“A new infinite family in ${}_2\pi_*^S$” by Mark Mahowald (1976)

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Abstract

On October 31st 2013 I spoke in the Thursday seminar on Mahowald's η_j paper. Nerves got the better of me in some places and I didn't say things how I would have liked. These are my cleaned up lecture notes. I hope that they will be useful for those who attended.

1 Introduction to the notes

The goal of Mahowald's paper is to prove the following theorem.

Theorem (Mahowald)**.** *For $j \geq 3$ there exist classes $\eta_j \in \pi_{2^j}(S^0)$ detected by*

$$h_1 h_j = \{[\xi_1^2 | \xi_1^{2^j}]\} \in \mathrm{Cotor}_{A_*}^{2,2^j+2}(\mathbb{F}_2, \mathbb{F}_2).$$

His proof proceeds by constructing two maps for each $j \geq 3$: $g_j : S^{2^j} \longrightarrow X_j$ and $f_j : X_j \longrightarrow S^0$. g_j is supposed to correspond to h_1 and f_j is supposed to correspond to h_j. The composite is the class

$$\eta_j : S^{2^j} \longrightarrow S^0.$$

It is clear that we have to say something about the spectra X_j appearing in the maps above. The Thursday seminar in the Fall of 2013 was based around Brown-Gitler spectra and sure enough, each X_j is a supension of a Brown-Gitler spectrum. This paper would have been much harder to talk about if we had not already devoted seven weeks to the study of these spectra!

In preparing the talk I tried to proceed linearly through Mahowald's paper and the order in which material was presented reflected the order in which I understood the various results. I had a hard time describing the first part of Mahowald's paper and I think that the material of the second and third parts of the talk is easier to digest.

2 Introduction to the talk

The main goal of this talk is to prove the following theorem of Mahowald.

Theorem (Mahowald)**.** *For $j \geq 3$ there exist classes $\eta_j \in \pi_{2^j}(S^0)$ detected by*

$$h_1 h_j = \{[\xi_1^2 | \xi_1^{2^j}]\} \in \mathrm{Cotor}_{A_*}^{2,2^j+2}(\mathbb{F}_2, \mathbb{F}_2).$$

The talk will proceed in three stages.

1. Firstly, we will focus attention on the "h_j part" of η_j. This involves some pretty classical constructions.

2. Along my journey through Mahowald's paper I became distracted by the cofibration sequences he uses:
$$B(2n-2) \longrightarrow B(2n) \longrightarrow \Sigma^n B(n).$$
In the second part of the talk I will discuss some of the observations I made about these cofibration sequences.

3. Finally, we will focus on the "h_1 part" of η_j. Mahowald's construction was complicated by the fact that he could not prove the spectra he used were Brown-Gitler spectra. His original arguments are insightful and so we will discuss them, but just as we reach the point at which more details are necessary, we will show how his argument can be simplified. [Mahowald was surely aware of this simplification; it was Frank Adams who pointed out that he had not proved the spectra he was using were Brown Gitler spectra and that he could not make such an argument without this fact.]

3 The "h_j part" of the map

For $j = 3$ we have a map $S^{2^j-1} \longrightarrow S^0$ detected by h_j: it is the Hopf invariant one class $\sigma : S^7 \longrightarrow S^0$. For $j > 3$ we do not have Hopf invariant one classes $S^{2^j-1} \longrightarrow S^0$ since $d_2 h_j = h_0 h_{j-1}^2$.

One might hope that we can succeed in constructing a "Hopf invariant one map" if we replace S^{2^j-1} by some other connected complex of dimension $2^j - 1$ which only has one $(2^j - 1)$-cell.

Notation. In this talk the Brown-Gitler spectra $B(n)$ are indexed so that $B(2n) = B(2n+1)$.

Definition. For $j \geq 3$ let $k(j) = 2^{j-3}$ and $X_j = \Sigma^{7k(j)} B(k(j))$.

Lemma. *X_j has dimension $2^j - 1$, $H^{2^j-1}(X_j) = \mathbb{Z}/2\langle e^{2^j-1}\rangle$ and $H^*(X_j) = 0$ for $* < 2^j - 2^{j-3}$.*

Proof. $X_3 = S^7$ and so the result is easily seen to be true when $j = 3$.

Assume $j \geq 4$. Then $B(k(j)) = B(2 \cdot 2^{j-4})$ and so by previous talks $H^*(B(k(j)))$ has basis
$$\{\chi \mathrm{Sq}^I : I = (i_1, i_2, \ldots) \text{ is admissable and } i_1 \leq 2^{j-4}\}.$$
The highest degree basis element is thus
$$\chi(\mathrm{Sq}^{2^{j-4}} \mathrm{Sq}^{2^{j-5}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1)$$
which has degree $2^{j-3} - 1$. $X_j = \Sigma^{7k(j)} B(k(j))$ and
$$7k(j) = (2^3 - 1)2^{j-3} = 2^j - 2^{j-3}$$
which completes the proof. $\square$

X_j is a candidate for a replacement of S^{2^j-1} on which we can construct a "Hopf invariant one map" $X_j \longrightarrow S^0$. The following proposition is the desired result.

Proposition. *We can construct a map $f_j : X_j \longrightarrow S^0$ so that the top and bottom cohomology classes in the cofiber of f_j are related by Sq^{2^j}.*

f_j is what we mean by the "h_j part" of the map; we will construct another map $g_j : S^{2^j} \longrightarrow X_j$, which will be the "$h_1$ part" of the map.

Let's construct the map f_j right away. We need to recall a theorem which Mike Hopkins spoke about a few weeks back.

Theorem. $\Sigma^\infty \Omega^2 S^9 = \bigvee_{k>0} \Sigma^{7k} B(k)$.

Definition. For $j \geq 3$ let $\iota_j : X_j \longrightarrow \Sigma^\infty \Omega^2 S^9$ denote the inclusion map.

We are now equipped to define f_j.

Definition. Let $d : S^7 \longrightarrow SO$ be a generator for $\pi_7(SO) = \mathbb{Z}$. SO is a double loop space and so d extends uniquely to double loop map $\overline{d}$ from $\Omega^2 S^9$. 2-locally we have the following diagram where σ is a generator for $\pi_7(S^0) = \mathbb{Z}/16$.

$$\begin{tikzcd}
S^7 \arrow[r, "d"] \arrow[d] \arrow[rrr, bend left, "\sigma"] & SO \arrow[r, hook] & QS^0 \arrow[r, "-1"] & QS^0 \\
\Omega^2 S^9 \arrow[ur, "\overline{d}"'] \arrow[urrr, bend right, "\overline{\sigma}"'] & & &
\end{tikzcd}$$

[Note that the composite $SO \longrightarrow QS^0 \xrightarrow{-1} QS^0$ induces the J-homomorphism.]

Adjoint to $\overline{\sigma}$ is a map $\tilde{\sigma} : \Sigma^\infty \Omega^2 S^9 \longrightarrow S^0$. For $j \geq 3$ let $f_j : X_j \longrightarrow S^0$ be the composite

$$X_j \xrightarrow{\iota_j} \Sigma^\infty \Omega^2 S^9 \xrightarrow{\tilde{\sigma}} S^0.$$

We state the proposition that we wish to prove more carefully.

Proposition. *Let C_j be the cofiber of f_j so that we have nonzero classes $e^0 \in H^0(C_j)$ and $e^{2^j} \in H^{2^j}(C_j)$. Then $Sq^{2^j} e^0 = e^{2^j}$.*

How might we go about proving the proposition? We have the following diagram of Puppe sequences.

$$\begin{tikzcd}
X_j \arrow[r, "f_j"] \arrow[d, "\iota_j"'] & S^0 \arrow[r] \arrow[d, "="] & C_j \arrow[r, "\partial"] \arrow[d, "l_j"] & \Sigma X_j \arrow[d, "\Sigma \iota_j"] \\
\Sigma^\infty \Omega^2 S^9 \arrow[r, "\tilde{\sigma}"] & S^0 \arrow[r] & D \arrow[r, "\partial"] & \Sigma^\infty \Sigma \Omega^2 S^9
\end{tikzcd}$$

We are asking about cohomology classes in C_j. These classes pull back from cohomology classes in D. Thus, we'd like to describe D and $\partial : D \longrightarrow \Sigma^\infty \Sigma \Omega^2 S^9$ in a tractible way. For this we need a classical result.

Lemma. *Suppose given a based CW complex M and a based map $f : M \longrightarrow SO$. We may view f as a clutching function and define a stable bundle V over ΣM. On the other hand the adjoint to the composite*

$$M \xrightarrow{f} SO \hookrightarrow QS^0 \xrightarrow{-1} QS^0$$

is a map $g : \Sigma^\infty M \longrightarrow S^0$.

The cofiber of g is homotopy equivalent to the Thom spectrum $(\Sigma M)^V$.

Let's indicate the prove for the finite dimensional version.

Lemma. *Suppose given a based CW complex M and a based map $f : M \longrightarrow SO(n)$. We may view f as a clutching function and define an n-bundle $V \longrightarrow \Sigma M$. On the other hand the adjoint to the composite*

$$M \xrightarrow{f} SO(n) \hookrightarrow \Omega^n S^n \xrightarrow{-1} \Omega^n S^n$$

is a map $g : \Sigma^n M \longrightarrow S^n$.

The cofiber of g is homotopy equivalent to the Thom space $(\Sigma M)^V$ as long as $n \geq 2$.

Proof. Firstly, let's identify $(\Sigma M)^V$ in terms of a homotopy pushout diagram. Let $F : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be adjoint to the composite

$$M \xrightarrow{f} SO(n) \hookrightarrow \mathrm{Map}(\mathbb{R}^n, \mathbb{R}^n)$$

and $i : M \to CM$ be the inclusion of M into the reduced cone. By definition of a clutching function we have the pushout diagram on the left. Thomifying gives the pushout diagram on the right.

$$\begin{tikzcd} M \times \mathbb{R}^n \arrow[r, "(i\pi_M, F)"] \arrow[d, "i\times\mathrm{id}"'] & CM \times \mathbb{R}^n \arrow[d] \\ CM \times \mathbb{R}^n \arrow[r] & V \end{tikzcd} \qquad \begin{tikzcd} M_+ \wedge S^n \arrow[r] \arrow[d] & CM_+ \wedge S^n \arrow[d] \\ CM_+ \wedge S^n \arrow[r] & (\Sigma M)^V \end{tikzcd}$$

Thus $(\Sigma M)^V$ is the homotopy pushout of a diagram

$$S^n \xleftarrow{I} M_+ \wedge S^n \xrightarrow{G} S^n.$$

I is the map $M_+ \wedge S^n \longrightarrow (*)_+ \wedge S^n$ and G corresponds to the composite $M \xrightarrow{f} SO(n) \hookrightarrow \Omega^n S^n$ under the identification $\mathrm{Top}_*(M_+ \wedge S^n, S^n) = \mathrm{Top}_*(M_+, \Omega^n S^n) = \mathrm{Top}(M, \Omega^n S^n)$.

The cofiber of g is given by the homotopy pushout of the diagram $* \longleftarrow M \wedge S^n \xrightarrow{g} S^n$ and so we now have two diagrams we wish to compare. It is crucial to note that we have a split cofibration sequence

$$(*)_+ \longrightarrow M_+ \longrightarrow M.$$

Since we assumed $n \geq 2$ this means that after smashing with S^n the inner space decomposes as a wedge of the outer spaces and so we obtain the following cofibration sequence

$$M \wedge S^n \longrightarrow M_+ \wedge S^n \longrightarrow (*)_+ \wedge S^n.$$

[We are using the fact that

$$(M_+) \wedge S^2 \longrightarrow ((M_+) \wedge S^2) \vee ((M_+) \wedge S^2) \longrightarrow ((*)_+ \wedge S^2) \vee (M \wedge S^2)$$

is a homology isomorphism between simply-connected CW complexes and thus a homotopy equivalence. This is one place where we need $n \geq 2$.]

We then consider the map of diagrams in which each column is a cofiber sequence.

$$\begin{tikzcd}
* \arrow[d] & M \wedge S^n \arrow[l] \arrow[r, "g"] \arrow[d] & S^n \arrow[d, "\mathrm{id}"] \\
S^n \arrow[d, "\mathrm{id}"'] & M_+ \wedge S^n \arrow[l, "I"'] \arrow[r, "G"] \arrow[d] & S^n \arrow[d] \\
S^n & (*)_+ \wedge S^n \arrow[l, "\mathrm{id}"'] \arrow[r] & *
\end{tikzcd}$$

Taking homotopy pushouts gives a cofibration sequence $\mathrm{cof}(g) \longrightarrow (\Sigma M)^V \longrightarrow *$. One sees that $\mathrm{cof}(g)$ and $(\Sigma M)^V$ are simply-connected (using $n \geq 2$ again) and so this completes the proof. $\square$

Before proving the proposition we introduce some notation that will be useful.

Notation.

$$\begin{tikzcd}
S^7 \arrow[r, "d"] \arrow[d] & SO \\
\Omega^2 S^9 \arrow[ur, "\overline{d}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
S^8 \arrow[r, "D"] \arrow[d] & BSO \\
\Omega S^9 \arrow[ur, "\overline{D}"'] &
\end{tikzcd}$$

1. Writing $SO = \Omega BSO$ we have $D : S^8 \longrightarrow BSO$ adjoint to d.

2. Writing $SO = \Omega^2 B^2 SO$ we have $e : S^9 \longrightarrow B^2 SO$ adjoint to d.

3. Then $\Omega e : \Omega S^9 \longrightarrow BSO$ is $\overline{D}$ and $\Omega^2 e : \Omega^2 S^9 \longrightarrow SO$ is $\overline{d}$.

4. Writing $SO = \Omega BSO$ the adjoint of $\overline{d}$ is

$$\Sigma\Omega^2 S^9 \xrightarrow{c} \Omega S^9 \xrightarrow{\Omega e} BSO.$$

Proposition. *$Sq^{2^j} e^0 = e^{2^j}$ in $H^*(C_j)$.*

Proof. $\overline{d} : \Omega^2 S^9 \longrightarrow SO$ can be viewed as a clutching function for a stable bundle V over $\Sigma\Omega^2 S^9$ classified by $(\Omega e) \circ c$. By the lemma applied to $\overline{d}$, our diagram of Puppe sequences becomes

$$\begin{tikzcd}
X_j \arrow[r, "f_j"] \arrow[d, "\iota_j"'] & S^0 \arrow[r] \arrow[d, "="] & C_j \arrow[r, "\partial"] \arrow[d, "l_j"] & \Sigma X_j \arrow[d, "\Sigma\iota_j"] \\
\Sigma^\infty \Omega^2 S^9 \arrow[r, "\tilde{\sigma}"] & S^0 \arrow[r] & (\Sigma\Omega^2 S^9)^V \arrow[r, "\partial"] & \Sigma^\infty \Sigma\Omega^2 S^9
\end{tikzcd}$$

The map $\partial : (\Sigma\Omega^2 S^9)^V \longrightarrow \Sigma^\infty \Sigma\Omega^2 S^9$ induces a map $\widetilde{H}^*(\Sigma\Omega^2 S^9) \longrightarrow H^*((\Sigma\Omega^2 S^9)^V)$ which is the restriction of the Thom isomorphism

$$(-) \cup U : H^*(\Sigma\Omega^2 S^9) \longrightarrow H^*((\Sigma\Omega^2 S^9)^V).$$

Here, $U \in H^0((\Sigma\Omega^2 S^9)^V)$ is the Thom class. Thus, in $H^*(C_j)$ we have

$$\mathrm{Sq}^{2^j} e^0 = \mathrm{Sq}^{2^j}(l_j^* U) = l_j^*(\mathrm{Sq}^{2^j} U) = l_j^* \partial^*(w_{2^j}(V)) = l_j^* \partial^* c^* (\Omega e)^* w_{2^j} = \partial^* (\Sigma \iota_j)^* c^* (\Omega e)^* w_{2^j}$$

and in order to check $\mathrm{Sq}^{2^j} e^0 = e^{2^j}$ it is enough to show that $(\Omega e)^* w_{2^j} \in H^{2^j}(\Omega S^9) = \mathbb{Z}_2$ is nonzero and $c_*(\Sigma\iota_j)_* : H_{2^j}(\Sigma X_j) \longrightarrow H_{2^j}(\Omega S^9)$ is an isomorphism.

At this point in the proof we recall and emphasize that $j \geq 3$.

Let's first consider $(\Omega e)^* w_{2^j} \in H^{2^j}(\Omega S^9)$. We have a commuting diagram.

$$\begin{tikzcd}
(S^8)^{k(j)} \arrow[r] \arrow[dr, "D^{k(j)}"'] & (\Omega S^9)^{k(j)} \arrow[r, "\mu"] \arrow[d, "(\overline{D})^{k(j)}"', "(\Omega e)^{k(j)}"] & \Omega S^9 \arrow[d, "\Omega e"] \\
 & (BSO)^{k(j)} \arrow[r, "\mu"] & BSO
\end{tikzcd}$$

It is a result related to the Hopf invariant one problem that $D^* w_8$ is a generator κ for $H^8(S^8)$. Thus, using the Whitney product formula together with the Künneth formula we see that $(D^{k(j)})^* \mu^* w_{2^j} = \kappa_1 \times \ldots \times \kappa_{k(j)} \neq 0$. We deduce that $(\Omega e)^* w_{2^j}$ is nonzero and thus a generator for $H^{2^j}(\Omega S^9) = \mathbb{Z}_2$.

We have the fundamental class $x \in H_7(\Omega^2 S^9)$. In Mike's talk, a few weeks back, he hinted at the fact that $Q^{j-3}x \in H_{2^j-1}(\Omega^2 S^9)$ is equal to $(\iota_j)_*(e_{2^j-1})$. Assuming this, we have $(\Sigma\iota_j)^*(e_{2^j}) = \Sigma Q^{j-3}x$ and so we are left with showing that $c_*(\Sigma Q^{j-3}x)$ is a generator for $H_{2^j}(\Omega S^9)$. Denote by y the fundamental class in $H_8(\Omega S^9)$. $y^{k(j)}$ is transgressive in the homology Serre spectral sequence for the fibration $\Omega^2 S^9 \longrightarrow * \longrightarrow \Omega S^9$ and the Kudo transgression theorem tells us that it transgresses to $Q^{j-3}x$. The following commutative diagram explains the relationship between the map induced by the counit $c : \Sigma\Omega^2 S^9 \longrightarrow \Omega S^9$ and the transgression. This completes the proof.

$$\begin{tikzcd}
 & H_{2^j-1}(\Omega^2 S^9) \arrow[r, "\cong"] \arrow[d] & H_{2^j}(\Sigma\Omega^2 S^9) \arrow[d] \arrow[dd, bend left, "c_*"] \\
H_{2^j}(\Omega S^9) \arrow[r, "\tau"] \arrow[drr, bend right, "\mathrm{id}"'] & E^{2^j}_{0,2^j-1} \arrow[r] & H_{2^j}(\Sigma\Omega^2 S^9)/\ker c_* \arrow[d] \\
 & & H_{2^j}(\Omega S^9)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Q^{j-3}x \arrow[r, mapsto] \arrow[d, mapsto] & \Sigma Q^{j-3}x \arrow[d, mapsto] \\
y^{k(j)} \arrow[r, mapsto] \arrow[drr, mapsto, bend right] & \bullet \arrow[r, mapsto] & \bullet \arrow[d, mapsto] \\
 & & y^{k(j)}
\end{tikzcd}$$

□

4 The cofibration sequences $B(2n-2) \longrightarrow B(2n) \longrightarrow \Sigma^n B(n)$

These cofibration sequences have driven me mad over the past week: I found myself awake at 5am on Saturday morning with them plaguing me. Here, I will talk about some observations that I have made about them. They are not deep but I hope that they will help consolidate and clarify some of what we have learned in previous weeks.

4.1 Thom complex observations

Definition. Let $\widetilde{C}_n(\mathbb{R}^2)$ be the space of n-ordered tuples of distinct points in $\mathbb{R}^2$ and let F_n be the n^{th} part in the May filtration of $\Omega^2 S^3$, i.e. $\coprod_{k \leq n} \widetilde{C}_k(\mathbb{R}^2) \times_{\Sigma_k} (S^1)^{\times k}/\sim$. Let $\gamma : \Omega^2 S^3 \longrightarrow BO$ be the double loop map which extends the generator for $\pi_1(BO)$.

Mark Behrens used the following theorem last week.

Theorem. *The Thom spectrum F_n^γ is $B(2n)$.*

Thus the inclusions $F_{n-1} \longrightarrow F_n$ give us maps $i : B(2n-2) \longrightarrow B(2n)$ for each $n \in \mathbb{N}$.

Definition. Let $D_n(X)$ be the n^{th} term in the Snaith splitting for $\Omega^2\Sigma^2 X$, i.e. $\Sigma^\infty \widetilde{C}_n(\mathbb{R}^2)_+ \wedge_{\Sigma_n} X^{\wedge n}$. Of course, $D_n(S^1) = \Sigma^\infty F_n/F_{n-1}$.

Mike Hopkins mentioned the following theorem.

Theorem. *$D_n(S^1) = \Sigma^n B(n)$. In fact, for odd k we have $D_n(S^k) = \Sigma^{kn} B(n)$.*

I found it very strange that the Brown-Gitler spectra have descriptions in terms of Thom spectra over the F_n and simultaneously as the quotients $\Sigma^\infty F_n/F_{n-1}$. I think I was trying to seek out which was the most 'useful' description. In the end I made a rather trivial observation but a nice enough one that I thought I'd tell you. Without supposing any of the theorems above we can prove the following lemma. Thank you to Søren Galatius for pointing out how to fix my original proof.

Lemma. *The cofiber of the map $F_{n-1}^\gamma \longrightarrow F_n^\gamma$ is $\Sigma^\infty F_n/F_{n-1}$.*

Proof. $F_n - F_{n-1} = \widetilde{C}_n(\mathbb{R}^2) \times_{\Sigma_n} (S^1 - \{1\})^{\times n}$, where we are thinking of S^1 as a subset of $\mathbb{C}$, and so we have a deformation retraction

$$F_n - F_{n-1} \longrightarrow \widetilde{C}_n(\mathbb{R}^2) \times_{\Sigma_n} \{-1\}^{\times n}.$$

Thus, the inclusion $F_n - F_{n-1} \longrightarrow \widetilde{C}_n(\mathbb{R}^2) \times_{\Sigma_n} (S^1)^{\times n}$ is homotopic to a map factoring through $\widetilde{C}_n(\mathbb{R}^2) \times_{\Sigma_n} \{1\}^{\times n}$. We deduce that the inclusion $F_n - F_{n-1} \subset F_n$ is nullhomotopic and so $\gamma|_{F_n - F_{n-1}}$ is trivial. This allows one to identify the cofiber of $F_{n-1}^\gamma \to F_n^\gamma$: it is the "one point compactification" of $\Sigma^\infty(F_n - F_{n-1})_+$ which is $\Sigma^\infty F_n/F_{n-1}$. □

Corollary. *Supposing only that $F_n^\gamma = B(2n)$, the following statements are equivalent:*

1. *$D_n(S^1) = \Sigma^n B(n)$ for each $n \in \mathbb{N}$;*
2. *There is a cofibration sequence $B(2n-2) \xrightarrow{i} B(2n) \xrightarrow{p} \Sigma^n B(n)$ for each $n \in \mathbb{N}$.*

Of course, we already know that each of these statements is true but it shows the connection between the two.

4.2 A Goodwillie perspective on $p : B(2n) \longrightarrow \Sigma^n B(n)$

We have another construction of the map $p : B(2n) \longrightarrow \Sigma^n B(n)$ using Goodwillie calculus.

Theorem (Arone-Goodwillie)**.** *The functor $\Sigma^\infty \Omega^2$ has n^{th}-layer $X \longmapsto D_n(\Sigma^{-2}X)$, i.e.*

$$\mathbb{D}_n(\Sigma^\infty \Omega^2)(X) = D_n(\Sigma^{-2}X).$$

Proposition (Arone-Mahowald, Behrens)**.** *The natural transformation of functors*

$$\Sigma^\infty \Omega H : \Sigma^\infty \Omega^2 \Sigma \longrightarrow \Sigma^\infty \Omega^2 \Sigma Sq$$

induces a map of Goodwillie towers. Evaluating the map on the $(2n)^{th}$ layer at S^2 and desuspending n times gives $p : B(2n) \longrightarrow \Sigma^n B(n)$.

Proof. A result of Arone-Mahowald, used explicitly by Behrens in his EHP/Goowillie paper, implies that

$$P_n(\Sigma^\infty \Omega^2 \Sigma \mathrm{Sq}) \simeq P_{\lfloor \frac{n}{2} \rfloor}(\Sigma^\infty \Omega^2 \Sigma)\mathrm{Sq}.$$

Thus the natural transformation induced by $\Sigma^\infty \Omega H$ on the $(2n)^{\text{th}}$ layer takes the form

$$\mathbb{D}_{2n}(\Sigma^\infty \Omega^2)\Sigma \longrightarrow \mathbb{D}_n(\Sigma^\infty \Omega^2)\Sigma \mathrm{Sq}.$$

Evaluating on X gives $D_{2n}(\Sigma^{-1}X) \longrightarrow D_n(\Sigma^{-1}X^{\wedge 2})$. Setting $X = S^2$ gives $D_{2n}(S^1) \longrightarrow D_n(S^3)$, which is $\Sigma^{2n}B(2n) \longrightarrow \Sigma^{3n}B(n)$. Desuspending n times gives a map $B(2n) \longrightarrow \Sigma^n B(n)$. Inspecting Mahowald's construction of the map $p : B(2n) \longrightarrow \Sigma^n B(n)$ will convince you that this is the same map. $\square$

Let's push this direction further. Setting $X = S^2$ above meant that we were really studying the map $\Sigma^\infty \Omega H : \Sigma^\infty \Omega^2 S^3 \longrightarrow \Sigma^\infty \Omega^2 S^5$. The Goodwillie tower gives the Snaith splitting and so the result above shows that we can write this map as the wedge of the maps on the right below. We have filled in what the fibers are supposed to be.

$$\begin{array}{ccccc}
\Sigma B(1) & \longrightarrow & \Sigma B(1) & \longrightarrow & * \\
\Sigma^2 B(0) & \longrightarrow & \Sigma^2 B(2) & \longrightarrow & \Sigma^3 B(1) \\
\Sigma^3 B(3) & \longrightarrow & \Sigma^3 B(3) & \longrightarrow & * \\
\Sigma^4 B(2) & \longrightarrow & \Sigma^4 B(4) & \longrightarrow & \Sigma^6 B(2) \\
\Sigma^5 B(5) & \longrightarrow & \Sigma^5 B(5) & \longrightarrow & * \\
\Sigma^6 B(4) & \longrightarrow & \Sigma^6 B(6) & \longrightarrow & \Sigma^9 B(3) \\
\Sigma^7 B(7) & \longrightarrow & \Sigma^7 B(7) & \longrightarrow & * \\
\Sigma^8 B(6) & \longrightarrow & \Sigma^8 B(8) & \longrightarrow & \Sigma^{12} B(4) \\
\Sigma^9 B(9) & \longrightarrow & \Sigma^9 B(9) & \longrightarrow & * \\
\Sigma^{10} B(8) & \longrightarrow & \Sigma^{10} B(10) & \longrightarrow & \Sigma^{15} B(5) \\
 & & \cdots & &
\end{array}$$

On the other hand we can analyze the map on homology. We know $H_*(\Omega^2 S^3) = \mathbb{F}_2[y, Qy, Q^2y, \ldots]$ and $H_*(\Omega^2 S^5) = \mathbb{F}_2[y', Qy', Q^2y', \ldots]$ where $|y| = 1$ and $|y'| = 3$ and the map induced by ΩH is given as follows. [The $\sim$ appearing here means "up to decomposables".]

$$\begin{aligned} \mathbb{F}_2[y, Qy, Q^2y, \ldots] &\longrightarrow \mathbb{F}_2[y', Qy', Q^2y', \ldots] \\ y &\longmapsto 0 \\ Q^n y &\overset{\sim}{\longmapsto} Q^{n-1}y' \qquad \text{for } n \in \mathbb{N}. \end{aligned}$$

Thus the homology of the fiber of $\Sigma^\infty \Omega H$ is the ideal in $\mathbb{F}_2[y, Qy, Q^2y, \ldots]$ generated by y. A few weeks ago Mike drew out the homology of the $B(n)$'s in a line to demonstrate that they build up the homology of $\Omega^2 S^3$. If one takes the ideal generated by y then this has the effect of picking out $H_*(\Sigma^{2n} B(2n-2)) \subset H_*(\Sigma^{2n} B(2n))$.

4.3 Some homology remarks

[I omitted this material form the talk.]

We have a map $B(n) \longrightarrow H\mathbb{F}_2$ which induces an inclusion on homology

$$H_*(B(n)) \hookrightarrow H_*(H\mathbb{F}_2) = A_* = \mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \ldots], \qquad |\zeta_j| = 2^j - 1\ .$$

If we let $\mathrm{wt}(\zeta_j) = 2^{j-1}$ then the image of this inclusion can be described as $\{\zeta \in A_* : \mathrm{wt}(\zeta) \leq \lfloor \frac{n}{2} \rfloor\}$.

We have a map $F_n \longrightarrow \Omega^2 S^3$ which induces an inclusion on homology

$$H_*(F_n) \hookrightarrow H_*(\Omega^2 S^3) = \mathbb{F}_2[y, Qy, Q^2y, Q^3y, \ldots], \qquad |Q^j y| = 2^{j+1} - 1\ .$$

If we let $\mathrm{wt}(Q^j y) = 2^j$ then the image of this map can be described as $\{z \in H_*(\Omega^2 S^3) : \mathrm{wt}(z) \leq n\}$.

Mark Behrens stated, last week, that $(\Omega^2 S^3)^\gamma$ is $H\mathbb{F}_2$ and so the Thom isomorphism gives an identification of A_* and $H_*(\Omega^2 S^3)$. Up to decomposables, the Thom isomorphism is given by

$$\zeta_j \longleftrightarrow Q^{j-1}y.$$

Mark also described the maps induced on homology by $B(2n-2) \to B(2n) \to \Sigma^n B(n)$. We have an equivalence $\Sigma^n B(n) \simeq \Sigma^\infty F_n / F_{n-1}$ which gives an isomorphism $\Sigma^n H_*(B(n)) \cong \widetilde{H}_*(F_n/F_{n-1})$. Using the description of the Thom isomorphism just given and Mark's description of the maps induced by the Mahowald cofibration sequence, we can see that under the identifications above this isomorphism takes the following form up to decomposables.

$$\Sigma^n \left\{ \zeta \in A_* : \mathrm{wt}(\zeta) \leq \left\lfloor \frac{n}{2} \right\rfloor \right\} \cong \left\{ z \in H_*(\Omega^2 S^3) : \mathrm{wt}(z) = n \right\}$$

$$\Sigma^n \zeta_1^{j_1} \zeta_2^{j_2} \cdots \zeta_k^{j_k} \longleftrightarrow y^{n - \sum_i 2^i j_i} (Qy)^{j_1} (Q^2y)^{j_2} \cdots (Q^k y)^{j_k}$$

5 The "h_1-part"

We'd like to prove the following result.

Proposition. *There exists a homotopy class* $g_j : S^{2^j} \longrightarrow X_j$ *such that composing with the collapse map* $X_j \longrightarrow S^{2^j-1}$ *gives* $\eta : S^{2^j} \longrightarrow S^{2^j-1}$.

The property stated in the proposition justifies calling g_j the "h_1-part" of η_j. Since $2^j = 8k(j)$ and $X_j = \Sigma^{7k(j)} B(k(j))$ it is sufficient to find a map $S^{k(j)} \longrightarrow B(k(j))$ with the required property. Desuspending further, we look for a map $S^1 \to \Sigma^{1-k(j)} B(k(j))$. We note that when $j = 3$, $k(j) = 1$ and we can take $\eta : S^1 \longrightarrow S^0$. We rewrite the proposition for clarity.

Proposition. *For each* $j \geq 0$, *there exists a homotopy class* $S^1 \longrightarrow \Sigma^{1-2^j} B(2^j)$ *such that composing with the collapse map* $\Sigma^{1-2^j} B(2^j) \longrightarrow S^0$ *gives* $\eta : S^1 \longrightarrow S^0$.

5.1 The Adams spectral sequence argument

We can consider all j's simultaneously by using the maps which we previously called p.

$$\ldots \longrightarrow \Sigma^{1-2^{j+1}} B(2^{j+1}) \longrightarrow \Sigma^{1-2^j} B(2^j) \longrightarrow \ldots \longrightarrow \Sigma^{-7} B(8) \longrightarrow \Sigma^{-3} B(4) \longrightarrow \Sigma^{-1} B(2) \longrightarrow B(1).$$

We'll take an Adams spectral sequence approach to this computation and so a first step is to find compatible elements of

$$\mathrm{Cotor}^{1,2}_{A_*}(\mathbb{F}_2, H_*(\Sigma^{1-2^j} B(2^j))).$$

The first few elements are easy to construct by hand using the cobar construction. The first term of each cocycle is $[\xi_1^2]\zeta_j$ which will project to $[\xi_1^2]$ when one collapses to the top cell; these cocycles were constructed by trying to extend this cochain to a cocycle.

$j = 0$	$[\xi_1^2]1$
$j = 1$	$[\xi_1^2]\zeta_1 + [\xi_2]1$
$j = 2$	$[\xi_1^2]\zeta_2 + [\xi_2]\zeta_1^2 + [\xi_1^4]\zeta_1 + [\xi_2\xi_1^2]1$
$j = 3$	$\ldots$

Since one understands the map induced on homology by $\Sigma^{1-2^{j+1}} B(2^{j+1}) \longrightarrow \Sigma^{1-2^j} B(2^j)$, the j^{th} cocycle determines a large chunk of the $(j+1)^{\text{st}}$ cocycle. However, more terms are always necessary and the process becomes tiresome (unless I'm failing to spot a pattern that is emerging). Mahowald uses the Λ-algebra, which is far more efficient than the cobar construction. The relevant result is the following.

Proposition. $\mathrm{Cotor}_{A_*}(\mathbb{F}_2, H_*(B(2n)))$ *is the homology of a complex* $\Lambda(n) = \Lambda/\Lambda_n$.

Here, Λ is the Λ-algebra, a DG-algebra with an $\mathbb{F}_2$-vector space basis given by

$$\{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s} : 2i_r \geq i_{r+1}\},$$

and Λ_n is the sub-DG-space with basis

$$\{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s} : 2i_r \geq i_{r+1},\ i_s < n\}.$$

[There was a mistake in the talk at this point, which Haynes pointed out afterwards.]

I think one should be able to construct a map $\Omega(A; H_*(B(2n))) \to \Lambda(n)$. It would be described by picking out the terms which looks like $[a]1$ and evaluating the the relevant Sq^j on $\chi(a)$. If one follows this procedure then we see that our cocycles above correspond to various λ's.

$j=0$	$[\xi_1^2]1$	λ_1
$j=1$	$[\xi_1^2]\zeta_1 + [\xi_2]1$	λ_2
$j=2$	$[\xi_1^2]\zeta_2 + [\xi_2]\zeta_1^2 + [\xi_1^4]\zeta_1 + [\xi_2\xi_1^2]1$	λ_4
$j=3$	???	λ_8
	$\dots$	
$j=k$	???	λ_{2^k}

Notice that a disadvantage of this operation is that it forgets about the $[\xi_1^2]\zeta_j$ terms. It is not so clear, in general, that a class represented by λ_{2^j} will project to h_1.

Mahowald uses the lambda algebra to his advantage. In particular, he credits Ed Brown with a lemma concerning the λ-algebra, which is crucial for proving the following results.

Proposition.

1. *We have compatible nonzero elements* $\{\lambda_{2^j}\} \in \mathrm{Cotor}^{1,2}_{A_*}(\mathbb{F}_2, H_*(\Sigma^{1-2^j}B(2^j)))$.
2. $p : E_2^{s,s}(\Sigma^{1-2^{j+1}}B(2^{j+1})) \longrightarrow E_2^{s,s}(\Sigma^{1-2^j}B(2^j))$ *is zero for all* $s > 0$.

Corollary. *The elements* $\{\lambda_{2^j}\}$ *are permanent cycles in the Adams spectral sequence.*

Proof. Let r_j be the first r for which $d_r\{\lambda_{2^j}\} \neq 0$; if no such r exists let $r_j = \infty$. We wish for the set $\{r_1, r_2, r_3 \dots\}$ to be $\{\infty\}$. Suppose for contradiction it is not and pick the minimum value, say r_j. Then $r_{j+1} \geq r_j$ so that $d_{r_j}\{\lambda_{2^{j+1}}\}$ is defined. Moreover,

$$d_{r_j}\{\lambda_{2^j}\} = d_{r_j}p\{\lambda_{2^{j+1}}\} = pd_{r_j}\{\lambda_{2^{j+1}}\}.$$

But $d_{r_j}\{\lambda_{2^{j+1}}\} \in E_2^{r_j+1, r_j+1}(\Sigma^{1-2^{j+1}}B(2^{j+1}))$ and so we deduce that $d_{r_j}\{\lambda_{2^j}\}$ is zero, which gives the required contradiction. $\square$

We obtain maps $g_j : S^{2^j} \longrightarrow X_j$ detected by $\{\lambda_{2^j-3}\}$. From our cobar construction argument it is plausible that on the level of Adams E_2-pages they map to h_1 and thus, our claim concerning η is true. As remarked above, this is less clear if one works with the λ-algebra entirely and so Mahowald has to do more to prove that this is the case. His method of doing this closely follows the argument he would have made if he had have known that the spectra he was working with were Brown-Gitler spectra. In particular, as long as we suppose the surjectivity property discussed in previous lectures (which will appear below shortly) there is a simpler argument. We turn to this presently.

5.2 The simpler argument

Proposition. *For each* $j \geq 0$, *there exists a homotopy class* $S^{2^j} \longrightarrow B(2^j)$ *factoring through* $D(\Sigma^{-2^j}P_1^{2^j})$ *such that composing with the collapse map* $B(2^j) \longrightarrow S^{2^j-1}$ *gives* $\eta : S^{2^j} \longrightarrow S^{2^j-1}$.

Proof. We construct maps $t : S^{2^j} \longrightarrow D(\Sigma^{-2^j} P_1^{2^j})$ and $e : D(\Sigma^{-2^j} P_1^{2^j}) \longrightarrow B(2^j)$ and then check that their composite has the requisite property.

1. We look for $\Sigma^{2^j} Dt : P_1^{2^j} \longrightarrow S^0$: we can take the composite of the inclusion $P_1^{2^j} \longrightarrow P_1^{\infty}$ with the transfer map $P_1^{\infty} \longrightarrow S^0$.

2. Let $e_{2^j} : S^0 \longrightarrow \Sigma^{-2^j} P_1^{2^j} \wedge H$ be a generator for $H_{2^j}(P_1^{2^j})$. The "surjectivity property of Brown-Gitler spectra" that Jacob Lurie stated at the start of his first lecture on Dieudonné theory tells us that there is a lift of e_{2^j} to an element $\tilde{e}_{2^j} \in B(2^j)_{2^j}(P_1^{2^j})$. Let e be the composite

$$D(\Sigma^{-2^j} P_1^{2^j}) \xrightarrow{\mathrm{id}\wedge\tilde{e}_{2^j}} D(\Sigma^{-2^j} P_1^{2^j}) \wedge (\Sigma^{-2^j} P_1^{2^j}) \wedge B(2^j) \xrightarrow{\text{(duality pairing)}\wedge\mathrm{id}} B(2^j).$$

The following commuting diagram shows that $e^*(e^0) = e^0$ and so $e^*(\mathrm{Sq}^{2^j-1} e^0) = \mathrm{Sq}^{2^j-1} e^0$.

$$\begin{tikzcd}
 & D(\Sigma^{-2^j} P_1^{2^j}) \arrow[r, "\mathrm{id}\wedge\tilde{e}_{2^j}"] \arrow[dr, "\mathrm{id}\wedge e_{2^j}"'] \arrow[rr, bend left, "e"] & D(\Sigma^{-2^j} P_1^{2^j}) \wedge (\Sigma^{-2^j} P_1^{2^j}) \wedge B(2^j) \arrow[r] \arrow[d] & B(2^j) \arrow[d] \\
S^0 \arrow[ur, "\text{bottom cell}"] \arrow[rr, "\text{(bottom cell)}\wedge\text{(top cell)}"] \arrow[rrr, bend right, "\text{unit}"'] & & D(\Sigma^{-2^j} P_1^{2^j}) \wedge (\Sigma^{-2^j} P_1^{2^j}) \wedge H \arrow[r, "\text{(duality pairing)}\wedge\mathrm{id}"] & H
\end{tikzcd}$$

[$\mathrm{Sq}^{2^j-1} = \chi(\mathrm{Sq}^{2^{j-1}}\mathrm{Sq}^{2^{j-2}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1)$ and so $\mathrm{Sq}^{2^j-1} e^0$ is nonzero in $H^{2^j-1}(B(2^j))$, although this is implied by what we are about to observe.] We have

$$D(\Sigma^{-2^j} P_1^{2^j}) = \Sigma^{2^j+1} P_{-2^j-1}^{-2} = \Sigma P_{-1}^{2^j-2}.$$

The total Steifel-Whitney class of the tautological line bundle ξ on $\mathbb{RP}^\infty$ is $(1+x)$. Thus the total Steifel-Whitney class of $-\xi$ is $(1+x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ Since the Thom complex of $-\xi$ is ΣP_{-1}^{∞} this shows that $\mathrm{Sq}^{2^j-1} e^0$ is nonzero in $H^{2^j-1}(\Sigma P_{-1}^{2^j-2}) = H^{2^j-1}(D(\Sigma^{-2^j} P_1^{2^j}))$. We conclude that the top cell is preserved by e, and this gives the following commuting diagram (in which maps are labelled correctly up to multiplication by a unit in $\mathbb{Z}_{(2)}$).

$$\begin{tikzcd}
S^{2^j} \arrow[r, "t"] & D(\Sigma^{-2^j} P_1^{2^j}) \arrow[r, "e"] \arrow[d, "\text{collapse}"'] & B(2^j) \arrow[dl, "\text{collapse}"] \\
 & S^{2^j-1} &
\end{tikzcd}$$

Applying $\Sigma^{2^j} D(-)$ to the composite $S^{2^j} \xrightarrow{t} D(\Sigma^{-2^j} P_1^{2^j}) \xrightarrow{\text{collapse}} S^{2^j-1}$ gives

$$S^1 \xrightarrow{\text{bottom cell}} P_1^{2^j} \xrightarrow{\text{inclusion}} P_1^{\infty} \xrightarrow{\text{transfer}} S^0.$$

This composite is η because the transfer map is surjective in homotopy (the Kahn-Priddy theorem) above dimension zero. This completes the proof. □

6 Completing the proof of the main theorem

[This was omitted from the talk.]

For each $j \geq 3$ we have constructed maps $g_j : S^{2^j} \longrightarrow X_j$ and $f_j : X_j \longrightarrow S^0$. We have also proved the following properties:

1. The composite of g_j with collapse map $X_j \longrightarrow S^{2^j-1}$ is $\eta : S^{2^j} \longrightarrow S^{2^j-1}$.

2. We have $\mathrm{Sq}^{2^j}(e^0) = e^{2^j}$ in $H^*(C_j)$ where C_j is the cofiber of f_j.

Geometrically one would have liked to construct maps detected by h_1 and h_j and taken their composite. The Hopf invariant one problem prevented us from doing this. We have shown g_j to be a suitable replacement for η and we have shown that $f_j : X_j \longrightarrow S^0$ is a suitable replacement for a Hopf invaraint one map. However, we have not completely justified it being the "h_j part" of the map η_j.

What must we verify? The picture below demonstrates the geometry we are trying to perform.

$$\begin{tikzcd}
S^{2^j} \arrow[r, "g_j"] \arrow[dr, "\eta"'] & X_j \arrow[r, "f_j"] \arrow[d, "c"] & S^0 \\
& S^{2^j-1} \arrow[ur, dotted, "``h_j"'"'] &
\end{tikzcd}$$

Algebraically the dashed map exists and we must verify that the right triangle commutes.

$$\begin{tikzcd}
E_2(S^{2^j}) \arrow[r, "(g_j)_*"] \arrow[dr, "h_1"'] & E_2(X_j) \arrow[r, "(f_j)_*"] \arrow[d, "c_*"] & E_2(S^0) \\
& E_2(S^{2^j-1}) \arrow[ur, "h_j"'] &
\end{tikzcd}$$

h_j is in filtration one and we have a concrete description of Ext^1 in terms of extensions, so we just need to check that there is a map of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & H^*(\Sigma S^{2^j-1}) \arrow[r] \arrow[d, "(\Sigma c)^*"'] & \mathbb{F}_2\langle e^0, e^{2^j} : \mathrm{Sq}^{2^j} e^0 = e^{2^j}\rangle \arrow[r] \arrow[d] & H^*(S^0) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H^*(\Sigma X_j) \arrow[r, "\partial"] & H^*(C_j) \arrow[r] & H^*(S^0) \arrow[r] & 0
\end{tikzcd}$$

The map is obvious on the level of $\mathbb{F}_2$-vector spaces. We just need to check that $\mathrm{Sq}^i(e^0) = 0$ in $H^*(C_j)$ unless $i = 0$ or 2^j. This is because the only indecomposables in the Steenrod algebra are of the form Sq^{2^i} and if $2^j - 2^{j-3} < 2^i \leq 2^j$, the range in which $H^*(\Sigma X_j)$ is concentrated, then $i = j$.

7 Comparing the Adams SS approach with a Dieudonné module approach

Recall that in section 5.1 we tried to construct a map from S^1 into the system

$$\ldots \longrightarrow \Sigma^{1-2^{j+1}} B(2^{j+1}) \longrightarrow \Sigma^{1-2^j} B(2^j) \longrightarrow \ldots \longrightarrow \Sigma^{-7} B(8) \longrightarrow \Sigma^{-3} B(4) \longrightarrow \Sigma^{-1} B(2) \longrightarrow B(1).$$

This is the same as asking for an element in the limit of the system

$$\ldots \longrightarrow B(2^{j+1})_{2^{j+1}}(S^0) \longrightarrow B(2^j)_{2^j}(S^0) \longrightarrow \ldots \longrightarrow B(2)_2(S^0) \longrightarrow B(1)_1(S^0).$$

Recalling the Dieudonné construction of the Brown-Gitler spectra that Jacob spoke about, this is the same as looking for an element in the limit of the system

$$DM(H_*(QS^0))_{2^{j+1}} \xrightarrow{V} DM(H_*(QS^0))_{2^j} \longrightarrow \ldots \longrightarrow DM(H_*(QS^0))_2 \longrightarrow DM(H_*(QS^0))_1.$$

We might even refine our search and look for an element in the limit of the system

$$DM(H_*(QS^0))^{j+1}_{2^j} \xrightarrow{V} DM(H_*(QS^0))^{j}_{2^{j-1}} \longrightarrow \ldots \longrightarrow DM(H_*(QS^0))^2_2 \longrightarrow DM(H_*(QS^0))^1_1.$$

Proposition. *Let $[n] \in H_0(QS^0)$ be "the n^{th} connected component of QS^0". Then*

1. $Q^{2^j}[1] \cdot [-2] \in DM(H_*(QS^0))^{j+1}_{2^j}$;
2. $V(Q^{2^j}[1] \cdot [-2]) = Q^{2^{j-1}}[1] \cdot [-2]$.

Proof. $\Delta(Q^{2^j}[1] \cdot [-2]) = \sum_{m+n=2^j} (Q^m[1] \cdot [-2]) \otimes (Q^n[1] \cdot [-2])$. By definition of V we immediately obtain $V(Q^{2^j}[1] \cdot [-2]) = Q^{2^{j-1}}[1] \cdot [-2]$, proving the second claim.

We prove the first case for $j = 0$ and $j = 1$. Firstly, we wish to show that $Q^1[1] \cdot [-2]$ is contained in $DM(H_*(QS^0))^1_1$, i.e. that $Q^1[1] \cdot [-2]$ is primitive. This is clear from the formula above since $Q^0[1] \cdot [-2] = [0] = 1$. Secondly, we wish to show that $Q^2[1] \cdot [-2] \in DM(H_*(QS^0))^2_2$. This comes down to the formula

$$\Delta(Q^2[1] \cdot [-2]) = (Q^2[1] \cdot [-2]) \otimes 1 + 1 \otimes (Q^2[1] \cdot [-2]) + V(Q^2[1] \cdot [-2]) \otimes V(Q^2[1] \cdot [-2])$$

which is precisely the formula above. $\square$

Jacob only claimed the equality $B(n)_n(X) = DM(H_*(\Omega^\infty X))_n$ for even n, so we should check the end of the sequence directly. The map $B(2)_2(S^0) \longrightarrow B(1)_1(S^0)$ is the map $\pi_2(S/2) \longrightarrow \pi_2(S^1)$, which can be identified with the surjection $\mathbb{Z}/4 \longrightarrow \mathbb{Z}/2$. We should check that $Q^2[1] \cdot [-2]$ has order 4 in $DM(H_*(QS^0))_2$. We see directly from the law for addition that

$$2 \cdot (Q^2[1] \cdot [-2]) = (Q^1[1] \cdot [-2])^2 = (Q^1[1])^2 \cdot [-4] \neq 0$$

and so we're done.