

# Diamonds, Compactness, and Measure Sequences

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## Abstract

We establish the consistency of the failure of the diamond principle on a cardinal  $\kappa$  which satisfies a strong simultaneous reflection property. The result is based on an analysis of Radin forcing, and further leads to a characterization of weak compactness of  $\kappa$  in a Radin generic extension.

## 1 Introduction

In pursuit of an understanding of the relations between compactness and approximation principles, we address the following question: To what extent do compactness principles assert the existence of a diamond sequence?

The compactness principles considered in this paper are stationary reflection and weak compactness. The main result of this paper shows that a strong form of stationary reflection does not imply  $\diamond_\kappa$ .

**Theorem 1.** *It is consistent relative to a certain hypermeasurability large cardinal assumption that there exists a cardinal  $\kappa$  satisfying the following properties.*

1. *For every sequence  $\vec{S} = \langle S_i \mid i < \kappa \rangle$  of stationary subsets in  $\kappa$ , there exists  $\delta < \kappa$  such that all sets in  $\vec{S} \upharpoonright \delta = \langle S_i \mid i < \delta \rangle$  reflect at  $\delta$ .*
2.  *$\diamond_\kappa$  fails.*

Let us recall the relevant definitions. Suppose that  $\kappa$  is a regular cardinal. The diamond principle at  $\diamond_\kappa$ , introduced by Jensen in [10], asserts the existence of a sequence  $\langle s_\alpha \mid \alpha < \kappa \rangle$  of sets  $s_\alpha \subset \alpha$ , such that for every  $X \subset \kappa$  the set  $\{\alpha < \kappa \mid s_\alpha = X \cap \alpha\}$  is stationary in  $\kappa$ .

We say that a stationary subset  $S$  of  $\kappa$  reflects at  $\delta < \kappa$  if  $S \cap \delta$  is stationary in  $\delta$ . A cardinal  $\kappa$  is reflecting if every stationary subset of  $\kappa$  reflects at some  $\delta < \kappa$ . The stronger reflection property in the statement of Theorem 1 will be called a strong simultaneous reflection property. It clearly implies

that every family of less than  $\kappa$  many stationary subsets of  $\kappa$  simultaneously reflect at some  $\delta < \kappa$ .

Reflecting is a compactness type property <sup>1</sup>. It well-known that a reflecting cardinal is greatly Mahlo, and that every weakly compact cardinal satisfies the strong simultaneous reflection property. Although reflection is a consequence of the strong simultaneous reflection property, the two properties may coincide: Jensen [10] has shown that in  $L$ , a reflecting cardinal is weakly compact, and therefore satisfies the strong principle. In contrast, Harrington and Shelah [7] proved that the existence of a Mahlo cardinal is equi-consistent with reflection of every stationary subset of  $\omega_2 \cap \text{cf}(\omega)$ , while Magidor [13] proved the stronger simultaneous reflection property for stationary subsets of  $\omega_2 \cap \text{cf}(\omega)$  is equi-consistent with the existence of a weakly compact cardinal.

The relations between compactness (large cardinal) axioms and  $\diamond$  type principles have been extensively studied. See [12] for a comprehensive discussion of the problem. It is well-known that every measurable cardinal  $\kappa$  carries a  $\diamond_\kappa$  sequence, and Jensen and Kunen [9] showed  $\diamond_\kappa$  holds at every subtle cardinal  $\kappa$ . In fact, they proved that a subtle cardinal  $\kappa$  satisfies the stronger approximation property -  $\diamond_\kappa(\text{Reg})$ . A  $\diamond_\kappa(\text{Reg})$  sequence is a diamond sequence which approximates subsets  $X \subset \kappa$  on the (tighter set) of regular cardinals  $\alpha < \kappa$ . Nevertheless, not every large cardinal assumption implies the existence of a diamond sequence. Woodin first showed the stronger principle  $\diamond_\kappa(\text{Reg})$  can fail at a weakly compact cardinal. The result was extended by Hauser [8] to indescribable cardinals, and by Džamonja and Hamkins [4] to strongly unfoldable cardinals. Each of these results is established from the minimal relevant large cardinal assumption <sup>2</sup>, which are all compatible with  $V = L$ , and have been shown to be insufficient for establishing the violation of the full diamond principle: Jensen [11] has shown  $\neg\diamond_\kappa$  at a Mahlo cardinal  $\kappa$  implies the existence of  $0^\#$ , and Zeman [16] improved the lower bound of the assumption to the existence of an inner model  $K$  with a Mahlo cardinal  $\kappa$ , such that for every  $\epsilon < \kappa$ , the set  $\{\alpha < \kappa \mid o^K(\alpha) \geq \epsilon\}$  is stationary in  $\kappa$ .

Zeman's argument indicates that establishing the consistency of  $\neg\diamond_\kappa$  via forcing, requires changing the cofinality of many cardinals below  $\kappa$ . Indeed, starting from certain hypermeasurability (large cardinal) assumptions, Woodin [2] has shown  $\neg\diamond_\kappa$  is consistent with  $\kappa$  being inaccessible, Mahlo, or

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<sup>1</sup>I.e., its contrapositive postulates that if  $A$  is a subset of  $\kappa$  and  $A \cap \alpha$  is non-stationary for each  $\alpha$ , then  $A$  is not stationary in  $\kappa$ .

<sup>2</sup>E.g., the existence of a weakly compact cardinal  $\kappa$  with  $\neg\diamond_\kappa(\text{Reg})$  is equi-consistent with the existence of a weakly compact cardinal.

greatly Mahlo cardinal. Woodin’s argument is based on Radin forcing  $\mathbb{R}(\vec{U})$  introduced in [15], which adds a closed unbounded subset to  $\kappa$  consisting of indiscernibles associated with ultrafilters on  $\kappa$  from a measure sequence  $\vec{U}$ . Theorem 1 is based on Woodin’s strategy, and relies on an analysis of Radin forcing. The analysis also leads to a characterization of weak compactness of  $\kappa$  in a generic extension by  $\mathbb{R}(\vec{U})$ .

Extending Woodin’s result in a different direction, it has been recently shown in [1] that the weak diamond principle,  $\Phi_\kappa$ , also fails in Woodin’s model. All the results of this paper concerning the failure of  $\diamond_\kappa$  are compatible with the argument for  $\neg\Phi_\kappa$ . Finally, in yet another direction, Golshani [6] has recently shown that  $\neg\Phi_\kappa$  is consistent with  $\kappa$  being the first inaccessible cardinal.

A brief summary of this paper. The rest of this Section is devoted to reviewing Radin forcing  $\mathbb{R}(\vec{U})$  and its basic properties. Section 2 is devoted to studying the ground model sets  $A \subset \kappa$  which remain stationary in a Radin generic extension. In Section 3 we extend the analysis to arbitrary stationary subsets of  $\kappa$  in a generic extension, and prove Theorem 1. Finally, in Section 4, we go beyond reflection and consider the weak compactness of  $\kappa$ . We introduce a property of a measure sequence  $\vec{U}$  called the weak repeat property (WRP), and prove  $\kappa$  is weakly compact in a  $\mathbb{R}(\vec{U})$  extension if and only if  $\vec{U}$  satisfies WRP.

## 1.1 Additional information - Radin forcing

We review Radin forcing and its basic properties. Our presentation follows Gitik’s Handbook chapter [5]. Thus, everything in Section 1.1 (except Proposition 13) can be found in [5]. Also, we shall follow the Jerusalem forcing convention of [5], that a condition  $p$  is stronger (more informative) than a condition  $q$  is denoted by  $p \geq q$ .

**Definition 2.** Let  $\kappa$  be a measurable cardinal and  $\vec{U} = \langle \kappa \rangle \frown \langle U_\alpha \mid \alpha < \ell(\vec{U}) \rangle$  be a sequence such that each  $U_\alpha$  is a measure on  $V_\kappa$  (i.e., a  $\kappa$ -complete normal ultrafilter on  $V_\kappa$ ). For each  $\beta < \ell(\vec{U})$ , let  $\vec{U} \upharpoonright \beta$  denote the initial segment  $\langle \kappa \rangle \frown \langle U_\alpha \mid \alpha < \beta \rangle$ . We say  $\vec{U}$  is a **measure sequence** on  $\kappa$  if there exists an elementary embedding  $j : V \rightarrow M$  such that for each  $\beta < \ell(\vec{U})$ ,  $\vec{U} \upharpoonright \beta \in M$  and  $U_\beta = \{X \subset V_\kappa \mid \vec{U} \upharpoonright \beta \in j(X)\}$ .

We will frequently use the following notations. Let  $\cap \vec{U}$  denote the filter  $\bigcap_{\alpha < \ell(\vec{U})} U_\alpha$ , and  $\mathcal{MS}$  denote the set of measure sequences  $\vec{\mu}$  on measurable cardinals below  $\kappa$ .  $\vec{\mu}$  is of the form  $\langle \nu \rangle \frown \langle u_i \mid i < \ell(\vec{\mu}) \rangle$ , where each  $u_i$  is a

measure on  $V_\nu$ . We denote  $\nu$  by  $\kappa(\bar{\mu})$ , and  $\cap\{u_i \mid i < \ell(\bar{\mu})\}$  by  $\cap\bar{\mu}$ . For each  $i < \ell(\bar{\mu})$ , we denote  $u_i$  by  $\bar{\mu}(i)$ .

Let  $\vec{U}$  be a measure sequence on  $\kappa$ . We proceed to define the Radin forcing  $\mathbb{R}(\vec{U})$ . Define first a sequence of sets  $A^n \subset \mathcal{MS}$ ,  $n < \omega$ . Let  $A^0 = \mathcal{MS}$ , and for every  $n < \omega$ ,  $A^{n+1} = \{\bar{\mu} \in A^n \mid A^n \cap V_{\kappa(\bar{\mu})} \in \cap\bar{\mu}\}$ . Finally, set  $\bar{A} = \bigcap_n A^n$ . Using the embedding  $j$  and the definition of the measures  $U_\alpha$ ,  $\alpha < \ell(\vec{U})$ , it is straightforward to verify  $\bar{A} \in \bigcap \vec{U}$ .

**Definition 3** (Radin forcing).  $\mathbb{R}(\vec{U})$  consists of finite sequences  $p = \langle d_i \mid i \leq k \rangle$  satisfying the following conditions.

1. For every  $i \leq k$ ,  $d_i$  is either of the form  $\langle \kappa_i \rangle$  for some  $\kappa_i < \kappa$ , or of the form  $d_i = \langle \vec{\mu}_i, a_i \rangle$  where  $\vec{\mu}_i$  is a measure sequence on a measurable cardinal  $\kappa_i = \kappa(\vec{\mu}_i) \leq \kappa$ , and  $a_i \in \cap\vec{\mu}_i$  is a subset of  $(\bar{A} \cap V_{\kappa_i}) \setminus V_{\kappa_{i-1}}$ .
2.  $\langle \kappa_i \mid i \leq k \rangle$  is a strictly increasing sequence and  $\kappa_k = \kappa$ .
3.  $d_k = \langle \vec{U}, A \rangle$  and  $A \subset \bar{A}$ .

For each  $i \leq k$  we denote  $\kappa_i$  by  $\kappa(d_i)$ ,  $\bar{\mu}_i$  by  $\bar{\mu}(d_i)$ , and  $a_i$  by  $a(d_i)$ . For  $m < k$ , we denote  $p^{\leq m} = \langle d_i \mid i \leq m \rangle$  and  $p^{> m} = \langle d_i \mid m < i \leq k \rangle$ . Given a condition  $p = \langle d_i \mid i \leq k \rangle$ , we frequently separate the top part  $d_k = \langle \vec{U}, A \rangle$  from rest, and write  $p = \vec{d} \frown \langle \vec{U}, A \rangle$  or  $p = p_0 \frown \langle \vec{U}, A \rangle$ , where  $p_0 = \vec{d} = \langle d_i \mid i < k \rangle$ . We denote the set of all lower parts of conditions by  $\mathbb{R}_{< \kappa}$ . A condition  $p^* = \langle d_i^* \mid i \leq k^* \rangle$  is a **direct extension** of  $p = \langle d_i \mid i \leq k \rangle$  if  $k^* = k$  and  $a(d_i^*) \subset a(d_i)$  whenever  $a(d_i)$  exists. A condition  $p'$  is a **one point extension** of  $p$  if there exists  $j \leq k$  and a measure sequence  $\vec{v} \in a(d_j)$  such that  $p' = p \frown \langle \vec{v} \rangle$  is either  $\langle d_i \mid i < j \rangle \frown \langle \vec{v} \rangle \frown \langle d_i \mid i \geq j \rangle$  if  $\vec{v} = \alpha$  is an ordinal, or  $\langle d_i \mid i < j \rangle \frown \langle \vec{v}, a(d_j) \cap V_{\kappa(\vec{v})} \rangle \frown \langle d_i \mid i \geq j \rangle$  if  $\vec{v}$  is a nontrivial measure sequence<sup>3</sup>. Let  $p, \bar{p}$  be two conditions of  $\mathbb{R}(\vec{U})$  and  $n < \omega$ . We say that  $\bar{p}$  is an  $n$ -extension of  $p$ , if there exists a sequence  $\vec{\eta} = \langle \vec{v}_1, \dots, \vec{v}_n \rangle \subset \mathcal{MS}$  such that  $\bar{p} = (\dots((p \frown \langle \vec{v}_1 \rangle) \frown \langle \vec{v}_2 \rangle) \dots) \frown \langle \vec{v}_n \rangle$ . We denote  $\bar{p}$  by  $p \frown \vec{\eta}$ . Given two conditions  $p, q \in \mathbb{R}(\vec{U})$  we say that  $q$  **extends**  $p$ , denoted  $q \geq p$ , if it is obtained from  $p$  by a finite sequence of one point extensions and direct extensions. Equivalently,  $q$  extends  $p$  if there exists a finite sequence  $\vec{\eta}$  so that  $q$  is a direct extension of  $p \frown \vec{\eta}$ .

**Definition 4.** Suppose that  $G \subset \mathbb{R}(\vec{U})$  is a  $V$  generic filter. Define  $\mathcal{MS}_G = \{\bar{\mu} \in \mathcal{MS} \mid \exists p \in G, p = \langle d_i \mid i \leq k \rangle \text{ and } \bar{\mu} = \bar{\mu}(d_i) \text{ for some } i < k\}$ , and  $C_G = \{\kappa(\bar{\mu}) \mid \bar{\mu} \in \mathcal{MS}_G\}$ .

<sup>3</sup>Note that implicitly, we are assuming here that  $\kappa(\vec{v}) > \kappa(d_{j-1})$  and  $a(d_j) \cap V_{\kappa(\vec{v})} \in \cap\vec{v}$ .

A standard density argument shows that  $\mathcal{MS}_G$  is almost contained in every ground model set  $A \in \bigcap \vec{U}$  and it completely determines  $G$ . In particular,  $V[G] = V[\mathcal{MS}_G]$ .  $C_G$  is called the generic Radin closed unbounded set. The definition of the forcing implies that if  $p = \vec{d} \frown \langle \vec{U}, A \rangle$  and  $q = \vec{e} \frown \langle \vec{U}, B \rangle$  are two conditions in  $\mathbb{R}(\vec{U})$  satisfying  $|\vec{d}| = |\vec{e}|$  and  $\bar{\mu}(d_i) = \bar{\mu}(e_i)$  for each  $i < |\vec{d}|$ , then  $p, q$  are compatible. Since  $|V_\kappa| = \kappa$ , it follows  $\mathbb{R}(\vec{U})$  satisfies  $\kappa^+$ .c.c

**Lemma 5.**

1.  $(\mathbb{R}(\vec{U}), \leq, \leq^*)$  satisfies the Prikry condition. Namely, for every condition  $p \in \mathbb{R}(\vec{U})$  and a statement of the forcing language  $\sigma$ , there exists  $p^* \geq^* p$  which decides  $\sigma$ .
2. For each  $p = \vec{d} \frown \langle \vec{U}, A \rangle \in \mathbb{R}(\vec{U})$  and  $m < |\vec{d}|$ , the forcing  $\mathbb{R}(\vec{U})/p$  is isomorphic to the product  $\mathbb{R}(\bar{\mu}(d_m))/p^{\leq m} \times \mathbb{R}(\vec{U})/p^{> m}$ .
3. For every condition  $p = \vec{d} \frown \langle \vec{U}, A \rangle \in \mathbb{R}(\vec{U})$  and  $m < |\vec{d}|$ , the direct extension order  $\leq^*$  of  $\mathbb{R}(\vec{U})/p^{> m}$  is  $\kappa_m^+$ -closed.

Combining the last two Lemmata with a standard factorization argument, it is routine to verify  $\mathbb{R}(\vec{U})$  preserves all cardinals. Further analysis of  $\mathbb{R}(\vec{U})$  relies on the notion of fat trees.

**Definition 6.** Let  $\bar{\mu}$  be a measure sequence on a cardinal  $\nu = \kappa(\bar{\mu})$ . A tree  $T \subset [V_\nu]^{\leq n}$ , for some  $n < \omega$ , is called  $\bar{\mu}$ -**fat** if it consists of sequences of measure sequences  $\vec{v} = \langle \bar{v}_1, \dots, \bar{v}_k \rangle$ ,  $k \leq n$ , satisfying the following two conditions.

1.  $\kappa(\bar{v}_1) < \dots < \kappa(\bar{v}_k)$ .
2. if  $o\vec{v}$  is not maximal in  $T$  then there exists some  $i < \ell(\bar{\mu})$  so that the set  $\text{succ}_T(\vec{v}) = \{\vec{v}' \mid \vec{v} \frown \langle \vec{v}' \rangle \in T\}$  belongs to  $\bar{\mu}(i)$ .

Let  $p \in \mathbb{R}(\vec{U})$  and  $\vec{\eta} = \langle \bar{v}_1, \dots, \bar{v}_n \rangle$  be a sequence of measure sequences such that  $p \frown \vec{\eta} \geq p$ . We say that a sequence of sets  $\vec{A} = \langle A_1, \dots, A_n \rangle$  is a  $\vec{\eta}$ -measure-one sequence, if for every  $1 \leq l \leq n$ , such that  $\ell(\bar{v}_l) > 0$ , then  $A_l \in \bar{v}_l$ . Let  $p \frown \langle \vec{\eta}, \vec{A} \rangle$  be the direct extension of  $p' = p \frown \vec{\eta}$  obtained by intersecting  $A_l$  with the measure-one set  $a^{p'}(\bar{v}_l)$  appearing in  $p'$ , for every  $l \leq n$  with  $\ell(\bar{v}_l) > 0$ .

**Lemma 7.** Suppose  $D$  is a dense open subset of  $\mathbb{R}(\vec{U})$  and  $p \in \mathbb{R}(\vec{U})$ . Then there are  $p_D^* = \langle d_1^*, \dots, d_n^* \rangle \geq^* p$ , a finite sequence of integers,  $1 \leq i_1 < \dots < i_m \leq n$ , and a sequence of trees  $\langle T_1, \dots, T_m \rangle$ , where each  $T_l \subset [V_{\kappa(d_{i_l}^*)}]^{\leq n_l}$  is a  $\bar{\mu}(d_{i_l}^*)$ -fat tree, satisfying the following condition: For every sequence

of maximal branches  $\langle \vec{\eta}_l \mid 1 \leq l \leq m \rangle$  with each  $\vec{\eta}_l$  maximal in  $T_l$ , there exists a sequence of sequences of sets  $\langle \vec{A}_l \mid 1 \leq l \leq m \rangle$  such that for every  $l$ ,  $\vec{A}_l$  is a  $\vec{\eta}_l$ -measure-one sequence, and  $p_D^* \frown \langle \vec{\eta}_1, \vec{A}_1 \rangle \frown \langle \vec{\eta}_2, \vec{A}_2 \rangle \frown \dots \frown \langle \vec{\eta}_m, \vec{A}_m \rangle$  belongs to  $D$ .

[5] utilizes Lemma 7 to prove the following result, originally due to Mitchell [14].

**Theorem 8** (Mitchell). *If  $\text{cf}(\ell(\vec{U})) \geq \kappa^+$  then  $\kappa$  is regular in any  $\mathbb{R}(\vec{U})$  generic extension.*

**Remark 9.** *It is also shown in [5] that the result of Theorem 8 is optimal in the sense that  $\kappa$  becomes singular in all  $\mathbb{R}(\vec{U})$  generic extensions when  $\ell(\vec{U}) < \kappa^+$ . A similar argument shows that if  $\vec{U}$  does not contain a repeat point (see the definition below) and that  $\text{cf}(\ell(\vec{U})) \leq \kappa$ , then  $\kappa$  becomes singular in the generic extension.*

**Definition 10.** A measure  $U_\rho \in \vec{U}$  is a repeat point if  $U_\alpha \subset \bigcup_{i < \rho} U_i$  for every  $\alpha \geq \rho$ . We say that  $\vec{U}$  satisfies the **Repeat Property** (RP) if it contains a repeat point.

**Theorem 11** (Mitchell). *If  $\vec{U}$  satisfies RP then  $\kappa$  remains measurable in a  $\mathbb{R}(\vec{U})$  generic extension.*

We conclude this Section with a proposition concerning fresh subsets of  $\kappa$  in a Radin/Magidor generic extensions. Although it will only be used in the last part of the paper (i.e., Lemma 31), we include it here as we believe it is of an independent interest.

**Definition 12** (Joel Hamkins). Let  $V[G]$  be a generic extension of  $V$ . A set  $X \subset \kappa$  in  $V[G]$  is fresh if  $X \cap \alpha \in V$  for every  $\alpha < \kappa$ .

The following result is originally due to Cummings and Woodin ([3]).

**Proposition 13.** *Let  $\mathbb{R}(\vec{U})$  be a Radin or a Magidor forcing on a cardinal  $\kappa$ . If the forcing  $\mathbb{R}(\vec{U})$  does not change the cofinality of  $\kappa$  to  $\omega$  then it does not add fresh subsets to  $\kappa$ .*

*Proof.* Let  $\tau$  be a name of a subset of  $\kappa$ , such that  $0_{\mathbb{R}}$  forces  $\tau \cap \beta \in V$  for every  $\beta < \kappa$ . We introduce the following terminology to prove that  $\tau$  must coincide with a set  $S \in V$ . For a condition  $p = \vec{d} \frown \langle \vec{U}, A \rangle$ ,  $\vec{d} = \langle d_i \mid i < k \rangle$ , let  $\text{supp}(p) = \{\kappa(d_i) \mid i < k\}$ ,  $\kappa_0(p) = \max(\text{supp}(p))$ , and  $\beta(p)$  denote the  $\mathbb{R}(\vec{U})$  name of the minimal ordinal on the Radin generic closed unbounded

set  $C_G$ , which is above  $\kappa_0(p)$ . We call the condition  $p = \vec{d} \frown \langle U, A \rangle$  good, if there exists  $S \subset \kappa$  in  $V$  such that  $p \Vdash \tau \cap \beta(p) = \check{S} \cap \beta(p)$ . We denote the set  $S$  by  $S_p$ . Let us first show that the set of good conditions is dense in  $\mathbb{R}(\vec{U})$ . Fix  $p = \vec{d} \frown \langle \vec{U}, A \rangle$  in  $\mathbb{R}(\vec{U})$ . For every  $\bar{v} \in A$  of order 0 (i.e.,  $\bar{v} = \langle \kappa(\bar{v}) \rangle$ ) then  $p \frown \langle \bar{v} \rangle$  has an extension  $q = \vec{d}(\bar{v}) \frown \langle \bar{v} \rangle \frown \langle \vec{U}, B(\bar{v}) \rangle$  forcing  $\tau \cap \kappa(\bar{v}) = s(\bar{v})$  for some  $s(\bar{v}) \subset \kappa(\bar{v})$  in  $V$ . Note that  $\vec{d}$  and  $\vec{d}(\bar{v})$  must have the same maximal ordinal  $\kappa(d_{k-1})$ . In particular  $\vec{d}(\bar{v}) \in V_{\kappa_0(p)+1}$ . Next, set  $\vec{d} = [\vec{d}(\bar{v})]_{U_0}$ ,  $B = \Delta_{\bar{v}} B(\bar{v})$ , and  $S_p = [s(\bar{v})]_{U_0}$ , then there exists  $A(0) \in U_0$  such that for each  $\bar{v} \in A(0)$ ,  $\vec{d} = \vec{d}(\bar{v})$ ,  $s(\bar{v}) = S_p \cap \kappa(\bar{v})$ , and  $B(\bar{v}) \subset B \setminus V_{\kappa(\bar{v})}$ . Let  $A^*$  be the set obtained from  $A \cap B$ , by reducing the order 0 measure sequences to  $A(0)$ . Then  $p^* = \vec{d} \frown \langle \vec{U}, A^* \rangle$  is good.

Let  $G \subset \mathbb{R}(\vec{U})$  be a generic filter and suppose  $\tau_G \neq S$  for every set  $S \subset \kappa$  in  $V$ . Working in  $V[G]$ , we define an increasing sequence of good conditions  $\langle p^n \mid n < \omega \rangle$  in  $G$ . Let  $p^0 \in G$  be a good condition and denote  $S_{p^0}$  by  $S^0$ . Given  $p^n \in G$  and  $S^n = S_{p^n}$ , we use the fact  $\tau_G \neq S^n$  to find  $p^{n+1} \geq p^n$  in  $G$ , such that  $\kappa_0(p^{n+1}) \cap \tau_G \neq \kappa_0(p^{n+1}) \cap S^n$ . We may also assume  $p^{n+1}$  is good and set  $S^{n+1} = S_{p^{n+1}}$ . Clearly,  $\kappa_0(p^{n+1}) > \kappa_0(p^n)$ ,  $S^{n+1} \cap \kappa_0(p^n) = S^n \cap \kappa_0(p^n)$ , and  $S^{n+1} \cap \kappa_0(p^{n+1}) \neq S^n \cap \kappa_0(p^{n+1})$ . Next, let  $\gamma = \bigcup_{n < \omega} \kappa_0(p^n)$ . Since  $\text{cf}(\kappa)^{V[G]} > \omega$ ,  $\gamma \in C_G$ . By the construction of the conditions  $p^n \in G$ ,  $\tau_G \cap \gamma \neq S^n \cap \gamma$  for all  $n < \omega$ . Let  $q \in G$  and  $X \subset \gamma$  in  $V$  such that  $\gamma \in \text{supp}(q)$  and  $q \Vdash \tau \cap \check{\gamma} = \check{X} \cap \check{\gamma}$ . Let us write  $q = \vec{d}^0 \frown \vec{d}^1 \frown \langle \vec{U}, A_q \rangle$ , where  $\vec{d}^0 = \langle d_i^0 \mid i \leq k^0 \rangle$ , and  $\kappa(d_{k^0}^0) = \gamma$ . Take  $n < \omega$  so that  $\kappa_0(p^n) > \text{supp}(q) \cap \gamma$ , and  $q^n \in G$  be the minimal common extension of  $p^n$  and  $q$ . Note that  $\max(\text{supp}(q^n) \cap \gamma) = \kappa_0(p^n)$ . Pick  $\bar{v} \in a_{k^0}(q)$  with  $\ell(\bar{v}) = 0$ , so that  $X \cap \kappa(\bar{v}) \neq S^n \cap \kappa(\bar{v})$  and consider the extension  $q^n \frown \langle \bar{v} \rangle$  of  $q^n$ . By the choice of  $q^n \geq p^n$ ,  $q^n \frown \langle \bar{v} \rangle \Vdash \kappa(\bar{v}) = \beta(p^n)$ , thus  $q^n \frown \langle \bar{v} \rangle \Vdash \tau \cap \kappa(\bar{v}) = S^n \cap \kappa(\bar{v})$ . This is an absurd as  $q^n \frown \langle \bar{v} \rangle \geq q$  and therefore  $q^n \frown \langle \bar{v} \rangle \Vdash \tau \cap \kappa(\bar{v}) = \check{X} \cap \kappa(\bar{v}) \neq S^n \cap \kappa(\bar{v})$ .  $\square$

## 2 Radin forcing and stationarity of ground model sets

We utilize Lemma 7 to determine which subsets of  $\kappa$  remain stationary in a generic extension by  $\mathbb{R}(\vec{U})$ . It is known that if  $\vec{U}$  is a  $\triangleleft$ -increasing sequence of measures of length  $\ell(\vec{U}) < \kappa$  such that  $\text{cf}(\ell(\vec{U}))$  is uncountable, then  $\kappa$  becomes singular of uncountable cofinality in a Magidor forcing extension by  $\vec{U}$ , and for every  $X \subset \kappa$  in  $V$ ,  $X$  remains a stationary subset of  $\kappa$  in a generic extension if and only if  $X \in U_\tau$  for closed unbounded many  $\tau < \ell(\vec{U})$ . This characterization of ground model sets which remain stationary does not

apply to  $\mathbb{R}(\vec{U})$  when  $\text{otp}(\ell(\vec{U})) \geq \kappa^+$ .

**Definition 14.** 1. Let  $Z \subset \mathcal{MS}$  in  $V$ . We say that  $Z$  is  $\vec{U}$ -positive if  $Z \in U_\tau$  for unbounded many ordinals  $\tau < \ell(\vec{U})$ .

2. For a set  $Z \subset \mathcal{MS}$  we define  $O(Z) = \{\kappa(\vec{\mu}) \mid \vec{\mu} \in Z\}$ .

**Proposition 15.** *Suppose that  $cf(\ell(\vec{U})) \geq \kappa^+$ . Then for every  $Z \subset \mathcal{MS}$  in  $V$ , if  $Z$  is  $\vec{U}$ -positive then  $O(Z)$  is stationary in  $V[G]$ .*

*Proof.* Let  $\tau$  be a  $\mathbb{R}(\vec{U})$ -name for a closed unbounded subset in  $\kappa$ . We show that every condition  $p$  has an extension forcing  $O(\check{Z}) \cap \tau \neq \emptyset$ . For a condition  $q = q_0 \frown \langle \vec{U}, B \rangle$  where  $q_0 = \langle d_i \mid i < k \rangle$ , we denote  $\text{supp}(q_0) = \{\kappa(d_i) \mid i < k\}$  and  $\kappa_0(q) = \max(\text{supp}(q_0))$ . For every  $i < \kappa$ , let  $D_i \subset \mathbb{R}(\vec{U})$  be the dense open set of all conditions  $q = q_0 \frown \langle \vec{U}, B \rangle$  such that  $q \Vdash \dot{\beta}_i < \kappa_0(q)$ , where  $\dot{\beta}_i$  is the name of the  $i$ -th element of  $\tau$ . By Lemma 7, for each  $\vec{d} \in \mathbb{R}_{<\kappa}$  there is a sequence of fat trees  $\langle T_1, \dots, T_m \rangle$  associated with  $\vec{d}$  and  $D_i$ . Following the notations of the Lemma, let  $T_{i,\vec{d}}$  denote the top tree  $T_m$  if it is a  $\vec{U}$ -fat tree, and  $A_{i,\vec{d}} \in \bigcap \vec{U}$  be the top measure-one set in the condition  $p_{D_i}^*$ . Since  $cf(\ell(\vec{U})) \geq \kappa^+$ , and there are at most  $\kappa$  many trees of the form  $T_{i,\vec{d}}$ , there exists some  $\alpha^* < \ell(\vec{U})$  which is greater than the indices of all measures associated with the splitting levels of the fat trees  $T_{i,\vec{d}}$ ,  $i, \vec{d} \in V_\kappa$ . Define  $\Gamma = \{\vec{v} \in \mathcal{MS} \mid \forall i, \vec{d} \in V_{\kappa(\vec{v})}. T_{i,\vec{d}} \cap V_{\kappa(\vec{v})} \text{ is a } \vec{v}\text{-fat tree}\}$ . It follows that  $\Gamma \in \bigcap_{\gamma \geq \alpha^*} U_\gamma$ , in particular there exists some  $\gamma \geq \alpha$  such that  $Z \in U_\gamma$ . Define  $A^* = \Delta_{i,\vec{d}} A_{i,\vec{d}}$  and  $p^* = p_0 \frown \langle \vec{U}, A^* \rangle$ .

**Claim\*:** For every  $\vec{v} \in \Gamma \cap Z \cap A^*$ ,  $p^* \frown \langle \vec{v} \rangle \Vdash \kappa(\vec{v}) \in \tau$ .

It is sufficient to verify  $p^* \frown \langle \vec{v} \rangle \Vdash \kappa(\vec{v}) = \beta_{\kappa(\vec{v})}$ . Clearly,  $q \Vdash \dot{\beta}_{\kappa(\vec{v})} \geq \kappa(\vec{v})$ , and since  $p \Vdash \tau$  is closed unbounded, it is actually sufficient to show that  $p^* \frown \langle \vec{v} \rangle \Vdash \dot{\beta}_i < \kappa(\vec{v})$  for all  $i < \kappa(\vec{v})$ . Fix  $i < \kappa(\vec{v})$ , and  $r \geq p^* \frown \langle \vec{v} \rangle$ . We claim  $r$  has an extension which forces that “ $\beta_i < \kappa(\vec{v})$ “. Suppose  $r = r_0 \frown \langle \vec{v}, b \rangle \frown r_1 \frown \langle \vec{U}, A_r \rangle$ . Our construction of  $p^*$  guarantees  $r_0 \in V_{\kappa(\vec{v})}$  and that every measure sequence  $\vec{v}$  in  $r_1$  belongs to  $A_{i,r_0}$ . Moreover, as  $\vec{v} \in \Gamma$ ,  $T_{i,r_0} \cap V_{\kappa(\vec{v})}$  is a fat  $\vec{v}$ -tree. Let  $T_1, \dots, T_m$  be the sequence of fat trees associated with  $\vec{d} = r_0$ . For every sequence of maximal branches  $\mathfrak{t} = \langle \vec{\eta}_l \mid 1 \leq l \leq m \rangle$  through  $\langle T_1, \dots, T_m \rangle$  respectively, there is a sequence of sequences of sets  $\mathfrak{a} = \langle \vec{A}_l \mid 1 \leq l \leq m \rangle$ , such that the extension  $(r_0 \frown \langle \vec{v}, b \rangle \frown \langle \vec{U}, A_{i,r_0} \rangle) \frown \langle \vec{\eta}_1, \vec{A}_1 \rangle \frown \langle \vec{\eta}_2, \vec{A}_2 \rangle \frown \dots \frown \langle \vec{\eta}_m, \vec{A}_m \rangle$  belongs to  $D_i$ . Denote the last condition by  $r_0^{+(\mathfrak{t}, \mathfrak{a})}$ . Since  $T_{i,\vec{d}} \cap V_{\kappa(\vec{v})}$  is  $\vec{v}$ -fat, there is



a sequence of maximal branches  $\mathbf{t} = \langle \vec{\eta}_l \mid 1 \leq l \leq m \rangle$  consisting only of  $V_{\kappa(\vec{\nu})}$  elements, resulting in a condition  $r_0^{+(\mathbf{t}, \mathbf{a})}$  which is compatible with  $r$ . Finally, as  $\kappa_0(r_0^{+(\mathbf{t}, \mathbf{a})}) = \kappa(\vec{\nu})$  and  $\kappa_0(r_0^{+(\mathbf{t}, \mathbf{a})}) \in D_i$ ,  $r_0^{+(\mathbf{t}, \mathbf{a})} \Vdash \dot{\beta}_i < \kappa(\vec{\nu})$ . Claim\* and the Proposition follow.  $\square$

As an immediate corollary of the Lemma, we obtain the following result of Woodin.

**Corollary 16** (Woodin). *For every  $\tau \leq \kappa^+$ , If  $\text{otp}(l(\vec{U}))$  is the ordinal exponent  $(\kappa^+)^{1+\tau}$  then  $\kappa$  is  $\tau$ -Mahlo in a  $\mathbb{R}(\vec{U})$  generic extension.*

The next result shows that assuming the sequence  $\vec{U}$  does not contain a repeat point, the sufficient condition given in Proposition 15 for  $O(Z)$  to be stationary is also necessary.

**Proposition 17.** *Suppose  $\vec{U}$  is a measure sequence of limit length which does not contain a repeat point. For every  $Z \subset \mathcal{MS}$ , if  $Z \in \bigcap (\vec{U} \setminus \tau) = \bigcap \{U_\rho \mid \tau \leq \rho < \ell(\vec{U})\}$  for some  $\tau < \ell(\vec{U})$  then  $O(Z)$  contains a closed unbounded set in a  $\mathbb{R}(\vec{U})$  generic extension.*

*Proof.* Fix  $Z, \tau < \ell(\vec{U})$  as in the statement of the Lemma. We show that for every  $p = p_0 \frown \langle \vec{U}, A^p \rangle \in \mathbb{R}(\vec{U})$  there is a direct extension  $p^* = p_0 \frown \langle \vec{U}, A^* \rangle$  forcing that  $O(Z)$  contains a closed unbounded set. Since  $\tau$  is not a repeat point, there exists  $B \in U_\tau \setminus (\bigcup \vec{U} \upharpoonright \tau)$ . Defining  $B' = \{\bar{\mu} \in \mathcal{MS} \mid \exists i < \ell(\bar{\mu}). B \cap V_{\kappa(\bar{\mu})} \in \bar{\mu}(i) \setminus (\cup \bar{\mu} \upharpoonright i)\}$ , it is routine to verify that for every  $\rho < \ell(\vec{U})$ ,  $\vec{U} \upharpoonright \rho \in j(B')$  if and only if  $\rho > \tau$ . By replacing  $Z$  with  $Z \cap B$  we may assume  $Z \in U_\rho$  only for  $\rho > \tau$ . Next, let  $Z^\leq = \{\bar{\mu} \in \mathcal{MS} \mid Z \cap V_{\kappa(\bar{\mu})} \notin \cup \bar{\mu}\}$ . It follows that  $Z^\leq \in U_\rho$  if and only if  $\rho \leq \tau$ . We define  $A^* = A^p \cap (Z \cup Z^\leq)$ ,  $p^* = p_0 \frown \langle \vec{U}, A^* \cap A \rangle$ , and  $\dot{D} = (\mathcal{MS}_G \setminus \max(p_0)) \cap Z$ . Let us show  $p^* \Vdash O(\dot{D})$  is closed unbounded in  $\kappa$ . Let  $q = \vec{d} \frown \langle \vec{U}, B \rangle$  be an extension of  $p^*$ , and  $\alpha < \kappa$ . Since  $Z$  is unbounded in  $V_\kappa$ ,  $q$  has a one point extension  $q \frown \langle \vec{\nu} \rangle$  where  $\vec{\nu} \in Z \setminus V_\alpha$ . Thus  $q \frown \langle \vec{\nu} \rangle \Vdash \kappa(\vec{\nu}) \in O(\dot{D}) \setminus \alpha$ . Finally, suppose  $\alpha < \kappa$  and  $q \Vdash \alpha$  is a limit point of  $O(\dot{D})$ . We may assume  $\alpha = \kappa(\vec{\nu})$  for some  $\vec{\nu} = \vec{\nu}(d_i)$  for some  $d_i \in \vec{d}$ . Since  $\kappa(\vec{\nu}) > \max(p_0)$ ,  $\vec{\nu} \in A^* \subset Z^\leq \cup Z$ .  $\vec{\nu}$  cannot be an element of  $Z^\leq$  as otherwise,  $Z \cap V_\alpha \notin \cup \vec{\nu}$  and by substituting  $a(d_i)$  with  $a(d_i) \setminus Z$ , we can form a direct extension  $q^* \geq^* q$  forcing  $\alpha = \kappa(\vec{\nu})$  is not a limit point of  $O(\dot{D})$ . Contradiction. It follows that  $\vec{\nu} \in Z$ , and  $q \Vdash \kappa(\vec{\nu}) \in O(\dot{D})$ .  $\square$

### 3 Stationary reflection and the failure of diamond

Woodin's construction of a model of set theory satisfying  $\neg\Diamond_\kappa$  on a Mahlo cardinal  $\kappa$ , is based on the following result.

**Theorem 18** (Woodin). *Suppose that  $\vec{U}$  is a measure sequence and  $2^\kappa > \ell(\vec{U})$ , and let  $G \subset \mathbb{R}(\vec{U})$  be generic over  $V$ . If  $\kappa$  remains regular in  $V[G]$  then  $\neg\Diamond_\kappa$  holds in  $V[G]$ .*

We include Woodin's elegant argument for completeness.

*Proof.* Let  $\dot{s} = \langle \dot{s}_\alpha \mid \alpha < \kappa \rangle$  be a  $\mathbb{R}(\vec{U})$ -name, and suppose that  $p = p_0 \frown \langle \vec{U}, A^p \rangle$  is a condition forcing  $\dot{s}_\alpha \subset \alpha$  for all  $\alpha < \kappa$ . For each  $\bar{v} \in A^p$ , the forcing  $\mathbb{R}(\vec{U})/(p \frown \langle \bar{v} \rangle)$  factors into  $\mathbb{R}(\bar{v})/(p_0 \frown \langle \bar{v} \rangle) \times R(\vec{U})/(\vec{U}, A^p \setminus V_{\kappa(\bar{v})+1})$ , where the direct extension order of the second component is  $(2^{\kappa(\bar{v})})^+$ -closed. It follows that the condition  $\langle \vec{U}, A^p \setminus V_{\kappa(\bar{v})+1} \rangle$  in second component has a direct extension  $\langle \vec{U}, A_{\bar{v}} \rangle$  which decides the value of the set  $s_{\kappa(\bar{v})}$ , hence reducing the  $\mathbb{R}(\vec{U})$  name  $s_{\kappa(\bar{v})}$  to a  $\mathbb{R}(\bar{v})$ -name  $\dot{s}'_{\bar{v}}$ . Now, let  $A^* = \Delta_{\bar{v} \in A^p} A_{\bar{v}}$ . Consider the condition  $p^* = p_0 \frown \langle \vec{U}, A^* \rangle$  and the function  $\vec{s}' : A^* \rightarrow V$ , defined by  $\vec{s}'(\bar{v}) = \dot{s}'_{\bar{v}}$ . It follows that  $p^* \frown \langle \bar{v} \rangle \Vdash s_{\kappa(\bar{v})} = \vec{s}'(\bar{v})$  for each  $\bar{v} \in A^*$ . Consequently, for every  $\tau < \ell(\vec{U})$ ,  $j(p^*) \frown \langle \vec{U} \upharpoonright \tau \rangle \Vdash j(\vec{s})_\kappa = j(\vec{s}')(\vec{U} \upharpoonright \tau)$ , where the last is a  $\mathbb{R}(\vec{U} \upharpoonright \tau)$  name of a subset of  $\kappa$ . Since  $\mathbb{R}(\vec{U} \upharpoonright \tau)$  satisfies  $\kappa^+$ .c.c and  $2^\kappa > \ell(\vec{U})$ , there must exist  $X \subset \kappa$  such that  $j(p^*) \frown \langle \vec{U} \upharpoonright \tau \rangle \Vdash j(\vec{s}')(\vec{U} \upharpoonright \tau) \neq \check{X}$  for every  $\tau < \ell(\vec{U})$ . It follows that  $p^*$  has a direct extension  $q = p_0 \frown \langle \vec{U}, B \rangle$  such that  $q \frown \langle \bar{v} \rangle \Vdash \dot{s}_{\kappa(\bar{v})} \neq \check{X} \cap \kappa(\bar{v})$  for every  $\bar{v} \in B$ . Hence  $q$  forces  $\dot{s}$  is not a  $\Diamond_\kappa$  sequence.  $\square$

Woodin's argument essentially implies that every large cardinal property of  $\kappa$ , obtainable in a  $\mathbb{R}(\vec{U})$  generic extension from assumptions concerning the length of  $\vec{U}$ , is consistent with  $\neg\Diamond_\kappa$ . Therefore, from Theorem 8 and Corollary 16, we can infer  $\neg\Diamond_\kappa$  is consistent when  $\kappa$  is inaccessible, or  $\tau$ -Mahlo for some  $\tau \leq \kappa^+$ . Indeed, it is well-known that under certain hypermeasurability large cardinal assumptions we can construct a model  $V$  in which  $2^\kappa = \kappa^{++}$  and  $\kappa$  carries a measure sequence  $\vec{U}$  of length  $\kappa^+$ , or  $(\kappa^+)^{1+\tau}$  for  $\tau \leq \kappa^+$ .<sup>4</sup> By extending the analysis of the stationary subsets of  $\kappa$  in  $\mathbb{R}(\vec{U})$  generic extensions we prove the following result.

<sup>4</sup>I.e., using Mitchell's version of Radin forcing [?], the assumption of a measurable cardinal with  $o(\kappa) = \kappa^{++} + (\kappa^+)^{1+\tau}$  suffices.

**Theorem 19.** *Let  $\vec{U}$  be a measure sequence . If  $\text{cf}(\ell(\vec{U})) \geq \kappa^{++}$  then  $\kappa$  satisfies the strong simultaneous reflection principle in every  $\mathbb{R}(\vec{U})$  generic extension.*

The following family of functions play a central role in the analysis of stationary subsets of  $\kappa$  in a Radin generic extension.

**Definition 20.** A **measure function** is a function  $b : \mathcal{MS} \rightarrow V_\kappa$  satisfying  $b(\bar{\mu}) \in \cap \bar{\mu}$  for every  $\bar{\mu} \in \mathcal{MS}$ . We denote the set of measure functions by  $\mathfrak{MF}$ .

**Proof.**(Theorem 19) First, if  $\vec{U}$  contains a repeat point then  $\kappa$  is measurable in any  $\mathbb{R}(\vec{U})$  generic extension, and in particular satisfies the strong simultaneous reflection property. Therefore, let us assume from now on that  $\vec{U}$  does not contain a repeat point. We commence with showing that every stationary subset  $S$  of  $\kappa$  in  $V[G]$  reflects. Let  $\dot{S}$  be a  $\mathbb{R}(\vec{U})$  name of  $S$ . Working in  $V$ , for each  $\vec{d} \in \mathbb{R}_{<\kappa}$  consider the condition  $p_{\vec{d}} = \vec{d} \frown \langle \vec{U}, \mathcal{MS} \setminus \max(\vec{d}) \rangle$ . For each  $\bar{\mu} \in \mathcal{MS} \setminus \max(\vec{d})$ , the condition  $p_{\vec{d}} \frown \langle \bar{\mu} \rangle$  has a direct extension of the form  $q_{\vec{d}}(\bar{\mu}) = e_{\vec{d}}(\bar{\mu}) \frown \langle \bar{\mu}, b_{\vec{d}}(\bar{\mu}) \rangle \frown \langle \vec{U}, A_{\vec{d}}(\bar{\mu}) \rangle$  deciding the statement “ $\kappa(\bar{\mu}) \in \dot{S}$ ”. Therefore, for each  $\vec{d}$ , we obtain three functions,  $e_{\vec{d}}$ ,  $b_{\vec{d}}$ , and  $A_{\vec{d}}$ . Let  $A_{\vec{d}} = \Delta_{\bar{\mu} \in \mathcal{MS}} A_{\vec{d}}(\bar{\mu})$  and  $A = \Delta_{\vec{d}} A_{\vec{d}}$ . Define  $b^* \in \mathfrak{MF}$  by  $b^*(\bar{\mu}) = \Delta_{\vec{d}} V_{\kappa(\bar{\mu})} b_{\vec{d}}(\bar{\mu}) = \{\bar{\nu} \in \mathcal{MS} \cap V_{\kappa(\bar{\mu})} \mid \forall \vec{d} \in V_{\kappa(\bar{\nu})} \cdot \bar{\nu} \in b_{\vec{d}}(\bar{\mu})\}$ . While independent of  $\vec{d} \in \mathbb{R}_{<\kappa}$ ,  $A, b^*$  capture the information given by the sets  $A_{\vec{d}}$  and measure functions  $b_{\vec{d}}$ . Next, for an element  $\vec{e} \in \mathbb{R}_{<\kappa}$  define  $Z_{\vec{e}} = \{\bar{\mu} \in \mathcal{MS} \mid \exists A \in \cap \vec{U} \cdot \vec{e} \frown \langle \bar{\mu}, b^*(\bar{\mu}) \rangle \frown \langle \vec{U}, A \rangle \Vdash \kappa(\bar{\mu}) \in \dot{S}\}$ . We say that  $\vec{e}$  is a stationary witness of  $\dot{S}$  if there exists  $B_{\vec{e}} \in \cap \vec{U}$  such that for every  $\vec{\eta} \in \mathcal{MS}^{<\omega}$ ,  $\vec{\eta} \subset B_{\vec{e}}$  the set  $Z_{\vec{e}} \upharpoonright \vec{\eta} = \{\bar{\mu} \in Z_{\vec{e}} \mid \vec{\eta} \subset b^*(\bar{\mu}) \text{ and } b^*(\bar{\mu}) \cap V_{\kappa(\bar{\mu}')} \in \cap \bar{\mu}' \text{ for every } \bar{\mu}' \in \vec{\eta}\}$  is  $\vec{U}$ -positive.

**Claim 1:** Suppose  $p = p_0 \frown \langle \vec{U}, A_p \rangle \in \mathbb{R}(\vec{U})$  forces that  $\dot{S}$  is a stationary subset of  $\kappa$ . Then  $p$  has an extension  $q = \vec{e} \frown \langle \vec{U}, A' \rangle$  such that  $\vec{e}$  is a stationary witness of  $\dot{S}$  and  $A' \subset B_{\vec{e}}$ .

By replacing  $p$  with a direct extension if needed, we may assume that  $A_p \subset A$ . It is sufficient to show that for every generic filter  $G \subset \mathbb{R}(\vec{U})$  which contains  $p$ , there exists  $q = \vec{e} \frown \langle \vec{U}, A_{\vec{e}} \rangle \in G$  such that  $\vec{e}$  is a stationary witness of  $\dot{S}$ . To this end, work in  $V[G]$  and for each  $\alpha \in S \cap C_G$  set  $\bar{\mu}_\alpha \in \mathcal{MS}_G$  to be the unique measure sequence in  $\mathcal{MS}_G$  satisfying  $\alpha = \kappa(\bar{\mu}_\alpha)$ . For each  $\alpha \in S$ ,  $b^*(\bar{\mu}_\alpha) \in \cap \bar{\mu}_\alpha$ . Therefore there exists a maximal ordinal  $\beta_\alpha < \alpha$  greater or equal to  $\max(\kappa(p_0))$  such that  $\mathcal{MS}_G \setminus V_{\beta_\alpha} \subset b(\bar{\mu}_\alpha)$ . By Fodor’s Lemma, there exist  $\beta^* \in C_G$  and  $S' \subset S$  stationary, so that  $\beta^* = \beta_\alpha$  for each

$\alpha \in S'$ . Let  $G_{<\kappa}$  denote the set of bottom parts of generic conditions.<sup>5</sup> By a standard density argument, for each  $\alpha \in S'$  there exists  $\vec{d}^\alpha = \langle d_i^\alpha \mid i < k_\alpha \rangle$  extending  $p_0$ , with  $\kappa(d_{k_\alpha-1}^\alpha) = \beta^*$ , such that  $e_{\vec{d}^\alpha}(\vec{\mu}_\alpha) \in G_{<\kappa}$ . By pressing down again, we can find  $\vec{d}^*, \vec{e} \in V_{\beta^*+1}$ , and a stationary set  $S^* \subset S'$  such that  $\vec{d}^* = \vec{d}^\alpha$  and  $\vec{e} = e_{\vec{d}^*}(\vec{\mu}_\alpha)$  for each  $\alpha \in S^*$ . It follows that for each  $\alpha \in S^*$ , the condition  $\vec{e} \frown \langle \vec{\mu}_\alpha, b^*(\vec{\mu}_\alpha) \rangle \frown \langle \vec{U}, A_p \rangle$  belongs to  $G$  and forces that “ $\alpha \in \dot{S}$ ”. This implies  $\{\vec{\mu}_\alpha \mid \alpha \in S^*\} \subset Z_{\vec{e}}$ , which in turn implies that  $Z_{\vec{e}}$  is  $\vec{U}$ -positive. To see the last, note that otherwise, Proposition 17 implies there is a closed unbounded set  $C \subset \kappa$  in  $V[G]$  which is disjoint from  $O(Z_{\vec{e}}) = \{\kappa(\vec{\mu}) \mid \vec{\mu} \in Z_{\vec{e}}\}$ . But this is an absurd as  $S^* \cap C \neq \emptyset$ .

Our next goal is to construct a set  $B_{\vec{e}}$  as described in the definition of  $\vec{e}$  being a stationary witness of  $\dot{S}$ . For this we consider sets of the form  $Z_{\vec{e}} \upharpoonright \vec{\eta}$  for various  $\vec{\eta} \in A^{<\omega} \subset \mathcal{MS}^{<\omega}$ . Let us say that  $\vec{\eta}$  is 0-positive if  $Z_{\vec{e}} \upharpoonright \vec{\eta}$  is  $\vec{U}$ -positive, and for an integer  $n \geq 0$ , say  $\vec{\eta}$  is  $(n+1)$ -positive if the set  $B_{\vec{\eta}}^n = \{\vec{v} \in \mathcal{MS} \mid \vec{\eta} \frown \langle \vec{v} \rangle \text{ is } n\text{-positive}\}$  belongs to  $\bigcap \vec{U}$ . Finally,  $\vec{\eta}$  is  $\omega$ -positive if it is  $n$ -positive for each  $n < \omega$  and  $B_{\vec{\eta}}^\omega = \bigcap_{n < \omega} B_{\vec{\eta}}^n$ . Let  $B^\omega = \{\vec{v} \in \mathcal{MS} \mid \langle \vec{v} \rangle \text{ is } \omega\text{-positive}\}$ .

**Sub-Claim 1.1:**  $B^\omega \in \bigcap \vec{U}$ . It is sufficient to show that for every  $n < \omega$ , the set  $B^n = \{\vec{v} \in \mathcal{MS} \mid \langle \vec{v} \rangle \text{ is } n\text{-positive}\}$  belongs to  $\bigcap \vec{U}$ . Suppose otherwise. Then there are  $\alpha_0 < \ell(\vec{U})$  and  $A_0 \in U_{\alpha_0}$  such that  $\langle \vec{v}_0 \rangle$  is not  $n$ -positive for each  $\vec{v}_0 \in A_0$ . That is, for each  $\vec{v}_0 \in A_0$  there are  $U_{\alpha(\vec{v}_0)}$  and  $A_{\langle \vec{v}_0 \rangle} \in U_{\langle \vec{v}_0 \rangle}$  such that for each  $\vec{v}_1 \in A_{\langle \vec{v}_0 \rangle}$ ,  $\langle \vec{v}_0, \vec{v}_1 \rangle$  is not  $(n-1)$ -positive. By continuing to unravel the statement in this manner, we can construct a  $\vec{U}$ -fat tree  $T \subset \mathcal{MS}^{\leq n}$  (see Definition 6) such that for every maximal branch  $\vec{\eta} = \langle \vec{v}_1, \dots, \vec{v}_n \rangle$  of  $T$ ,  $Z_{\vec{e}} \upharpoonright \vec{\eta}$  is not  $\vec{U}$ -positive. Since  $T$  is  $\vec{U}$ -fat, a standard density argument shows there exists a condition  $e^* \frown \vec{d} \frown \langle \vec{U}, A' \rangle$  in  $G$  such that  $\vec{d} = \langle d_1, \dots, d_n \rangle$ , where  $\vec{\eta} = \langle \vec{\mu}(d_1), \dots, \vec{\mu}(d_n) \rangle$  is a maximal branch of  $T$ . Now by Proposition 17, there exists a closed unbounded set  $C \subset \kappa$  in  $V[G]$  which is disjoint from  $O(Z_{\vec{e}} \upharpoonright \vec{\eta})$ . To get a contradiction, we take  $\alpha \in S^* \cap C$  which is above  $\max(\kappa(\vec{v}_n))$ . By the definition of  $S^*$ , we have that  $e^* \frown \langle \vec{\mu}_\alpha, b^*(\vec{\mu}_\alpha) \rangle \frown \langle \vec{U}, A^* \rangle$  belongs to  $G$  and therefore forces “ $\check{\alpha} = \kappa(\vec{\mu}_\alpha) \in \dot{S}$ ”. Also, since the ordinals in  $\vec{\eta}$  are all above  $\max(\vec{e}) = \beta^*$ ,  $\vec{\eta} \subset b^*(\vec{\mu}_\alpha)$ . But this means  $\alpha \in O(Z_{\vec{e}} \upharpoonright \vec{\eta}) \cap C$ . Contradiction.  $\square$ (Sub-Claim 1.1)

We can now define  $B_{\vec{e}}$ . First, let  $\Delta^1 = B^\omega$  and for each  $n < \omega$ , let  $\Delta^{n+1} = \Delta^n \cap \{\vec{\mu} \in \Delta^n \mid \forall \vec{\eta} = \langle \vec{\mu}_1, \dots, \vec{\mu}_n \rangle \subset \Delta^n \cap V_{\kappa(\vec{\mu})} \cdot \vec{\mu} \in B_{\vec{\eta}}^\omega\}$ . We then set  $B_{\vec{e}}$  to be  $A \cap (\bigcap_n \Delta^n)$ . It is routine to verify  $B_{\vec{e}} \in \bigcap \vec{U}$  and that for every

<sup>5</sup>Namely, the set of all  $\vec{d} \in \mathbb{R}_\kappa$  such that  $\vec{d} \frown \langle \vec{U}, \mathcal{MS} \rangle \in G$ .

increasing sequence  $\vec{\eta} = \langle \bar{\mu}_1, \dots, \bar{\mu}_m \rangle \subset B_{\vec{e}}$ ,  $Z_{\vec{e}} \upharpoonright \vec{\eta}$  is  $\vec{U}$ -positive. Finally, let  $A' = B_{\vec{e}} \cap A_p$ . Then  $\vec{e}$  is stationary witness of  $\dot{S}$  and  $q = \vec{e} \frown \langle \vec{U}, A' \rangle$  is an extension of  $p$ .  $\square$ (Claim 1)

Let us show how a stationary witness of  $\dot{S}$ ,  $\vec{e} \in \mathbb{R}_{<\kappa}$ , can be used to find a reflection point of  $\dot{S}$ . Let  $B_{\vec{e}} \in \bigcap \vec{U}$  as in the definition of a stationary witness. For each  $\vec{\eta} \in B_{\vec{e}}^{<\omega}$  define  $E_{\vec{e}}(\vec{\eta}) \subset \ell(\vec{U})$  to be the set of accumulation points of all  $\tau < \ell(\vec{U})$  such that  $Z_{\vec{e}} \upharpoonright \vec{\eta} \in U_\tau$ . Since each  $E_{\vec{e}}(\vec{\eta})$  is closed unbounded in  $\ell(\vec{U})$  and  $\text{cf}(\ell(\vec{U})) \geq \kappa^{++}$ ,  $E_{\vec{e}} = \bigcap \{E_{\vec{e}}(\vec{\eta}) \mid \vec{\eta} \subset A_{\vec{e}}\}$  is also closed unbounded and there exists  $\tau \in \kappa^{++} \cap \text{cf}(\kappa^+)$  which is a limit point of  $E_{\vec{e}}$ . It follows that there exists  $X \in U_\tau$  such that every  $\bar{v} \in X$  satisfies the following conditions:

1.  $\text{cf}(\ell(\bar{v})) = \kappa(\bar{v})^+$ ,
2. for every  $\vec{\eta} \subset A_{\vec{e}} \cap V_{\kappa(\bar{v})}$ ,  $Z_{\vec{e}} \upharpoonright \vec{\eta}$  is  $\bar{v}$ -positive.

**Claim 2:** For every  $\bar{v} \in X$ , the condition  $\vec{e} \frown \langle \bar{v}, A_{\vec{e}} \cap V_{\kappa(\bar{v})} \rangle \frown \langle \vec{U}, A_{\vec{e}} \rangle$  forces  $\dot{S} \cap \kappa(\bar{v})$  is stationary in  $\kappa(\bar{v})$ .

Let us denote the condition  $\vec{e} \frown \langle \bar{v}, A_{\vec{e}} \cap V_{\kappa(\bar{v})} \rangle \frown \langle \vec{U}, A_{\vec{e}} \rangle$  by  $t$ . Suppose that  $\sigma$  is a name for a subset of  $\kappa(\bar{v})$  and  $q$  is an extension of  $t$ , forcing that  $\sigma$  is a closed unbounded subset of  $\kappa(\bar{v})$ . We separate  $q$  into parts and write  $q = q_0 \frown q_1 \frown \langle \bar{v}, b \rangle \frown q_2 \frown \langle \vec{U}, A_q \rangle$ , where  $q_0 \geq \vec{e}$ ,  $q_1 \frown \langle \bar{v}, b \rangle \geq \langle \bar{v}, A_{\vec{e}} \cap V_{\kappa(\bar{v})} \rangle$ , and  $q_2 \frown \langle \vec{U}, A_q \rangle \geq \langle \vec{U}, A_{\vec{e}} \rangle$ . By further extending  $q_2 \frown \langle \vec{U}, A_q \rangle$  if necessary, we may assume  $\sigma$  is a  $\mathbb{R}(\bar{v})$  name of a closed unbounded subset of  $\kappa(\bar{v})$ . The rest of the proof follows the argument of the proof of Proposition 15, applied to the forcing  $\mathbb{R}(\bar{v})$ . For each  $i < \kappa(\bar{v})$  and  $\vec{d} \in R_{<\kappa(\bar{v})}$  we define a  $\bar{v}$ -fat tree  $T_{i,\vec{d}}$  and a  $\vec{U}$  set  $A_{i,\vec{d}}$ , associated with the set  $D_i$  of all  $\mathbb{R}(\bar{v})$  conditions  $r = r_0 \frown \langle \bar{v}, a_r \rangle$  which force the  $i$ -th element of  $\sigma$  to be bounded in  $\kappa_0(r) = \max(r_0)$ . We then define  $\Gamma = \{\bar{\mu} \in \mathcal{MS} \cap V_{\kappa(\bar{v})} \mid \forall i, \vec{d} \in V_{\kappa(\bar{\mu})}. T_{i,\vec{d}} \cap V_{\kappa(\bar{\mu})} \text{ is a fat-}\bar{\mu} \text{ tree}\}$ . Since  $\text{cf}(\ell(\bar{v})) \geq \kappa(\bar{v})^+$ , there exists  $\alpha^* < \ell(\bar{v})$  so that  $\Gamma \in \bigcap_{i \geq \alpha^*} \bar{v}(i)$ . Let  $\vec{\eta} \in \mathcal{MS}^{<\omega}$  be an increasing enumeration of the measure sequences in  $q_1$ .<sup>6</sup> Since  $q$  extends  $t$ ,  $\vec{\eta} \subset A_{\vec{e}} \cap V_{\kappa(\bar{v})}$ , and by our assumption  $\bar{v} \in X$ ,  $Z_{\vec{e}} \upharpoonright \vec{\eta}$  must be a  $\bar{v}$ -positive. Hence, there must exist  $\bar{\mu} \in (Z_{\vec{e}} \upharpoonright \vec{\eta}) \cap b$  such that  $T_{i,\vec{d}} \cap V_{\kappa(\bar{\mu})}$  is  $\bar{\mu}$ -fat for each  $i, \vec{d} \in V_{\kappa(\bar{\mu})}$ . By Claim\* of Proposition 15,  $q \frown \langle \bar{\mu} \rangle$  forces  $\kappa(\bar{\mu}) \in \sigma$ . Furthermore, the fact  $\bar{\mu} \in Z_{\vec{e}} \upharpoonright \vec{\eta}$  implies  $q \frown \langle \bar{\mu} \rangle$  is compatible with the

<sup>6</sup>Namely, if  $q_1 = \langle d_1, \dots, d_k \rangle$  then  $\vec{\eta} = \langle \bar{\mu}(d_1), \dots, \bar{\mu}(d_k) \rangle$ .

condition  $\vec{e} \frown \langle \bar{\mu}, b^*(\bar{\mu}) \rangle \frown \langle \vec{U}, A \rangle$ , which forces  $\kappa(\bar{\mu}) \in \dot{S}$ . Hence  $q$  has an extension which forces  $\sigma \cap \dot{S} \neq \emptyset$ .  $\square$ (Claim 2)

Claims 1,2 imply that if  $p = p_0 \frown \langle \vec{U}, A_p \rangle$  is a condition which forces  $\dot{S}$  is a stationary subset of  $\kappa$ , then  $p$  has an extension of the form  $\vec{e} \frown \langle \bar{\nu}, B_{\vec{e}} \cap V_{\kappa(\bar{\nu})} \rangle \frown \langle \vec{U}, B_{\vec{e}} \rangle$  forcing that  $\dot{S} \cap \kappa(\bar{\nu})$  is stationary. It follows that in a  $\mathbb{R}(\vec{U})$  generic extension  $V[G]$ , every stationary subset of  $\kappa$  reflects.

For the final part of the proof we extend the argument to obtain the strong simultaneous reflection property at  $\kappa$ . Suppose  $\langle \dot{S}_i \mid i < \kappa \rangle$  is a sequence of names of subsets of  $\kappa$  and  $p = p_0 \frown \langle \vec{U}, A_p \rangle$  is a condition of  $\mathbb{R}(\vec{U})$  forcing that each  $\dot{S}_i$  is a stationary in  $\kappa$ . For each  $i < \kappa$  let  $W(\dot{S}_i)$  denote the set of all  $\vec{e} \in \mathbb{R}_{<\kappa}$  which are stationary witnesses of  $\dot{S}_i$ . As shown above, for each  $\vec{e} \in W(\dot{S}_i)$  there exists  $B_{\vec{e}}^i \in \cap \vec{U}$  and a closed unbounded set  $E_{\vec{e}}^i \subset \ell(\vec{U})$ , such that for every limit point  $\tau \in E_{\vec{e}}^i$  of cofinality  $\kappa^+$ , there exists a set  $X \in U_\tau$  which consists of  $\bar{\nu}$  for which the condition  $\vec{e} \frown \langle \bar{\nu}, B_{\vec{e}} \cap V_{\kappa(\bar{\nu})} \rangle \frown \langle \vec{U}, B_{\vec{e}} \rangle$  forces  $\dot{S}_i$  reflects at  $\kappa(\bar{\nu})$ . For each  $i < \kappa$ , define  $A^i = \Delta_{\vec{e} \in W(\dot{S}_i)} B_{\vec{e}}^i = \{ \bar{\nu} \in \mathcal{MS} \mid \forall \vec{e} \in W(\dot{S}_i) \cap V_{\kappa(\bar{\nu})} \cdot \bar{\nu} \in B_{\vec{e}}^i \}$  and  $E^i = \cap_{\vec{e} \in W(\dot{S}_i)} E_{\vec{e}}^i$ . Finally, define  $A^* = \Delta_{i < \kappa} A^i$  and  $E^* = \cap_{i < \kappa} E^i$ . We conclude that there exists a set  $X \subset \mathcal{MS}$  which belongs to each  $U_\tau$  where  $\tau$  is a limit point of  $E^*$  of cofinality  $\kappa^+$ , such that for every  $\bar{\nu} \in X$ ,  $i < \kappa(\bar{\nu})$ , and  $\vec{e} \in W(\dot{S}_i) \cap V_{\kappa(\bar{\nu})}$ , the condition  $\vec{e} \frown \langle \bar{\nu}, A^* \cap V_{\kappa(\bar{\nu})} \rangle \frown \langle \vec{U}, A^* \rangle$  forces  $\dot{S} \cap \kappa(\bar{\nu})$  is stationary in  $\kappa(\bar{\nu})$ . Note that  $X$  is  $\vec{U}$ -positive. Let  $G \subset \mathbb{R}(\vec{U})$  be a generic filter containing  $p^* = p_0 \frown \langle \vec{U}, A^* \rangle$ . By Proposition 15, the set  $O(X) = \{ \kappa(\bar{\nu}) \mid \bar{\nu} \in X \}$  is a stationary subset of  $\kappa$  in  $V[G]$ . For each  $i < \kappa$ , let  $S_i = (\dot{S}_i)_G$ . By Claim 1 above,  $W(\dot{S}_i) \cap G_{<\kappa} \neq \emptyset$ . Let  $\vec{e}_i$  be the lexicographic minimal sequence in  $W(\dot{S}_i) \cap G_{<\kappa}$ , and  $\kappa(\vec{e}_i)$  denote its maximal critical point. In  $V[G]$ , define  $f : \kappa \rightarrow \kappa$  in  $V[G]$  by  $f(i) = \kappa(\vec{e}_i) + 1$ . Since  $O(X)$  is stationary, there exists  $\bar{\nu} \in \mathcal{MS}_G \cap X$  such that  $\alpha = \kappa(\bar{\nu})$  is a closure point of  $f$ . It follows that for each  $i < \alpha$ ,  $\vec{e} \frown \langle \bar{\nu}, A^* \cap V_{\kappa(\bar{\nu})} \rangle \frown \langle \vec{U}, A^* \rangle$  belongs to  $G$ , hence  $S_i \cap \alpha$  is a stationary subset of  $\alpha$ .  $\square$ (Theorem 19)

## 4 Weak compactness and Radin forcing

It is natural to ask whether the Radin forcing machinery can be extended to establish the consistency of  $\neg \diamond_\kappa$  at a weakly compact cardinal  $\kappa$ . One necessary step required towards giving an affirmative answer to this question, is to find a reasonably weak assumption of a measure sequence  $\vec{U}$  which

implies  $\kappa$  is weakly compact in a  $\mathbb{R}(\vec{U})$  generic extension. The section will be mostly devoted to providing a property of  $\vec{U}$ , called the weak repeat property (WRP), which characterizes weak compactness of  $\kappa$  in a  $\mathbb{R}(\vec{U})$  generic extension. In the last part of the section, we return to the violation of the diamond question and discuss some natural obstructions raised by the weak compactness characterization.

**Definition 21.**

1. We say that a filter  $W \subset \mathcal{P}(\mathcal{MS})$  measures a set  $X \subset \mathcal{MS}$  if  $X \in W$  or  $\mathcal{MS} \setminus X \in W$ . If  $F \subset \mathcal{P}(\mathcal{MS})$  is a family of sets, then we say  $W$  measures  $F$  if it measures each  $X \in F$ . For every  $b \in \mathfrak{MF}$  and  $\bar{\mu} \in \mathcal{MS}$  let  $X_{b, \bar{\mu}} = \{\bar{\nu} \in \mathcal{MS} \mid \bar{\mu} \in b(\bar{\nu})\}$ . We say that a filter  $W$  measures a function  $b \in \mathfrak{MF}$  if it measures the family  $\mathcal{F}_b = \{X_{b, \bar{\mu}} \mid \bar{\mu} \in \mathcal{MS}\}$ . Whenever  $W$  measures  $b \in \mathfrak{MF}$ , we define  $[b]_W = \{\bar{\mu} \in \mathcal{MS} \mid X_{b, \bar{\mu}} \in W\} \subset \mathcal{MS}$ .
2. Let  $b \in \mathfrak{MF}$  and  $W \subset \mathcal{P}(\mathcal{MS})$  be a filter. We say that  $W$  is a **repeat filter** of  $b$  with respect to  $\vec{U}$  if it satisfies the following conditions.
  - a.  $W$  is a  $\kappa$ -complete filter extending the co-bounded filter on  $\mathcal{MS}$ ,<sup>7</sup>
  - b.  $W \subset \bigcup \vec{U}$ ,
  - c.  $W$  measures  $b$ ,
  - d.  $[b]_W \in \bigcap \vec{U}$ .
3. We say that  $\vec{U}$  satisfies the **Weak Repeat Property** (WRP) if every  $b \in \mathfrak{MF}$  has a repeat filter with respect to  $\vec{U}$ .

Let us say  $\vec{U}$  satisfies the repeat property (RP) if it contains a repeat point measure.

**Lemma 22.** *RP implies WRP. Moreover, if  $U_\rho$  is a repeat point of  $\vec{U}$  then  $\{\bar{\mu} \in \mathcal{MS} \mid \bar{\mu} \text{ satisfies WRP}\}$  belongs to  $U_\rho$  and there exists  $\tau < \rho$  such that  $\vec{U} \upharpoonright \tau$  satisfies WRP.*

*Proof.* Let  $\rho$  be the first repeat point on  $\vec{U}$ . Then  $\vec{U} \upharpoonright \rho$  does not satisfy RP. Nevertheless,  $\bigcap \vec{U} \upharpoonright \rho = \bigcap \vec{U}$  and so  $\kappa$  remains regular (and even measurable) in a generic extension by  $\mathbb{R}(\vec{U} \upharpoonright \rho) = \mathbb{R}(\vec{U})$ . By Remark 9, it follows that  $\text{cf}(\rho) \geq \kappa^+$ . To establish the first assertion, note that  $W = U_\rho$  is a repeat filter of every  $b \in \mathfrak{MF}$ . Indeed,  $[b]_{U_\rho} \in \bigcap \vec{U} \upharpoonright \rho = \bigcap \vec{U}$ . Next, we claim that for each  $b \in \mathfrak{MF}$  there exists  $\tau < \rho$  such that  $U_\tau$  is a repeat filter

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<sup>7</sup>Namely, for every  $\alpha < \kappa$ , the set  $\{\bar{\mu} \in \mathcal{MS} \mid \kappa(\bar{\mu}) > \alpha\} \in W$ .

of  $b$  with respect to both  $\vec{U}$  and  $\vec{U} \upharpoonright \rho$ . Fix  $b \in \mathfrak{MF}$  and an enumeration  $\langle Y_i \mid i < \kappa \rangle$  of  $\mathcal{F}_b = \{X_{b, \bar{\mu}} \mid \bar{\mu} \in \mathcal{MS}\}$ . For each  $i < \kappa$  let

$$Y'_i = \begin{cases} Y_i & \text{if } Y_i \in U_\rho \\ \mathcal{MS} \setminus Y_i & \text{otherwise} \end{cases}$$

Let  $Y' = \Delta_{i < \kappa} Y'_i$ .  $Y' \in U_\rho$  since  $U_\rho$  is normal. Since  $U_\rho$  is a repeat point there exists some  $\tau < \rho$  such that  $Y' \in U_\tau$ . It follows that  $[b]_{U_\tau} = [b]_{U_\rho} \in \bigcap (\vec{U} \upharpoonright \rho)$ , and thus  $W = U_\tau \in \vec{U} \upharpoonright \rho$  is a repeat filter of  $b$ . As these witnesses are known to  $M$ ,  $M \models \vec{U} \upharpoonright \rho$  satisfies WRP, and  $\{\bar{\mu} \in \mathcal{MS} \mid \bar{\mu} \text{ satisfies WRP}\} \in U_\rho$ . The fact  $U_\rho$  is a repeat point implies there exists  $\tau < \rho$  such that  $\{\bar{\mu} \in \mathcal{MS} \mid \bar{\mu} \text{ satisfies WRP}\} \in U_\tau$ , which in turn, implies  $M \models \vec{U} \upharpoonright \tau$  satisfies WRP. Since  $\mathfrak{MF} \subset M$ , it follows that  $\vec{U} \upharpoonright \tau$  satisfies WRP in  $V$ .  $\square$

**Theorem 23.**  *$\kappa$  is weakly compact in a  $\mathbb{R}(\vec{U})$  generic extension if and only if  $\vec{U}$  satisfies the Weak Repeat Property.*

#### 4.1 From the Weak Repeat Property to Weak Compactness

Suppose that  $\vec{U} \in V$  is a measure sequence on  $\kappa$ , satisfying WRP. Let  $G \subset \mathbb{R}(\vec{U})$  be a generic filter over  $V$ . To show  $\kappa$  weakly compact in  $V[G]$ , it is sufficient to prove that for every sufficiently large regular cardinal  $\theta > \kappa$  and  $N' \prec H_\theta[G]$  satisfying  ${}^{<\kappa}N' \subset N'$ ,  $G, \vec{U} \in N'$ , and  $|N'| = \kappa$ , there exists a  $\kappa$ -complete  $N'$ -ultrafilter  $U'$  on  $\kappa$ . That is,  $U'$  measures all the sets in  $\mathcal{P}(\kappa) \cap N$  and is closed under intersection of sequences of its elements of length less than  $\kappa$ . Since  $\mathbb{R}(\vec{U})$  satisfies  $\kappa^+$ .c.c.,  $N'$  has an elementary extension of the form  $N[G]$  (i.e.,  $N' \prec N[G] \prec H_\theta[G]$ ) for some  $N \prec H_\theta$  in  $V$ , such that  $|N| = \kappa$ ,  $N^{<\kappa} \subset N$ , and  $\vec{U} \in N$ . We therefore focus on models  $N[G]$  of this form.

**Lemma 24.** *Let  $\theta > \kappa$  be a regular cardinal, and  $N \prec H_\theta$  be an elementary substructure of cardinality  $\kappa$  with  $V_\kappa \subset N$ . If  $\vec{U}$  satisfies WRP then there exists a  $\kappa$ -complete filter  $W \subset \bigcup \vec{U}$ , which measures all the subsets of  $\mathcal{MS}$  in  $N$  and all  $b \in N \cap \mathfrak{MF}$ , and which satisfies  $[b]_W \in \bigcap \vec{U}$  for every  $b \in N \cap \mathfrak{MF}$ .*

*Proof.* Fix an enumeration  $\langle b_i \mid i < \kappa \rangle$  of  $\mathfrak{MF} \cap N$ . Define  $b' \in \mathfrak{MF}$  by  $b'(\bar{\mu}) = \Delta_{i < \kappa(\bar{\mu})} b_i(\bar{\mu}) = \{\bar{\nu} \in V_{\kappa(\bar{\mu})} \mid \forall i < \kappa(\bar{\nu}) \bar{\nu} \in b_i(\bar{\mu})\}$ . It follows that for every filter  $W$ , if  $W$  is a repeat filter of  $b'$  then it measures each  $b_i$  and  $[b_i]_W \supset [b']_W \setminus V_{i+1} \in \bigcap \vec{U}$ . Therefore if  $W$  is a repeat filter of  $b'$  then it is also a repeat filter of each  $b_i$ ,  $i < \kappa$ . Next, we tweak  $b'$  to obtain  $b^* \in \mathfrak{MF}$



such that every filter  $W$  which measures  $b^*$  also measures  $\mathcal{P}(\mathcal{MS}) \cap N$ . Let  $\{A_i \mid i < \kappa\}$  be an enumeration of  $\mathcal{P}(\mathcal{MS}) \cap N$  and fix an auxiliary set  $X \subset \mathcal{MS}$  such that  $|X| = \kappa$  and  $O(X) \cap \rho$  is nonstationary in  $\rho$  for every regular cardinal  $\rho \leq \kappa$ . Therefore, any modification in the measure function  $b'$  which is restricted to  $X$  will not affect its key properties of  $b'$  established above. Fix an enumeration  $\{\bar{\mu}_i \mid i < \kappa\}$  of  $X$  and define  $b^* : \mathcal{MS} \rightarrow V_\kappa$  as follows. For every  $\bar{\nu} \in \mathcal{MS}$  let  $b^*(\bar{\nu}) = (b'(\bar{\nu}) \setminus X) \uplus \{\bar{\mu}_i \in V_{\kappa(\bar{\nu})} \mid \bar{\nu} \in A_i\}$ . Clearly,  $b^*(\bar{\nu}) \setminus X = b'(\bar{\nu}) \setminus X \in \bigcap \bar{\nu}$  for each  $\bar{\nu} \in \mathcal{MS}$ , thus  $b^* \in \mathfrak{M}\mathfrak{F}$ . Furthermore, for each  $i < \kappa$ ,  $A_i \setminus V_i = \{\bar{\nu} \mid \bar{\mu}_i \in b^*(\bar{\nu})\}$ . It follows that if  $W$  is a repeat filter of  $b^*$  then  $W$  is a repeat filter of  $b'$  and it measures all the sets  $A_i \in \mathcal{P}(\mathcal{MS}) \cap N$ .  $\square$

Let  $N \prec H_\theta$  such that  ${}^{<\kappa}N \subset N$  and  $\vec{U} \in N$ , and fix a repeat filter  $W \subset \mathcal{P}(\mathcal{MS})$  given by Lemma 24. Working in  $V[G]$ , we define an  $N[G]$ -filter  $U_W$ .

**Definition 25.** Let  $U_W$  be the set of all  $X \in \mathcal{P}(\kappa) \cap N[G]$ , for which there exists a name  $\dot{X} \in N$  such that  $X = \dot{X}_G$ , and there are  $p = p_0 \wedge \langle \vec{U}, A_p \rangle \in G$  and  $b \in \mathfrak{M}\mathfrak{F} \cap N$  such that

- $A_p \subset [b]_W$ , and
- $\{\bar{\mu} \in \mathcal{MS} \mid \exists A(\bar{\mu}) \in \bigcap \vec{U}. p_0 \wedge \langle \bar{\mu}, b(\bar{\mu}) \rangle \wedge \langle \vec{U}, A(\bar{\mu}) \rangle \Vdash \kappa(\bar{\mu}) \in \dot{X}\} \in W$ .

Given  $\dot{X}, p, b$  as in the definition, we say  $p$  and  $b$  **witness**  $X \in U_W$ , and denote the set  $\{\bar{\mu} \in \mathcal{MS} \mid \exists A(\bar{\mu}) \in \bigcap \vec{U}. p_0 \wedge \langle \bar{\mu}, b(\bar{\mu}) \rangle \wedge \langle \vec{U}, A(\bar{\mu}) \rangle \Vdash \kappa(\bar{\mu}) \in \dot{X}\}$  by  $Z(\dot{X}, p, b)$ .

Note that the definition of  $Z(\dot{X}, p, b)$  depends only on  $p_0, b, \dot{X} \in N$ . This implies that the set  $Z(\dot{X}, p, b)$  belongs to  $N$  and thus is measured by  $W$ . The following two Lemmata show that  $U_W$  is a  $\kappa$ -complete  $N[G]$  ultrafilter.

**Lemma 26.**

1. Suppose  $p, b$  witness  $X \in U_W$ . Then for every  $q \geq p$  there exists some  $b' \in \mathcal{MS} \cap N$  so that  $q, b'$  witness  $X \in U_W$  as well.
2. If  $X, Y \in N[G] \cap \mathcal{P}(\kappa)$  with  $X \in U_W$  and  $X \subset Y$ , then  $Y \in U_W$ .

*Proof.*

1. Let  $\dot{X} \in N$  be a  $\mathbb{R}(\vec{U})$ -name of  $X$  such that  $Z(\dot{X}, p, b) \in W$ . Given  $q \geq p = p_0 \wedge \langle \vec{U}, A_p \rangle$ , we split  $q$  into three parts,  $q = q_0 \wedge q_1 \wedge \langle \vec{U}, A_q \rangle$ , where  $q_0 \geq p_0$  and  $q_1 \wedge \langle \vec{U}, A_q \rangle \geq \langle \vec{U}, A_p \rangle$ . We have that  $A_q \subset A_p \setminus \max(q_1) \subset [b]_W \setminus \max(q_1)$ , and note that since  $q_1 \in N$ , the set  $Z = \{\bar{\mu} \in \mathcal{MS} \mid q_1 \wedge \langle \bar{\mu}, b(\bar{\mu}) \rangle \in Z(\dot{X}, p, b)\}$

$\max(q_1) \geq \langle \bar{\mu}, b(\bar{\mu}) \rangle$  belongs to  $N$ . Therefore  $Z$  is measured by  $W$ , and furthermore, since  $A_p \subset [b]_W$  and  $\text{supp}(q_1) \subset A_p$ ,  $Z$  must be a member of  $W$ . Define a function  $b' \in \mathfrak{M}\mathfrak{F}$  by setting  $b'(\bar{\nu})$  to be  $b(\bar{\nu}) \setminus V_{\max(q_1)+1}$  if  $\kappa(\bar{\nu}) > \max(q_1)$ , and  $b(\bar{\nu})$  otherwise. It follows that  $b' \in \mathfrak{M}\mathfrak{F} \cap N$ ,  $[b']_W = [b]_W \setminus \max(q_1)$ , and  $A_q \subset [b']_W$ . We conclude that for each  $\bar{\mu} \in Z(\dot{X}, p, b) \cap Z$ ,  $q_0 \frown q_1 \frown \langle \bar{\mu}, b'(\bar{\mu}) \rangle \geq p_0 \frown \langle \bar{\mu}, b(\bar{\mu}) \rangle$ , hence, by the definition of  $Z(\dot{X}, p, b)$ , there exists  $A(\bar{\mu}) \in \bigcap \vec{U}$  so that  $q_0 \frown q_1 \frown \langle \bar{\mu}, b'(\bar{\mu}) \rangle \frown \langle \vec{U}, A(\bar{\mu}) \rangle \Vdash \kappa(\bar{\mu}) \in \dot{X}$ . As the last applies to every  $\bar{\mu} \in Z(\dot{X}, p, b) \cap Z \in W$ , where  $Z \cap Z(\dot{X}, p, b) \in W$ , we conclude  $q, b'$  witness  $X \in U_W$ .

**2.** Suppose  $p, b$  witness  $X = \dot{X}_G \in U_W$  and  $\dot{Y} \in N$  is a name such that  $X \subset \dot{Y}_G$ . Since  $\mathbb{R}(\vec{U})$  satisfies  $\kappa^+.c.c$  and  $|N| = \kappa$ , and  $\kappa \subset N$ , there must exist some  $t \in N \cap G$  forcing  $\dot{X} \subset \dot{Y}$ . Writing  $t = \vec{t}_0 \frown \langle \vec{U}, A_t \rangle$  we have that  $A_t \in \bigcap \vec{U} \cap N$  must belong to  $W$  and  $t, p \in G$  must be compatible. Let  $q \geq p, t$  be a common extension in  $G$ , and let  $b \in \mathcal{MS} \cap N$  so that  $q, b$  witness  $X \in U_W$  via  $\dot{X}$ , namely,  $Z(\dot{X}, q, b) \in W$ . For every  $\bar{\mu} \in Z(\dot{X}, q, b)$ , there exists some  $A(\bar{\mu}) \in \bigcap \vec{U}$  such that  $q_0 \frown \langle \bar{\mu}, b(\bar{\mu}) \rangle \frown \langle \vec{U}, A(\bar{\mu}) \rangle \Vdash \kappa(\bar{\mu}) \in \dot{X}$ . Furthermore, if  $\bar{\mu} \in A_t \setminus \max(q_0)$  then  $q_0 \frown \langle \bar{\mu}, b(\bar{\mu}) \rangle \frown \langle \vec{U}, A(\bar{\mu}) \cap A_t \rangle$  is an extension of  $t$  and forces  $\kappa(\bar{\mu}) \in \dot{Y}$ . It follows that  $Z(\dot{X}, q, b) \cap A_t \subset Z(\dot{Y}, q, b)$ , thus  $Z(\dot{Y}, q, b) \in W$ . We conclude that  $q, b$  witness  $Y \in U_W$   $\square$

It follows from the first part of Lemma 26 that  $U_W$  is closed under intersections of its sets, and by the second part of the Lemma that it is upwards closed under inclusions. Hence,  $U_W$  is a filter on  $\mathcal{P}(\kappa) \cap N[G]$ . It remains to show that it is  $\kappa$ -complete. We first introduce the following terminology.

**Definition 27.**

**1.** Let  $D \subset \mathbb{R}(\vec{U})$  be a dense set. We say  $D$  is **strongly dense** if for every  $p \in \mathbb{R}(\vec{U})$ ,  $p = p_0 \frown \langle \vec{U}, A \rangle$ , there exists some  $q \in D$ ,  $q \geq p$  such that  $q = q_0 \frown \langle \vec{U}, A' \rangle$ , and  $\kappa(q_0) = \kappa(p_0)$  (i.e.  $q_0 \geq p_0$  in  $\mathbb{R}_{<\kappa}$ ).

**2.** Let  $\vec{D} = \langle D_\nu \mid \nu < \kappa \rangle$  be a sequence of strongly dense sets, and  $p_0 \in \mathbb{R}_{<\kappa}$ . Define three functions  $b_{p_0, \vec{D}}, B_{p_0, \vec{D}}, r_{p_0, \vec{D}}$  with domain  $\mathcal{MS}$ : Fix some well ordering of  $V_{\kappa+1}$  and consider the condition  $p = p_0 \frown \langle \vec{U}, \mathcal{MS} \setminus V_{\kappa(p_0)} \rangle$ . For every  $\bar{\nu} \in \mathcal{MS} \setminus V_{\kappa(p_0)}$ , Let  $q$  be the first extension of  $p \frown \bar{\nu}$  which belongs to  $D_{\kappa(\bar{\nu})}$ . Writing  $q = r' \frown \langle \bar{\nu}, a' \rangle \frown \langle \vec{U}, A' \rangle$ , we set  $b_{p_0, \vec{D}}(\bar{\nu}) = a'$ ,  $B_{p_0, \vec{D}}(\bar{\nu}) = A'$ , and  $r_{p_0, \vec{D}}(\bar{\nu}) = r'$ . Since  $N \prec H_\theta$ , it follows that for every sequence of strongly dense sets  $\vec{D} \in N$  and  $p_0 \in \mathbb{R}_{<\kappa} \subset N$ ,  $b_{p_0, \vec{D}}, B_{p_0, \vec{D}}, r_{p_0, \vec{D}}$  all belong to  $N$  as well.

**Lemma 28.** *Let  $\lambda < \kappa$  and suppose that  $\langle X_i \mid i < \lambda \rangle$  is a partition of  $\kappa$  in  $N[G]$ . Then there exists  $i^* < \lambda$  such that  $X_{i^*} \in U_W$ .*

*Proof.* Since  $N^{<\kappa} \subset N$ , there is a sequence of names  $\langle \dot{X}_i \mid i < \lambda \rangle$  in  $N$  such that  $X_i = (\dot{X}_i)_G$  for each  $i < \lambda$ . The claim will follow from a density argument once we show that for every  $p = p_0 \frown \langle \vec{U}, A_p \rangle \in \mathbb{R}(\vec{U})$ , there are  $r \geq p$ ,  $i^* < \lambda$ , and  $b \in \mathcal{MS} \cap N$ , such that  $r, b$  witness  $X_{i^*} \in U_W$ .

For every  $\nu < \kappa$  let  $D_\nu = \{p' \in \mathbb{R}(\vec{U}) \mid \exists i < \lambda, p' \Vdash \nu \in \dot{X}_i\}$ . Each  $D_\nu$  is strongly dense,<sup>8</sup> and  $\vec{D} = \langle D_\nu \mid \nu < \kappa \rangle$  belongs to  $N$ . Let  $b_{p_0, \vec{D}}, r_{p_0, \vec{D}}, B_{p_0, \vec{D}} \in N$  be the associated functions defined above. For each  $i < \lambda$ , let  $Z_i = \{\bar{\mu} \in \mathcal{MS} \mid r_{p_0, \vec{D}}(\bar{\mu}) \frown \langle \bar{\mu}, b_{p_0, \vec{D}}(\bar{\mu}) \rangle \frown \langle \vec{U}, B_{p_0, \vec{D}}(\bar{\mu}) \rangle \Vdash \kappa(\bar{\mu}) \in \dot{X}_i\}$ . As the sets  $Z_i$ ,  $i < \lambda$ , are pairwise disjoint and belong to  $N$ , there exists a unique  $i^* < \lambda$  such that  $Z_{i^*} \in W$ . Furthermore, since  $W$  is  $\kappa$ -complete and measures  $N$ , there exists  $r_0 \geq p_0$  such that  $\{\bar{\mu} \in Z_{i^*} \mid r_{p_0, \vec{D}}(\bar{\mu}) = r_0\} \in W$ . Define  $A_r = A_p \cap [b_{p_0, \vec{D}}]_W \cap \Delta_{\bar{\mu} \in Z_{i^*}} B_{p_0, \vec{D}}(\bar{\mu})$ , and  $r = r_0 \frown \langle \vec{U}, A_r \rangle$ . Then  $r \geq p$  and  $A_r \subset [b]_W$ , where  $b = b_{p_0, \vec{D}}$  is in  $N$ . Furthermore, for every  $\bar{\mu} \in Z_{i^*}$ ,  $r_0 \frown \langle \bar{\mu}, b(\bar{\mu}) \rangle \frown \langle \vec{U}, B_{p_0, \vec{D}} \rangle \Vdash \kappa(\bar{\mu}) \in \dot{X}_{i^*}$ . It follows that  $Z_{i^*} \subset Z(\dot{X}_{i^*}, r, b)$ , and thus  $Z(\dot{X}_{i^*}, r, b) \in W$ .  $\square$

## 4.2 From Weak Compactness to the Weak Repeat Property

Let  $G \subset \mathbb{R}(\vec{U})$  be a generic filter. Recall  $G$  is completely determined by its induced sequence of measure sequences,  $\mathcal{MS}_G = \{\bar{\mu} \in \mathcal{MS} \mid \exists p = \langle d_i \mid i \leq k \rangle \in G, \bar{\mu} = \bar{\mu}(d_i) \text{ for some } i < k\}$ .

Suppose  $\kappa$  is weakly compact in  $V[G]$ , and fix a measure function  $b$  in  $V$ . We would like to show  $b$  has a repeat filter  $W$  in  $V$ . If  $\vec{U}$  satisfies RP there is nothing to show. We therefore assume  $\vec{U}$  does not contain a repeat point. Then, by Remark 9,  $\text{cf}(\ell(\vec{U}))$  must be at least  $\kappa^+$  for  $\kappa$  to be weakly compact in a generic extension.

To accomplish this, we construct a  $\Pi_1^1$  statement  $\varphi$  of the structure  $M_b = \langle V_\kappa[G], \in, b, V_\kappa, \mathcal{MS}_G \rangle$  such that  $M_b \models \varphi$  if and only if  $b$  does not have a repeat filter in  $V$ , and show that the reflections of  $\varphi$  to  $\alpha < \kappa$  fail on a closed unbounded set of cardinals  $\alpha < \kappa$ . Since  $\kappa$  is weakly compact, it follows that  $M_b$  must satisfy  $\neg\varphi$ , thus  $b$  has a repeat filter.

We commence by observing that the existence of a repeat filter for  $b$  is witnessed by a family of  $\kappa$  many subsets of  $\mathcal{P}(\mathcal{MS})$ . Recall that for every

<sup>8</sup>Every  $p \in \mathbb{R}(\vec{U})$  has a direct extension in  $D_\nu$ .

$b \in \mathfrak{M}\mathfrak{F}$ , we define  $\mathcal{F}_b = \{X_{b,\bar{\mu}} \mid \bar{\mu} \in \mathcal{MS}\}$ , where for each  $\bar{\nu} \in \mathcal{MS}$ ,  $X_{b,\bar{\mu}} = \{\bar{\nu} \in \mathcal{MS} \mid \bar{\mu} \in b(\bar{\nu})\}$ . Clearly  $|\mathcal{F}_b| = \kappa$ .

**Definition 29.** A subset  $P$  of  $\mathcal{P}(\mathcal{MS})$  is called a **repeat Prefilter** of  $b$  (with respect to  $\vec{U}$ ) if it satisfies the following properties:

- a. For every  $\lambda < \kappa$  and every sequence  $\langle X_i \mid i < \lambda \rangle \subset P$ , the intersection  $\bigcap_{i < \lambda} X_i \in \bigcup \vec{U}$ .
- b.  $P \subset \mathcal{F}_b \cup \{\mathcal{MS} \setminus X \mid X \in \mathcal{F}_b\}$ .
- c.  $P$  measures  $b$ . In particular,  $[b]_P = \{\bar{\mu} \in \mathcal{MS} \mid X_{b,\bar{\mu}} \in P\}$  is defined.
- d.  $[b]_P \in \bigcap \vec{U}$ .

It is easy to see that if  $W$  is a repeat filter of  $b$  then  $P = W \cap (\mathcal{F}_b \cup \{\mathcal{MS} \setminus X \mid X \in \mathcal{F}_b\})$  is a prefilter of  $b$ , and that if  $P$  is a prefilter of  $b$  then its upwards closure  $W = \{Y \subset \mathcal{MS} \mid \exists X \in P. X \subset Y\}$  is a repeat filter of  $b$ .

**Definition 30.** Working in  $V[G]$ , let  $\varphi$  be the following statement:

For every  $P \subset \mathcal{P}(\mathcal{MS})$  of cardinality  $\kappa$ , at least one of the following conditions hold.

- $\varphi_1$ .  $P \notin V$
- $\varphi_2$ . There exists  $\lambda < \kappa$  and a sequence  $\langle X_i \mid i < \lambda \rangle \subset P$  such that  $\bigcap_{i < \lambda} X_i \notin \bigcup \vec{U}$
- $\varphi_3$ .  $P \not\subset \mathcal{F}_b \cup \{\mathcal{MS} \setminus X \mid X \in \mathcal{F}_b\}$
- $\varphi_4$ .  $P$  does not measure  $b$
- $\varphi_5$ .  $P$  measures  $b$  and  $[b]_P \notin \bigcap \vec{U}$ .

It is clear that  $M_b \models \varphi$  if and only if  $b$  has a repeat prefilter.

**Lemma 31.**  $\varphi$  is equivalent to a  $\Pi_1^1$  statement over  $M_b = \langle V_\kappa[G], \in, b, V_\kappa, \mathcal{MS}_G \rangle$ .

*Proof.* Since any family  $P \subset \mathcal{P}(\mathcal{MS})$  of size  $\kappa$  can be enumerated as a subset of  $\mathcal{MS} \times \kappa$ , we identify  $P \subset \mathcal{MS} \times \kappa$  with a sequence  $\langle X_i \mid i < \kappa \rangle$ , where  $X_i = \{\bar{\nu} \in \mathcal{MS} \mid (\bar{\nu}, i) \in P\}$ .  $\varphi$  is clearly equivalent to a statement of the form  $\forall P \subseteq (\mathcal{MS} \times \kappa). (\varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \varphi_4 \vee \varphi_5)$ . It is therefore sufficient to verify each  $\varphi_i$  is equivalent to a  $\Sigma_\omega^0$  statement over  $M_b$ . We take each  $\varphi_i$  at a time.

( $\varphi_1$ ). By Proposition 13,  $\mathbb{R}(\vec{U})$  does not add fresh subsets to  $\kappa$ , and hence, neither to  $\mathcal{MS} \times \kappa$ . Therefore,  $\varphi_1$  is equivalent to  $\exists \alpha < \kappa. P \cap (V_\alpha \times \alpha) \notin V_\kappa$  which is clearly equivalent to a  $\Sigma_\omega^0$  statement over  $M_b$ . ( $\varphi_2$ ). An easy density argument shows that for every  $A \subset \mathcal{MS}$  in  $V$ ,  $A \in \bigcup \vec{U}$  if and only if  $A \cap \mathcal{MS}_G$  is not bounded in some  $V_\alpha$ ,  $\alpha < \kappa$ . Therefore,  $\varphi_2$  is equivalent to  $\exists \lambda < \kappa \exists \alpha < \kappa. (\bigcap_{i < \lambda} X_i \setminus V_\alpha) = \emptyset$ , which is clearly equivalent to a  $\Sigma_\omega^0$  statement over  $M_b$ . ( $\varphi_3 + \varphi_4$ ). It is straightforward to verify  $\varphi_3$  and  $\varphi_4$  are

equivalent to  $\Sigma_\omega^0$  statement over  $M_b$ , using the fact  $b \subset \mathcal{MS} \times \mathcal{MS}$  is a part of the augmented structure  $M_b$ . ( $\varphi_5$ ). It is easy to see that the first part of the assertion, “ $P$  measures  $b$ “, is equivalent to a  $\Sigma_\omega^0$  statement of  $M_b$ . Considering the second part, “ $[b]_P \notin \vec{U}$ “ of  $\varphi_5$ , note that the same density argument used for the description of  $\varphi_1$  shows that a  $V$  set  $A \subset \mathcal{MS}$  belongs to  $\bigcap \vec{U}$  if and only if  $\mathcal{MS}_G$  is almost contained in  $A$ . Therefore the second part of  $\varphi_5$  is equivalent to the  $M_b$  statement  $\forall \alpha < \kappa \exists \bar{\mu} \in \mathcal{MS}_G \setminus V_\alpha \exists i < \kappa \forall \bar{\nu} \in \mathcal{MS}. (\bar{\nu} \in X_i \iff \bar{\mu} \notin b(\bar{\nu}))$ .  $\square$

**Lemma 32.** *In  $V[G]$ , there exists a closed unbounded set of  $\alpha < \kappa$  for which  $\langle V_\alpha[G], \alpha, b, V_\alpha, \mathcal{MS}_G \rangle \models \neg \varphi$ .*

*Proof.* Assuming  $\vec{U}$  does not contain a repeat point, Proposition 17 implies it is sufficient to show there exists  $\tau < \ell(\vec{U})$  and  $Z \in \bigcap(\vec{U} \setminus \tau)$ , such that in  $V$ , for every  $\bar{\nu} \in Z$ , the restriction  $b \upharpoonright V_{\kappa(\bar{\nu})}$  has a weak repeat filter with respect to  $\bar{\nu}$ . Equivalently, it is sufficient to check  $b = j(b) \upharpoonright V_\kappa$  has a weak repeat filter with respect to  $U_\eta$ , for every  $\eta \in [\tau, \ell(\vec{U}))$ . Let  $\mathcal{F}_b = \{X_{b, \bar{\mu}} \mid \bar{\mu} \in \mathcal{MS}\}$  and  $F^*$  be the family of all intersections of length  $\lambda < \kappa$  of sets in  $\mathcal{F}_b \cup \{\mathcal{MS} \setminus X \mid X \in \mathcal{F}_b\}$ . Fix an enumeration  $\langle Y_i \mid i < \kappa \rangle$  of  $F^*$ , and for each  $Y_i$  which does not belong to  $\bigcap \vec{U}$ , let  $\tau_i < \ell(\vec{U})$  be the first  $\tau < \ell(\vec{U})$  such that  $Y_i \notin U_\tau$ . Since  $\text{cf}(\ell(\vec{U})) \geq \kappa^+$ ,  $\tau = \sup_{i < \kappa} \tau_i + 1$  is below  $\ell(\vec{U})$ . Fix an ordinal  $\eta \in [\tau, \ell(\vec{U}))$ . We have that for every  $i < \kappa$ ,  $Y_i \notin U_\eta$  implies that  $Y_i \notin \bigcap \vec{U}$ , which in turn, implies  $Y_i \notin \vec{U} \upharpoonright \eta$ . Define a prefilter  $P = U_\eta \cap F^*$ . Since  $\eta \geq \tau$ , every intersection of  $\lambda < \kappa$  sets of  $P$  belongs to  $\bigcup(U \upharpoonright \tau) \subset \bigcup(\vec{U} \upharpoonright \eta)$ . It is also clear that  $[b]_P = [b]_{U_\eta} \in \bigcap \vec{U} \upharpoonright \eta$ . It follows that in  $V$ ,  $P$  is a repeat prefilter for  $b$  with respect to  $\vec{U} \upharpoonright \eta$ . Considering  $j : V \rightarrow M$ , it is clear that  $P, b \in M$  and that  $M \models P$  is a repeat prefilter with respect to  $b$  as well.  $\square$

### 4.3 Towards the failure of diamond at a weakly compact cardinal

In light of Theorem 18 and Theorem 23, the following question is prominent.

**Question 33.** Is it consistent there exists a measure sequence  $\vec{U}$  on a cardinal  $\kappa$  such that  $\vec{U}$  satisfies the weak repeat property and  $2^\kappa > \ell(\vec{U})$ ?

We conclude this Section with a discussion describing some of the obstructions to a positive answer to the above question.

The definition of the weak repeat property (WRP) and the proof of Lemma 22 suggest WRP has a natural (seemingly) stronger property which is still weaker than the existence of a repeat point (RP).

**Definition 34.** Let us say that a measure sequence  $\vec{U}$  satisfies the **Local Repeat Property** (LRP) if for every  $b \in \mathfrak{MF}$  there exists  $\tau < \ell(\vec{U})$  such that  $[b]_{U_\tau} \in \bigcap \vec{U}$ .

Clearly,  $\text{RP} \implies \text{LRP} \implies \text{WRP}$ , and the proof of Lemma 22 implies that if  $\vec{U}$  has a repeat point  $U_\rho$ , then  $\{\bar{\mu} \in \mathcal{MS} \mid \bar{\mu} \text{ satisfies LRP}\} \in U_\rho$ . Moreover, it is not difficult to see LRP is equivalent to the variant of WRP which restricts the possible repeat filters  $W$  to the normal ones.

**Observation 35.** Let us say that a measure sequence  $\vec{U}$  satisfies  $\text{WRP}^+$  if every  $b \in \mathfrak{MF}$  has a repeat filter  $W$  which is normal (i.e., closed under diagonal intersections). Then  $\text{WRP}^+$  is equivalent to LRP.

*Proof.* Suppose  $\vec{U}$  satisfies  $\text{WRP}^+$ , and let  $b \in \mathfrak{MF}$  and  $W$  a normal repeat filter of  $b$ . Let  $P$  be the restriction of  $W$  to  $\mathcal{F}_b \cup \{\mathcal{MS} \setminus X \mid X \in F_b\}$  where  $\mathcal{F}_b = \{X_{\bar{v},b} \mid \bar{v} \in \mathcal{MS}\}$ .  $P$  has size  $\kappa$  and measures  $b$  and  $[b]_P = [b]_W \in \bigcap \vec{U}$ . Let  $X^*$  be a diagonal intersection of the sets in  $P$ . Then  $X^* \in W \subset \bigcup \vec{U}$ , thus  $X^* \in U_\rho$  for some  $\rho < \ell(\vec{U})$ . Since every  $X \in P$  is almost contained in  $X^*$ , it follows that  $[b]_{U_\rho} = [b]_P \in \bigcap \vec{U}$ .  $\square$

Albeit natural, the LRP cannot be targeted to provide an affirmative answer to Question 33.

**Proposition 36.** Let  $\vec{U}$  be a measure sequence on a cardinal  $\kappa$ . If  $2^\kappa \geq \ell(\vec{U})$  then  $\vec{U}$  fails to satisfy the local repeat property.

*Proof.* Let  $j : V \rightarrow M$  be an embedding which generates the measure sequence  $\vec{U}$ . Denote  $2^\kappa$  by  $\lambda$ . Let  $x^\kappa = \langle x_\alpha^\kappa \mid \alpha < \lambda \rangle$  be an enumeration of  $\mathcal{P}(\kappa)$  in  $M$ . We may assume there is a sequence  $\vec{x} = \langle x^\alpha \mid \alpha < \kappa \rangle$  so that  $x^\alpha$  enumerates  $\mathcal{P}(\alpha)$  and  $j(\vec{x})(\kappa) = x^\kappa$ . Since  $\lambda \geq \ell(\vec{U})$ ,  $\lambda > \ell(\vec{U} \upharpoonright \alpha)$  for every  $\alpha < \ell(\vec{U})$ , hence the set  $A = \{\bar{\mu} \in \mathcal{MS} \mid 2^{\kappa(\bar{\mu})} > l(\bar{\mu})\}$  belongs to  $\bigcap \vec{U}$ . Define a measure function  $b \in \mathfrak{MF}$  by taking  $b(\bar{\mu})$  to be the set of all  $\bar{v} \in \mathcal{MS} \cap V_{\kappa(\bar{\mu})}$  for which  $x_{\ell(\bar{\mu})}^{\bar{\mu}} \cap \kappa(\bar{v}) = x_\beta^{\bar{v}}$  for some  $\beta > \ell(\bar{v})$ . Then for each  $\alpha < \ell(\vec{U})$ ,  $[b]_{U_\alpha}$  is the set of all  $\bar{v} \in \mathcal{MS}$  so that  $x_\alpha^\kappa \cap \kappa(\bar{v}) = x_\beta^{\bar{v}}$  for some  $\beta \geq \ell(\bar{v})$ . Denoting this set by  $X$ , it is easy to see that  $\vec{U} \upharpoonright \beta \in j(X)$  for every  $\beta < \alpha$ . Hence  $[b]_{U_\alpha} \in \bigcap (\vec{U} \upharpoonright \alpha)$ . The same argument, applied to  $\bar{\mu} \in \mathcal{MS}$  instead of  $\vec{U} \upharpoonright \alpha$  shows that  $b \in \mathfrak{MF}$ . We claim that this set does not belong to  $U_\alpha$ . Indeed,  $\vec{U} \upharpoonright \alpha \in j(X)$  if and only if  $j(x_\alpha^\kappa) \cap \kappa = x_\alpha^\kappa$  is of the form  $j(\vec{x})_\beta^\kappa$  for some  $\beta > \ell(\vec{U} \upharpoonright \alpha) = \alpha$  which is absurd.  $\square$

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