Disassociated Indiscernibles

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Abstract

This work defines a variant of Prikry forcing and applies it to increase the number of normal measures on a single cardinal, relying on a large cardinal property of consistency strength between measurable and Mitchell order two. In particular, from an assumption weaker than $o(\kappa) = 2$, there is a forcing extension in which the first measurable cardinal κ has any number $\lambda \leq \kappa$ of normal measures.

1 Introduction

This paper is generally directed to a variant of Prikry Forcing [3]. Moti Gitik provides a survey of other variants in [5].

The following is an outline. Section 1 contains preliminary material from which the author draws inspiration and much technique. It reviews Karel Prikry's notion of forcing for altering the cofinality of a measurable cardinal and Magidor's iteration of Prikry's technique, which leaves some of the cardinals measurable. It states the Prikry Property, which is essential for showing that cardinals are preserved. And it discusses some intuition behind the term "indiscernibles." Section 2 introduces a forcing for what is essentially the union of the range of a function generic for Magidor's iterated Prikry forcing. As an application, Section 3 proves that exactly two normal measures on the least measurable cardinal requires less consistency strength than that of Mitchell order two. Section 4 defines an iterated version of this forcing, and Section 5 uses it to produce any number $\lambda \leq \kappa$ of measures on κ , which becomes the *least measurable cardinal*, in the presence of less consistency strength than that of a measurable cardinal whose measure concentrates on measurable cardinals.

The motivation of this work was the folk question: How many normal measures can a measurable cardinal have? Kunen and Paris [2] forced a measurable cardinal κ to have the maximal number, 2^{2^κ}, of normal measures. Relative to a measurable cardinal κ of order $o(\kappa) = 2$, Mitchell [8] proved that an inner model with exactly two normal measures κ is possible where one of the measures concentrates on measurable cardinals below κ. Stewart Baldwin [1] showed that under the same assumption of, it is possible to construct an inner model where the *only* measurable cardinal κ has two normal measures. Recently, Magidor and Friedman [4], achieved models with arbitrary number normal measures

$\lambda \leq \kappa^{++}$ on the first measurable cardinal κ from the miminal assumption of one measurable cardinal.

This work is based on the first author's doctoral dissertation at the University of Florida (1999). The author wishes to thank his advisor, Professor William J. Mitchell, for indispensable support and insight throughout the process. The second author had merely made some corrections and clerafication in some of the ideas and proofs in order complete a publishable version of this fascinating work.

1.1 Ultrafilters and Ultrapowers

Definition 1. A *measure* is a κ-complete nonprincipal ultrafilter on a cardinal κ.

If Lebesgue measure is relaxed to remove its translation invariance requirement and restricted to two values instead of non-negative real values, then a measure as defined above corresponds to the set of Lebesgue measure-one sets.

The main reason that set theorists are interested in measures is that the corresponding ultrapowers (Definition 2) of absolutely well-founded models of set theory are absolutely well-founded. Hence the ultrapower of the class of all sets $\mathbf{V}$ is embeddable into $\mathbf{V}$ by Mostowski's collapse isomorphism. This gives an elementary embedding of the universe into a proper subclass of itself.

Definition 2. Let U be a measure on κ. Consider the equivalence relation on functions $f, g : \kappa \to \kappa$

$$f \sim g \text{ iff } \{\eta < \kappa : f(\eta) = g(\eta)\} \in U.$$

Let $\mathrm{Ult}(\mathbf{V}, U)$ be the model whose universe consists of equivalence classes $[f]$ of functions $f : \kappa \to \kappa$ under the relation $\sim$. Interpret "$\in$" in $\mathrm{Ult}(\mathbf{V}, U)$ as

$$[f] \in [g] \text{ iff } \{\eta < \kappa : f(\eta) \in g(\eta)\} \in U.$$

Let $j_U : \mathbf{V} \to \mathfrak{M} \approx \mathrm{Ult}(\mathbf{V}, U)$ be the natural map given by $x \mapsto [c_x]$ where c_x is the constant function $c_x(\eta) = x$, and $\mathfrak{M}$ is Mostowski's transitive isomorph of the ultrapower.

An element of $\mathrm{Ult}(\mathbf{V}, U)$ is a class of functions that are equal "almost everywhere," to use the measure theoretic terminology, modulo U. Elements are included in other elements if their representative functions are pointwise included almost everywhere.

Definition 3. A measure U on κ is *normal* if every regressive ($\forall \alpha\ f(\alpha) < \alpha$) function $f : \kappa \to \kappa$ is constant on some set $X \in U$.

The property of normality gives a clear description of which equivalence class of functions represents which set in the ultrapower. In fact, for normal measures, the equivalence class of the identity function $f(\alpha) = \alpha$ represents the cardinal κ.

Definition 4. Let $(X_\eta)_{\eta<\kappa}$ be a collection of subsets of κ. The *diagonal intersection* of these sets is the set $\triangle_{\eta<\kappa} X_\eta = \{\gamma < \kappa : \forall \eta < \gamma\, (\gamma \in X_\eta)\}$.

Proposition 1. *Let U be a measure on κ. The following are equivalent.*

1. *U is normal.*
2. *$\forall X \subset \kappa (X \in U \iff \kappa \in j_U(X))$.*
3. *U is closed under diagonal intersection.*

Definitions 14 and 21 are essentially forcing-syntactic versions of property 2 above.

This work uses Theorem 2, due to Rowbottom, infinitely often.

Theorem 2. *Let U be a normal measure on κ. Let $f : [\kappa]^{<\omega} \to \eta$, where $\eta < \kappa$. There exists a set $X \in U$ such that f is constant on $[X]^n$ for each $n < \omega$.*

The set X above is said to be *homogeneous for f*.

1.2 Prikry Forcing

Prikry forces an unbounded sequence of order type ω through a measurable cardinal in his dissertation [3].

Let κ be a measurable cardinal and let U_κ be a normal measure on κ.

Definition 5. *Prikry forcing,* $\mathbb{P}(U_\kappa)$, has as *conditions* pairs $\langle s, X\rangle$ such that

1. $s \in [\kappa]^{<\omega}$
2. $X \in U_\kappa$.

A condition *extends* another, $\langle s, X\rangle \leq \langle s', X'\rangle$, iff

1. s is an end extension of s'
2. $X \subset X'$
3. $s \setminus s' \subset X'$.

If G is $\mathbb{P}(U_\kappa)$ generic, let $\mathfrak{S} = \cup\{s : \exists X \in U_\kappa \langle s, X\rangle \in G\}$. The set $\mathfrak{S}$ has order type ω, and a standard density argument proves that it is unbounded in κ. After forcing with $\mathbb{P}(U_\kappa)$ the cardinal κ has cofinality ω. A crucial component in the theory is the following.

Theorem 3. (Prikry Property) *Let σ be a formula in the language of $\mathbb{P}(U_\kappa)$ and let $\langle t, X\rangle$ be any condition. Then there exists a condition $\langle t, X'\rangle \leq \langle t, X\rangle$ that decides σ, that is, such that either $\langle t, X\rangle \Vdash \sigma$ or $\langle t, X\rangle \Vdash \neg\sigma$.*

The notation "$p \parallel \sigma$" means "p decides σ".

Proof. In interest of notational simplicity we assume that $t = 0$. Proof for the general version can be easily carried out by applying the same statements above $\max(t)$. For each $t \in [\mu]^{<\omega}$, let $f^s : \kappa \setminus \max(t) + 1 \to 3$ via

$$f_s(\gamma) \quad = \quad \begin{cases} 0 & \text{if } \exists X \ \langle s \cup \{\gamma\}, X\rangle \Vdash \sigma, \\ 1 & \text{if } \exists X \ \langle s \cup \{\gamma\}, X\rangle \Vdash \neg\, \sigma, \\ 2 & \text{otherwise.} \end{cases}$$

Let $H_s \in U_\kappa$ be homogeneous for f^s. Thus, for every γ, γ' in $H = \triangle_{s\in[\kappa]^{<\omega}} H_s$ and $s \in [\kappa]^{<\omega}$ below both ordinals, we have

$$\exists X \ \langle s \cup \{\gamma\}, X\rangle \parallel \sigma \text{ if and only if } \exists X \ \langle s \cup \{\gamma'\}, X\rangle \parallel \sigma.$$

Moreover, these conditions decide σ the same way. We attend to show that the condition $\langle 0, X \cap H\rangle$ has a direct extension which decides σ. Suppose this is not ture. Take a condition $\langle s^\frown\{\delta\}, Z\rangle$ with the shortest entry (i.e. $|s^\frown\{\delta\}|$ is minimal) which decides σ. We obtain a contradiction by showing that by changing the measure one set Z, we can obtain a condition of the form $\langle s, Z'\rangle$ which decides σ as well. WLG suppose that $\langle s^\frown\{\delta\}, Z\rangle \Vdash \sigma$. $\delta \in H$ and $s \ll \delta$ implies that $\delta \in H_s$. Thus, for any $\gamma \in H_s$ there is some $X_\gamma \in U_\kappa$ for which, $\langle s^\frown\{\gamma\}, X_\gamma\rangle \Vdash \sigma$. Set $Y = \triangle_{\gamma<\kappa} X_\gamma$ and let $Z' = X \cap H \cap Y$. It follows that a condition $\langle s^\frown\{\gamma\}^\frown t, Z''\rangle$ which is a non-direct extension of $\langle s, Z'\rangle$ (here $s \ll \gamma \ll t$ and t might be empty), cannot force $\neg\sigma$ because it is compatible with $\langle s^\frown\{\gamma\}, X_\gamma\rangle$ which force σ. This promises that $\langle s, Z'\rangle \Vdash \sigma$ which stands in contradiction to the minimality of $|s^\frown\{\delta\}|$. □

Prikry conditions $\langle s, X\rangle, \langle s, Y\rangle$ are incompatible only if $s \neq s'$. Hence $\mathbb{P}(\mathcal{U}_\kappa)$ has the κ^+-chain condition. This fact viewed in light of Theorem 3 yields:

Corollary 4. *Prikry forcing preserves cardinals.*

The rest of this section to explores some properties of $\mathfrak{S}$.

Definition 6. Let $\mathfrak{M} = \langle M, <, =\rangle$ be a model and let I be a simply ordered subclass of M. I consists of *first-order indiscernibles over* $\mathfrak{M}$ iff for any formula $\varphi(x_1, \ldots, x_n)$ and any sequences $\gamma_1 < \cdots < \gamma_n$ and $\gamma'_1 < \cdots < \gamma'_n$ from I,

$$\mathfrak{M} \models \varphi[\gamma_1, \ldots, \gamma_n] \iff \mathfrak{M} \models \varphi[\gamma'_1, \ldots, \gamma'_n].$$

The next theorem is intended to explain the term "indiscernibles" in referring to a Prikry sequence. Kunen noticed the first part, and Solovay, Mathias, and Kunen independently proved the second.

Theorem 5. *Let U_0 be a normal ultrafilter on κ_0 in an inner model $\mathfrak{M}_0$ of ZFC. Let $i_{n,n+1} : \mathfrak{M}_n \to \mathfrak{M}_{n+1} \approx \mathrm{Ult}(\mathfrak{M}_n, U_n)$ be the map from $\mathfrak{M}_n$ to the transitive collapse of the ultrapower of $\mathfrak{M}_n$ by U_n. Denote $U_{n+1} = i_{n,n+1}(U_n)$ and $\kappa_{n+1} = i_{n,n+1}(\kappa_n)$. Let $\mathfrak{M}_\omega$ be the transitive collapse of the direct limit of the above system of ultrapowers, and let κ_ω and U_ω be the images of κ_0 and U_0 respectively. Finally, denote $\mathfrak{S} = \{\kappa_n : n < \omega\}$. Then:*

1. $\mathfrak{S}$ *is a set of first-order indiscernibles over* $\mathfrak{M}_\omega$.

2. $\mathfrak{S}$ *is* $\mathbb{P}(U_\omega)$*-generic over* $\mathfrak{M}_\omega$.

Although Prikry sequences do not generally consist of first-order indiscernibles in the above sense, this is nearly the case.

Theorem 6. *Let* $\mathfrak{S}$ *be* $\mathbb{P}(U)$*-generic, where* U *is a measure on* κ*. Then for each formula* $\varphi(x_1, \dots, x_n)$ *there is a* $\lambda_\varphi < \kappa$ *such that for any sequences* $\gamma_1 < \dots < \gamma_n$*,* $\gamma'_1 < \dots < \gamma'_n$ *from* $\mathfrak{S} \setminus \lambda_\varphi$*,*

$$\varphi[\gamma_1, \dots, \gamma_n] \iff \varphi[\gamma'_1, \dots, \gamma'_n].$$

Proof. First, let U^n be the ultrafilter consisting of sets X such that for some $H \in U$, $[H]^n \subset X$. Consider the sets

$$T = \{(\gamma_1, \dots, \gamma_n) : \gamma_1 < \dots < \gamma_n < \kappa \text{ and } \varphi[\gamma_1, \dots, \gamma_n]\},$$

$$F = \{(\gamma_1, \dots, \gamma_n) : \gamma_1 < \dots < \gamma_n < \kappa \text{ and } \neg\, \varphi[\gamma_1, \dots, \gamma_n]\}.$$

Because T is the complement of F, one of these sets must be in U^n. Without loss of generality, suppose $T \in U^n$. Thus, for some $H \in U$, $[H]^n \subset T$. It follows that the set $\{\langle s, X\rangle : [X]^n \subset T\}$ is dense in $\mathbb{P}(U)$. Hence all but finitely many elements of $\mathfrak{S}$ come from H. The theorem follows. □

The above theorem is likely close to the best possible, in that there are Prikry generic sequences that are not first order indiscernibles over the ground model.

Theorem 7. *If there is a transitive model of "ZFC + there is a measurable cardinal" then there is a model* $\mathfrak{M}$ *of the same theory and a class of parameter-free formulas* $\{\varphi_x(v)\}_{x\in\mathfrak{M}}$ *such that each* φ_x *is true in* $\mathfrak{M}$ *for* x *and only* x*.*

Proof. Let $\mathfrak{N}$ be the set of elements definable over $L_\alpha[U]$ by parameter-free formulas. That is, let:

$$\mathfrak{N} = \{x \in L_\alpha[U] : \text{for some formula } \phi_x,\ L_\alpha[U] \models \phi_x[x] \ \&\ \forall y\, (\phi_x[y] \implies y = x)\}.$$

The claim is that $\mathfrak{N}$ is an elementary submodel of $L_\alpha[U]$. Proving this involves first proving that existential formulas with parameters from $\mathfrak{N}$ that hold in $L_\alpha[U]$ also hold in $\mathfrak{N}$. The converse is similar. Suppose

$$L_\alpha[U] \models \exists v\ \varphi[v, a_1, \dots, a_n],$$

where $a_1, \dots, a_n \in \mathfrak{N}$. Because $a_1, \dots, a_n$ are definable without parameters, assume that φ has only one free variable. The model $L_\alpha[U]$ has a definable well-ordering $\leq_{L[U]}$ of its elements. Let ψ be the definable Skolem function for $\exists v\, \varphi(v)$. That is, let $\psi(v) \equiv \varphi(v) \wedge \forall v'(\varphi(v') \to v \leq_{L[U]} v')$. This formula defines an element x in $L_\alpha[U]$ without parameters, and x satisfies φ. It follows that $x \in \mathfrak{N}$, and moreover, $\mathfrak{N} \models \varphi[x]$. Thus $\mathfrak{N} \models \exists v\, \varphi(v)$. This establishes the claim.

Let $i : \mathfrak{N} \to \mathfrak{M}$ be the transitive collapse. Then $\mathfrak{M}$, together with the set of formulas $\{\phi_{i^{-1}(x)}(v)\}_{x\in\mathfrak{M}}$, satisfy the theorem. □

The property of the above class of formulas precludes getting any class of first-order indiscernibles for this model. In particular, no Prikry generic sequence can consist of first-order indiscernibles, because for each element γ of such a sequence there is a formula ϕ_γ satisfied only by γ.

1.3 Iterated Prikry Forcing

Magidor [6] formulated his iterated Prikry forcing to obtain results comparing compact cardinals with both measurable and supercompact cardinals. Starting with a compact cardinal κ, he uses forcing to kill all the measurable cardinals below while preserving the compactness of κ, thereby establishing that that the first compact cardinal can be the first measurable cardinal. He also proves that the first supercompact cardinal can be the first compact cardinal. Although Magidor defines the forcing as an iterated Prikry forcing, it suffices to use the following definition, which avoids using forcing names; see the proof of his Theorem 4.3.

Let Δ be a set of measurable cardinals. For each $\mu \in \Delta$, pick a normal measure U_μ on μ not containing the set of measurable cardinals below μ. Denote $\mathcal{U} = (U_\mu)_{\mu\in\Delta}$.

Definition 7. The *iterated Prikry forcing for $\mathcal{U}$,* denoted $\mathbb{M}(\mathcal{U})$, has as *conditions* pairs $\langle s_\mu, X_\mu\rangle_{\mu\in\Delta}$ such that

1. for each $\mu \in \Delta$, $s_\mu \in [\mu]^{<\omega}$
2. for each $\mu \in \Delta$, $X_\mu \in U_\mu$
3. $\cup_{\mu\in\Delta} s_\mu$ is finite.

A condition *extends* another, $\langle s_\mu, X_\mu\rangle_{\mu\in\Delta} \leq \langle s'_\mu, X'_\mu\rangle_{\mu\in\Delta}$, iff for each $\mu \in \Delta$, $\langle s_\mu, X_\mu\rangle \leq \langle s'_\mu, X'_\mu\rangle$ in the $\mathbb{P}(U_\mu)$ order.

If G is $\mathbb{M}(\mathcal{U})$ generic, then for each $\mu \in \Delta$, let $\mathfrak{S}_\mu = \cup\{s_\mu : \exists\langle s_\mu, X_\mu\rangle_{\mu\in\Delta} \in G\}$. Each $\mathfrak{S}_\mu$ has order type ω. Magidor's forcing adds a function $\vec{\mathfrak{S}} = \{\mathfrak{S}_\mu\}_{\mu\in\Delta}$, which associates to each measurable cardinal μ in the domain a Prikry sequence $\mathfrak{S}_\mu \subset \mu$. This work describes a forcing that adds a set of indiscernibles that are essentially the union of the range of this function, $\cup_{\mu\in\Delta}\mathfrak{S}_\mu$.

Theorem 8. *$\mathbb{M}(\mathcal{U})$ satisfies the Prikry Property.*

Theorem 9. *Iterated Prikry forcing preserves cardinals.*

2 Disassociated Indiscernible Forcing

Let Δ be a set of measurable cardinals. For each $\mu \in \Delta$, pick a normal measure U_μ on μ that gives measure zero to the set of measurable cardinals below μ. Denote

$$\mathcal{U} = (U_\mu)_{\mu\in\Delta},$$

$$\kappa = \sup(\Delta).$$

These conventions will remain fixed throughout the rest of this paper. For convenience, assume that the only normal measures in the universe are those that appear in $\mathcal{U}$. The core model [9] for this sequence provides a suitable ground model.

The following definitions are relative to $\mathcal{U}$.

Definition 8. Let the *filter of long measure one sets* be

$$\mathfrak{L}(\mathcal{U}) = \{\mathcal{X} \subset \kappa : \mathcal{X} \cap \mu \in U_\mu \text{ for all } \mu \in \Delta\}.$$

Definition 9. The *disassociated indiscernible forcing*, $\mathbb{D}_\lambda(\mathcal{U})$*, has as conditions* pairs $\langle s, \mathcal{X}\rangle$ such that

1. $s \in [\lambda]^{<\omega}$
2. $\mathcal{X} \in \mathfrak{L}(\mathcal{U})$.

A condition *extends* another, $\langle s, \mathcal{X}\rangle \leq \langle s', \mathcal{X}'\rangle$, iff

1. $s \supset s'$,
2. $\mathcal{X} \subset \mathcal{X}'$
3. $s \setminus s' \subset \mathcal{X}'$.

For $p = \langle s, \mathcal{X}\rangle \in \mathbb{D}_\lambda(\mathcal{U})$, let the *support of* p*, denoted by* $\text{supt}(p)$*, be the set* s. A condition p is a *direct extension* of q, denoted $p \leq^* q$, if $p \leq q$ and $\text{supt}(p) = \text{supt}(q)$.

Whenever $\kappa = \lambda$ we shortly write $\mathbb{D}(\mathcal{U})$ for $\mathbb{D}_\kappa(\mathcal{U})$.

The conditions and extension criteria of $\mathbb{D}(\mathcal{U})$ closely resemble those for iterated Prikry forcing [6]. The main difference is that instead of getting a function associating cardinals with sets of indiscernibles, the present forcing provides a *single* set of indiscernibles: denote $\mathfrak{S} = \{\eta < \kappa : \exists p \in G \text{ such that } \eta \in \text{supt}(p)\}$.

Note that when $\lambda < \kappa$ we have $\mathbb{D}_\lambda(\mathcal{U}) = \mathbb{D}(\mathcal{U}\restriction_{\lambda+1})$ (that is, forcing with long measure one set for the measurable cardinals $\leq \lambda$).
Also note that if $\lambda > \kappa$, then by a simple density argument $\mathfrak{S} \setminus \lambda$ is finite.

This work generally abides by the following conventions. Lower case letters denote natural numbers $(i, j, n, \dots)$, finite sets of ordinals $(s, t, u, \dots)$, or conditions $(p, q, r, \dots)$. Upper case letters $(U, X, Z, \dots)$ denote measures or measure one sets. Upper case script letters $(\mathcal{U}, \mathcal{X}, \mathcal{Z}, \dots)$ denote sequences of measures or long measure one sets. Lower case Greek letters $(\delta, \gamma, \eta, \dots)$ usually represent ordinals.

2.1 Basic Structure

Definition 10. Relative to $\mathcal{U}$, a measure U_κ is a *vertical repeat point* if for all $X \in U_\kappa$ there exists some $\mu < \kappa$ such that $X \cap \mu \in U_\mu$.

The idea is that the measure one sets of U_κ are entangled in measure one sets from cardinals below. The following proposition shows that a vertical repeat point has strictly less consistency strength than a measure of Mitchell order two.

Proposition 10. *Let $L[\mathcal{U}]$ be the core model for the existence of a measure of Mitchell order two on some cardinal κ (see* **??** *for an extensive survey) . Then $\mathcal{U}(\kappa, 1)$ contains the set of cardinals $\mu < \kappa$ such that $\mathcal{U}(\mu, 0)$ is a vertical repeat point.*

Proof. The class $\mathcal{U}$ is a function such that each $\mathcal{U}(\mu, \gamma)$ is a normal measure on μ and γ is its Mitchell order index.

First note that $\mathcal{U}(\kappa, 0)$ itself is a vertical repeat point. In fact, consider the ultrapower $j_{\mathcal{U}(\kappa,1)} : L[\mathcal{U}] \to L[\mathcal{U}'] \approx \mathrm{Ult}(L[\mathcal{U}], \mathcal{U}(\kappa, 1))$. Note that in $L[\mathcal{U}']$, $j_{\mathcal{U}(\kappa,1)}(\kappa)$ is measurable. Let $X \in \mathcal{U}(\kappa, 0)$. By coherence, $\mathcal{U}(\kappa, 0) = \mathcal{U}'(\kappa, 0)$. But then $L[\mathcal{U}'] \models X = \kappa \cap j_{\mathcal{U}(\kappa,1)}(X) \in \mathcal{U}'(\kappa, 0)$. By elementarity, there exists some $\eta < \kappa$ such that $\eta \cap X \in \mathcal{U}(\eta, 0)$. Hence $\mathcal{U}(\kappa, 0)$ is a vertical repeat point. Now note that $\mathcal{U}(\kappa, 0)$ remains a vertical repeat point in $L[\mathcal{U}']$. By normality, $\kappa = [I]_{\mathcal{U}(\kappa,1)}$, where I is the identity function. The proposition then follows from Łoś' Theorem. □

Proposition 11. *Suppose* $\sup \Delta = \kappa \in \Delta$ *and* $\lambda \geq \kappa$*.*

1. *If U_κ is a vertical repeat point, then $\mathbb{D}_\lambda(\mathcal{U}) = \mathbb{D}_\lambda(\mathcal{U} \restriction \kappa)$.*
2. *If U_κ is not a vertical repeat point, then forcing with $\mathbb{D}_\lambda(\mathcal{U})$ is equivalent to forcing with $\mathbb{D}_\lambda(\mathcal{U} \restriction \kappa) \times \mathbb{P}(U_\kappa)$, where $\mathbb{P}(U_\kappa)$ is Prikry forcing for U_κ.*

"Equivalent" in the second part above means that a generic object for one forcing is constructible from a generic object for the other.

Proof.

1. It suffices to show that any set long measure one for $\mathcal{U} \restriction \kappa$ is long measure one for $\mathcal{U}$. Let $\mathcal{X}$ be long measure one for $\mathcal{U} \restriction \kappa$. Suppose, for the sake of a contradcution, that $\mathcal{X} \notin U_\kappa$. Then $\kappa \setminus \mathcal{X}$ has measure one. Because U_κ is a vertical repeat point, for some $\gamma < \kappa$, $\gamma \cap (\kappa \setminus \mathcal{X}) \in U_\gamma$. But then $\gamma \cap \mathcal{X} \notin U_\gamma$. This contradicts that $\mathcal{X}$ is long measure one for $\mathcal{U} \restriction \kappa$.

2. Let N be a set in U_κ witnessing that κ is not a vertical repeat point. In other words, let $N \in U_\kappa$ be such that for all $\mu < \kappa$, $\mu \cap N \notin U_\mu$. Consider the map
$$i : \mathbb{D}_\lambda(\mathcal{U}) \to \mathbb{D}_{\lambda \setminus N}(\mathcal{U} \restriction \kappa) \times \mathbb{P}^{\subset}_N(U_\kappa),$$
given by $\langle s, \mathcal{X} \rangle \mapsto \langle \langle s \setminus N, \mathcal{X} \setminus N \rangle, \langle s \cap N, \mathcal{X} \cap N \rangle \rangle$, where $\mathbb{D}_{\lambda \setminus N}(\mathcal{U} \restriction \kappa)$ is the disassociated indiscernible forcing for $\mathcal{U} \restriction \kappa$ below the condition $\langle 0, \lambda \setminus$

$N\rangle$, and $\mathbb{P}^{\subseteq}_{N}(U_\kappa)$ is Prikry forcing below the condition $\langle 0, N\rangle$ with the extension of conditions, $\langle s, \mathcal{X}\rangle \leq \langle s', \mathcal{X}'\rangle$, broadened to allow $s \supset s'$ instead of just s an end extension of s'. It is simple to confirm that i is a complete embedding. That is, i preserves incompatibility and extension of conditions, and i"$\mathbb{D}(\mathcal{U})$ is a dense subset of $\mathbb{D}_{\lambda\setminus N}(\mathcal{U}\restriction\kappa) \times \mathbb{P}^{\subseteq}_{N}(U_\kappa)$. It follows that the domain and target of i are equivalent as notions of forcing. The first term in the target product is obviously equivalent to $\mathbb{D}_\lambda(\mathcal{U}\restriction\kappa)$, so it remains to prove that $\mathbb{P}^{\subseteq}_{N}(U_\kappa)$ is equivalent to Prikry forcing. Consider the map

$$\iota : \mathbb{P}(U_\kappa) \to \mathbb{P}^{\subseteq}_{N}(U_\kappa)$$

given by $\langle s, X\rangle \overset{\iota}{\mapsto} \langle s \cap N, (X \cap N) \setminus \max(s) + 1\rangle$. This map is also a complete embedding, and this concludes the proof of the proposition.

□

The above proposition provides a clue of how disassociated indiscernibles forced from vertical repeat points might behave.

2.2 Prikry Property

Definition 11. The relation $s \ll t$ means that s, t are ordinals or finite sets of ordinals (they need not be the same type of object) and that $\sup(s) < \sup(t)$.

Definition 12. Let $(\mathcal{X}_s)_{s\in[\kappa]^{<\omega}}$ be a sequence of sets. The *diagonal intersection* of this collection is the set

$$\triangle_{s\in[\kappa]^{<\omega}}\mathcal{X}_s = \{\gamma < \kappa : \forall s \ll \gamma(\gamma \in \mathcal{X}_s)\}.$$

Proposition 12. *Let $\mathcal{X}_s \in \mathfrak{L}(\mathcal{U})$ for $s \in [\kappa]^{<\omega}$. Then $\triangle_{s\in[\kappa]^{<\omega}}\mathcal{X}_s \in \mathfrak{L}(\mathcal{U})$.*

Proof. Let $\mathcal{A}$ be the diagonal intersection. Fix $\mu \in \Delta$. Then $\mathcal{A}\cap\mu \in U_\mu$. To see this, consider the ultrapower map $i_{U_\mu} : \mathbf{V} \to \mathfrak{M} \approx Ult(\mathbf{V}, U_\mu)$. The proposition will follow from a proof that $\mu \in i_{U_\mu}(\mathcal{A})$. Denote $\vec{\mathcal{X}} = (\mathcal{X}_s)_{s\in[\kappa]^{<\omega}}$. It suffices to show that for $s \ll \mu$, $\mu \in i_{U_\mu}(\vec{\mathcal{X}})_s$. Notice that for $s \ll \mu$, $i_{U_\mu}(\vec{\mathcal{X}})_s = i_{U_\mu}(\mathcal{X}_s)$. Because $\mathcal{X}_s \cap \mu \in U_\mu$ for $s \ll \mu$, $\mu \in i_{U_\mu}(\mathcal{X}_s)$. The theorem follows. □

W describe here some differences between Prikry's original forcing notion and the new Disassociated Indiscernible forcing. We will conclude what are the significant changes needed to be carried out in the original Prikry property proof (see 3) in order to achieve this property for the later. Both forcings use large sets which are closed under diagonal intersections, thus enhance our ability to produce a direct extension which is *strongly compatible* with a certain collection of conditions. By *strongly compatible* we mean that every extension of this new condition is compatible with almost every condition from the large collection. Therefor, the new condition has to agree with every statement forced by the conditions of the collection. This fact is highly used in the original proof for

Prikry condition for Prikry forcing. Indeed, following the proof of 3 we see that in order to achieve contradiction to the minimality for the size of an entry which decide σ we cooked up U_κ-measure one conditions, $\langle s^\frown\{\gamma\}, X_\gamma\rangle$ above the same entry $s \in [\kappa]^{<\omega}$, all force σ. For the next step we took the diagonal intersection $X = \triangle_{\gamma<\kappa} X_\gamma$ and intended to prove $\langle s, X\rangle \Vdash \sigma$. The latter condition achieved property that after adding some γ from the measure one set to s, any new possible members for the entry of $\langle s^\frown\{\gamma\}, X\rangle$ must be above γ and therefor be taken from X_γ. The same argument will not hold in our new forcing since in $\mathbb{D}(\mathcal{U})$ it possible that ordinals *below* γ will be added to the entry $s \cap \{\gamma\}$. Therefore the diagonal intersection will not be sufficient. We need a stronger version of uniformization property in order to produce a direct extension which is strongly compatible with our collection. The following lemma is responsible for producing us with one.

Lemma 13. *Fix a measurable cardinal μ with a normal measure U_μ. For every list of sets $\{\mathcal{X}_\gamma\}_{\gamma<\mu}$ There is a set x such that for $U_\mu-$almost all $\gamma < \mu$, $x \cap \gamma = \mathcal{X}_\gamma \cap \gamma$.*

Proof. Let $x = \{\alpha < \mu : \{\gamma < \mu : \alpha \in \mathcal{X}_\gamma\} \in U_\mu\}$ (Note that x is not other then the set of $\mathrm{Ult}(\mathbf{V}, U_\mu)$ which is represented by the function $\gamma \longmapsto \mathcal{X}_\gamma$). We claim that the set $Y = \{\gamma < \mu : x \cap \gamma = \mathcal{X}_\gamma \cap \gamma\}$ belongs to U_μ. If not, then let $f : \mu \setminus Y \to \mu$ be such that $f(\gamma) = \nu$, where ν witnesses that $\mathcal{X}_\gamma \cap \gamma \neq x \cap \gamma$. Then there is some $\bar{\nu}$ and $C \in U_\mu$ such that $f(\gamma) = \bar{\nu}$ for all $\gamma \in C$. Whether or not $\bar{\nu} \in x$, a contradiction establishes the claim. □

Theorem 14. *Let σ be a formula in the language of $\mathbb{D}_\lambda(\mathcal{U})$ and let $\langle t, \mathcal{Z}\rangle$ be any condition. Then there exists a direct extension of $\langle t, \mathcal{Z}\rangle$ that decides σ.*

Proof. In the interest of notational simplicity assume that $\lambda = \kappa$ ($\mathbb{D}(\mathcal{U}, \lambda) = \mathbb{D}(\mathcal{U})$) and that $t = 0$. The proof can be easily modified to accommodate the more general case by starting from intersecting the long measure one set with κ and carrying a constant entry through in some of the arguments.

Let σ and $\langle 0, \mathcal{Z}\rangle$ be given.

For each $\mu \in \Delta$ and each $s \in [\mu]^{<\omega}$, let $f^s_\mu : \mu \setminus max(s) + 1 \to 3$ via

$$f^s_\mu(\gamma) \quad = \quad \begin{cases} 0 & \text{if } \exists \mathcal{X}\ \langle s \cup \{\gamma\}, \mathcal{X}\rangle \Vdash \sigma, \\ 1 & \text{if } \exists \mathcal{X}\ \langle s \cup \{\gamma\}, \mathcal{X}\rangle \Vdash \neg\, \sigma, \\ 2 & \text{otherwise.} \end{cases}$$

Let $H^s_\mu \in U_\mu$ be homogeneous for f^s_μ. Let $H_\mu = \triangle_{s\in[\mu]^{<\omega}} H^s_\mu$. So, for any $s \in [\mu]^{<\omega}$ and for $\gamma, \gamma' \in H_\mu \setminus \max(s) + 1$,

$$\exists \mathcal{X}\ \langle s \cup \{\gamma\}, \mathcal{X}\rangle \parallel \sigma \text{ if and only if } \exists \mathcal{X}\ \langle s \cup \{\gamma'\}, \mathcal{X}\rangle \parallel \sigma,$$

and moreover, these conditions decide σ the same way.

Let $\mathcal{Z}' = \mathcal{Z} \cap \cup_{\mu\in\Delta} H_\mu$. A direct extension of $\langle 0, \mathcal{Z}'\rangle$ decides σ. Suppose not, and a contradiction shall appear. Fix $\langle s \cup \{\delta\}, \mathcal{Z}''\rangle \leq \langle 0, \mathcal{Z}'\rangle$ with $|s \cup \{\delta\}|$ the least possible such that $\langle s \cup \{\delta\}, \mathcal{Z}''\rangle \parallel \sigma$, where $s \ll \delta$. It is possible that s

is empty. Without loss of generality, assume $\langle s \cup \{\delta\}, \mathcal{Z}'' \rangle \Vdash \sigma$. Fix μ such that $\delta \in H_\mu$.

Now, for each $\gamma \in H_\mu \setminus \max(s)$, there is some $\mathcal{X}$ such that $\langle s \cup \{\gamma\}, \mathcal{X} \rangle$ forces σ. Let $\mathcal{X}_\gamma$ be some such $\mathcal{X}$. Let $\mathcal{X} = \triangle_{\gamma<\mu} \mathcal{X}_\gamma$.

By the previous lemma there is some x for which the set $Y = \{\gamma < \mu : \mathcal{X}_\gamma \cap \gamma = x \cap \gamma\}$ belongs to U_μ.

The fact $x \cap \gamma = \mathcal{X}_\gamma \cap \gamma$ for U_μ-almost all γ implies that $x \in \mathbb{D}(\mathcal{U}, \mu)$.

Let $\mathcal{Q} = ((Y \cup x) \cap (\mathcal{X} \cap \mu)) \cup (\mathcal{X} \smallsetminus \mu)$. Proving that $\langle s, \mathcal{Q} \rangle$ forces σ will contradict the minimality of $|s \cup \{\delta\}|$. It suffices to show that any extension of $\langle s, \mathcal{Q} \rangle$ is compatible with a condition $\langle s \cup \{\gamma\}, \mathcal{X}_\gamma \rangle$ for some $\gamma \in H_\mu \smallsetminus max(s)$.

Let $\langle s', \mathcal{Q}' \rangle \leq \langle s, \mathcal{Q} \rangle$ be arbitrary. By extending if necessary, assume that $s' \cap H \neq 0$. Denote $\gamma = \min(s' \cap Y)$. Now, for each $\eta \in s' \smallsetminus s$, if $\eta > \gamma$ then $\eta \in \mathcal{Q} \setminus \gamma \subset \mathcal{X}_\gamma$. Otherwise, $\eta \in x \cap Y \cap \gamma$, but by the minimality of γ in $s' \cap Y$ we must have $\eta \in x \cap \gamma = \mathcal{X}_\gamma \cap \gamma$. This ensures that $\langle s', \mathcal{Q}' \rangle$ is compatible with $\langle s \cup \{\gamma\}, \mathcal{X}_\gamma \rangle$ which force σ. Accordingly, $\langle s, \mathcal{Q} \rangle$ forces σ, contradicting the minimality of $|s \cup \{\delta\}|$ and finishing the proof. □

2.3 More Structure

Definition 13. Let $p = \langle s, \mathcal{X} \rangle \in \mathbb{D}(\mathcal{U})$ and let $\nu < \kappa$. Let

1. $p \upharpoonright (\nu + 1) = \langle s \cap (\nu + 1), \mathcal{X} \cap (\nu + 1) \rangle$
2. $p \downharpoonleft \nu = \langle s \setminus (\nu + 1), \mathcal{X} \setminus (\nu + 1) \rangle$
3. $\mathbb{D}^\nu(\mathcal{U}) = \{\langle s, \mathcal{X} \rangle \in \mathbb{D}(\mathcal{U} \upharpoonright (\kappa \setminus (\nu + 1))) : (s \cup \mathcal{X}) \cap (\nu + 1) = 0\}$.

The terms $p \upharpoonright (\nu + 1)$ and $p \downharpoonleft \nu$ are read as "p up through $\nu + 1$" and "p down past ν", respectively.

Proposition 15. *For any $\nu < \kappa$, $\mathbb{D}(\mathcal{U})$ is isomorphic to the product $\mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$ by the map $p \mapsto (p \upharpoonright (\nu + 1), p \downharpoonleft \nu)$.*

Proposition 16. *Fix $\nu \leq \kappa$.*

1. *$\mathbb{D}_{\nu+1}(\mathcal{U})$ has the ν^+-chain condition.*
2. *The $\leq^*$ relation is 2^ν-closed in $\mathbb{D}^\nu(\mathcal{U})$.*

Proof. For the first assertion, note that incompatible elements must differ in their support. Because there are only ν choices for the support of a condition in $\mathbb{D}_{\nu+1}(\mathcal{U})$, there can be at most ν incompatible conditions.

For the second assertion, note that each member of $\Delta \setminus (\nu + 1)$ is greater than ν and a strong limit. Fix $\eta < \lambda$. The claim follows from the fact that if $\mathcal{X}_\gamma \in \mathfrak{L}(\mathcal{U} \upharpoonright (\kappa \setminus (\nu + 1)))$ for $\gamma < \eta$, then $\cap_{\gamma<\eta} \mathcal{X}_\gamma \in \mathfrak{L}(\mathcal{U} \upharpoonright (\kappa \setminus (\nu + 1)))$. □

Theorem 17. *Fix $\nu < \kappa$. Let σ be a formula in the language of $\mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$. Let $(p, q) \in \mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$. There is a $q' \leq^* q$ such that for any $(p', q'') \leq (p, q')$ such that (p', q'') decides σ, (p', q') decides σ (the same way that (p', q'') does).*

Proof. For each $a \in \mathbb{D}_{\nu+1}(\mathcal{U})$, there is a $q(a) \leq^* q,\ q(a) \in \mathbb{D}^\nu(\mathcal{U})$ such that $q(a)$ decides the formulas

$$\exists q \in \dot{G}^\nu\ (a, q) \Vdash_{\mathbb{D}^\nu(\mathcal{U})} \sigma \tag{$\mathbb{T}_a$}$$

$$\exists q \in \dot{G}^\nu\ (a, q) \Vdash_{\mathbb{D}^\nu(\mathcal{U})} \neg\, \sigma. \tag{$\mathbb{F}_a$}$$

Because, in $\mathbb{D}^\nu(\mathcal{U})$, the direct extension relation is 2^ν-closed, there is some q' such that $q' \leq^* q(a)$ for all $a \in \mathbb{D}_{\nu+1}(\mathcal{U})$. This q' satisfies the theorem. In fact, suppose $(p', q'') \leq (p, q')$ such that (p', q'') decides σ. Without loss of generality, suppose $(p', q'') \Vdash \sigma$. It follows that $q'' \Vdash_{\mathbb{D}^\nu(\mathcal{U})} \mathbb{T}_{p'}$. Because $q'' \leq q'$ and q' decides $\mathbb{T}_{p'}$, it follows that $q' \Vdash_{\mathbb{D}^\nu(\mathcal{U})} \mathbb{T}_{p'}$.

With this fact in hand, suppose for a contradiction that (p', q') does not force σ. Because (p', q') does not force σ, there is a $(p'', q'') \leq (p', q')$ such that $(p'', q'') \Vdash \neg\, \sigma$. Because q'' extends q', q'' also forces $\mathbb{T}_{p'}$. This means that for some q''' compatible with q'', the condition (p', q''') forces σ. But (p'', q'') is compatible with (p', q'''), and the former forces $\neg\,\sigma$ while the latter forces σ. This contradiction establishes the theorem. □

Theorem 18. *Disassociated indiscernible forcing preserves cardinals.*

Proof. Suppose for a contradiction that λ is the least cardinal in $\mathbf{V}$ that is not a cardinal in $\mathbf{V}[G]$.

First note that λ is a successor cardinal because if all cardinals below a limit cardinal are preserved then the limit cardinal must be preserved as well. Hence the least collapsed cardinal cannot be a limit.

Next, $\lambda < \kappa$. In fact, $\mathbb{D}(\mathcal{U})$ has the κ^+ chain condition, thus preserving cardinals above κ.

Now, λ is not a cardinal in $\mathbf{V}[G{\restriction}(\lambda+1)]$. It suffices to show that the power set of λ is the same in $\mathbf{V}[G]$ and $\mathbf{V}[G{\restriction}(\lambda+1)]$. Let $\dot{S}$ be a $\mathbb{D}(\mathcal{U})$-name for a subset of λ. Suppose $p \Vdash \dot{S} \subset \lambda$. For each $\nu < \lambda$, apply Theorem 17 to p at $\lambda + 1$ for the formula $\ulcorner \nu \in \dot{S} \urcorner$ and get a condition $q(\nu) \in \mathbb{D}_{\lambda+1}(\mathcal{U})$. Because the $q(\nu)$ have identical support, by Proposition 16 there is a condition q extending all of them. Consider $\dot{T}$, the name for the set $\{\nu < \lambda : (p \restriction (\lambda+1), q) \Vdash \nu \in \dot{S}\}$. It follows from the definition that $\dot{S}^G = \dot{T}^{G\restriction(\lambda+1)}$.

But note that $\mathbb{D}_{\lambda+1}(\mathcal{U})$ has the λ chain condition. Hence λ must remain a cardinal in $\mathbf{V}[G{\restriction}(\lambda+1)]$, which is a contradiction.

□

3 Two Normal Measures

3.1 The Measures

In this section we assume $\mathcal{U}$ contains only measures below κ and that U_κ is a vertical repeat point. We show that after disassociated indiscernible forcing, κ remains the first measurable cardinal and has exactly two normal measures. This section requires the assumption that the ground model is the minimal core model with a vertical repeat point.

Proposition 19. *Forcing $\mathbb{D}(\mathcal{U})$ over a core model for a vertical repeat point κ, kills all instances of measurability below κ*

Proof. The core model assumption promises us that every measurable cardinal in the generic extension must be measurable in the ground model (see [9] for more details). Thus, it is sufficient to show that every measure cardinal $\mu < \kappa$ looses his measurability in the generic extension. Fix such $\mu < \kappa$. By 15, $\mathbb{D}(\mathcal{U})$ is isomorphic to $\mathbb{D}_{\mu+1}(\mathcal{U}) \times \mathbb{D}^{\mu}(\mathcal{U})$.

As μ is not a vertical repeat point, proposition 11 implies , $\mathbb{D}_{\mu+1}(\mathcal{U}) \cong \mathbb{D}_{\mu+1}(\mathcal{U} \restriction \mu) \times \mathbb{P}(U_\mu)$ which in turn implies μ becomes singular in a generic extension. □

Definition 14. Define $\dot{U}^i_\kappa$ for $i \in \{0,1\}$. Let $j : \mathbf{V} \to \mathfrak{M} \approx \mathrm{Ult}(\mathbf{V}, U_\kappa)$ be the ultrapower map via the normal measure U_κ. *The name $\dot{U}^0_\kappa$ is consistent of pairs $\langle \dot{X}, p \rangle$ for which:*

1. *$\dot{X}$ is a name for a subset of κ and $p = \langle s, \mathcal{X} \rangle \in \mathbb{D}(\mathcal{U})$.*
2. *For some $\mathcal{X}' \in \mathfrak{L}(j(\mathcal{U}))$ with $\mathcal{X}' \cap \kappa = \mathcal{X}$, $\langle s, \mathcal{X}' \rangle \Vdash \kappa \in j(\dot{X})$.*

Similarly, the name $\dot{U}^1_\kappa$ is consistent of pairs $\langle \dot{X}, p \rangle$ with the demand in the second part is changed to,

2. *For some $\mathcal{X}' \in \mathfrak{L}(j(\mathcal{U}))$ with $\mathcal{X}' \cap \kappa = \mathcal{X}$, $\langle s \cup \{\kappa\}, \mathcal{X}' \rangle \Vdash \kappa \in j(\dot{X})$.*

In either case, we say p *testifies* $\dot{X} \in \dot{U}^i$ and the $j(\mathbb{D}(\mathcal{U}))$-condition which forces κ to be in $j(\dot{X})$ is said to be a *witness* of p testifying X in U^i_κ.0

The key point behind the notation of *testification* is that on the one hand it poses some strong closure property (This emerges from the strong completeness of the "witness" part of $j(p)$ which belongs to $(\mathbb{D}^\kappa(\mathcal{U}), \leq^*)$) and one the other, it is rich enough to decide statements in a similar fashion to the forcing notation. The combination of this two properties will give us the ability to decide from a large number of complex statements and thereby achieve the desired properties of κ - complete normal ultrafilter for U^0_κ, U^1_κ (see Theorem 21 below).

Lemma 20. *A condition $p \in \mathbb{D}(\mathcal{U})$ forces $\dot{X} \in \dot{U}^i_\kappa$* iff *it testifies the same.*

Proof. It is clear from the definition of the names $\dot{U}^i_\kappa$ above, that testification implies forcing. In the opposite direction, let p be a condition forcing $\dot{X} \in \dot{U}^i_\kappa$. Appealing to theorem 17, take a suitable direct extension $q \in \mathbb{D}^\kappa(\mathcal{U})$ of $(j(p) \downharpoonright \kappa)$ with respect to the statement "$\kappa \in j(\dot{X})$".

We claim that the condition $(p^i, q) \in \mathbb{D}_{\kappa+1}(\mathcal{U}) \times \mathbb{D}^\kappa(j(\mathcal{U}))$ (here p is treated as a condition in $\mathbb{D}_{\kappa+1}(\mathcal{U})$ with $i = 0, 1$ indicates κ being (not added)/added to the support of the condition) forces $\kappa \in j(\dot{X})$ thus making q a witness for testification of p.

Otherwise, there is a condition $(t, q') \leq (p^i, q)$ forcing $\kappa \notin j(\dot{X})$ thus $(t, q) \Vdash \kappa \notin j(\dot{X})$ as well. Next, p forces $\dot{U}^i_\kappa$ implies $t \restriction \kappa$ forces the same which in turn, implies that it has an extension $p' \in \mathbb{D}(\mathcal{U})$ which testifies this statement. Let $r \leq$

$j(p')$ be a witness for p' testification. By our choice of q we get $(r \restriction (\kappa+1), q) \Vdash \kappa \in j(\dot{X})$. Noticing that $r \restriction (\kappa+1)$ is just $(p')^i$ which is an extension of t, we get that $(r \restriction (\kappa+1), q) \Vdash \kappa \notin j(\dot{X})$ and a contradiction appears. $\square$

From now on we will not distinguish between the notation *testifying* and *forcing*.

Here is an informal discussion of the motivation behind the definition and why a vertical repeat point is required. Note that a condition q that witnesses that p forces X to be in U^i_κ is compatible with $j(p)$. In fact the definition requires that the long measure one part of p matches the long measure one part of q up to κ. Because U_κ is a vertical repeat point, long measure one sets in ultrapowers by any $U^*_\kappa \supset U_\kappa$ are measure one for U_κ (even though κ cannot be measurable in any such ultrapower). Hence the long measure one part of q is measure one for U_κ. This is key. Because U_κ is a vertical repeat point, $q \restriction (\kappa+1)$ is in the original forcing for $\mathcal{U}$. In other words, the fact that U_κ is a vertical repeat point ensures that the restriction to $\kappa+1$ of images of conditions in $\mathbb{D}(\mathcal{U})$ remain in $\mathbb{D}(\mathcal{U})$. Further, if U_κ were not a vertical repeat point, some q (compatible with the image of p) would force the part of the set of indiscernibles below κ to be disjoint from a set in U_κ. But the set of indiscernibles must be identical to its image up to κ under any ultrapower with critical point κ. In other words, for a condition p to force the set of indiscernibles to be in some normal measure extending U_κ, conditions compatible with the image of p must have a long measure one part that ensures that the initial segment of the indiscernibles comes from sets in U_κ. Otherwise, the part of the image of the indiscernibles below the critical point could differ from the original set of indiscernibles.

Theorem 21. *In the generic extension, U^0_κ and U^1_κ are normal κ-complete nonprincipal ultrafilters with $\mathfrak{S} \notin U^0_\kappa$ and $\mathfrak{S} \in U^1_\kappa$.*

Proof. First, the U^i_κ are well-defined. That is, $p \Vdash \dot{X} \in U^i_\kappa$ does not depend on choice of name for X. Suppose $p \Vdash \dot{X} = \dot{Y}$ and $p \Vdash \dot{X} \in U^i_\kappa$. Then $j(p) \Vdash j(\dot{X}) = j(\dot{Y})$, and if $q \leq j(p)$ with $q \Vdash \kappa \in \dot{X}$, then $q \Vdash \kappa \in \dot{Y}$. So the definition is name-independent.

Next, each U^i_κ is a nonprincipal ultrafilter. U^i_κ is closed under enlargements; the proofs of closure under intersection, nonprincipality, and maximality are similar.

Suppose $p \Vdash \ulcorner X \in U^i_\kappa \ \wedge \ X \subset Y \urcorner$. There is a witness $q \leq j(p)$ of p forcing X in U^i_κ. So $q \Vdash \ulcorner \kappa \in j(X) \ \wedge \ j(X) \subset j(Y) \urcorner$. It follows that $q \Vdash \kappa \in j(Y)$. Hence q is also a witness of p forcing Y in U^i_κ.

Each U^i_κ is κ-complete. Suppose for some $\eta < \kappa$, $p \Vdash \cup_{\gamma<\eta} X_\gamma = \kappa$. The following produces a $p' \leq p$ and a $\gamma' < \eta$ such that $p' \Vdash X_{\gamma'} \in U^i_\kappa$. By elementarity, $j(p) \Vdash \cup_{\gamma<\eta} j(X_\gamma) = j(\kappa)$. For each $\gamma < \eta$, apply Theorem 17 to $j(p)$ at κ to get a pair $(r_\kappa, r^\kappa(\gamma)) \in \mathbb{D}_{\kappa+1}(j(\mathcal{U})) \times \mathbb{D}^\kappa(j(\mathcal{U}))$ such that $r_\kappa = j(p) \restriction (\kappa+1)$, $r^\kappa(\gamma) \leq^* j(p) \downharpoonright \kappa$ and such that if $(a,b) \leq (r_\kappa, r^\kappa(\gamma))$ with (a,b) deciding $\kappa \in j(X_\gamma)$, then $(a, r^\kappa(\gamma))$ decides $\kappa \in j(X_\gamma)$ the same way. Because $(\mathbb{D}^\kappa(j(\mathcal{U})), \geq^*)$ is κ^+-closed and the $r^\kappa(\gamma)$ are $\geq^*$-compatible, there is a condition $r^\kappa \leq r^\kappa(\gamma)$ for all $\gamma < \eta$. Note that $r_\kappa = j(p) \restriction (\kappa+1)$

for all $\gamma < \eta$, so that $(j(p) \restriction (\kappa+1), r^\kappa) \leq (r_\kappa, r^\kappa(\gamma))$ for all $\gamma \leq \eta$. Because $(r_\kappa, r^\kappa) \Vdash \cup_{\gamma<\eta} j(X_\gamma) = j(\kappa)$, there is some $q \leq (p', r^\kappa)$ and $\gamma' < \eta$ such that $q \Vdash \kappa \in j(X_{\gamma'})$. Denote $p' = q \restriction (\kappa+1)$. Because U_κ is a vertical repeat point, $p' \in \mathbb{D}(\mathcal{U})$. The claim is that (p', r^κ) is a witness to p' forcing $X_{\gamma'}$ in U^i_κ, where the i depends on whether or not κ is in the support of q. It is clear from the fact that $r^\kappa \leq^* j(p) \downharpoonleft \kappa$ that the support of (p', r^κ) is equal to the support of p'. Also $j((p', r^\kappa)) \restriction (\kappa+1) = j(p') \restriction (\kappa+1)$. And by definition of the (r_κ, r^κ), $(p', r^\kappa) \Vdash \kappa \in (X_{\gamma'})$. Thus (p', r^κ) is a witness as claimed. It follows that $p' \Vdash X_{\gamma'} \in U^i_\kappa$ so that U^i_κ is κ-complete.

Finally, the U^i_κ are normal. Suppose $p \Vdash \ulcorner f : \kappa \to \kappa \ \wedge \ \forall \gamma < \kappa \ f(\gamma) < \gamma \urcorner$. The following produces a $p' \leq p$, a $\gamma' < \kappa$ and a set X such that $p' \Vdash \ulcorner X \in U^i_\kappa \ \wedge \ f\text{"}X = \gamma' \urcorner$. For every $\gamma < \kappa$, apply Theorem 17 to $j(p)$ at κ to get a pair $(r_\kappa, r^\kappa(\gamma)) \in \mathbb{D}_{\kappa+1}(j(\mathcal{U})) \times \mathbb{D}^\kappa(j(\mathcal{U}))$ such that $r_\kappa = j(p) \restriction (\kappa+1)$, $r^\kappa(\gamma) \leq^* j(p) \downharpoonleft \kappa$, and such that if $(a,b) \leq (r_\kappa, r^\kappa(\gamma))$ with (a,b) deciding $j(f)(\kappa) = \gamma$, then $(a, r^\kappa(\gamma))$ decides $j(f)(\kappa) = \gamma$ the same way. Because $\mathbb{D}^\kappa(j(\mathcal{U}))$ is κ-closed, there is a condition $r^\kappa \leq r^\kappa(\gamma)$ for all $\gamma < \eta$. Now, because $j(p) \geq (r_\kappa, r^\kappa) \Vdash j(f) : j(\kappa) \to j(\kappa)$, there is some $\gamma' < \kappa$ and some $q \leq (r_\kappa, r^\kappa)$ such that $q \Vdash j(f)(\kappa) = \gamma'$. Denote $p' = q \restriction (\kappa+1)$, which is in $\mathbb{D}(\mathcal{U})$ because U_κ is a vertical repeat point. Then (p', r^κ) is a witness to p' forcing $\{\gamma < \kappa : f(\gamma) = \gamma'\}$ in U^i_κ. It is clear from the fact that $r^\kappa \leq^* j(p) \downharpoonleft \kappa$ that the support of (p', r^κ) is equal to the support of p'. Also $j((p', r^\kappa)) \restriction (\kappa+1) = j(p') \restriction (\kappa+1)$. And by definition of the (r_κ, r^κ), $(p', r^\kappa) \Vdash j(f)(\kappa) = \gamma'$. In other words, $(p', r^\kappa) \Vdash \kappa \in \{\gamma < j(\kappa) : j(f)(\gamma) = \gamma'\}$. That (p', r^κ) is the witness as claimed follows by elementarity.
Finally, it is a trivial to verify that $1 = \langle \emptyset, \kappa \rangle$ testifies both $\kappa \setminus \mathfrak{S} \in U^0_\kappa$ and $\mathfrak{S} \in U^1_\kappa$. □

Theorem 22. *The only normal ultrafilters on κ in the generic extension are U^0_κ and U^1_κ.*

Proof. Suppose $p = \langle s, \mathcal{X} \rangle$ forces both $\dot{X} \in \dot{W}$ and "$\dot{W}$ is a normal measure on κ". There is an $r \leq p$ such that $r \Vdash \dot{X} \in U^i_\kappa$ for some $i \in \{0,1\}$.

In the generic extension there is an ultrapower map j_W. The restriction of this function to the ground model factors through the ultrapower j_{U_κ}. That is, $j_W \restriction \mathbf{V} = k \circ j_{U_\kappa}$ for some iterated ultrapower k with critical point greater than κ, relying on the minimality of the core model for a vertical repeat point.

Appealing to Theorem 17, for each $\nu < \kappa$ there is a $q_\nu = \langle s, \mathcal{Q}_\nu \rangle \leq^* p \downharpoonleft \nu$ in $\mathbb{D}^\nu(\mathcal{U})$ such that, if $(a,b) \leq (1, q_\nu)$ and $(a,b) \in \mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$ decides $\nu \in \dot{X}$, then (a, q_ν) decides $\nu \in \dot{X}$ the same way. Let $\mathcal{Q} = \triangle_{\nu<\kappa} \mathcal{Q}_\nu$ and let $p' = \langle s, \mathcal{X} \cap \mathcal{Q} \rangle \leq^* p$.

Now $p' \Vdash \dot{X} \in \dot{W}$. So in the generic extension there is some q extending $j_W(p')$ such that $q \Vdash \kappa \in \dot{j_W}(\dot{X})$. But p' is in the ground model. Hence q extends $k \circ j_{U_\kappa}(p')$ and $q \Vdash \kappa \in k \circ j_{U_\kappa}(\dot{X})$. By elementarity, there is some $q' \leq j_{U_\kappa}(p')$ such that $q' \Vdash \kappa \in j_{U_\kappa}(\dot{X})$. Denote $r = q' \restriction \kappa$ and $q'' = (q' \restriction (\kappa+1), j(p') \downharpoonleft \kappa)$.

Then q'' is a witness to $r \Vdash \dot{X} \in U^i_\kappa$, where

$$i = \begin{cases} 0 & \text{if } \kappa \in \operatorname{supt}(q' \restriction (\kappa+1)), \\ 1 & \text{if } \kappa \notin \operatorname{supt}(q' \restriction (\kappa+1)). \end{cases}$$

Now suppose for some $\bar{p}$ compatible with p, $\bar{p} \Vdash \dot{Y} \in \dot{W}$. By repeating the proof with a common extension of p and $\bar{p}$, $\dot{Y}^G$ and $\dot{X}^G$ must be in the same U^i_κ. □

3.2 Properties

Proposition 23. *Let $G \subset \mathbb{D}(\mathcal{U})$ be a generic set. The following holds in the generic extension $V[G]$.*

1. *For each $X \in U^0_\kappa$ there is some $p = \langle s, \mathcal{X} \rangle \in G$ for which the condition $\langle s, j(\mathcal{X}) \setminus \{\kappa\} \rangle$ forces $\kappa \in \dot{X}$.*

2. *For each $X \in U^1_\kappa$ there is some $p = \langle s, \mathcal{X} \rangle \in G$ such that $\langle s \cup \{\kappa\}, j(\mathcal{X}) \rangle$ forces $\kappa \in \dot{X}$.*

Proof. Suppose $p = \langle s, \mathcal{X} \rangle \Vdash X \in U^i_\kappa$ for some $i \in \{0,1\}$. The two cases are not exclusive, but without loss of generality the reader can assume that exactly one holds. By Theorem 17, for each $\nu < \kappa$, there are $q_\nu = \langle s \setminus (\nu+1), \mathcal{Q}_\nu \rangle \in \mathbb{D}^\nu(\mathcal{U})$ such that if $(a, q') \leq (1, q_\nu)$, $(a, q') \in \mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$ and (a, q') decides $\nu \in X$, then (a, q_ν) decides $\nu \in X$ the same way (we will say that q_ν decides $\nu \in X$ above ν). Set $\mathcal{Q} = \triangle_{\nu<\kappa} \mathcal{Q}_\nu$ and let $q = \langle s, \mathcal{X} \cap \mathcal{Q} \rangle$. It follows that whenever $(1, q_\nu) \geq (a, q') \in \mathbb{D}_{\nu+1}(\mathcal{U}) \times \mathbb{D}^\nu(\mathcal{U})$ and (a, q') decides the statement "$\nu \in \dot{X}$", then $(a, q \downharpoonright \nu)$ decides this statement in the same manner.

Appealing to lemma 20, we get that $q \Vdash X \in U^i_\kappa$ implies that there is some $r \leq j(q)$ with $r \restriction (\kappa+1) = j(q) \restriction (\kappa+1)$, $r \Vdash \kappa \in j(X)$, and $\operatorname{supt}(r) \setminus \operatorname{supt}(q)$ is either $\emptyset$ or $\{\kappa\}$, depending on whether X is forced to be in U^0_κ or U^1_κ. We may assume that κ is not included in the long measure one set of r.

By elemntarity of j, the fact $(1, j(q) \downharpoonright \kappa) \geq (r \restriction \kappa+1, r \downharpoonright \kappa) \Vdash \kappa \in \dot{X}$ implies that $(r \restriction \kappa+1, j(q) \downharpoonright \kappa)$ force the same. However, as noted, $r \restriction \kappa+1$ is just $j(q) \restriction \kappa$ with an optional addition of κ to its entry for the U^1_κ case and with κ removed from its long measure one set. We conclude that for each $p \in \mathbb{D}(\mathcal{U})$ which force a name $\dot{x}$ to be in U^i_κ there is a suitable extension $q \leq p$ such that stands in the criterion of the proposition and the proposition follows by a simple density argument. □

As a corollary, we receive the following esthetic characterization for our new normal measures.

Theorem 24. *In the generic extension, the following hold.*

1. *For each $X \in U^0_\kappa$ there is some $Y \in U_\kappa$ such that $Y \setminus \mathfrak{S} \subset X$.*

2. *For each $X \in U^1_\kappa$ there is some $Y \in U_\kappa$ such that $Y \cap \mathfrak{S} \subset X$.*

Proof. Let us describe the case for U^1_κ. This proof can be easily modified to accommodate the desired result for U^0_κ. Let G be a generic set for our forcing. In $V[G]$, let $X \in U^1_\kappa$. Thus, By the previous proposition there is some $p = \langle s, \mathcal{X}\rangle$ in G such that $\langle s \cup \{\kappa\}, j(\mathcal{X})\rangle \Vdash \kappa \in j(\dot{X})$. In V we define the set $Y = \{\nu < \kappa : \langle s \cup \{\nu\}, \mathcal{X}\rangle \Vdash \nu \in \dot{X}\}$. Clearly, $\kappa \in j(Y)$ and therefore $Y \in U_\kappa$. Notice that for all $\nu \in Y$, if $\langle s \cup \{\nu\}, j(\mathcal{X})\rangle$ is in G, then $\nu \in X$. But, $\langle s \cup \{\nu\}, \mathcal{X}\rangle \in G$ iff $\nu \in \mathfrak{S}$. The claim follows. □

Compare the following with Theorem 5.

Theorem 25. *For $i \in \{0, 1\}$, consider the restriction of the ultrapower*

$$j_{U^i_\kappa} : L[\mathcal{U}][\mathfrak{S}] \to L[\mathcal{U}'][\mathfrak{S}'] \approx \mathrm{Ult}(L[\mathcal{U}][\mathfrak{S}], U^i_\kappa)$$

to the ground model

$$j_i : L[\mathcal{U}] \to L[\mathcal{U}'].$$

Then

1. *j_i is an iterated ultrapower along measures from $\mathcal{U}$*
2. *every element of $\mathfrak{S}' \setminus \mathfrak{S}$ is a critical point of the iterated ultrapower.*

Proof. The first point is established in [9]; a proof of the second follows. Suppose for a contradiction that $\nu \in \mathfrak{S}' \setminus \mathfrak{S}$ is not a critical point in the iteration. Then for some $n < \omega$ and a function $F : [\kappa]^n \to \kappa$ in V such that $\nu = j_{U^i_\kappa}(F)(s)$ for some $s \in [\nu]^n$. Define $Z = \{\eta < \kappa : \eta \in F``[\eta]^n\}$, so $\nu \in j_{U^i_\kappa}(Z)$. Note that for each $\mu \in \Delta$, $Z \cap \mu \notin U_\mu$. Otherwise we would have some $s \in [\mu]^n$ with $\mu = j_{U_\mu}(F)(s)$. But $s \in [\mu]^n$ implies that $s = j_{U_\mu}(s)$ thus, $\mu = j_{U_\mu}(F(s)) \in j_{U_\mu}``Ord$ which is impossible. It follows that the set $D = \{\langle s, X \cap Z\rangle : \langle s, X\rangle \in \mathbb{D}(\mathcal{U})\}$ is dense, so $G \cap D \neq 0$ for all generic G. Suppose p is a condition in the intersection. Then $j_{U^i_\kappa}(p) \in j_{U^i_\kappa}(G)$. But because p has finite support bounded below κ, ν is not in the support of $\mathrm{j}_{U^i_\kappa}(p)$. Neither is ν in the long measure one set. That $\mathfrak{S}' = j_{U^i_\kappa}(\mathfrak{S})$ prevents ν from being in $\mathfrak{S}'$. This contradiction establishes the theorem. □

4 Iterated Disassociated Indiscernible Forcing

Definition 15. Define functions $f_\gamma : \kappa \to \kappa$ for $\gamma < \kappa^+$ by recursion. Let $f_0(\eta) = 0$. Let $f_{\gamma+1}(\eta) = f_\gamma(\eta) + 1$. For limit ordinals γ, let $f_\gamma(\eta) = \cup_{\nu<\eta} f_{\gamma_\nu}(\eta)$ where $(\gamma_\nu)_{\nu<\kappa}$ is onto γ.

The definition of f_γ for limit γ is the diagonal union. Note that $[f_\gamma]_{U_\kappa} = \gamma$. That is, these functions are canonical representations of the ordinals less than κ^+ in the ultrapower by U_κ.

The following is relative to our fixed sequence of measures $\mathcal{U} = (U_\mu)_{\mu\in\Delta}$.

Definition 16. Define a *vertical repeat point of order* $\gamma \leq \kappa^+$ by recursion on γ. Let $ord(U_\kappa) = 0$ if for some $X \in U_\kappa$ and for all $\mu < \kappa$, $X \cap \mu \notin U_\mu$. Let $ord(U_\kappa) > \gamma$ if for every $X \in U_\kappa$, the set $\{\mu < \kappa : ord(U_\mu) = f_\gamma(\mu)$ and $X \cap \mu \in U_\mu\}$ is stationary in κ.

The definition for successor ordinals is an analogy to being Mahlo for the predecessor order.

Theorem 26. *The consistency strength of a vertical repeat point of order* $\gamma \leq \kappa^+$ *is strictly less than that of a measure concentrating on measurable cardinals.*

Define iterated disassociated indiscernible forcing of order up to and including κ as follows.

For each $\gamma \leq \kappa$, let $\Delta_\gamma = \{\mu \in \Delta : ord(U_\mu) = \gamma\}$. For each $\gamma \leq \kappa$, let

$$\mathcal{U}^\gamma = (U_\mu)_{\mu \in \Delta_\gamma}.$$

Note that by this definition, we get that if $\mu \in \Delta_\gamma$ then for every $\nu < \gamma$,

$$\mathfrak{L}(\mathcal{U}^\nu \restriction \mu) \subset U_\mu.$$

Definition 17. Let I be an interval of ordinals and ν an ordinal. The *I-iterated disassociated indiscernible forcing for* $\mathcal{U}$, ν, *denoted* $\mathbb{D}_\nu(\mathcal{U}, I)$, *has as conditions* sequences $\langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma \in I}$ such that

1. for all $\gamma \in I$, $s_\gamma \in [\nu \setminus (\gamma + 1)]^{<\omega}$
2. all but finitely many s_γ are empty
3. for all $\gamma \in I$, $\mathcal{X}_\gamma \in \mathfrak{L}(\mathcal{U}^\gamma)$
4. if $\gamma \neq \gamma'$, then $s_\gamma \cap s_{\gamma'} = 0$.

A condition *extends* another, $\langle s'_\gamma, \mathcal{X}'_\gamma \rangle_{\gamma \in I} \leq \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma \in I}$, if for all $\gamma \in I$ $\langle s'_\gamma, \mathcal{X}'_\gamma \rangle \leq \langle s_\gamma, \mathcal{X}_\gamma \rangle$ in the $\mathbb{D}_\nu(\mathcal{U}^\gamma)$ order.

Following the notations of the previous sections, we shorten and write $\mathbb{D}(\mathcal{U}, I)$ *for* $\mathbb{D}_\kappa(\mathcal{U}, I)$.

The definition of direct extension is analogous to that for disassociated indiscernible forcing. The following uses the notation $\langle \vec{s}, \vec{\mathcal{X}} \rangle$ for conditions when the index is obvious and uses the adjoin operator "$\frown$" to display the contents of $\vec{s}$, regarding the partial function as a set of pairs:

$$\langle i_0, \gamma_0^0, \rangle^\frown \langle i_0, \gamma_1^0, \rangle^\frown ...^\frown \langle i_0, \gamma_{k_0}^0 \rangle^\frown \langle i_1, \gamma_0^1, \rangle^\frown ...^\frown \langle i_n, \gamma_0^n, \rangle^\frown \langle i_n, \gamma_{k_n}^n \rangle.$$

Where for each i_m, $s_{i_m} = \langle \gamma_0^m, .., \gamma_{k_m}^m \rangle$.

Assume from now on that $\max(\Delta) = \kappa$ and that U_κ is a vertical repeat point of order $\lambda \leq \kappa$.

4.1 Basic Structure

Definition 18. Fix $\nu < \kappa$. Let $p = \langle s_\gamma, \mathcal{X}_\gamma\rangle_{\gamma\in\lambda} \in \mathbb{D}(\mathcal{U}, \lambda)$. Let

1. $p \restriction (\nu + 1) = \langle s_\gamma \cap (\nu + 1), \mathcal{X}_\gamma \cap (\nu + 1)\rangle_{\gamma<\lambda}$
2. $p \downharpoonright \nu = \langle s_\gamma \setminus (\nu + 1), \mathcal{X}_\gamma \setminus (\nu + 1)\rangle_{\gamma<\lambda}$
3. $\mathbb{D}^\nu(\mathcal{U}, \lambda) = \{\langle s_\gamma, \mathcal{X}_\gamma\rangle_{\gamma<\lambda} \in \mathbb{D}((U_\mu)_{\mu\in\Delta\setminus(\nu+1)}, \lambda) : \forall\gamma < \lambda((s_\gamma \cup \mathcal{X}_\gamma) \cap (\nu + 1) = 0)\}$.

Some of the long measure one entries may be empty in $p \restriction (\nu + 1)$.

Proposition 27. *For any $\nu < \kappa$, $\mathbb{D}(\mathcal{U}, \lambda)$ is isomorphic to the product $\mathbb{D}_{\nu+1}(\mathcal{U}, \lambda)\times \mathbb{D}^\nu(\mathcal{U}, \lambda)$ by the map $p \mapsto (p \restriction (\nu + 1), p \downharpoonright \nu)$.*

Note that this forcing and its conditions can be factored in two ways: "vertically", as above, and although not used herein, horizontally:

Definition 19. Let $p = \langle s_\gamma, \mathcal{X}_\gamma\rangle_{\gamma\in\lambda} \in \mathbb{D}(\mathcal{U}, \lambda)$. Fix $\zeta < \lambda$. Denote

1. $p\dagger \zeta = \langle s_\gamma, \mathcal{X}_\gamma\rangle_{\gamma<\zeta}$
2. $p\ddagger \zeta = \langle s_\gamma, \mathcal{X}_\gamma\rangle_{\gamma\in\lambda\setminus\zeta}$.

Proposition 28. *For any $\zeta < \lambda$, $\mathbb{D}(\mathcal{U}, \lambda)$ is isomorphic to the product $\mathbb{D}(\mathcal{U}, \zeta)\times \mathbb{D}(\mathcal{U}, \alpha \setminus \zeta)$ by the map $p \mapsto (p\dagger \zeta, p\ddagger \zeta)$.*

4.2 Prikry Property

Definition 20. If $\vec{s}$ is a set of pairs, denote $\sup \vec{s} = \max\{i, j : \langle i, j\rangle \in \vec{s}\}$. The relation $\vec{s} \ll \vec{t}$ means that $\vec{s}$, $\vec{t}$ are either ordinals or finite partial functions from κ to $[\kappa]^{<\omega}$ and that $\sup(\vec{s}) < \sup(\vec{t})$.

Theorem 29. *Let σ be a formula in the language of $\mathbb{D}(\mathcal{U}, \lambda)$ and let $p = \langle s_\eta, \mathcal{X}_\eta\rangle_{\eta<\lambda}$ be any condition, where $\lambda \le \kappa$. There is a direct extension of p that decides σ.*

Proof. The proof is essentially the same as the proof for the disassociated indiscernible Prikry Property with some slight modifications to handle the iterations. Select some bijection $e : \kappa \to [\kappa \times \kappa]^{<\omega}$ such that for each cardinal ζ, $e"\zeta = [\zeta \times \zeta]^{<\omega}$. The proof uses e to define diagonal intersections. To simplify the notation, assume that each $s_\eta = 0$.

Let σ and $\langle 0, \vec{\mathcal{X}}\rangle$ be given.

For each $\mu \in \Delta$, $\vec{s}$, and $i < \lambda$ let $f^{\vec{s}}_{\mu,i} : \mu \to 3$ via

$$f^{\vec{s}}_{\mu,i}(\gamma) = \begin{cases} 0 & \text{if } \exists\vec{\mathcal{X}}\ \langle \vec{s}\frown\langle i, \gamma\rangle, \vec{\mathcal{X}}\rangle \Vdash \sigma, \\ 1 & \text{if } \exists\vec{\mathcal{X}}\ \langle \vec{s}\frown\langle i, \gamma\rangle, \vec{\mathcal{X}}\rangle \Vdash \neg\sigma, \\ 2 & \text{otherwise.} \end{cases}$$

Let $H^{\vec{s}}_{\mu,i} \in U_\mu$ be homogeneous for $f^{\vec{s}}_{\mu,i}$. Let $H_{\mu,i} = \triangle_{\vec{s}} H^{\vec{s}}_{\mu,i}$. For any $\vec{s}$, any $\gamma, \gamma' \in H_{\mu,i} \setminus (\sup \vec{s} + 1)$,

$$\exists \vec{\mathcal{X}}\ \langle \vec{s} \frown \langle i, \gamma \rangle, \vec{\mathcal{X}} \rangle \parallel \sigma \text{ if and only if } \exists \vec{\mathcal{X}}\ \langle \vec{s} \frown \langle i, \gamma' \rangle, \vec{\mathcal{X}} \rangle \parallel \sigma,$$

and these conditions decide σ the same way.

For each $i < \lambda$, let $H_i = \cup_{\mu \in \Delta_i} H_{\mu,i}$. Let $\vec{\mathcal{Z}}' = (H_i \cap \mathcal{Z}_i)_{i<\lambda}$. A direct extension of $\langle 0, \vec{\mathcal{Z}}' \rangle$ decides σ. Suppose not, toward reaching a contradiction. Fix $\langle \vec{s} \frown \langle j, \delta \rangle, \vec{\mathcal{Z}}'' \rangle \leq \langle 0, \vec{\mathcal{Z}}' \rangle$ with $|\vec{s} \frown \langle j, \delta \rangle|$ the least out of all conditions deciding σ, where $\vec{s} \ll \delta$. Without loss of generality, assume $\langle \vec{s} \frown \langle j, \delta \rangle, \vec{\mathcal{Z}}'' \rangle \Vdash \sigma$. Let μ be such that $\delta \in H_{\mu,j}$.

It follows that for every $\gamma \in H_{\mu,j}$ with $\vec{s} \ll \gamma$ there is some $\vec{\mathcal{X}}^\gamma$ such that $\langle \vec{s} \frown \langle j, \gamma \rangle, \vec{\mathcal{X}}^\gamma \rangle$ force σ. Let $\vec{\mathcal{X}} = (\triangle_{\gamma \in H_{\mu,j}} \mathcal{X}^\gamma_k)_{k<\lambda}$.

For each $k < \lambda$, we use lemma 13 to produce sets $x_k \subseteq \mu$ and $Y_k \in U_\mu$ such that $\mathcal{X}^\gamma_k \cap \gamma = x_k \cap \gamma$ for every $\gamma \in Y_k$.
It follows that $x_k \in \mathfrak{L}(\mathcal{U}^k \restriction \mu)$ so for every $k \neq j = Ord(U_\gamma)$. The set

$$\mathcal{Q}_k = (x_k \cap \mathcal{X}_k \cap \mu) \cup (\mathcal{X}_k \setminus \mu)$$

belongs to $\mathfrak{L}(\mathcal{U}^k)$. For Q_j we add the U_μ set $Y = \triangle_{k<\mu} Y_k \in U_\mu$, thus

$$\mathcal{Q}_j = ((x_j \cup Y) \cap \mathcal{X}_j \cap \mu) \cup (\mathcal{X}_k \setminus \mu)\,.$$

We claim that $\langle \vec{s}, \vec{\mathcal{Q}} \rangle \Vdash \sigma$.

Let $\langle \vec{t}, \vec{\mathcal{R}} \rangle \leq \langle \vec{s}, \vec{\mathcal{Q}} \rangle$. Without loss of generality, $\vec{t}$ contains a pair $\langle j, \xi \rangle$ such that $\xi \in Y$. Consider the least such ξ. The claim is that $\langle \vec{t}, \vec{\mathcal{R}} \rangle$ is compatible with $\langle \vec{s} \frown \langle j, \xi \rangle, \vec{\mathcal{X}}^\xi \rangle$. Indeed, let $\langle k, \gamma \rangle \in \vec{t}$. If $\gamma > \xi$ then $\gamma \in \mathcal{Q}_k \setminus \xi + 1 \subset \mathcal{X}^{\langle j,\xi \rangle}_k$. Suppose $\gamma < \xi$. Because $\xi \in Y$ and $k < \gamma < \xi$, it follows that $\mathcal{X}^\xi_k \cap \xi = x_k \cap \xi$, so $\gamma in \mathcal{X}^\xi_k$. Either way $\vec{t} \setminus \vec{s}$ belongs to $\vec{\mathcal{X}}^\xi$. This establishes the claim. Now, since the conditions $\langle \vec{s} \frown \langle j, \xi \rangle, \vec{\mathcal{X}}^\xi \rangle$ forces σ then so does $\langle \vec{s}, \vec{\mathcal{Q}} \rangle$. This contradicts the minimality of $|\vec{s} \frown \langle j, \delta \rangle|$, finishing the proof. □

4.3 More Structure

Theorem 30. *Fix $\nu < \kappa$. Let σ be a formula in the language of $\mathbb{D}_\nu(\mathcal{U}, \lambda) \times \mathbb{D}^\nu(\mathcal{U}, \lambda)$. Let $(p, q) \in \mathbb{D}_\nu(\mathcal{U}, \lambda) \times \mathbb{D}^\nu(\mathcal{U}, \lambda)$. There is a $q' \leq^* q$ such that for any $(p', q'') \leq (p, q')$ such that (p', q'') decides σ, (p', q') also decides σ.*

Proof. (outline)
The proof is virtually the same as the proof of Theorem 17. At first, there seems to be a problem with applying the same argument since unlike the situation occurs in proof 17, the size of $D_\nu(\mathcal{U}, \lambda)$ need not to be smaller than the closer rate of $\langle D^\nu(\mathcal{U}, \lambda), \geq^* \rangle$ (This was the key for constructing the new "dominant" direct extension in $D^\nu(\mathcal{U})$). However, this problem is only virtual because for every $p = \langle s_\gamma, \mathcal{X}_\gamma \rangle_{\gamma<\lambda} \in D(\mathcal{U}, \lambda)$ the demand $s_\gamma \in [\nu \setminus (\gamma + 1)]^{<\omega}$ for all $\gamma < \lambda$ implies $p \restriction \nu + 1$ has empty support in coordinates $\gamma \geq \nu$. It follows that $D_\nu(\mathcal{U}, \lambda) \cong D_\nu(\mathcal{U}, \nu)$. As the size of the forcing notion $D_\nu(\mathcal{U}, \nu)$ is 2^ν, it is possible to appeal to the argument appears in proof 17. □

Theorem 31. *Iterated disassociated indiscernible forcing preserves cardinals.*

Proof. The obvious alteration of Theorem 18 works. □

5 Many Normal Measures

Assume that κ is a vertical repeat point of order $\lambda \leq \kappa$. After λ-iterated disassociated indiscernible forcing, κ has exactly λ normal measures.

Definition 21. *Let j be the ultrapower map via the normal measure U_κ.* For $\eta < \lambda$, let $\dot{U}_\kappa^{(1,\eta)}$ be the name consists of all pairs $\langle p, \dot{X}\rangle$ for which the following holds:

1. $p = \langle s_\gamma, \mathcal{X}_\gamma\rangle_{\gamma<\lambda}$ belongs to $\mathbb{D}(\mathcal{U}, \lambda)$ and $\dot{X}$ is a name for a subset of κ.
2. The condition $j(\langle s_\gamma, \mathcal{X}_\gamma\rangle_{\gamma<\lambda})$ has a *direct extension* $\langle s'_\gamma, \mathcal{X}'_\gamma\rangle_{\gamma<\lambda} \in \mathbb{D}(\mathcal{U}, \lambda)$ with $s'_\gamma = s_\gamma$ for $\gamma \neq \eta$, $s'_\eta = s_\eta \cup \{\kappa\}$ and $\mathcal{X}_\gamma = \mathcal{X}'_\gamma \cap \kappa$ for all $\gamma < \lambda$ such that,
$$\langle s'_\gamma, \mathcal{X}'_\gamma\rangle_{\gamma<\lambda} \Vdash \kappa \in \dot{X}.$$

In addition, as in the two measure case, we define the name $\dot{U}^0_\kappa$ in a similar fashion to definition 14.

For each $\eta < \lambda$, denote $\mathfrak{S}_\eta = \{\zeta < \kappa : \exists\langle s_\gamma, \mathcal{X}_\gamma\rangle_{\gamma\in\lambda} \in G \text{ such that } \zeta \in s_\eta\}$.

Theorem 32. *In the generic extension, the normal measures on κ are exactly the $U_\kappa^{(1,\eta)}; \eta < \lambda$ and U^0_κ.*

Proof. The proof is almost identical to those for Theorems 21 and 22 with Theorem 30 used in place of Theorem 17. □

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