}_\gamma$ generic object G'_γ. This follows from (c) and the description of the measures $U_{\gamma,\tau}(s)$, given in Corollary 3.4. To see this, let $X = \mathrm{Succ}_S(s)$ and fix $\tau < o(\gamma)$. We know there are $p \in G_\gamma$, $Y \in U_{\gamma,\tau}$, and a tree $S \subset [\gamma]^{<\omega}$ so that $Y \cap A_u(p, S, G_\gamma) \subset X$, where $A_u(p, S, G_\gamma) = \{\nu < \kappa \mid p \restriction \nu \frown \langle u \restriction o(\nu), S \restriction \nu\rangle \frown p \setminus (\nu + 1) \in G_\gamma\}$. Moreover, by Remark 3.5 we may assume the tree S comes from the $\leq^*$ dense set D_γ. Then by (c), we see that there exists some $Z \in U_{\gamma,\tau}$ so that $A_u(p, S, G_\gamma) \cap Z = A_{u'}(\pi \restriction \gamma(p), S, G'_\gamma) \cap Z$. Hence $Y \cap Z \cap A_{u'}(\pi \restriction \gamma(p), S, G'_\gamma) \subset \mathrm{Succ}_S(s)$.

To prove the main theorem of this section (Theorem 4.6) we will need the ability to amalgamate certain automorphisms. For this, we introduce the notions of *essentially trivial ordinals* and the *support* of π.

Definition 4.3. *Let π be an automorphism of $\mathbb{P}_\delta$ for some $\delta \leq \kappa$. We define by induction on $\alpha \leq \delta$, the support of $\pi \restriction \alpha$ and whether or not α is* essentially trivial *for π. Suppose these have been defined for every $\beta < \alpha$. We define support of $\pi \restriction \alpha$ to be the set $\mathrm{s}(\pi \restriction \alpha) = \{\beta < \alpha \mid \beta$ is not essentially trivial for $\pi \restriction \alpha\}$. We now define when α is essentially trivial.*

1. *If $o(\alpha) = 0$ then α is essentially trivial for π.*
2. *If $o(\alpha) > 0$ then α is essentially trivial for π if $0_{\mathbb{P}_\alpha}$ forces that for every maximal coherent sequence t and condition $q = \langle t, T\rangle \in \mathbb{Q}_\alpha$, there is a direct extension $q^* = \langle t, T^*\rangle$ satisfying $\pi_\alpha(q^*) = q^*$ and $T^* \subset [\alpha \setminus s(\pi \restriction \alpha)]^{<\omega}$ (i.e., T^* consists only of essentially trivial ordinals).*

Note that the support $s(\pi)$ is a set in V. The next lemma is the key technical piece of the argument that $\mathbb{P}$ is weakly homogeneous.

Lemma 4.4. *Suppose that $\alpha < \kappa$ is a V-measurable cardinal and $p(\alpha) = \langle t, T\rangle$ and $p'(\alpha) = \langle t', T'\rangle$ are two $\mathbb{P}_\alpha$-names for conditions in $\mathbb{Q}_\alpha$. For every closed unbounded set $C \subset \alpha$ there are direct extensions $q(\alpha), q'(\alpha)$ of $p(\alpha), p'(\alpha)$ and an automorphism π of $\mathbb{P}_{\alpha+1}$ with $s(\pi) \cap \alpha \subset C$ and $\pi(0_{\mathbb{P}_\alpha} \frown q(\alpha)) = 0_{\mathbb{P}_\alpha} \frown q'(\alpha)$.*

Proof. If $o(\alpha)$ is a successor ordinal, then by the previous Lemma, $\mathbb{Q}_\alpha$ is weakly homogeneous and there are direct extensions $q(\alpha), q'(\alpha)$ of $p(\alpha), p'(\alpha)$ respectively, and an automorphism π of $\mathbb{P}_{\alpha+1}$ so that $\pi(0_{\mathbb{P}_\alpha} \frown q(\alpha)) = 0_{\mathbb{P}_\alpha} \frown q'(\alpha)$ and $s(\pi) = \{\alpha\}$. So we may assume that $o(\alpha)$ is a limit ordinal.

We define an automorphism π by taking $\pi \restriction \mathbb{P}_\alpha$ to be trivial and defining its further restrictions $\pi \restriction \gamma$ to $\mathbb{P}_\gamma$ for $\gamma \leq \alpha + 1$ by induction on γ. We ensure the following conditions. First, $s(\pi) \subset C$, in particular $s(\pi \restriction \gamma) \subset C \cap \gamma$ for all γ. Second, if $G_\gamma \subset \mathbb{P}_\gamma$ is a generic filter and $\vec{c} = \langle c_\beta \mid \beta < \gamma \rangle$ is the induced generic sequence of Prikry/Magidor sequences, then for each measurable $\beta < \gamma$,

(1) if $\beta \notin C$ then β is an essentially trivial for $\pi \restriction \beta$.
(2) if $\beta \in C \cap \gamma$ then the following hold.
   (a) if c_β end extends c_t and $\min(c_\beta \setminus c_t) > \max(t')$ then $\pi_\beta(c_\beta) = (c_\beta \setminus c_t) \cup c_{t'}$, and
   (b) if c_β end extends $c_{t'}$ and $\min(c_\beta \setminus c_{t'}) > \max(t)$ then $\pi_\beta(c_\beta) = (c_\beta \setminus c_{t'}) \cup c_t$.

   We identify here π_β with naturally induced map on the generic Prikry/Magidor sequence at β.

Suppose that $\pi \restriction \gamma$ has been defined and satisfies the above conditions. We split the definition into three cases.

Case 1: If $\gamma \notin C$ then $C \cap \gamma$ is bounded in γ so $\gamma \setminus s(\pi \restriction \gamma)$ belongs to $U_{\gamma,\tau}(s)$ for each τ, s. For each $\tau \leq o(\gamma)$ define D^τ_γ to be the set of all conditions $\langle u, S \rangle \in \mathbb{Q}^\tau_\gamma$ which satisfy that for every $r \in S$, the sequence $u \frown r$ is coherent with respect to both G_γ and G'_γ. It is easy to show by induction on $\tau \leq o(\gamma)$ that D^τ_γ is $\leq^*$ dense open and that every $\langle u, S \rangle \in D^\tau_\gamma$ belongs to both $\mathbb{Q}^\tau_\gamma(G_\gamma)$, $\mathbb{Q}^\tau_\gamma(G'_\gamma)$. For a similar argument see the proof of Lemma 3.11 in [4]. Further inductive arguments of this form are similar and will be omitted. We can therefore define π_γ to be the identity automorphism on $D_\gamma = D^{o(\gamma)}_\gamma$. Clearly, $\gamma \notin s(\pi \restriction (\gamma + 1))$.

Case 2: Suppose that $\gamma \in C$ and $o(\gamma) \leq \max(\{o(\nu) \mid \nu \in t \cup t'\})$. Fix $\tau \leq o(\gamma)$ and $q = \langle u, S \rangle \in \mathbb{Q}^\tau_\gamma$. Let $S^* \subset S$ be the subtree which consists of all sequences $r \in S$ with containing only ordinals $\nu < \gamma$ which satisfy $(t_{o(\nu)}(\nu))_{G_\gamma} = t$ and $(t'_{o(\nu)}(\nu))_{G_\gamma} = t'$, and for which $u \frown r$ is coherent with respect to both G_γ and G'_γ. Since we may assume $o(\nu) < \tau$ for every ν in S, it is easy to verify by induction on τ that $q^* = \langle u, S^* \rangle$ is a direct extension of q. It follows that for $\tau = o(\gamma)$ the set D_γ of all $\langle u, S^* \rangle$ as above is $\leq^*$ dense open. It follows as in the preceding case that $\pi \restriction (\gamma + 1) = \pi_\gamma \frown \mathrm{Id}_{D_\gamma}$ is an automorphism of $\mathbb{P}_{\gamma+1}$. Note that γ need not be essentially trivial in this case as $0_{\mathbb{P}_\gamma}$ may not decide whether $o(\gamma) \leq \max(\{o(\nu) \mid \nu \in t \cup t'\})$.

Case 3: Suppose that $\gamma \in C$ and $o(\gamma) > \max(\{o(\nu) \mid \nu \in t \cup t'\})$. Let $\rho = \max(\{o(\nu) \mid \nu \in t \cup t'\})$. We split the construction into two subcases.

Subcase 3.1: If $o(\gamma) = \rho + 1$, then by the result of Lemma 4.2, $\mathbb{Q}_\gamma$ is weakly homogeneous and we can find a $\mathbb{P}_\gamma$-name for an automorphism π_γ to form $\pi \restriction \gamma + 1$ satisfying the induction hypotheses (2) parts (a) and (b) above.

Subcase 3.2: $o(\gamma) > \rho + 1$. By the remark following Lemma 4.2, $\mathbb{Q}_\gamma$ is isomorphic to the forcing $\mathbb{Q}_\gamma^{>\rho}$ which consists of conditions containing ordinals, measures, and measure one sets associated with $U_{\gamma,\tau}$ for $\tau > \rho$. For each ordinal τ with $\rho < \tau < o(\gamma)$ let $\mathbb{Q}_\gamma^{(\rho,\tau)}$ be the similar poset isomorphic to $\mathbb{Q}_\gamma^\tau$. It straightforward to verify by induction on $\tau > \rho$ that that every condition $q = \langle u^*, S\rangle \in \mathbb{Q}_\gamma^\tau$ has a direct extension $\langle u^*, S^*\rangle$ so that all ordinals appearing in S^* are above $\max(t \cup t')$ and every $u \in \{u^*\} \cup S^*$ satisfies one of the following conditions.

(1) c_u does not end extend c_t nor $c_{t'}$ and u is coherent with respect to both G_γ and G'_γ.
(2) c_u end extends c_t but $\min(c_u \setminus c_t) \leq \max(t')$ and u is coherent with respect to both G_γ and G'_γ,
(3) c_u end extends $c_{t'}$ but $\min(c_u \setminus c_{t'}) \leq \max(t)$, and u is coherent with respect to both G_γ and G'_γ,
(4) u is equivalent to a coherent sequence of the form $t \frown r$ which is coherent with respect to G_γ, and $t' \frown r$ is coherent with respect to G'_γ,
(5) u is equivalent to a coherent sequence of the form $t' \frown r$ which is coherent with respect to G_γ, and $t \frown r$ is coherent with respect to G'_γ.

Let D_γ denote the set of conditions $q^* = \langle u^*, S^*\rangle \in \mathbb{Q}_\gamma$ as above. We define the π_γ as follows. If u^* satisfies one of the first three cases above we define $\pi_\gamma(q^*) = q^*$. If u^* satisfies the fourth condition above then it is equivalent to a coherent sequence of the form $t \frown r$. We define $\pi_\gamma(q^*) = \langle t' \frown r, S^*\rangle$. Similarly, If u^* satisfies the last condition then it is equivalent to a sequence of the form $t' \frown r$ and we define $\pi_\gamma(q^*) = \langle t \frown r, S^*\rangle$. As in the previous cases, it is routine to check $\pi \restriction (\gamma + 1) = \pi \restriction \gamma \frown \pi_\gamma$ defines an automorphism of $\mathbb{P}_\gamma$.
At limit stages $\delta \leq \alpha$ we st $\pi \restriction \delta$ to be the limit of $\pi \restriction \gamma$, $\gamma < \delta$. This concludes the inductive construction.
Finally, let $\pi = \pi \restriction \alpha$ and $q^* = \langle t, S^*\rangle$ be a direct extension of $p(\alpha)$ which belongs to D_α. Define $S = T' \cap S^*$, $q(\alpha) = \langle t, S\rangle$, and $q'(\alpha) = \langle t', S\rangle$. Both $q(\alpha)$ and $q'(\alpha)$ are forced to be direct extensions of $p(\alpha)$ and $p'(\alpha)$ respectively and $\pi(0_{\mathbb{P}_\alpha} \frown q(\alpha)) = 0_{\mathbb{P}_\alpha} \frown q'(\alpha)$. □

The following Corollary of Lemma 4.4 shows that π can be amalgamated with an automorphism π' with a disjoint support.

Corollary 4.5. *Let $\alpha < \kappa$ be a measurable cardinal in V. Suppose that π' is an automorphism of $\mathbb{P}_\alpha$ with $\mathrm{s}(\pi') \subset \alpha$ is nonstationary and $w, w' \in \mathbb{P}_\alpha$ so that $\pi'(w) = w'$. Then for every two $\mathbb{P}_\alpha$-names for conditions in $\mathbb{Q}_\alpha$, $p(\alpha)$ and $p'(\alpha)$ and every $\beta < \alpha$ there are strong direct extensions q, q' of w w' respectively and an isomorphism π of $\mathbb{P}_{\alpha+1}$ satisfying the following conditions.*

- $q \restriction \beta + 1 = w \restriction \beta + 1$ *and* $q' \restriction \beta + 1 = w' \restriction \beta + 1$,
- $\pi \restriction (\beta + 1) = \pi' \restriction \beta + 1$,
- $\pi(q \frown p(\alpha)) = q' \frown p'(\alpha)$.

Proof. Let $\langle D'_\gamma \mid \gamma < \kappa \rangle$ be the sequence of $\leq^*$ dense open sets defining the domain of π'. and $C \subset \alpha$ be a closed unbounded set disjoint from $s(\pi') \cup s(w) \cup \beta$. The construction of π is similar to the one given in the proof of Lemma 4.4 above where here, we would also like to have $\pi_\gamma = \pi'_\gamma$ at $\gamma \in s(\pi')$. This is possible since C is disjoint from $s(\pi')$. To guarantee that the action of π' does not interfere with the construction of π along $\gamma \in C$ designed to swap t and t', we shrink the domain sets D'_γ for $\gamma > \beta$. We set $\pi \restriction (\beta + 1) = \pi' \restriction (\beta + 1)$. For every nontrivial stage $\gamma > \beta$, we define $D_\gamma \subset D'_\gamma$ and π_γ as follows. If $\gamma \notin C$ then we define D_γ to be the set of all $\langle u, S \rangle \in D'_\gamma$ so that $S \cap [C]^{<\omega} = \emptyset$ and set $\pi_\gamma = \pi'_\gamma$. If $\gamma \in C$ and $o(\gamma) \leq \max(\{o(\nu) \mid \nu \in t \cup t'\})$ then $D_\gamma = D'_\gamma$ and define π_γ as in the preceding proof. If $\gamma \in C$ and $o(\gamma) > \max(\{o(\nu) \mid \nu \in t \cup t'\})$. Then set D_γ to be the set of all $\langle u, S \rangle \in D'_\gamma$ so that $S \subset [C]^{<\omega}$ and define π_γ as in the preceding proof. It is easy to verify by induction that π is an automorphism with the desired properties. Finally, we define a strong direct extension q of w. For each $\gamma \in s(w)$, $w(\gamma) = \langle u, S \rangle$ is a $\mathbb{P}_\gamma$-name for a condition in $\mathbb{Q}_\gamma$. We set $q(\gamma) = \langle u, S \setminus [C]^\omega \rangle$. □

We are now ready to prove the main result of this section.

Theorem 4.6. *$\mathbb{P}$ is weakly homogeneous.*

Proof. We prove by induction on $\alpha \leq \kappa$ that for every $p, p' \in \mathbb{P}_\alpha$ an automorphism $\bar{\pi}$ of $\mathbb{P}_{\beta+1}$ for some β with $\beta + 1 < \alpha$ satisfying $\bar{\pi}(p \restriction \beta + 1) = p' \restriction \beta + 1$, there are strong direct extensions q, q' of p, p' respectively and an automorphism π of $\mathbb{P}_\alpha$ which satisfy the following conditions.

(1) $q \restriction \beta + 1 = p \restriction \beta + 1$, $q' \restriction \beta + 1 = p' \restriction \beta + 1$.
(2) $\pi \restriction \beta + 1 = \bar{\pi}$.
(3) $\pi(q) = q'$.

(4) If α is inaccessible in V then $\mathrm{s}(\pi)$ is nonstationary in α.

Successor step: Suppose the statement holds for α and let $p, p' \in \mathbb{P}_{\alpha+1}$, $\beta < \alpha$, and $\bar{\pi}$ as above. If α is not measurable in V then $\bar{\pi}$ trivially extends to an automorphism of $\mathbb{P}_{\alpha+1}$. Suppose α is measurable. By applying the inductive assumption to $p \restriction \alpha$, $p' \restriction \alpha$, we can find strong direct extensions w, w' of $p \restriction \alpha$, $p' \restriction \alpha$ respectively and an automorphism π' of $\mathbb{P}_\alpha$ extending $\bar{\pi}$ so that $s(\pi') \subset \alpha$ is nonstatioanry and $\pi'(w) = w'$. By Corollary 4.5 applied to π', there are strong direct extensions $q, q' \in \mathbb{P}_{\alpha+1}$ of $w \frown p(\alpha)$ and $w' \frown p'(\alpha)$ respectively and an automorphism π of $\mathbb{P}_{\alpha+1}$ which extends $\bar{\pi}$. Furthermore, if $\alpha \notin s(p) \cap s(p')$ then we can take π to be a trivial extension of $\bar{\pi}$ for which α is an essentially trivial ordinal.

Limit step: Let $\delta \leq \kappa$ be a limit ordinal and $s = s(p) \cup s(p')$. Let $\rho = cf(\delta)$, $C \subseteq \delta$ be a club disjoint from s and $\langle \alpha_i \mid i < \rho \rangle$ be an increasing continuous enumeration of C with $\delta = \alpha_\rho$ and $\alpha_0 = \beta+1$. We construct our automorphism π by induction on its restrictions to $\mathbb{P}_{\alpha_i+1}$ for $i \leq \rho$. We also construct two sequences $\langle p_i \mid i \leq \rho \rangle$, $\langle p'_i \mid i \leq \rho \rangle$ of direct extensions of p, p' respectively.

Set $p_0 = p$, $p'_0 = p'$ and $\pi \restriction \alpha_0 = \bar{\pi}$. Suppose p_i, p'_i, $\pi \restriction \alpha_i + 1$ have been defined. As $\alpha_{i+1} \notin s$, the inductive assumption implies that we can find conditions p_{i+1}, p'_{i+1} and an automorphism $\pi \restriction \alpha_{i+1} + 1$ of $\mathbb{P}_{\alpha_{i+1}+1}$ satisfying

- p_{i+1} and p'_{i+1} are direct strong extensions of p_i and p'_i respectively, with $p_{i+1} \restriction \alpha_i + 1 = p_i \restriction \alpha_i + 1$, $p'_{i+1} \restriction \alpha_i + 1 = p'_i \restriction \alpha_i + 1$,
- $\mathrm{s}(p_{i+1}) \setminus \alpha_i = \mathrm{s}(p_i) \setminus \alpha_i$, and $\mathrm{s}(p'_{i+1}) \setminus \alpha_i = \mathrm{s}(p'_i) \setminus \alpha_i$,
- $\alpha_{i+1} \notin s(\pi \restriction \alpha_{i+1} + 1)$.

Next, let $j < \delta$ be a limit ordinal and suppose that $\pi \restriction \alpha_i + 1, p_i, p'_i$ have been defined for each $i < j$. Let $\pi \restriction \alpha_j$ be the natural limit of $\pi \restriction \alpha_i + 1$. As in the proof of the Prikry Lemma, we can find natural $\leq^*$ upper bounds p_j and p'_j for the sequences $\langle p_i \mid i < j \rangle$ and $\langle p'_i \mid i < j \rangle$ respectively. It follows that $\pi \restriction \alpha_j$ is an automorphism of $\mathbb{P}_{\alpha_j}$ with $s(\pi \restriction \alpha_j) \cap \{\alpha_i \mid i < j\} = \emptyset$. As $\alpha_j \notin s$, we can apply the result of Corollary 4.5 and extend $\pi \restriction \alpha_j$ to an automorphism $\pi \restriction \alpha_j + 1$ of $\mathbb{P}_{\alpha_j+1}$ so that $\alpha_j \notin s(\pi \restriction \alpha_j + 1))$. This concludes the inductive construction of the three sequences. It is easy to see that π the limit of $\pi \restriction \alpha_i + 1$ for $i < \rho$, $q = p_\rho$ and $q' = p'_\rho$ are as required. □

We can now complete the proof of Theorem 1.3.

Proof. Suppose that for each $\tau < \kappa$ the set $\Delta_\tau = \{\alpha < \kappa \mid o^{\mathcal{U}}(\alpha) = \tau\}$ is stationary in κ, and let $\Delta = \bigcup_{\tau < \kappa} \Delta_\tau$. By standard coding methods, we may assume $V = \mathrm{HOD}^V$ and V is contained in HOD as computed in the further generic extensions we consider.

Let $\mathbb{P}$ be the nonstationary support iteration of Prikry forcing which singularizes cardinals α in Δ with $o(\alpha) > 0$. Let $G \subset \mathbb{P}$ be a generic filter. Since $\mathbb{P}$ preserves stationary subsets of κ, Δ is a fat stationary set in $V[G]$. Therefore by [1] the forcing $\mathbb{C}$ for adding a closed unbounded set to Δ by forcing with closed bounded sets is $< \kappa$ distributive. The forcing $\mathbb{C}$ is clearly weakly homogeneous in $V[G]$. Furthermore, as a $\mathbb{P}$-name in V, $\dot{\mathbb{C}}$ is a fixed point of all automorphisms π on $\mathbb{P}$ constructed in the proof of Theorem 4.6 above. Thus, by [3] is follows that $\mathbb{P} * \dot{\mathbb{C}}$ is weakly homogeneous. Let $K \subset \mathbb{C}$ be a generic filter in $V[G]$ and let $C \subset \Delta$ be its induced closed unbounded set. We conclude that $\mathrm{HOD}^{V[G*K]} = \mathrm{HOD}^V = V$ therefore every ordinal $\alpha \in C$ is measurable in $\mathrm{HOD}^{V[G*K]}$. □

Remark 4.7. *By increasing the large cardinal assumption for the cardinals $\alpha < \kappa$ in the nontrivial domain of the coherent sequence $\mathcal{U}$ (i.e., $o(\alpha) > 0$), it is easy to see how the same construction yields a generic extension $V[G * K]$ in which the ordinals in C have the same large cardinal properties in $\mathrm{HOD}^{V[G*K]}$. For example, if each α in the nontrivial domain of $\mathcal{U}$ is supercompact then it is also supercompact in $\mathrm{HOD}^{V[G*K]}$.*

5. Radin/Mitchell forcing with application to HOD

In this section we give the proof of Theorem 1.4. The proof goes by comparing the Radin/Mitchell forcing from two different models and proving that the quotient is homogeneous. Let $\kappa < \theta$ be cardinals where κ is 2^θ-supercompact and θ is the least cardinal μ above κ which is μ^+ supercompact. Let $j : V \to M$ be an elementary embedding with $\mathrm{cp}(j) = \kappa$, ${}^{2^\theta}M \subset M$.

For each $\alpha < \kappa$, let θ_α be the least cardinal μ above α which is μ^+ supercompact and let A be the set of inaccessible β which are closed under the function $\alpha \mapsto \theta_\alpha$. We note that A is in the normal measure derived from j. We define $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \kappa \rangle$ to be a backwards Easton support iteration where for each $\alpha \leq \kappa$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the Levy collapse $\mathrm{Coll}(\alpha, < \theta_\alpha)$ if $\alpha \in A$ and is trivial otherwise.

We start by constructing a coherent sequence of supercompact measures $\vec{W} = \langle W_{\alpha,\tau} \mid \alpha \leq \kappa, \tau < o^*(\alpha) \rangle$ in V which extends into a sequence of normal measures $\vec{U}^* = \langle U^*_{\alpha,\tau} \mid \alpha \leq \kappa, \tau < o^*(\alpha) \rangle$ after forcing with $\mathbb{P}$.

We then force with the Radin/Mitchell poset $\mathbb{R}(\vec{U}^*)$ to add a generic club $C \subset \kappa$ preserving the regularity of κ. We compare the Radin forcing to the Radin forcing over the ground model V (prior to the collapses) via a sequence $\vec{U} = \langle U_{\alpha,i} \mid \alpha \leq \kappa, i < o(\alpha)\rangle$. To construct $\vec{U}$, we take the sequence $\vec{U}^W = \langle U^W_{\alpha,\tau} \mid \alpha \leq \kappa, \tau < o^*(\alpha)\rangle$ where each $U^W_{\alpha,\tau}$ is the normal measure on α derived from the supercompact measure $W_{\alpha,\tau}$, and remove all weak repeat points (in V) from this sequence. By analyzing the quotient of $\mathbb{P} * \mathbb{R}(\vec{U}^*)$ and $\mathbb{R}(\vec{U})$ we show that HOD as computed in $V^{\mathbb{P}*\mathbb{R}(\vec{U}^*)}$ is contained in HOD as computed in $V^{\mathbb{R}(\vec{U})}$.

5.1. **Coherent Sequences.** In V, we define by induction on $\alpha < \kappa$ three functions $g, g', o' : \kappa \to V$ satisfying the following properties.

(1) For every $\alpha < \kappa$, $g(\alpha) = \langle W_i \mid i < o'(\alpha)\rangle$ is a $\triangleleft$-increasing sequence of θ_α-supercompact measures on α;
(2) $g'(\alpha) = \langle \dot{U}_i{}' \mid i < o'(\alpha)\rangle$ is a $\mathbb{P}_{\alpha+1}$-name for a $\triangleleft$-increasing sequence of normal measures on α.

Suppose that $\alpha < \kappa$ and $g \restriction \alpha, g' \restriction \alpha, o' \restriction \alpha$ have been defined. Let Σ_α be the set of all $\triangleleft$-increasing sequences $\langle W_i \mid i < \eta\rangle$ of θ_α-supercompact measures on α such that for every $\tau < \eta$, if $j_\tau : V \to M_\tau \cong \mathrm{Ult}(V, W_\tau)$ then $\langle W_i \mid i < \tau\rangle = j_\tau(g)(\alpha)$, and that there is an increasing sequence of $\mathbb{P}_{\alpha+1}$-names $\langle \dot{U}'_i \mid i < \eta\rangle$, so that the following hold for each $\tau < \eta$:

- $\dot{U}'_\tau$ is a $\mathbb{P}_{\alpha+1}$-name for a normal measure on α;
- There is a $\mathbb{P}_{\alpha+1}$-name for a $j_\tau(\mathbb{P}_{\alpha+1})/\mathbb{P}_{\alpha+1}$ generic filter G_τ over $M_\tau[G(\mathbb{P}_{\alpha+1})]$, definable from W_τ and $\mathbb{P}_{\alpha+1}$ satisfying $j_\tau``G(\mathbb{P}_{\alpha+1}) \subset G(\mathbb{P}_{\alpha+1}) * G_\tau$, and $\dot{U}'_\tau$ is the normal measure on α defined via the extension $j^*_\tau : V[G(\mathbb{P}_{\alpha+1})] \to M_\tau[G(\mathbb{P}_{\tau+1}) * G_\tau]$;
- $j^*_\tau(g' \restriction \alpha)(\alpha) = \langle \dot{U}'_i \mid i < \tau\rangle$.

If Σ_α is not empty, let $g(\alpha) = \langle W_i \mid i < \eta\rangle$ be a maximal sequence in Σ_α (chosen by a fixed well order). Let $o'(\alpha) = \eta$ and $g'(\alpha) = \langle \dot{U}'_i \mid i < o'(\alpha)\rangle$ be the compatible $\mathbb{P}_{\alpha+1}$-name.

Let $\vec{W}'_\kappa = \langle W_i \mid i < o'(\kappa)\rangle = j(g)(\kappa)$ and $\vec{U}'_\kappa = \langle \dot{U}'_i \mid i < o'(\kappa)\rangle = j(g')(\kappa)$ be the resulting sequences on κ. We claim that $o'(\kappa) \geq \theta^+$. Suppose otherwise. Let W' be the θ-supercompact measure on κ derived from j. Since j is a θ^+-supercompact embedding then $j(\alpha \mapsto \theta_\alpha)(\kappa) = \theta$. Moreover, if $j' : V \to M' \cong \mathrm{Ult}(V, W')$ and $k : M' \to M$ are the obvious ultrapower and connecting embeddings then $\mathrm{cp}(k) > \theta^+$. Thus $o'(\kappa) = j(o')(\kappa) = j'(o')(\kappa)$ and for every $\tau < o'(\kappa)$, $W_\tau = j(g)(\kappa)_\tau = j'(g)(\kappa)_\tau$. Similarly, since $\mathbb{P}_{\kappa+1}$ satisfies θ-c.c and has cardinality θ then $\dot{U}'_\tau = j(g')(\kappa)_\tau = j'(g')(\kappa)_\tau$. Let $G_{\kappa+1}$ be a $\mathbb{P}_{\kappa+1}$-name for a $j'(\mathbb{P})/\mathbb{P}_{\kappa+1}$ generic filter over $M'[G(\mathbb{P}_{\kappa+1})]$, and let $\dot{U}'$ be the

$\mathbb{P}_{\kappa+1}$-name for the induced normal measure on κ. Then $\vec{W}'_\kappa {}^\frown \langle W' \rangle$ and $\vec{U}'_\kappa {}^\frown \langle U' \rangle$ extend $\vec{W}'_\kappa$ and $\vec{U}'_\kappa$ respectively, and they are members of M. This contradicts the maximality of $j(g)(\kappa)$, $j(g')(\kappa)$ in M.

Next, we thin out the coherent sequences given by g and g' to define, for G which is $\mathbb{P}$-generic, coherent sequences $\vec{W}$ and $\vec{U}^*$ in $V[G]$ with $o^{\vec{W}}(\kappa) = \theta = (\kappa^+)^{V[G]}$. First, define a function $o^* : \kappa \to \kappa$ in V by

$$o^*(\alpha) = \begin{cases} o'(\alpha) & \text{if } o'(\alpha) < \theta_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

We then set $o^*(\kappa) = \theta$. Next, define in V a sequence $\vec{W} = \langle W_{\alpha,\tau} \mid \alpha \leq \kappa, \tau < o^*(\alpha) \rangle$ as follows. For $\alpha < \kappa$, if $o^*(\alpha) > 0$ then set $\langle W_{\alpha,\tau} \mid \tau < o^*(\alpha) \rangle = g(\alpha) \restriction o^*(\alpha)$. Finally, define $\langle W_{\kappa,\tau} \mid \tau < \theta \rangle = j(g)(\kappa) \restriction \theta$.

Similarly, we define a $\mathbb{P}$-name for a coherent sequence of normal measures $\vec{U}^*$ from g'. Let $G \subset \mathbb{P}$ be a generic filter over V. For every $\alpha \leq \kappa$ let $G_{\alpha+1} = G \restriction \mathbb{P}_{\alpha+1}$. Define $\vec{U}^* = \langle U^*_{\alpha,\tau} \mid \alpha \leq \kappa, \tau < o^*(\alpha) \rangle$ by setting $U^*_{\alpha,\tau} = (g'(\alpha)_\tau)_{G_{\alpha+1}}$ for every $\alpha \leq \kappa$ and $\tau < o^*(\alpha)$.

The fact $\vec{W}$ is a coherent sequence is immediate from the construction. Let us show that $\vec{U}^*$ is coherent.

Lemma 5.1. *The sequence $\vec{U}^*$ is coherent in $V[G]$.*

Proof. Note that for every $\alpha \leq \kappa$, $\vec{U}^* \restriction (\alpha+1)$ is definable from $g' \restriction \alpha+1$ and $G_{\alpha+1}$. Suppose that $\alpha \leq \kappa$ and $\tau < o^*(\alpha)$. Let $j_\tau : V \to M_\tau \cong \mathrm{Ult}(V, W_\tau)$ and $j^*_\tau : V[G_{\alpha+1}] \to M_\tau[G^*_\tau]$ be its extension to $V[G_{\alpha+1}]$ by which $U^*_{\alpha,\tau}$ is defined. Note that $j^*_\tau(\vec{U}^*) \restriction \alpha+1$ is defined from $j_\tau(g') \restriction \alpha+1 = g' \restriction (\alpha,\tau)$ and $G^*_\tau \restriction \alpha+1 = G_{\alpha+1}$. Thus $j^*_\tau(\vec{U}^*) \restriction \alpha+1 = \vec{U}^* \restriction (\alpha,\tau)$.

We now compare j^*_α with the $V[G_{\alpha+1}]$ ultrapower embedding constructed by $U^*_{\alpha,\tau}$. Let $i_\tau : V[G_{\alpha+1}] \to M^*_\tau \cong \mathrm{Ult}(V[G_{\alpha+1}], U^*_{\alpha,\tau})$ and $k_\tau : M^*_\tau \to M_\tau[G^*_\tau]$ defined by $k_\tau([f]_{U^*_{\alpha,\tau}}) = j^*_\tau(f)(\alpha)$. Clearly, $j^*_\tau(\vec{U}^*) = k_\tau(i_\tau(\vec{U}^*))$ and $\tau = j_\tau(o^*)(\alpha) < \alpha^+ < \mathrm{cp}(k_\tau)$. It follows that $\mathrm{dom}(j^*_\tau(\vec{U}^*) \restriction \alpha+1) = k_\tau(\mathrm{dom}(i_\tau(\vec{U}^*) \restriction \alpha+1))$. As $U^*_{\alpha,\tau'} = k_\tau(U^*_{\alpha,\tau'})$ for every $\tau' < \tau$ we conclude that

$$i_\tau(\vec{U}^*) \restriction \alpha+1 = k_\tau(i_\tau(\vec{U}^*) \restriction \alpha+1) = j^*_\tau(\vec{U}^*) \restriction \alpha+1 = \vec{U}^* \restriction (\alpha,\tau).$$

□

For every $\alpha \leq \kappa$ and $\tau < o^*(\alpha)$ let $U^W_{\alpha,\tau}$ be the normal measure on α derived from $W_{\alpha,\tau}$. In V, set $\vec{U}^W = \langle U^W_{\alpha,\tau} \mid \alpha \leq \kappa, \tau < o^*(\alpha) \rangle$. It is clear that $U^*_{\alpha,\tau} \cap V = U^W_{\alpha,\tau}$ for every $\alpha \leq \kappa$ and $\tau < o^*(\alpha)$.

Corollary 5.2. *For every $\alpha \le \kappa$ with $o^*(\alpha) > 0$ let $\mathcal{F}^*_\alpha = \bigcap_{\tau < o^*(\alpha)} U^*_{\alpha,\tau} \in V[G]$ and $\mathcal{F}_\alpha = \bigcap_{\tau < o^*(\alpha)} U^W_{\alpha,\tau} \in V$. Then $\mathcal{F}_\alpha \subset \mathcal{F}^*_\alpha$.*

We would like to argue that the Radin forcing defined by $\vec{U}^*$ over $V[G]$ provides a club C which is Radin generic by $\vec{U}^W$ over V. The problem is that the sequnece $\vec{U}^W$ in V is too long to be coherent. Instead, we define a coherent sequence $\vec{U}$ in V whose corresponding filter at each $\alpha \le \kappa$ coincide with the filters $\mathcal{F}_\alpha$ derived from $\vec{U}^W$. Working in V, we obtain $\vec{U}$ from $\vec{U}^W$ by removing all its weak repeat points. Weak repeat points were introduced in [10]. We say that a measure $U^W_{\alpha,\tau} \in \vec{U}^W$ is a weak repeat point if for every $X \in U^W_{\alpha,\tau}$ there is some $\delta < \tau$ so that $X \in U^W_{\alpha,\delta}$. Equivalently, $U^W_{\alpha,\tau}$ is a weak repeat point if $\bigcap_{\delta<\tau} U^W_{\alpha,\delta} \subset U^W_{\alpha,\tau}$.

Definition 5.3 (The filtered sequence $\vec{U}$ and $o(\alpha)$)**.** *Work in V. For each $\alpha \le \kappa$, let $\langle U^W_{\alpha,\tau_i} \mid i < \rho\rangle$ be an increasing enumeration of all the measures in $\vec{U}^W(\alpha) = \langle U^W_{\alpha,\tau} \mid \tau < o^*(\alpha)\rangle$ which are not weak repeat points. We define $o(\alpha) = \rho$, and for each $i < o(\alpha)$ denote U^W_{α,τ_i} by $U_{\alpha,i}$. We call $\vec{U} = \langle U_{\alpha,i} \mid \alpha \le \kappa, i < o(\alpha)\rangle$ the filtered sequence of $\vec{U}^W$.*

It is clear from the definition of $\vec{U}$ that $\mathcal{F}_\alpha = \bigcap_{i<o(\alpha)} U_{\alpha,i}$ for every $\alpha \le \kappa$.

Lemma 5.4. *$\vec{U}$ is coherent in V.*

Proof. It is shown by induction on $\delta \le \kappa + 1$ that the sequence $\vec{U} \restriction \delta = \langle U_{\alpha,i} \mid \alpha < \delta, i < o(\alpha)\rangle$ is coherent. Limit stages are trivial. For notational simplicity, let us deal with the successor stage $\delta = \kappa+1$. Fix $i < o(\kappa)$ and let $\tau = \tau_i < o^*(\kappa)$ so that $U_{\kappa,i} = U^W_{\kappa,\tau}$. Let $j_\tau : V \to M_\tau \cong \mathrm{Ult}(V, W_\tau)$, $e_\tau : V \to N_\tau = \mathrm{Ult}(V, U^W_{\kappa,\tau})$, and $k_\tau : N_\tau \to M_\tau$ defined by $k_\tau([f]_{U^W_{\kappa,\tau}}) = j_\tau(f)(\kappa)$. As usual, we have that k_τ is elementary, $j_\tau = k_\tau \circ e_\tau$, $\mathcal{P}(\kappa)^V = \mathcal{P}(\kappa)^{N_\tau} = \mathcal{P}(\kappa)^{M_\tau}$, and $\mathrm{cp}(k_\tau) = (\kappa^{++})^{N_\tau} > \kappa^+$. Denote $e_\tau(\vec{U})(\kappa)$ by $\bar{u}$ and let us verify $\bar{u} = \vec{U}(\kappa) \restriction i = \langle U_{\kappa,\nu} \mid \nu < i\rangle$. Define first $\bar{w} = e_\tau(\vec{W})(\kappa)$ and $\bar{u}^W = e_\tau(\vec{U}^W)(\kappa)$. The coherence of $\vec{W}$ implies $k_\tau(\bar{w}) = j_\tau(\vec{W})(\kappa) = \langle W_{\kappa,\rho} \mid \rho < \tau\rangle$ and thus $k_\tau(\bar{u}^w) = \langle U^W_{\kappa,\rho} \mid \rho < \tau\rangle$. $k_\tau(\bar{u})$ is the sequence obtained from $\langle U^W_{\kappa,\rho} \mid \rho < \tau\rangle$ by removing all its weak repeat points in M_τ which conincide with its weak repeat points in V since $\mathcal{P}(\kappa)^{M_\tau} = \mathcal{P}(\kappa)$. Therefore $k_\tau(\bar{u}) = \vec{U}(\kappa) \restriction i$. It is easy to see that $k_\tau(u) = u$ for each $u \in \bar{u}$, thus $\bar{u}$ is a subsequence of $\vec{U}(\kappa) \restriction i$. For the reverse inclusion, take $\nu < i$ and $U_{\kappa,\nu} \in \vec{U}(\kappa) \restriction i$. Let $\rho < \tau$ so that $U_{\kappa,\nu} = U^W_{\kappa,\rho}$. Since $U^W_{\kappa,\rho}$ is not a weak repeat point there exists $X \subset \kappa$ so that $U^W_{\kappa,\rho}$ is the first measure on $\vec{U}^W(\kappa) \restriction \tau$ which does not contain X. By elementarity and the fact $\mathcal{P}(\kappa) = \mathcal{P}(\kappa)^{N_\tau}$, there exists a measure on the sequence $\bar{u}^w(\kappa) \in N_\tau$ which does not contain X. Let u denote the first such measure. Then $u \in \bar{u}$ and $k_\tau(u)$ is the first measure in $\vec{U}^W$ which does not contain X. Hence $k_\tau(u) = U_{\kappa,\nu}$. $\square$

5.2. **A Radin/Mitchell Forcing.** We recall the definition of the Radin/Mitchell forcing $\mathbb{R}^* = \mathbb{R}(\vec{U}^*)$ associated with a coherent sequence of measures $\vec{U}^*$. We refer the reader to [6] for an extensive survey on Radin forcing an its properties.

Definition 5.5. *Let $\mathbb{R}^*$ denote the Radin/Magidor forcing in $V[G]$ via $\vec{U}^*$. We use the notation from* [6]. *Conditions in $\mathbb{R}^*$ are finite sequences of the form $r = \langle d_0, d_1, \ldots, d_n\rangle$, $n < \omega$ where each d_i is either an ordinal $\alpha_i = \kappa_i(r)$ with $o(\alpha_i) = 0$, or a pair $d_i = \langle \alpha_i, a_i\rangle$, $a_i = a_i(r)$, so that $o(\alpha_i) > 0$ and $a_i \in \mathcal{F}^*_{\alpha_i}$. $\kappa(d_n) = \kappa$.*
We denote n by $n(r)$ and define the support of r to be the finite set $\mathrm{s}(r) = \{\kappa(d_i) \mid i \leq n(r)\}$.
If r, r' are two conditions in $\mathbb{R}^*$ then r is a direct extension of r', $r \geq^* r'$ if $\mathrm{s}(r) = \mathrm{s}(r')$ *and $a_i(r) \subset a_i(r')$ for every $i \leq n(r) = n(r')$. We say that r is a one point extension of r' if there exists some $i \leq n(r')$ and $\alpha \in a_i(r) \setminus \kappa_{i-1}$ so that $r = \langle d_0, d_1, \ldots, d_{i-1}, d^*, d_i, \ldots, d_n\rangle$ where $d^* = \langle \alpha\rangle$ if $o(\alpha) = 0$, and $d^* = \langle \alpha, a_i \cap V_\alpha\rangle$ with $a_i \cap V_\alpha \in \mathcal{F}^*_\alpha$ otherwise. We say that r extends r', $r \geq r'$, if r is obtained from r' by a finite sequence of direct extensions and one point extensions.*

The following are established in [6].

Lemma 5.6. *Let $H \subset \mathbb{R}^*$ be a generic filter, then*
(1) *H adds a generic closed unbounded set $C \subset \kappa$ of order type κ to $V[G]$. Furthermore, H is uniquely determined by C.*
(2) *$\mathbb{R}^*$ satisfies the Prikry condition and no cardinals are collapsed in $V[G][H] = V[G][C]$.*
(3) *Since $o(\kappa) = \kappa^+$, then κ remains inaccessible in $V[G][C]$.*

Next, we recall Mitchell's characterization of genericity for a club $C \subset \kappa$.

Definition 5.7. *Let $\vec{U}$ be a coherent sequence of measures at κ and $\mathbb{R}$ be the Radin forcing defined from $\vec{U}$. We say that a club $C \subseteq \kappa$ is* geometric *for $\mathbb{R}$ if*
- *For every limit $\alpha \in C$, $C \cap \alpha$ generates a generic for the Radin forcing defined from $\vec{U} \restriction \alpha + 1$ and,*
- *$\mathcal{F}_\kappa = \{X \subseteq \kappa \mid \exists \beta < \kappa\ C \setminus \beta \subseteq X\}$.*

Theorem 5.8 (Mitchell [10])**.** *With $\mathbb{R}$ and $\vec{U}$ as in the previous definition, a club $C \subseteq \kappa$ generates an $\mathbb{R}$-generic filter if and only if it is geometric.*

Remark 5.9. *For $\mathbb{R}$ and $\mathbb{R}^*$ defined as above from $\vec{U}$ and $\vec{U}^*$ respectively, if C generates an $\mathbb{R}^*$-generic filter over $V[G]$ then it also generates an $\mathbb{R}$-generic filter over V by Corollary 5.2.*

5.3. **Homogeneity.** We proceed to analyze the quotient $\mathbb{P}*\mathbb{R}^*/\mathbb{R}$ with a focus on its homogeneity properties. We use the following fact about the homogeneity of $\mathbb{P}$ due to Dobrinen and Friedman [3].

Lemma 5.10. *For all $p, q \in \mathbb{P}$, there are $p^* \geq p$ and $q^* \geq q$ such that $\mathbb{P} \restriction p^* \simeq \mathbb{P} \restriction q^*$.*

Definition 5.11. *Let $\mathbb{D} \subset \mathbb{P} * \mathbb{R}^*$ consists of conditions $\langle p, \dot{r}\rangle$ so there exists some $n < \omega$ and $\alpha_0 < \alpha_1 < \cdots < \alpha_n = \kappa$ such that $p \Vdash n(\dot{r}) = \check{n}$ and $\kappa(\dot{d}_i) = \check{\alpha}_i$ for every $i \leq n$.*

Clearly $\mathbb{D}$ is a dense subset of $\mathbb{P} * \mathbb{R}^*$. Note that any automorphism π of $\mathbb{P}$ extends naturally to an automorphism $\hat{\pi}$ of the name space $V^{\mathbb{P}}$.

Lemma 5.12. *For all $(p_0, \dot{r}_0), (p_1, \dot{r}_1) \in \mathbb{D}$ such that $n = n(\dot{r}_0) = n(\dot{r}_1)$ and for all $i < n$, $\kappa(d_i^{\dot{r}_0}) = \kappa(d_i^{\dot{r}_1})$, there are $p_0^* \geq p_0$, $p_1^* \geq p_1$ and an automorphism π of $\mathbb{P}$ such that $\pi(p_0^*) = p_1^*$ and $p_1^* \Vdash \hat{\pi}(\dot{r}_0)$ is direct extension compatible with $\dot{r}_1$.*

Proof. Using Fact 5.10, we can find p_0^*, p_1^* and an isomorphism $\pi : \mathbb{P} \restriction p_0^* \to \mathbb{P} \restriction p_1^*$. It is straightforward to see that $\hat{\pi}$ is an isomorphism between the name spaces $V^{\mathbb{P}\restriction p_0^*}$ and $V^{\mathbb{P}\restriction p_1^*}$.

Since $(p_0^*, \dot{r}_0)$ and $(p_1^*, \dot{r}_1)$ are in $\mathbb{D}$, we have that p_1^* forces that $\hat{\pi}(\dot{r}_0)$ and $\dot{r}_1$ have the same ordinal parts. It follows that p_1^* forces that $\hat{\pi}(\dot{r}_0)$ and $\dot{r}_1$ are direct extension compatible. □

Let C be club in κ which generates an $\mathbb{R}$-generic filter. In $V[C]$ we define $D_C = \{(p, \dot{r}) \in \mathbb{D} \mid \forall b \in [C]^{<\omega}$ if $\mathrm{s}(\dot{r}) \subseteq b$ then $\exists (p', \dot{r}') \in \mathbb{D}$ with $(p', \dot{r}') \geq (p, \dot{r})$ such that $\mathrm{s}(\dot{r}') = b\}$. Let $\mathbb{D}/C$ be the usual quotient forcing as defined in the $\mathbb{R}$ generic extension $V[C]$. For $(p, \dot{r}) \in \mathbb{D}$, it is easy to see that if $(p, \dot{r}) \notin D_C$ then for some $b \in [C]^{<\omega}$, $(p, \dot{r})$ forces that b is not contained in the generic club. So $(p, \dot{r}) \notin \mathbb{D}/C$, since there cannot be a generic object for $\mathbb{P}*\mathbb{R}^*$ containing $(p, \dot{r})$ whose Radin club is C. It follows that $\mathbb{D}/C \subset D_C$.

Claim 5.13. *If $G * C$ is generic for $\mathbb{P} * \mathbb{R}^*$, then* $\mathrm{HOD}^{V[G*C]} \subseteq V[C]$.

Proof. Let $\varphi(x, \vec{\alpha})$ be a formula with ordinal parameters. We will show that $A = \{\beta \mid \varphi(\beta, \vec{\alpha})^{V[G*C]}\}$ is in $V[C]$. Suppose $\varphi(\beta, \vec{\alpha})$ holds in

$V[G * C]$ for some β. Then there is a $(p_0, \dot{r}_0) \in \mathbb{D}/C$ (in particular in D_C) such that $(p_0, \dot{r}_0) \Vdash \varphi(\beta, \vec{\alpha})$.

We claim that there is no $(p_1, \dot{r}_1) \in D_C$ which forces $\neg\varphi(\beta, \vec{\alpha})$. Otherwise, we fix a counterexample $(p_1, \dot{r}_1)$ and by the definition of D_C, for $i \in 2$ we can extend $(p_i, \dot{r}_i)$ to $(p'_i, \dot{r}'_i) \in \mathbb{D}$ such that $\mathrm{s}(\dot{r}'_0) = \mathrm{s}(\dot{r}'_1)$. Now we can apply Lemma 5.12 to find automorphisms π and $\hat{\pi}$ and extensions p^*_i of p'_i such that $\pi(p^*_0) = p^*_1$ and p^*_1 forces that $\hat{\pi}(\dot{r}'_0)$ is direct extension compatible with $\dot{r}'_1$. This is a contradiction since $(p^*_1, \hat{\pi}(\dot{r}'_0))$ forces $\varphi(\beta, \vec{\alpha})$ and $(p^*_1, \dot{r}'_1)$ forces $\neg\varphi(\beta, \vec{\alpha})$. A similar argument applies to $\neg\varphi(x, \vec{\alpha})$.

So we can define the set A in $V[C]$ as the set $\{\beta \mid \exists t \in \mathbb{D}_C \ t \Vdash \varphi(\beta, \vec{\alpha})\}$. □

We are now ready to prove Theorem 1.4.

Proof. By standard coding methods, we may assume $\mathrm{HOD}^V = V$ and V is contained in HOD of as computed in the further generic extensions that we consider. Let $G * C \subset \mathbb{P} * \mathbb{R}^*$ be generic over V and $\langle \kappa_i \mid i < \kappa \rangle$ be a continuous increasing enumeration of C. Working in $V[G * C]$, let $\mathbb{Q}$ be an iteration of Levy collapses for collapsing each κ_{i+1} to $\kappa_i^+ = \theta_{\kappa_i}$. It is easy to see that $\pi(\mathbb{Q}) = \mathbb{Q}$ for every automorphism π of $\mathbb{P}$, hence, if $H \subset \mathbb{Q}$ is a $V[G * C]$ generic filter then by the results of [3], $\mathrm{HOD}^{V[G*C*H]} \subset V[C]$. Since $o(\kappa) = \kappa^+$ then by standard arguments about Radin forcing (see [6]) all limit points in C are singular and κ is regular in $V[G*C*H]$. It follows that each regular uncountable cardinal θ below κ in $V[G * C * H]$ is of the form $\theta = \theta_{\kappa_i}$ for some $i < \kappa$. Since the Radin club C is bounded below each θ_{κ_i} and $\min(C \setminus \theta_{\kappa_i}) > 2^{\theta^+_{\kappa_i}}$, it follows that θ_{κ_i} is $\theta^+_{\kappa_i}$-supercompact in HOD. □

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