The v_1-periodic homotopy of the sphere spectrum at an odd prime and the classical Adams spectral sequence

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January 22, 2014

Abstract

We compute the v_1-periodic homotopy of the sphere spectrum at an odd prime using a direct limit of localized modified Adams spectral sequences. We show the E_2-page of our spectral sequence is isomorphic, in a range, to that of the classical mod p Adams spectral sequence. As a consequence we obtain a very good understanding of the classical Adams spectral sequence above a line of slope $1/(p^2 - p - 1)$.

Thesis Supervisor: Haynes Miller
Title: Professor of Mathematics

Acknowledgements

Without the support of my mother and my advisor, Haynes, I have no doubt that this thesis would have ceased to exist.

There are many things I should thank my mother for. Most relevant is the time she dragged me to Oxford. I had decided, at sixteen years of age, that I was not interested in going to Oxbridge for undergrad; she knew better. Upon visiting Oxford, I experienced for the first time the wonder of being able to speak to others who love maths as much as I do. My time there was mathematically fulfilling and the friends I made, I hope, will be lifelong. Secondly, it was her who encouraged me to apply to MIT for grad school. There's no other way to put it, I was terrified of moving abroad and away from the friends I had made but she was right: I would come to be the happiest I could ever have been at MIT. Cambridge is a beautiful place to live and the energy of the faculty and students at MIT is untouched by many institutes. Her support during the first year or so, during the struggle of quals, from over $3,000$ miles away, and throughout the rest of my life is never forgotten.

Haynes picked up the pieces many times in my first year at MIT. His emotional support and kindness in those moments are the reasons I chose him to be my advisor. He has always been a pleasure to talk with and I am particularly appreciative of how he has adapted to my requirements, always giving me the level of detail he knows I need, while holding back enough to make our conversations remain exciting. His mathematical influence is evident thoughout this thesis but on the other hand, it is wonderful to have an advisor that I consider a friend and who I can talk to about things other than math. I will always remember him giving me strict orders to go out and buy a guitar amp when he could tell I was suffering without and I am thankful that he saved his conjecture [16] for me to prove! Thank you, Haynes.

There are many other people to thank for their support in the moments leading up to writing this thesis. For their support in my first year at MIT, I am particularly grateful to Michael, Dana, Jiayong and Saul. I think they know best the tough time that I had.

The three best courses I have been lucky to be a part of were taught by Haynes, Mark Behrens and Emily Riehl. Haynes' course on the Adams spectral sequence was the birthplace for this thesis.

Mark taught 18.906 and it was the best introductory algebraic topology course that you can imagine. I wish to thank him for the advice he gave me during the microteaching workshop, the chats we've had about topology and for the energy he brings to everything he is involved in.

Emily made sure that I finally learned some categorical homotopy theory. Each one of her classes was like watching Usain Bolt run the 100m over and over again for an hour. It was incredible. I thank her for her rigour, her energy and for showing me that abstract nonsense done right is beautiful. Her influence is all over section 2 of chapter III.

Finally, I wish to thank Jessica Barton for helping me during my first year at MIT, John Wilson who made it possible for me to take my GRE exams, and Yan Zhang who helped plot figure 1.

Contents

Introduction

Throughout this thesis p will denote an odd prime, $q = 2(p-1)$ and S^n will denote the n-fold suspension of the sphere spectrum completed at p.

Chromatic homotopy theory was born when Adams [2, theorem 1.7] constructed a self map on the mod p^n Moore spectrum S/p^n inducing an isomorphism on K-theory. S/p^n is defined as the cofiber of the multiplication by p^n map on S^0 and the map Adams constructed takes the form

$$v_1^{p^n} : S/p^{n+1} \longrightarrow \Sigma^{-p^n q} S/p^{n+1}. \tag{0.1}$$

Moreover, Crabb and Knapp [6, proposition 1.1] have shown that these maps can be chosen so that

$$S^0 \longrightarrow S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-p^n q} S/p^{n+1} \longrightarrow S^{1-p^n q}$$

generates the p-component of the image of J in $\pi_{p^n q-1}(S^0)$ and

$$\begin{tikzcd}
S/p^{n+1} \arrow[r, "v_1^{p^n}"] \arrow[d] & \Sigma^{-p^n q} S/p^{n+1} \arrow[d] \\
S/p^n \arrow[r, "\left(v_1^{p^{n-1}}\right)^p"] \arrow[d, "p"'] & \Sigma^{-p^n q} S/p^n \arrow[d, "p"] \\
S/p^{n+1} \arrow[r, "v_1^{p^n}"] & \Sigma^{-p^n q} S/p^{n+1}
\end{tikzcd}$$

commutes. The commutative diagram tells us that the naming convention is sensible. It also allows us to form the third of the following telescopes.

$$\begin{aligned}
v_1^{-1} S/p^{n+1} &= \mathrm{hocolim}(S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-p^n q} S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-2p^n q} S/p^{n+1} \longrightarrow \ldots) \\
S/p^\infty &= \mathrm{hocolim}(S/p \xrightarrow{p} S/p^2 \xrightarrow{p} S/p^3 \longrightarrow \ldots) \\
v_1^{-1} S/p^\infty &= \mathrm{hocolim}(v_1^{-1} S/p \xrightarrow{p} v_1^{-1} S/p^2 \xrightarrow{p} v_1^{-1} S/p^3 \longrightarrow \ldots)
\end{aligned}$$

The goal of this thesis is to compute the homotopy of $v_1^{-1} S/p^\infty$ using classical Adams spectral sequence methods and to use the following zig-zag of maps to obtain information about the classical mod p Adams spectral sequence for S^0. This gives the odd prime analogue of Davis and Mahowald's work in [7, theorem 1.2].

$$S^0 \longleftarrow \Sigma^{-1} S/p^\infty \longrightarrow \Sigma^{-1} v_1^{-1} S/p^\infty \tag{0.2}$$

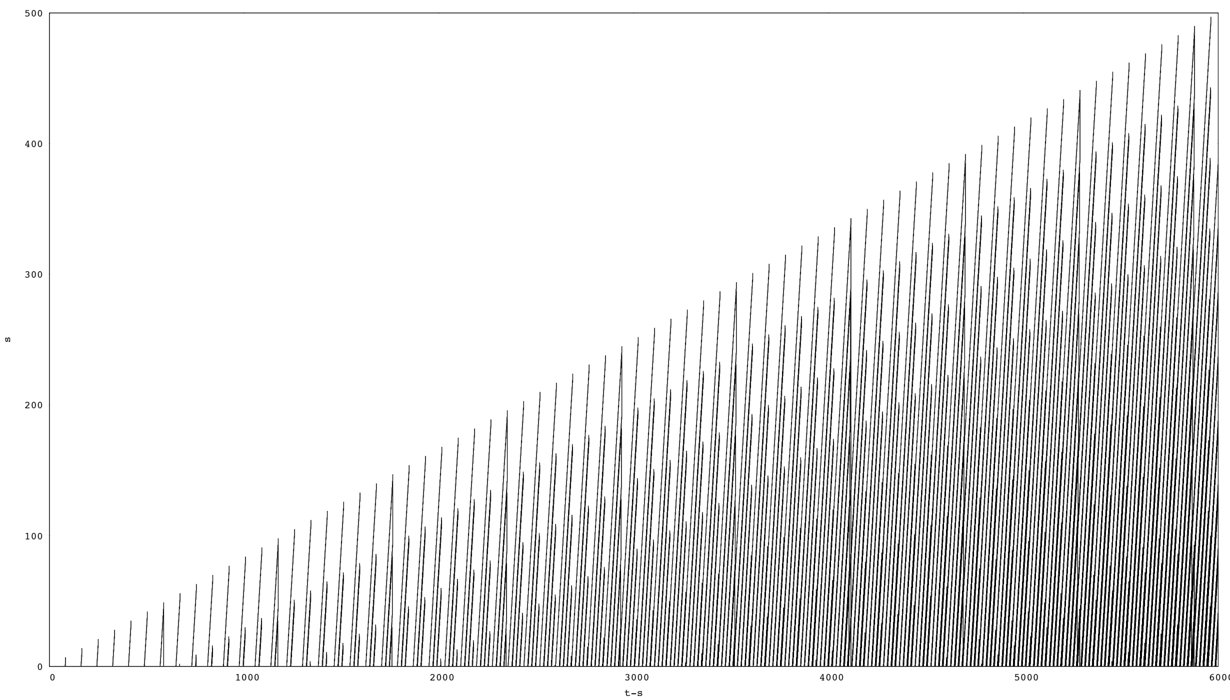

Figure 1: A portion of $E_2^{s,t}$(LASS-∞) when $p = 7$

1 Outline of the approach taken

Before stating some of our results we outline the approach taken.

The spectral sequence we use to compute the homotopy of $v_1^{-1}S/p^\infty$ has an involved construction. It is a direct limit of localized modified Adams spectral sequences. There are many spectral sequences in our account and this spectral sequence is referred to as the *localized Adams spectral sequence for the v_1-periodic sphere* $v_1^{-1}S/p^\infty$, LASS-∞ for short.

While setting up the spectral sequence, we identify its E_2-page algebraically. Our main tool for computing this E_2-page is the q_1^{-1}-*Bockstein spectral sequence* (q_1^{-1}-BSS). Miller [16] had previously made a conjecture about a family of differentials in this spectral sequence. We prove his conjecture and then use the multiplicative structure of the spectral sequences to determine the entire spectral sequence.

$$E_1(q_1^{-1}\text{-BSS}) \Longrightarrow E_2(\text{LASS-}\infty) \Longrightarrow \pi_*(v_1^{-1}S/p^\infty)$$

When we compute the LASS-∞ a miracle happens: the E_2-page is enormous (see figure 1 for when $p = 7$), whereas the E_3-page is modest in size. In fact, the E_3-page is small enough that we can deduce the rest of the spectral sequence using knowledge of the image of J.[1] Thus, once we have determined the E_2-page, the remainder of the work lies in computing the E_3-page. We use two more spectral sequences. They are more straightforward to set up than the LASS-∞, although the existence of what we call the *Mahowald spectral seqence* (MAHSS) relies on work of Miller in [17].

[1] $v_1^{-1}S/p^\infty$ is also known as M_1S^0, the first monochromatic component of S^0 (see [21, 5.7, 5.11]) and $\pi_*(M_1S^0)$ is well-known (see, for example, the proof of theorem 8.10(b) in [21]).

E_3(LASS-∞) is given by the homology of (E_2(LASS-∞), d_2) and E_2(LASS-∞) has the filtration associated with the q_1^{-1}-BSS. Since the differential d_2 respects this filtration we obtain the *q_0-filtration spectral sequence* (q_0-FILT); it has E_1-page given by $H(E_\infty(q_1^{-1}\text{-BSS}), \overline{d_2})$. We find that E_1(q_0-FILT) is a good approximation to E_3(LASS-∞) and the bulk of the work is in computing this object. For this we use the MAHSS which has as its E_1-page, the E_∞-page of the q_1^{-1}-BSS. It degenerates at the E_2-page and we compute this using our complete understanding of E_∞(q_1^{-1}-BSS) and the work of Miller in [17].

$$\begin{aligned} & E_1(\text{MAH}) \Longrightarrow E_1(q_0\text{-FILT}) \Longrightarrow E_3(\text{LASS-}\infty) \\ = \; & E_\infty(q_1^{-1}\text{-BSS}) \Longrightarrow H(E_\infty(q_1^{-1}\text{-BSS}), \overline{d_2}) \Longrightarrow H(E_2(\text{LASS-}\infty), d_2) \end{aligned}$$

Finally, we have the *Adams spectral sequence for the sphere* (ASS-S^0), the *modified Adams spectral sequence for the Prüfer sphere* S/p^∞ (MASS-∞) and the zig-zag of maps (0.2) gives rise to a zig-zag of maps of spectral sequences. We identify these maps at the E_2-pages algebraically and show that they are isomorphisms in a range. In this way our computation gives us a very good understanding of the classical Adams spectral sequence for S^0 above a line of slope $1/(p^2 - p - 1)$.

$$E_2(\text{ASS-}S^0) \longleftarrow E_2(\text{MASS-}\infty) \longrightarrow E_2(\text{LASS-}\infty)$$

Algebraically these maps fit into the "chromatic spectral sequence" as set up in [15, section 5] and we note that we have performed the first part of the program set up there.

2 Main results

We state our main results.

As with all Bockstein spectral sequences the q_1^{-1}-BSS has an E_1-page which consists of multiple copies of one algebraic object. In this case, it is an $\mathbb{F}_p$-algebra which shows up in [15]. We have

$$E_1(q_1^{-1}) = \bigoplus_{v<0} \left[\mathbb{F}_p \left[q_1, q_1^{-1}\right] \otimes E[h_{i,0} : i > 0] \otimes \mathbb{F}_p[b_{i,0} : i > 0] \right].$$

As is normal practice with Bockstein spectral sequences, we write $d_r x = y$ to denote a *family* of differentials indexed by v. It is explained later (I.3.3.1), precisely what we mean by this.

Notation 2.1. Let $p^{[0]} = 0$ and $p^{[i]} = \frac{p^i - 1}{p - 1}$ for $i \geq 1$. Write $[\![k]\!]$ for q_1^k, h_i for $h_{i,0}$ and b_i for $b_{i,0}$. If x is a nonzero scalar multiple of y write $x \doteq y$.

Suppose given $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$ and $K = (k_1, \ldots, k_r)$ such that $i_1 > \ldots > i_r \geq 1$, $j_1 > \ldots > j_s \geq 1$ and $k_a \geq 0$ for $a \in \{1, \ldots, r\}$. Write

1. $b_I^K h_J$ for the monomial $b_{i_1}^{k_1} \cdots b_{i_r}^{k_r} h_{j_1} \cdots h_{j_s}$;
2. $N_{I,J,K}$ for $\sum_a k_a(1 - p^{[i_a+1]}) - \sum_c p^{[j_c - 1]}$;
3. $J - 1$ for $(j_1, \ldots, j_{s-1})$ if $s \geq 1$;
4. $K - 1$ for $(k_1, \ldots, k_r - 1)$ if $r \geq 1$ and $k_r \geq 1$.

All the non-trivial differentials in the q_1^{-1}-BSS are described by the following theorem.

Theorem 2.2. *Suppose that we are given $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$ and $K = (k_1, \ldots, k_r)$ such that $i_1 > \ldots > i_r \geq 1$, $j_1 > \ldots > j_s \geq 1$ and $k_a \geq 1$ for $a \in \{1, \ldots, r\}$.*

Suppose $s \geq 1$, that either $r = 0$ or $r \geq 1$ and $i_r \geq j_s$, and that $p \nmid k \in \mathbb{Z}$. We have

$$d_{p^{[j_s]}} [\![kp^{j_s-1}]\!] \Big[[\![N_{I,J-1,K}]\!] b_I^K h_{J-1} \Big] \doteq [\![(k-1)p^{j_s-1}]\!] \Big[[\![N_{I,J,K}]\!] b_I^K h_J \Big]. \tag{2.3}$$

Suppose $r \geq 1$, that either $s = 0$ or $s \geq 1$ and $i_r < j_s$ and that $k \in \mathbb{Z}$. We have

$$d_{p^{i_r}-1} [\![kp^{i_r}]\!] \Big[[\![-p^{[i_r]}]\!] h_{i_r} \Big] \Big[[\![N_{I,J,K-1}]\!] b_I^{K-1} h_J \Big] \doteq [\![kp^{i_r}]\!] \Big[[\![N_{I,J,K}]\!] b_I^K h_J \Big]. \tag{2.4}$$

From this, we obtain an explicit description of $E_\infty(q_1^{-1}\text{-BSS})$.

Corollary 2.5. *$E_\infty(q_1^{-1}$-BSS) has $\mathbb{F}_p$-basis*

$$\begin{aligned} &\{\langle 1 \rangle_v : v < 0\} \\ &\cup \left\{ \left\langle [\![kp^{j_s-1}]\!] \Big[[\![N_{I,J-1,K}]\!] b_I^K h_{J-1} \Big] \right\rangle_v : \ I, J, K, k \textit{ satisfy the conditions in } (2.3), \ -p^{[j_s]} \leq v < 0 \right\} \\ &\cup \left\{ \left\langle [\![kp^{i_r}]\!] \Big[[\![-p^{[i_r]}]\!] h_{i_r} \Big] \Big[[\![N_{I,J,K-1}]\!] b_I^{K-1} h_J \Big] \right\rangle_v : \ I, J, K, k \textit{ satisfy } (2.4), \ 1 - p^{i_r} \leq v < 0 \right\}. \end{aligned}$$

Recall that $E_1(\text{MAH}) = E_\infty(q_1^{-1}\text{-BSS})$. The following theorem essentially computes E_3(MASS-∞).

Theorem 2.6. *E_2(MAH) has $\mathbb{F}_p$-basis*

$$\begin{aligned} \{\langle 1 \rangle_v : v < 0\} &\cup \left\{ \left\langle [\![kp^{j-1}]\!] \right\rangle_v : \ p \nmid k \in \mathbb{Z}, \ j \geq 1, \ -p^{[j]} \leq v < 0 \right\} \\ &\cup \left\{ \left\langle [\![kp^{i}]\!] \Big[[\![-p^{[i]}]\!] h_i \Big] \right\rangle_v : \ k \in \mathbb{Z}, \ i \geq 1, \ 1 - p^i \leq v < 0 \right\}. \end{aligned}$$

As we explain in the bulk of the text the MAHSS degenerates at the E_2-page for degree reasons. Running the q_0-FILT gives a complete description of the E_3(LASS-∞).

Corollary 2.7. *$E_\infty(q_0$-FILT) has an $\mathbb{F}_p$-basis given by*

$$\begin{aligned} \{\langle 1 \rangle_v : v < 0\} &\cup \left\{ \left\langle [\![kp^{j-1}]\!] \right\rangle_v : \ p \nmid k \in \mathbb{Z}, \ j \geq 1, \ -p^{[j-1]} - 1 \leq v < 0 \right\} \\ &\cup \left\{ \left\langle [\![kp^{i+1}]\!] \Big[[\![-p^{[i]}]\!] h_i \Big] \right\rangle_v : \ k \in \mathbb{Z}, \ i \geq 1, \ 1 - p^i \leq v < 0 \right\}, \end{aligned}$$

where we have abused notation and written the elements detecting the basis in the MAHSS.

3 Outline of thesis

This thesis is divided into three chapters.

The heart of the computation lies in the algebra, systematically identifying the E_2-page of the LASS-∞. After proving Miller's conjecture [16], writing down all the nontrivial differentials in the q_1^{-1}-BSS is a matter of combinatorics (theorem 2.2). Conceptually, the remaining theorems and corollaries follow quickly although there are many details to address.

The first chapter deals with the algebraic part of the computation and highlights what makes the computation work out. The second is devoted to setting up the topological spectral sequences used and verifying various properties of them. This becomes a technical affair and to make the exposition cleaner the most technical results are delayed until the third chapter.

Two very different methods are employed to obtain differentials in the q_1^{-1}-BSS. There are two classes of differentials in Miller's conjecture. The first is tackled head on. We compute, almost explicitly, the zig-zags that determine the differentials. The second class is obtained from the first by a Kudo transgression formula after transferring to a related spectral sequence (the $Q(0)$-BSS).

Chapter I proceeds by introducing relevant notation, setting up a convenient language to talk about spectral sequences and then setting up various Bockstein spectral sequences and examining their structure. We investigate the localization map which is used in obtaining information about the ASS-S^0 and three sections are devoted to the computation resulting in theorem 2.2. In section I.6, we make the observation that is crucial for proving theorem 2.6. The final section makes note of some differentials in the $Q(0)$-BSS which will be useful for future work on the ASS-S^0.

The bulk of Chapter II consists of setting up, one by one, all the Adams spectral sequences that we need and proving the properties that they have. Then we set up the q_0-FILT and the MAHSS and use them to complete the computation of the LASS-∞. We also use this result to deduce some information about the ASS-S^0. Chapter III deals with the most technical results which are omitted in chapter II.

Many of the spectral sequences appearing are known to experts in the field and some might not blink an eye if the material of chapter III were omitted. However, there seem to be gaps in the literature. Two examples stand out which give the motivation for chapter III and we explain them briefly.

In chapter II we set up a *modified Adams spectral sequence for* S/p^n (MASS-n), its E_2-page has an algebraic description as $\mathrm{Cotor}_P(\mathbb{F}_p, Q(0)/q_0^n)$ and it converges to the homotopy of S/p^n. There is a permanent cycle in the MASS-$(n+1)$

$$q_1^{p^n} \in \mathrm{Cotor}_P(\mathbb{F}_p, Q(0)/q_0^{n+1})$$

which detects an element $S^0 \longrightarrow \Sigma^{-p^n q} S/p^{n+1}$ such that the induced map $S/p^{n+1} \longrightarrow \Sigma^{-p^n q} S/p^{n+1}$ is a K-theory isomorphism. During the verfication of this result we realise that such a K-theory isomorphism can only have (unmodified) Adams filtration at most $p^n - n$. However, it is claimed in [8, page 156] that it has Adams filtration p^n. This suggests that the result above had not been considered carefully in the literature before now.

In constructing the MASS-n we make use of the "smash product of resolutions" defined in [5, chapter IV, definition 4.2]. We need some degree of functoriality of this construction to obtain the multiplicative structure of the MASS-n. It seems that a similar result is required in [3, proposition 3.2] although this issue is not addressed there. It is resolved, to some extent, in section 3 of [23]. However, it is more satisfying to have a functorial smash product on the category of towers in the stable homotopy category. This is what we construct.

We remark that our results are not necessarily presented in the order in which we prove them. The reader is assured that there are no circular arguments. Referenced results in the same chapter have their chapter number omitted; otherwise, we include the chapter number for clarity.

Chapter I

Algebra

This chapter contains the main ideas behind all of the results stated in the introduction. In chapter II we construct the *localized Adams spectral sequence for the* v_1*-periodic sphere* $v_1^{-1}S/p^\infty$ (LASS-∞) and we find it has E_2-page given by

$$\mathrm{Cotor}_P(\mathbb{F}_p, q_1^{-1}Q(0)/q_0^\infty).$$

The main goal of this chapter is to understand a spectral sequence which computes this object.

1 Introducing notation

First, we must introduce the relevant notation.

Recall that throughout this thesis p is an odd prime. All Hopf algebra and comodules have $\mathbb{F}_p$ as their ground field.

Definition 1.1. Let P denote the polynomial algebra on generators $\{\xi_n : n \geq 1\}$ where $|\xi_n| = (0, 2(p^n - 1))$. P is a Hopf algebra when equipped with the Milnor diagonal

$$P \longrightarrow P \otimes P, \quad \xi_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i, \quad (\xi_0 = 1).$$

Definition 1.2. Let $Q(0)$ denote the polynomial algebra on generators $\{q_n : n \geq 0\}$ where $|q_n| = (1, 2(p^n - 1))$. $Q(0)$ is an algebra in P-comodules when equipped with the coaction map

$$Q(0) \longrightarrow P \otimes Q(0), \quad q_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes q_i.$$

Note that the multiplication on $Q(0)$ is commutative; it is graded commutative with respect to the second grading. It is always understood that graded commutative ignores the first grading, which Miller [15] refers to as the "Cartan degree".

Definition 1.3. $q_0 \in Q(0)$ is a comodule primitive and so we may define $Q(1)$ via the following short exact sequence of P-comodules. $Q(1)$ is an algebra in P-comodules.

$$0 \longrightarrow Q(0) \xrightarrow{q_0} Q(0) \longrightarrow Q(1) \longrightarrow 0$$

Definition 1.4. Define $Q(0)/q_0^\infty$ by the following short exact sequence of P-comodules. $Q(0)/q_0^\infty$ is a $Q(0)$-module in P-comodules.

$$0 \longrightarrow Q(0) \longrightarrow q_0^{-1}Q(0) \longrightarrow Q(0)/q_0^\infty \longrightarrow 0$$

We find that $q_1 \in Q(1)$ is a comodule primitive so we may define $q_1^{-1}Q(1)$ which is an algebra in P-comodules. We may also define $q_1^{-1}Q(0)/q_0^\infty$, a $Q(0)$-module in P-comodules but this requires a more sophisticated construction, which we now outline.

Definition 1.5. For $k \geq 1$, M_k is the sub-P-comodule of $Q(0)/q_0^\infty$ defined by the following short exact sequence of P-comodules. M_k is a $Q(0)$-module in P-comodules.

$$0 \longrightarrow Q(0) \longrightarrow Q(0)\langle q_0^{-k}\rangle \longrightarrow M_k \longrightarrow 0.$$

Lemma 1.6. $q_1^{p^{k-1}} : M_k \longrightarrow M_k$ *is a homomorphism of* $Q(0)$*-modules in* P*-comodules.*

Definition 1.7. For each $l \geq 0$ let $M_k(l) = M_k$. $q_1^{-1}M_k$ is defined to be the colimit of the following diagram.

$$M_k(0) \xrightarrow{q_1^{p^{k-1}}} M_k(1) \xrightarrow{q_1^{p^{k-1}}} M_k(2) \xrightarrow{q_1^{p^{k-1}}} M_k(3) \xrightarrow{q_1^{p^{k-1}}} \dots$$

Definition 1.8. We have homomorphisms $q_1^{-1}M_k \longrightarrow q_1^{-1}M_{k+1}$ induced by the inclusions $M_k \longrightarrow M_{k+1}$. $q_1^{-1}Q(0)/q_0^\infty$ is defined to be the colimit of following diagram.

$$q_1^{-1}M_1 \longrightarrow q_1^{-1}M_2 \longrightarrow q_1^{-1}M_3 \longrightarrow q_1^{-1}M_4 \longrightarrow \dots$$

Notation 1.9. If Q is a P-comodule then we write $\Omega(P;Q)$ for the cobar construction on P with coefficients in Q. In particular, we have

$$\Omega^s(P;Q) = \overline{P}^{\otimes s} \otimes Q$$

where $P = \mathbb{F}_p \oplus \overline{P}$ as $\mathbb{F}_p$-modules and we write $[p_1|\dots|p_s]q$ for $p_1 \otimes \dots \otimes p_s \otimes q$. We set $\Omega P = \Omega(P;\mathbb{F}_p)$.

We recall (see [15, page 75]) that the differentials are given by an alternating sum making use of the diagonal and coaction maps. We also recall that if Q is an algebra in P-comodules then $\Omega(P;Q)$ is a DG-$\mathbb{F}_p$-algebra and if Q' is a Q-module in P-comodules then $\Omega(P;Q')$ is a DG-$\Omega(P;Q)$-module.

Definition 1.10. If Q is a P-comodule then $\mathrm{Cotor}_P(\mathbb{F}_p, Q) = H^*(\Omega(P;Q))$.

Notation 1.11. We abbreviate $\mathrm{Cotor}_P(\mathbb{F}_p, Q)$ and write $\mathrm{Cotor}_P(Q)$ instead. We use this notation for the rest of the thesis.

We remark that in our setting Cotor has three gradings. P and $Q(0)$ are bigraded and we write (t,u) for the bigrading, which is preserved by the differentials in the cobar complex, $\Omega(P;Q(0))$. Thus, $\mathrm{Cotor}_P(\mathbb{F}_p, Q(0))$ has the cohomological grading s as well as the gradings t and u. The same is true for $Q(1)$, $q_1^{-1}Q(1)$, $Q(0)/q_0^\infty$ and $q_1^{-1}Q(0)/q_0^\infty$ in place of $Q(0)$.

2 Spectral sequence terminology

Spectral sequences are used in abundance throughout this thesis. The purpose of this short section is to fix some potentially unconventional terminology which is used in subsequent sections.

The reader is probably familiar with the notion of an exact couple which is one of the most common ways in which a spectral sequence arises.

Definition 2.1. An *exact couple* consists of abelian groups A and E together with homomorphisms i, j and k such that the following triangle is exact:

$$\begin{tikzcd}
A \arrow[d, "j"'] & A \arrow[l, "i"'] \\
E \arrow[ur, "k"'] &
\end{tikzcd}$$

Given an exact couple one can form the associated derived exact couple; iterating this process gives rise to a spectral sequence. An alternative approach, more useful for what we have in mind, exploits correspondences. We find that the picture becomes clearer, especially once gradings are introduced, when we 'spread out' the exact couple:

$$\begin{tikzcd}
\cdots & A \arrow[l] \arrow[d] & A \arrow[l] & \cdots \arrow[l, "i"'] & A \arrow[l, "i"'] \arrow[d, "j"] & \cdots \arrow[l] \\
& E \arrow[ur, "k"'] & & & E &
\end{tikzcd}$$

Let $\pi : E \times A \times A \times E \longrightarrow E \times E$ be the projection map. Then we make the following definitions.

Definition 2.2. For each $r \geq 1$ let $\widetilde{d_r} = \{(x, \tilde{x}, \tilde{y}, y) \in E \times A \times A \times E : kx = \tilde{x} = i^{r-1}\tilde{y} \text{ and } j\tilde{y} = y\}$ and $d_r = \pi(\widetilde{d_r})$.

$$\begin{tikzcd}
& \tilde{x} & \cdots \arrow[l, mapsto, "i"'] & \tilde{y} \arrow[l, mapsto, "i"'] \arrow[d, mapsto, "j"] \\
x \arrow[ur, mapsto, "k"'] & & & y
\end{tikzcd}$$

Since i, j, k and π are homomorphisms of abelian groups $\widetilde{d_r}$ and d_r are subgroups of $E \times A \times A \times E$ and $E \times E$, respectively.

Notation 2.3. We write $d_r x = y$ if $(x, y) \in d_r$.

We see that d_1 is the function jk. We have the convention that d_0 is the zero function (so that $\ker d_0 = E$ and $\operatorname{im} d_0 = 0$). We also have the following useful observations.

Lemma 2.4. *For $r > 1$, $d_r x$ is defined if and only if $d_{r-1}x = 0$, i.e.*

$$(x, 0) \in d_{r-1} \iff \exists y : (x, y) \in d_r.$$

Lemma 2.5. *For $r > 1$, $d_r 0 = y$ if and only if there exists an x with $d_{r-1}x = y$, i.e.*

$$(0, y) \in d_r \iff \exists x : (x, y) \in d_{r-1}.$$

Corollary 2.6. *For $r > 1$, the following conditions are equivalent:*

1. *$d_r x = y$ and $d_r x = y'$;*
2. *$d_r x = y$ and there exists an x' with $d_{r-1}x' = y' - y$.*

Lemma 2.7. *Suppose $r \geq 1$ and that $d_r x = y$. Then $d_s y = 0$ for any $s \geq 1$.*

For all $r \geq 1$ we see that d_r is a correspondence. We can define kernels and images of correspondences. The preceding lemmas show that d_r defines a homomorphism

$$\ker d_{r-1}/\bigcup_s \operatorname{im} d_s \longrightarrow \bigcap_s \ker d_s/\operatorname{im} d_{r-1}.$$

Let $E_r = \ker d_{r-1}/\operatorname{im} d_{r-1}$. Then precomposing by $E_r \longrightarrow \ker d_{r-1}/\bigcup_s \operatorname{im} d_s$ and postcomposing by $\bigcap_s \ker d_s/\operatorname{im} d_{r-1} \longrightarrow E_r$ gives a homomorphism $E_r \longrightarrow E_r$. This is usually how d_r is defined.

In this chapter we use the correspondence perspective on d_r most frequently. We use the more common perspective that $d_r : E_r \to E_r$ and $E_{r+1} = H(E_r, d_r)$ more frequently in the other chapters.

Definition 2.8. Suppose $d_r x = y$; then x is a said to *support* a d_r. If, in addition, $y \notin \operatorname{im} d_{r-1}$, x is said to support a nontrivial differential. Elements of $\bigcap_s \ker d_s$ are called *permanent cycles.*

Definition 2.9. We write E_∞ for $\bigcap_s \ker d_s/\bigcup_s \operatorname{im} d_s$.

3 Bockstein spectral sequences

In this section we set up the q_1^{-1}*-Bockstein spectral sequence* (definition 3.4.1). One of the main goals of this chapter is to compute this spectral sequence completely. This computation makes use of the multiplicative structure that this spectral sequence has and we transfer differentials between it and other related spectral sequences. This section sets up all the Bockstein spectral sequences which we use and proves the various properties that we require of them.

3.1 The $Q(0)$-Bockstein

Applying $\operatorname{Cotor}_P(-)$ to the short exact sequence of P-comodules

$$0 \longrightarrow Q(0) \xrightarrow{q_0} Q(0) \longrightarrow Q(1) \longrightarrow 0$$

gives a long exact sequence. We also have a trivial long exact sequence consisting of the zero group every three terms and $\operatorname{Cotor}_P(Q(0))$ elsewhere. Intertwining these long exact sequences gives an exact couple:

$$\begin{tikzcd}
\operatorname{Cotor}_P^{s,t-v,u}(Q(0)) \arrow[d] & \operatorname{Cotor}_P^{s,t-v-1,u}(Q(0)) \arrow[l] & \dots \arrow[l, "q_0"'] & \operatorname{Cotor}_P^{s,t-v-r,u}(Q(0)) \arrow[l, "q_0"'] \arrow[d] \\
\operatorname{Cotor}_P^{s,t-v,u}(Q(1))_v \arrow[ur, dotted, "\partial"'] & & & \operatorname{Cotor}_P^{s,t-v-r,u}(Q(1))_{v+r}.
\end{tikzcd}$$

Here ∂ raises the degree of s by one relative to what is indicated and the subscripts on the copies of $\operatorname{Cotor}_P(Q(1))$ are used to distinguish them from one another.

Definition 3.1.1. The spectral sequence arising from this exact couple is called the *$Q(0)$-Bockstein spectral sequence* ($Q(0)$-BSS). It has E_1-page

$$E_1^{s,t,u,v}(Q(0)) = \begin{cases} \mathrm{Cotor}_P^{s,t-v,u}(Q(1))_v & \text{if } v \geq 0 \\ 0 & \text{if } v < 0 \end{cases}$$

and d_r has degree $(1, 0, 0, r)$. The spectral sequence converges to $\mathrm{Cotor}_P(Q(0))$ and the filtration degree is given by v. In particular, we have an identification

$$E_\infty^{s,t,u,v}(Q(0)) = F^v \,\mathrm{Cotor}_P^{s,t,u}(Q(0))/F^{v+1}\,\mathrm{Cotor}_P^{s,t,u}(Q(0))$$

where $F^v \,\mathrm{Cotor}_P(Q(0)) = \mathrm{im}(q_0^v : \mathrm{Cotor}_P(Q(0)) \longrightarrow \mathrm{Cotor}_P(Q(0)))$ for $v \geq 0$. The identification is given by lifting an element of $F^v \,\mathrm{Cotor}_P(Q(0))$ to the v^{th} copy of $\mathrm{Cotor}_P(Q(0))$ and mapping this lift down to $\mathrm{Cotor}_P(Q(1))$ to give a permanent cycle.

3.2 The q_0^∞-Bockstein

Applying $\mathrm{Cotor}_P(-)$ to the short exact sequence of P-comodules

$$0 \longrightarrow Q(1) \longrightarrow Q(0)/q_0^\infty \xrightarrow{q_0} Q(0)/q_0^\infty \longrightarrow 0 \tag{3.2.1}$$

gives a long exact sequence. We also have a trivial long exact sequence consisting of the zero group every three terms and $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$ elsewhere. Intertwining these long exact sequences gives an exact couple:

$$\begin{tikzcd}
\mathrm{Cotor}_P^{s,t-v+r-1,u}(Q(0)/q_0^\infty) & \dots \arrow[l, "q_0"'] & \mathrm{Cotor}_P^{s,t-v,u}(Q(0)/q_0^\infty) \arrow[l, "q_0"'] \arrow[dr, dotted, "\partial"] & \mathrm{Cotor}_P^{s,t-v-1,u}(Q(0)/q_0^\infty) \arrow[l] \\
\mathrm{Cotor}_P^{s,t-v+r,u}(Q(1))_{v-r} \arrow[u] & & & \mathrm{Cotor}_P^{s,t-v,u}(Q(1))_v. \arrow[u]
\end{tikzcd}$$

Here ∂ raises the degree of s by one relative to what is indicated.

Definition 3.2.2. The spectral sequence arising from this exact couple is called the *q_0^∞-Bockstein spectral sequence* (q_0^∞-BSS). It has E_1-page

$$E_1^{s,t,u,v}(q_0^\infty) = \begin{cases} \mathrm{Cotor}_P^{s,t-v,u}(Q(1))_v & \text{if } v < 0 \\ 0 & \text{if } v \geq 0 \end{cases}$$

and d_r has degree $(1, 0, 0, r)$. The spectral sequence converges to $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$ and the filtration degree is given by v. In particular, we have an identification

$$E_\infty^{s,t,u,v}(q_0^\infty) = F^v \,\mathrm{Cotor}_P^{s,t,u}(Q(0)/q_0^\infty)/F^{v+1}\,\mathrm{Cotor}_P^{s,t,u}(Q(0)/q_0^\infty)$$

where $F^v \,\mathrm{Cotor}_P(Q(0)/q_0^\infty) = \ker(q_0^{-v} : \mathrm{Cotor}_P(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P(Q(0)/q_0^\infty))$ for $v \leq 0$. The identification is given by taking a permanent cycle in the v^{th} copy of $\mathrm{Cotor}_P(Q(1))$, mapping it up to $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$ and pulling this element back to the $(-1)^{\text{th}}$ copy of $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$.

3.3 Conventions and a relationship between the $Q(0)$-BSS and the q_0^∞-BSS

The two Bockstein spectral sequences described so far and every other Bockstein spectral sequence we use have the feature that infinite families of differentials are determined by one differential. This feature allows us to omit the v grading in a systematic way. We illustrate this phenomenum with the following definition.

Definition 3.3.1. Suppose $x \in \mathrm{Cotor}_P^{s,t,u}(Q(1))$, $y \in \mathrm{Cotor}_P^{s+1,t-r,u}(Q(1))$ and we say that $d_r x = y$. This statement has precise interpretations in the $Q(0)$-BSS and the q_0^∞-BSS.

1. In the $Q(0)$-BSS this statement encodes the fact that for any $w \geq 0$ we may view

$$x \in \mathrm{Cotor}_P^{s,t,u}(Q(1))_w, \quad y \in \mathrm{Cotor}_P^{s+1,t-r,u}(Q(1))_{w+r}$$

   and in each case we have $d_r x = y$.

2. In the q_0^∞-BSS this statement encodes the fact that for any $w \leq -1$ we may view

$$x \in \mathrm{Cotor}_P^{s,t,u}(Q(1))_{w-r}, \quad y \in \mathrm{Cotor}_P^{s+1,t-r,u}(Q(1))_w$$

   and in each case we have $d_r x = y$.

It appears, a priori, that the truth of the statement $d_r x = y$ depends on which spectral sequence we are working in. However, we have the following lemma.

Lemma 3.3.2. *Suppose $x \in \mathrm{Cotor}_P^{s,t,u}(Q(1))$ and $y \in \mathrm{Cotor}_P^{s+1,t-r,u}(Q(1))$. Then $d_r x = y$ in the $Q(0)$-BSS if and only if $d_r x = y$ in the q_0^∞-BSS.*

Proof. There is a conceptual proof of this fact using Verdier's axiom because the $Q(0)$-BSS and the q_0^∞-BSS are both truncations of a spectral sequence converging to $q_0^{-1}\mathrm{Cotor}_P(Q(0))$. We provide a more direct proof.

Suppose that $d_r x = y$ in the $Q(0)$-BSS. By definition 2.2 there exist $\tilde{x}$ and $\tilde{y}$ fitting into the following diagram.

$$\begin{tikzcd}
 & \mathrm{Cotor}_P^{s+1,t-1,u}(Q(0)) & \dots \arrow[l, "q_0"'] & \mathrm{Cotor}_P^{s+1,t-r,u}(Q(0)) \arrow[l, "q_0"'] \arrow[d] \\
\mathrm{Cotor}_P^{s,t,u}(Q(1)) \arrow[ur, "\partial"'] & & & \mathrm{Cotor}_P^{s+1,t-r,u}(Q(1))
\end{tikzcd}$$

$$\begin{tikzcd}
 & \tilde{x} & \dots \arrow[l, mapsto, "q_0"'] & \tilde{y} \arrow[l, mapsto, "q_0"'] \arrow[d, mapsto] \\
x \arrow[ur, mapsto, "\partial"'] & & & y
\end{tikzcd}$$

Let $a \in \Omega(P; Q(1))$ be a representative for x and $\tilde{b} \in \Omega(P; Q(0))$ be a representative $\tilde{y}$. There exists an $\tilde{a} \in \Omega(P;Q(0))$ representing $\tilde{x}$, and an a'' and α fitting into the following diagram.

$$\begin{tikzcd}
\Omega(P;Q(0)) \arrow[r, "q_0"] \arrow[d] & \Omega(P;Q(0)) \arrow[r] \arrow[d, "d"] & \Omega(P;Q(1)) \arrow[d] \\
\Omega(P;Q(0)) \arrow[r, "q_0"] & \Omega(P;Q(0)) \arrow[r] & \Omega(P;Q(1))
\end{tikzcd}$$

$$\begin{tikzcd}
 & a'' \arrow[r, mapsto] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
\tilde{a} \arrow[r, mapsto] & \alpha \arrow[r, mapsto] & 0
\end{tikzcd}$$

There exists $c \in \Omega(P;Q(0))$ such that $\tilde{a} = q_0^{r-1}\tilde{b} + dc$. Let $a' = a'' - q_0 c$. Then

$$\begin{tikzcd}
 & a' \arrow[r, mapsto] \arrow[d, mapsto] & a \arrow[d, mapsto] \\
q_0^{r-1}\tilde{b} \arrow[r, mapsto] & \alpha - q_0 dc \arrow[r, mapsto] & 0.
\end{tikzcd}$$

In particular, $a' \in \Omega(P;Q(0))$ gives a lift of $a \in \Omega(P;Q(1))$ and $da' = q_0^r\tilde{b}$. Let b be the image of $\tilde{b}$ in $\Omega(P;Q(1))$. Then we have

$$\begin{tikzcd}
\Omega(P;Q(1)) \arrow[r] \arrow[d] & \Omega(P;Q(0)/q_0^\infty) \arrow[r, "q_0"] \arrow[d, "d"] & \Omega(P;Q(0)/q_0^\infty) \arrow[d] \\
\Omega(P;Q(1)) \arrow[r] & \Omega(P;Q(0)/q_0^\infty) \arrow[r, "q_0"] & \Omega(P;Q(0)/q_0^\infty)
\end{tikzcd}$$

$$\begin{tikzcd}
 & a'/q_0^{r+1} \arrow[r, mapsto] \arrow[d, mapsto] & a'/q_0^r \arrow[d, mapsto] \\
b \arrow[r, mapsto] & \tilde{b}/q_0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Cotor}_P^{s,t-1,u}(Q(0)/q_0^\infty) & \dots \arrow[l, "q_0"'] & \mathrm{Cotor}_P^{s,t-r,u}(Q(0)/q_0^\infty) \arrow[l, "q_0"'] \arrow[dr, "\partial"] & \\
\mathrm{Cotor}_P^{s,t,u}(Q(1)) \arrow[u] & & & \mathrm{Cotor}_P^{s+1,t-r,u}(Q(1))
\end{tikzcd}$$

$$\begin{tikzcd}
\{a'/q_0\} & \dots \arrow[l, mapsto, "q_0"'] & \{a'/q_0^r\} \arrow[l, mapsto, "q_0"'] \arrow[dr, mapsto, "\partial"] & \\
x \arrow[u, mapsto] & & & y
\end{tikzcd}$$

so that $d_r x = y$ in the q_0^∞-BSS.

We prove the converse using induction on r. The following map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Q(0) \arrow[r, "q_0"] \arrow[d, "="] & Q(0) \arrow[r] \arrow[d, "/q_0"] & Q(1) \arrow[r] \arrow[d, "/q_0"] & 0 \\
0 \arrow[r] & Q(0) \arrow[r] \arrow[d] & q_0^{-1}Q(0) \arrow[r] \arrow[d, "/q_0"] & Q(0)/q_0^{\infty} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & Q(1) \arrow[r, "/q_0"] & Q(0)/q_0^{\infty} \arrow[r, "q_0"] & Q(0)/q_0^{\infty} \arrow[r] & 0
\end{tikzcd}$$

gives a commuting square

$$\begin{tikzcd}
\mathrm{Cotor}_P^{s,t,u}(Q(1)) \arrow[r, "\partial"] \arrow[d] & \mathrm{Cotor}_P^{s+1,t-1,u}(Q(0)) \arrow[d] \\
\mathrm{Cotor}_P^{s,t-1,u}(Q(0)/q_0^{\infty}) \arrow[r, "\partial"] & \mathrm{Cotor}^{s+1,t-1,u}(Q(1)),
\end{tikzcd}$$

which proves the result for $r = 1$. For $r > 1$ we have

$$\begin{array}{rrclrllr}
 & d_r x & = & y & \text{in the} & q_0^{\infty}\text{-BSS} & & \\
\Longrightarrow & d_{r-1} x & = & 0 & \text{in the} & q_0^{\infty}\text{-BSS} & & \text{(Lemma 2.4)} \\
\Longrightarrow & d_{r-1} x & = & 0 & \text{in the} & Q(0)\text{-BSS} & & \text{(Induction)} \\
\Longrightarrow & d_r x & = & y' & \text{in the} & Q(0)\text{-BSS} & \text{for some } y' & \text{(Lemma 2.4)} \\
\Longrightarrow & d_r x & = & y' & \text{in the} & q_0^{\infty}\text{-BSS} & & \text{(1}^{\text{st}}\text{ half of proof)} \\
\Longrightarrow & d_{r-1} x' & = & y' - y & \text{in the} & q_0^{\infty}\text{-BSS} & \text{for some } x' & \text{(Corollary 2.6)} \\
\Longrightarrow & d_{r-1} x' & = & y' - y & \text{in the} & Q(0)\text{-BSS} & & \text{(Induction)} \\
\Longrightarrow & d_r x & = & y & \text{in the} & Q(0)\text{-BSS} & & \text{(Corollary 2.6)}
\end{array}$$

which completes the proof. □

We extend definition 2.8.

Definition 3.3.3. Suppose $x \in \mathrm{Cotor}_P^{s,t,u}(Q(1))$, $y \in \mathrm{Cotor}_P^{s+1,t-r,u}(Q(1))$ and that $d_r x = y$. x is a said to *support* a d_r. If, in addition, $y \notin \mathrm{im}\, d_{r-1}$ (which we may check for any of the differentials in the corresponding infinite families), x is said to support a nontrivial differential.

3.4 The q_1^{-1}-Bockstein

We can mimic the construction of the q_0^{∞}-BSS using the short exact sequence of P-comodules

$$0 \longrightarrow q_1^{-1}Q(1) \longrightarrow q_1^{-1}Q(0)/q_0^{\infty} \xrightarrow{q_0} q_1^{-1}Q(0)/q_0^{\infty} \longrightarrow 0.$$

Definition 3.4.1. The spectral sequence arising from the corresponding exact couple is called the q_1^{-1}-*Bockstein spectral sequence* (q_1^{-1}-BSS). It has E_1-page

$$E_1^{s,t,u,v}(q_1^{-1}) = \begin{cases} \mathrm{Cotor}_P^{s,t-v,u}(q_1^{-1}Q(1))_v & \text{if } v < 0 \\ 0 & \text{if } v \geq 0 \end{cases}$$

and d_r has degree $(1,0,0,r)$. The spectral sequence converges to $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$ and the filtration degree is given by v. In particular, we have an identification

$$E_\infty^{s,t,u,v}(q_1^{-1}) = F^v\,\mathrm{Cotor}_P^{s,t,u}(q_1^{-1}Q(0)/q_0^\infty)/F^{v+1}\,\mathrm{Cotor}_P^{s,t,u}(q_1^{-1}Q(0)/q_0^\infty)$$

where, as in the q_0^∞-BSS, $F^v = \ker q_0^{-v}$ for $v \le 0$. The identification is given by taking a permanent cycle in the v^{th} copy of $\mathrm{Cotor}_P(q_1^{-1}Q(1))$, mapping it up to $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$ and pulling this element back to the $(-1)^{\text{th}}$ copy of $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$.

We have an evident map of spectral sequences $E_*^{*,*,*,*}(q_0^\infty) \longrightarrow E_*^{*,*,*,*}(q_1^{-1})$.

3.5 Multiplicativity of the Bockstein spectral sequences

The $Q(0)$-BSS is multiplicative because $\Omega(P;Q(0)) \longrightarrow \Omega(P;Q(1))$ is a map of DG algebras.

Lemma 3.5.1. *Suppose $d_r x = y$ and $d_r x' = y'$ in the $Q(0)$-BSS. Then*

$$d_r(xx') = yx' + (-1)^{|x|}xy'.$$

Here $|x|$ and $|y|$ denote the cohomological gradings of x and y, respectively, since every element of P, $Q(0)$ and $Q(1)$ lies in even u grading.

Proof. Suppose $d_r x = y$ and $d_r x' = y'$.

We saw in the proof of Lemma 3.3.2 that there exist $a, a', b, b' \in \Omega(P;Q(0))$ such that their images in $\Omega(P;Q(1))$ represent x, x', y, y', respectively, and such that $da = q_0^r b$, $da' = q_0^r b'$. The image of $aa' \in \Omega(P;Q(0))$ in $\Omega(P;Q(1))$ represents xx' and the image of $ba' + (-1)^{|a|}ab' \in \Omega(P;Q(0))$ in $\Omega(P;Q(1))$ represents $yx' + (-1)^{|x|}xy'$. Since $d(aa') = q_0^r(ba' + (-1)^{|a|}ab')$, the proof is complete. ☐

Corollary 3.5.2. *We have a multiplication*

$$E_1^{s,t,u,v}(Q(0)) \otimes E_1^{s',t',u',v'}(Q(0)) \longrightarrow E_1^{s+s',t+t',u+u',v+v'}(Q(0))$$

restricting to

$$\begin{tikzcd}
\ker d_r \otimes \operatorname{im} d_r \arrow[r] \arrow[d] & \operatorname{im} d_r \arrow[d] \\
\ker d_r \otimes \ker d_r \arrow[r] & \ker d_r \\
\operatorname{im} d_r \otimes \ker d_r \arrow[r] \arrow[u] & \operatorname{im} d_r \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigcap_s \ker d_s \otimes \bigcup_s \operatorname{im} d_s \arrow[r] \arrow[d] & \bigcup_s \operatorname{im} d_s \arrow[d] \\
\bigcap_s \ker d_s \otimes \bigcap_s \ker d_s \arrow[r] & \bigcap_s \ker d_s \\
\bigcup_s \operatorname{im} d_s \otimes \bigcap_s \ker d_s \arrow[r] \arrow[u] & \bigcup_s \operatorname{im} d_s \arrow[u]
\end{tikzcd}$$

Thus we have induced maps

$$E_r^{s,t,u,v}(Q(0)) \otimes E_r^{s',t',u',v'}(Q(0)) \longrightarrow E_r^{s+s',t+t',u+u',v+v'}(Q(0))$$

for $1 \le r \le \infty$. Moreover,

$$E_\infty^{s,t,u,*}(Q(0)) \otimes E_\infty^{s',t',u',*}(Q(0)) \longrightarrow E_\infty^{s+s',t+t',u+u',*}(Q(0))$$

is the associated graded of the map

$$\mathrm{Cotor}_P^{s,t,u}(Q(0)) \otimes \mathrm{Cotor}_P^{s',t',u'}(Q(0)) \longrightarrow \mathrm{Cotor}_P^{s+s',t+t',u+u'}(Q(0)).$$

Lemma 3.3.2 gives the following corollary to lemma 3.5.1.

Corollary 3.5.3. *Suppose $d_r x = y$ and $d_r x' = y'$ in the q_0^∞-BSS. Then*

$$d_r(xx') = yx' + (-1)^{|x|} xy'.$$

The q_0^∞-BSS is not multiplicative, in the sense that we do not have a strict analogue of Corollary 3.5.2. This is unsurprising because $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$ does not have an obvious algebra structure. However, we do have a pairing between the $Q(0)$-BSS and the q_0^∞-BSS converging to the $\mathrm{Cotor}_P(Q(0))$-module structure map of $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$.

An identical result to lemma 3.5.1 holds for the q_1^{-1}-BSS.

Lemma 3.5.4. *Suppose $d_r x = y$ and $d_r x' = y'$ in the q_1^{-1}-BSS. Then*

$$d_r(xx') = yx' + (-1)^{|x|} xy'.$$

Proof. Suppose that $d_r x = y$ and $d_r x' = y'$ in the q_1^{-1}-BSS. For large enough s we obtain differentials in the q_0^∞-BSS:

$$d_r q_1^{p^s} x = q_1^{p^s} y, \quad d_r q_1^{p^s} x' = q_1^{p^s} y'.$$

[One sees this using the definition of $q_1^{-1}Q(0)/q_0^\infty$ and the fact that filtered colimits commute with tensor products and homology.] By corollary 3.5.3 we have

$$d_r((q_1^{p^s} x)(q_1^{p^s} x')) = (q_1^{p^s} y)(q_1^{p^s} x') + (-1)^{|x|}(q_1^{p^s} x)(q_1^{p^s} y')$$

i.e.

$$d_r(q_1^{2p^s}(xx')) = q_1^{2p^s}(yx' + (-1)^{|x|} xy')$$

in the q_0^∞-BSS. Inspecting the proof of lemma 3.3.2 shows that this formula can be validated using elements in $\Omega(P; M_{r+1})$ (see definition 1.5). Thus, we can divide through by $q_1^{2p^s}$ to obtain

$$d_r(xx') = yx' + (-1)^{|x|} xy'$$

(as long as we chose $s \geq r$). $\square$

4 The localization map

This section investigates the localization map

$$\mathrm{Cotor}_P(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty).$$

The results that we record are useful for transferring differentials between the $Q(0)$-BSS and the q_1^{-1}-BSS. They also allows us to obtain information about the *Adams spectral sequence for the sphere* (ASS-S^0) from the LASS-∞.

4.1 The trigraded perspective

Firstly, we make use of all three of our gradings to obtain information about the localization map.

Throughout this section $s \geq 0$. Recall from [15] the definition of $U(s)$ (which will be given in II.6.1) and the following proposition.

Proposition 4.1.1 ([15, page 81])**.** *The localization map* $\mathrm{Cotor}_P^{s,t,u}(Q(1)) \longrightarrow \mathrm{Cotor}_P^{s,t,u}(q_1^{-1}Q(1))$

1. *is injective if* $u < U(s-1) + 2(p^2-1)(t+1) - 2(p-1)$*;*
2. *is surjective if* $u < U(s) + 2(p^2-1)(t+1) - 2(p-1)$*.*

This allows us to prove the following lemma which explains how we can transfer differentials between the q_0^∞-BSS and the q_1^{-1}-BSS.

Lemma 4.1.2. *Suppose* $u < U(s) + 2(p^2-1)(t+2) - 2(p-1)$ *so that proposition 4.1.1 gives a surjection* $E_1^{s,t,u,*}(q_0^\infty) \longrightarrow E_1^{s,t,u,*}(q_1^{-1})$ *and an injection* $E_1^{s+1,t,u,*}(q_0^\infty) \longrightarrow E_1^{s+1,t,u,*}(q_1^{-1})$*.*

Suppose $x \in E_1^{s,t,u,*}(q_0^\infty)$ *maps to* $\overline{x} \in E_1^{s,t,u,*}(q_1^{-1})$*,* $y \in E_1^{s+1,t,u,*}(q_1^{-1})$ *and that* $d_r\overline{x} = y$ *in the* q_1^{-1}*-BSS. Then* y *lies in* $E_1^{s+1,t,u,*}(q_0^\infty)$ *and* $d_r x = y$ *in the* q_0^∞*-BSS.*

Proof. We proceed by induction on r. The result is true in the case $r = 1$ where d_r is a function. Suppose $r > 1$. Then

$$
\begin{array}{rrcllll}
 & d_r\overline{x} & = & y & \text{in the} & q_1^{-1}\text{-BSS} & & \\
\Longrightarrow & d_{r-1}\overline{x} & = & 0 & \text{in the} & q_1^{-1}\text{-BSS} & & \text{(Lemma 2.4)} \\
\Longrightarrow & d_{r-1}x & = & 0 & \text{in the} & q_0^\infty\text{-BSS} & & \text{(Induction)} \\
\Longrightarrow & d_r x & = & y' & \text{in the} & q_0^\infty\text{-BSS} & \text{for some } y' & \text{(Lemma 2.4)} \\
\Longrightarrow & d_r\overline{x} & = & y' & \text{in the} & q_1^{-1}\text{-BSS} & & \text{(Map of SSs)} \\
\Longrightarrow & d_{r-1}\overline{x}' & = & y'-y & \text{in the} & q_1^{-1}\text{-BSS} & \text{for some } x' & \text{(Corollary 2.6)} \\
\Longrightarrow & d_{r-1}x' & = & y'-y & \text{in the} & q_0^\infty\text{-BSS} & & \text{(Induction)} \\
\Longrightarrow & d_r x & = & y & \text{in the} & q_0^\infty\text{-BSS} & & \text{(Corollary 2.6)}
\end{array}
$$

We remark that the map $E_1^{s+1,t,u,*}(q_0^\infty) \longrightarrow E_1^{s+1,t,u,*}(q_1^{-1})$ is an isomorphism since $s \geq 0$ implies $U(s) < U(s+1)$. This means the statement about y lying in $E_1^{s+1,t,u,*}(q_0^\infty)$ is actually trivial. ☐

We obtain the following corollary, which can also be proved using lemma 1.5.2 of chapter III. Our main proposition of the section follows quickly.

Corollary 4.1.3. $E_\infty^{s,t,u,*}(q_0^\infty) \longrightarrow E_\infty^{s,t,u,*}(q_1^{-1})$ *is*

1. *injective if* $u < U(s-1) + 2(p^2-1)(t+2) - 2(p-1)$*;*
2. *surjective if* $u < U(s) + 2(p^2-1)(t+2) - 2(p-1)$*.*

Proposition 4.1.4. *The localization map* $\mathrm{Cotor}_P^{s,t,u}(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P^{s,t,u}(q_1^{-1}Q(0)/q_0^\infty)$ *is*

1. *injective if* $u < U(s-1) + 2(p^2-1)(t+2) - 2(p-1)$*;*
2. *surjective if* $u < U(s) + 2(p^2-1)(t+2) - 2(p-1)$*.*

Proof. We have $\mathrm{Cotor}_P(N) = \bigcup_v F^v \mathrm{Cotor}_P(N)$ and $F^0 \mathrm{Cotor}_P(N) = 0$ when $N = Q(0)/q_0^\infty$ and $N = q_1^{-1}Q(0)/q_0^\infty$ and so the result follows from corollary 4.1.3. ☐

4.2 The bigraded perspective

When we use the LASS-∞ to obtain information about the ASS-S^0 the three gradings are combined into two gradings. We prove analogous results to that of the previous section in a bigraded setting.

Definition 4.2.1. We have spectral sequences called the *bigraded q_0^∞-Bockstein spectral sequence* (bi-q_0^∞-BSS) and the *bigraded q_1^{-1}-Bockstein spectral sequence* (bi-q_1^{-1}-BSS). They are reindexed versions of the q_0^∞-BSS and q_1^{-1}-BSS, respectively. They have E_1-pages

$$E_1^{\sigma,\lambda,v}(\text{bi-}q_0^\infty) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} E_1^{s,t,u,v}(q_0^\infty), \qquad E_1^{\sigma,\lambda,v}(\text{bi-}q_1^{-1}) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} E_1^{s,t,u,v}(q_1^{-1})$$

and d_r has degree $(1, 0, r)$ in both spectral sequences.

They converge to $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$ and $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$, respectively, which are given bigradings (σ, λ) by summing over the (s, t, u) with $s + t = \sigma$ and $u + t = \lambda$. The filtration degree is given by v in both spectral sequences and we have a map of spectral sequences $E_*^{*,*,*}(\text{bi-}q_0^\infty) \longrightarrow E_*^{*,*,*}(\text{bi-}q_1^{-1})$.

We note that the bigrading (σ, λ) reappears later on in II.4.1.2. The bigraded version of proposition 4.1.1 is also given in [15].

Proposition 4.2.2 ([15, 4.7(*a*)])**.** *The localization map* $\mathrm{Cotor}_P^{\sigma,\lambda}(Q(1)) \longrightarrow \mathrm{Cotor}_P^{\sigma,\lambda}(q_1^{-1}Q(1))$ *is*

1. *a surjection if $\sigma \geq 0$ and $\lambda < U(\sigma + 1) - 2p + 1$;*
2. *an isomorphism if $\sigma \geq 0$ and $\lambda < U(\sigma) - 2p + 1$.*

Corollary 4.2.3. *The localization map* $\mathrm{Cotor}_P^{\sigma,\lambda}(Q(1)) \longrightarrow \mathrm{Cotor}_P^{\sigma,\lambda}(q_1^{-1}Q(1))$ *is*

1. *a surjection if $\lambda < p(p-1)\sigma - 1$;*
2. *an isomorphism if $\lambda < p(p-1)(\sigma - 1) - 1$.*

Proof. Consider $g(\sigma) = p(p-1)\sigma - U(\sigma)$ for $\sigma \geq 0$. We have $g(1) = p(p-3) + 2 > 0 = g(0)$ and $g(\sigma + 2) = g(\sigma)$. Thus $p(p-1)\sigma - U(\sigma) \leq p(p-3) + 2$ and so

$$p(p-1)(\sigma - 1) - 1 \leq [U(\sigma) + p(p-3) + 2] - p(p-1) - 1 = U(\sigma) - 2p + 1.$$

Together with proposition 4.2.2 this proves the claim for $\sigma \geq 0$. When $\sigma < 0$, $\mathrm{Cotor}_P^{\sigma,\lambda}(Q(1)) = 0$ and so the localization map is injective. We just need to prove that $\mathrm{Cotor}_P^{\sigma,\lambda}(q_1^{-1}Q(1)) = 0$ whenever $\sigma < 0$ and $\lambda < p(p-1)\sigma - 1$. We can only have $[(\lambda - \sigma) + 1]/(p^2 - p - 1) < \sigma < 0$ if $(\lambda - \sigma) + 1 < 0$. But then $[(\lambda - \sigma) + 1]/(2p - 2) < \sigma < 0$ and the vanishing line of corollary II.6.4 gives the result. $\square$

This allows us to prove bigraded versions of all the results of the previous subsection.

Lemma 4.2.4. *Suppose $\lambda + 1 < p(p-1)(\sigma + 1) - 1$ so that proposition 4.2.2 gives a surjection $E_1^{\sigma,\lambda,*}(bi\text{-}q_0^\infty) \longrightarrow E_1^{\sigma,\lambda,*}(bi\text{-}q_1^{-1})$ and an isomorphism $E_1^{\sigma+1,\lambda,*}(bi\text{-}q_0^\infty) \longrightarrow E_1^{\sigma+1,\lambda,*}(bi\text{-}q_1^{-1})$.*

Suppose $x \in E_1^{\sigma,\lambda,*}(bi\text{-}q_0^\infty)$ maps to $\overline{x} \in E_1^{\sigma,\lambda,*}(bi\text{-}q_1^{-1})$, $y \in E_1^{\sigma+1,\lambda,*}(q_0^\infty) = E_1^{\sigma+1,\lambda,*}(q_1^{-1})$ and that $d_r\overline{x} = y$ in the q_1^{-1}-BSS. Then $d_r x = y$ in the q_0^∞-BSS.

Corollary 4.2.5. $E_\infty^{\sigma,\lambda,*}(bi\text{-}q_0^\infty) \longrightarrow E_\infty^{\sigma,\lambda,*}(bi\text{-}q_1^{-1})$ *is*

1. a surjection if $\lambda < p(p-1)(\sigma+1)-2$;

2. an isomorphism if $\lambda < p(p-1)\sigma - 2$.

Proposition 4.2.6. *The localization map* $\mathrm{Cotor}_P^{\sigma,\lambda}(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P^{\sigma,\lambda}(q_1^{-1}Q(0)/q_0^\infty)$ *is*

1. a surjection if $\lambda < p(p-1)(\sigma+1)-2$;

2. an isomorphism if $\lambda < p(p-1)\sigma - 2$.

5 Calculating $E_\infty(q_1^{-1})$

In this section we compute $E_\infty(q_1^{-1})$ explicitly and thus we understand $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$. The four subsections proceed linearly through the argument although some of the more involved proofs are omitted. Two sections are devoted to filling in the gaps: sections 7 and 8.

Let's outline the argument we use.

1. We give an explicit description of $\mathrm{Cotor}_P(q_1^{-1}Q(1))$ (corollary 5.1.4).
2. We compute two classes of differentials in the q_1^{-1}-BSS (theorem 5.2.10):
   (a) We compute the first class of differentials by direct computation at the level of cochains (proposition 5.2.4, section 7);
   (b) We prove a Kudo transgression theorem for the $Q(0)$-BSS (proposition 5.2.6, section 8);
   (c) We transfer differentials between the q_1^{-1}-BSS and the $Q(0)$-BSS (lemma 5.2.8) so that the first class of differentials together with the Kudo transgression theorem determine the second class of differentials (proposition 5.2.9).
3. By using the multiplicative properties of the q_1^{-1}-BSS we find an $\mathbb{F}_p$-basis $\{1\} \cup \{x_\alpha\}_{\alpha\in A} \cup \{y_\alpha\}_{\alpha\in A}$ of $\mathrm{Cotor}_P(q_1^{-1}Q(1))$ such that for each $\alpha \in A$, x_α supports a differential $d_{r_\alpha}x_\alpha = y_\alpha$ in the q_1^{-1}-BSS (corollary 5.3.3, lemma 5.3.6).
4. We prove lemma 5.4.1 which shows how the result of (3) gives rise to an $\mathbb{F}_p$-basis of $E_\infty(q_1^{-1})$.

5.1 An explicit description of $\mathrm{Cotor}_P(q_1^{-1}Q(1))$

We address the first item of our list in the form of corollary 5.1.4. We need to make a definition and recall a result of Miller (theorem 5.1.2).

Definition 5.1.1. Let I be the ideal generated by the image of p^{th}-power map $P \longrightarrow P$, $x \longmapsto x^p$. $P(1)$ is the quotient Hopf algebra P/I.

We can make $\mathbb{F}_p[q_1]$ into an algebra in $P(1)$-comodules by defining q_1 to be a comodule primitive. The algebra map $Q(1) \longrightarrow Q(1)/(q_2, q_3, \ldots) = \mathbb{F}_p[q_1]$ makes the following diagram commute

$$\begin{array}{ccc} q_1^{-1}Q(1) & \longrightarrow & P \otimes q_1^{-1}Q(1) \\ \downarrow & & \downarrow \\ \mathbb{F}_p[q_1, q_1^{-1}] & \longrightarrow & P(1) \otimes \mathbb{F}_p[q_1, q_1^{-1}] \end{array}$$

and so we have an induced map $\Omega(P; q_1^{-1}Q(1)) \longrightarrow \Omega(P(1); \mathbb{F}_p[q_1, q_1^{-1}])$. The following is a theorem of Miller (see [15, corollary 4.4]).

Theorem 5.1.2. *The map* $\mathrm{Cotor}_P(q_1^{-1}Q(1)) \longrightarrow \mathrm{Cotor}_{P(1)}(\mathbb{F}_p[q_1, q_1^{-1}])$ *is an isomorphism.*

$[\bar{\xi}_i]$ and $\sum_{j=1}^{p-1} \frac{(-1)^{j-1}}{j}[\bar{\xi}_i^j|\bar{\xi}_i^{p-j}]$ are cocycles in $\Omega(P(1))$ and so they define elements $h_{i,0}$ and $b_{i,0}$ in $\mathrm{Cotor}_{P(1)}(\mathbb{F}_p)$. The cohomology of a primitively generated Hopf algebra is well understood and the following lemma is a consequence.

Lemma 5.1.3. $\mathrm{Cotor}_{P(1)}(\mathbb{F}_p) = E[h_{i,0} : i > 0] \otimes \mathbb{F}_p[b_{i,0} : i > 0]$.

Thus, we have the following corollary to theorem 5.1.2.

Corollary 5.1.4. $\mathrm{Cotor}_P(q_1^{-1}Q(1)) = \mathbb{F}_p[q_1, q_1^{-1}] \otimes E[h_{i,0} : i > 0] \otimes \mathbb{F}_p[b_{i,0} : i > 0]$. *The trigradings are as follows:*

$$|q_1| = (0, 1, 2(p-1)), \quad |h_{i,0}| = (1, 0, 2(p^i - 1)), \quad |b_{i,0}| = (2, 0, 2p(p^i - 1)).$$

5.2 Miller's conjecture [16]

We turn to the second item on our list, proving theorem 5.2.10. Firstly, we introduce some notation.

Notation 5.2.1. Let $p^{[0]} = 0$ and $p^{[i]} = \frac{p^i - 1}{p-1}$ for $i \geq 1$. Note that $p^{[i]} = p^{i-1} + p^{[i-1]} = p \cdot p^{[i-1]} + 1$ for $i \geq 1$.

Notation 5.2.2. Write $[\![k]\!]$ for q_1^k, h_i for $h_{i,0}$ and b_i for $b_{i,0}$.

Notation 5.2.3. If x is a nonzero scalar multiple of y write $x \doteq y$.

The first class of differentials is described by the following proposition.

Proposition 5.2.4. *In the* q_1^{-1}*-BSS we have, for* $j \geq 1$, $d_{p^{[j]}}[\![p^{j-1}]\!] \doteq [\![-p^{[j-1]}]\!]h_j$.

Proof. Postponed until section 7. □

Using the multiplicative structure of the q_1^{-1}-BSS (lemma 3.5.4) we obtain the following corollary to propostion 5.2.4.

Corollary 5.2.5.

1. *Whenever* $j \geq 1$ *we have* $d_{p^{[j]}}[\![p^j + p^{j-1}]\!] \doteq [\![p^j - p^{[j-1]}]\!]h_j$ *in the* q_1^{-1}*-BSS;*
2. *The following are equivalent:*
   (a) *whenever* $i \geq 1$ *we have* $d_{p^i-1}[\![-p^{[i]}]\!]h_i \doteq [\![1 - p^{[i+1]}]\!]b_i$ *in the* q_1^{-1}*-BSS,*
   (b) *whenever* $i \geq 1$ *we have* $d_{p^i-1}[\![p^{i+1} + p^i - p^{[i]}]\!]h_i \doteq [\![p \cdot [p^i - p^{[i-1]}]]\!]b_i$ *in the* q_1^{-1}*-BSS.*

We now introduce the Kudo transgression result together with a useful lemma.

Proposition 5.2.6 (Kudo transgression)**.** *Suppose* $x \in \mathrm{Cotor}_P^{0,*,*}(Q(1))$, $y \in \mathrm{Cotor}_P^{1,*,*}(Q(1))$ *and that* $d_r x = y$ *in the* $Q(0)$*-BSS. Then we have* $d_{(p-1)r}x^{p-1}y \doteq \langle y\rangle^p$. $\langle y\rangle^p$ *will be defined in the course of the proof.*

Proof. Postponed until section 8. □

Lemma 5.2.7. $\langle[\![p^i - p^{[i-1]}]\!]h_i\rangle^p = [\![p \cdot [p^i - p^{[i-1]}]]\!]b_i$ *in* $\mathrm{Cotor}_P(Q(1))$.

Proof. Postponed until section 8 (see subsection 8.7). $\square$

We now introduce the lemma, which allows us to transfer differentials between the q_1^{-1}-BSS and the $Q(0)$-BSS.

Lemma 5.2.8. *For $i, j \geq 1$, $[\![p^j + p^{j-1}]\!]$, $[\![p^j - p^{[j-1]}]\!]h_j$, $[\![p^{i+1} + p^i - p^{[i]}]\!]h_i$ and $[\![p \cdot [p^i - p^{[i-1]}]]\!]b_i$, elements of $\mathrm{Cotor}_P(q_1^{-1}Q(1))$, have unique lifts to $\mathrm{Cotor}_P(Q(1))$. In addition,*

1. *whenever $j \geq 1$ we have $d_{p^{[j]}}[\![p^j + p^{j-1}]\!] \doteq [\![p^j - p^{[j-1]}]\!]h_j$ in the $Q(0)$-BSS;*
2. *the following are equivalent:*
   (a) *whenever $i \geq 1$ we have $d_{p^i-1}[\![-p^{[i]}]\!]h_i \doteq [\![1 - p^{[i+1]}]\!]b_i$ in the q_1^{-1}-BSS;*
   (b) *whenever $i \geq 1$ we have $d_{p^i-1}[\![p^{i+1} + p^i - p^{[i]}]\!]h_i \doteq [\![p \cdot [p^i - p^{[i-1]}]]\!]b_i$ in the $Q(0)$-BSS.*

Proof. We have

$$\begin{aligned}
[\![p^j + p^{j-1}]\!] &\in E_1^{0,p^j-p^{[j-1]}-1,2p^{j-1}(p^2-1),-p^{[j]}-1}(q_1^{-1});\\
[\![p^j - p^{[j-1]}]\!]h_j &\in E_1^{1,p^j-p^{[j-1]}-1,2p^{j-1}(p^2-1),-1}(q_1^{-1});\\
[\![p^{i+1} + p^i - p^{[i]}]\!]h_i &\in E_1^{1,p^{i+1}-p^{[i]},2p^i(p^2-1),-p^i}(q_1^{-1});\\
[\![p \cdot [p^i - p^{[i-1]}]]\!]b_i &\in E_1^{2,p^{i+1}-p^{[i]},2p^i(p^2-1),-1}(q_1^{-1});
\end{aligned}$$

and so by proposition 4.1.1, corollary 5.2.5, lemma 4.1.2 and lemma 3.3.2 it is enough to show that

$$\begin{aligned}
&2p^{j-1}(p^2-1) < U(0) + 2(p^2-1)(p^j - p^{[j-1]} + 1) - 2(p-1)\\
\text{and}\qquad &2p^j(p^2-1) < U(0) + 2(p^2-1)(p^{j+1} - p^{[j]} + 2) - 2(p-1)
\end{aligned}$$

These inequalities are equivalent to $p^{j-1} + (p+1)p^{[j-1]} < p^{j+1} + p$ and $p^j + (p+1)p^{[j]} < p^{j+2} + 2p + 1$, respectively, so we are done. $\square$

The second class of differentials is described by the following proposition.

Proposition 5.2.9. *In the q_1^{-1}-BSS we have, for $i \geq 1$, $d_{p^i-1}[\![-p^{[i]}]\!]h_i \doteq [\![1 - p^{[i+1]}]\!]b_i$*

Proof. By lemma 5.2.8 part 2 it is equivalent to show that whenever $i \geq 1$, we have

$$d_{p^i-1}[\![p^{i+1} + p^i - p^{[i]}]\!]h_i \doteq [\![p \cdot [p^i - p^{[i-1]}]]\!]b_i$$

in the $Q(0)$-BSS. By lemma 5.2.8 part 1 we have $d_{p^{[i]}}[\![p^i + p^{i-1}]\!] \doteq [\![p^i - p^{[i-1]}]\!]h_i$ in the $Q(0)$-BSS whenever $i \geq 1$.

Let $x = [\![p^i + p^{i-1}]\!]$ and $y = [\![p^i - p^{[i-1]}]\!]h_i$ then $x^{p-1}y = [\![p^{i+1} + p^i - p^{[i]}]\!]h_i$. Applying the Kudo transgression (proposition 5.2.6) and using lemma 5.2.7 finishes the proof. $\square$

Together, proposition 5.2.4 and proposition 5.2.9 give the following theorem, a conjecture of Miller's (see [16]).

Theorem 5.2.10. *In the q_1^{-1}-BSS we have, for $i, j \geq 1$, the following differentials:*

1. $d_{p^{[j]}}[\![p^{j-1}]\!] \doteq [\![-p^{[j-1]}]\!]h_j$;
2. $d_{p^i-1}[\![-p^{[i]}]\!]h_i \doteq [\![1 - p^{[i+1]}]\!]b_i$.

5.3 Generating all the nontrivial differentials in the q_1^{-1}-BSS

In this subsection we use theorem 5.2.10 together with the multiplicative structure of the q_1^{-1}-BSS (lemma 3.5.4) to obtain all the nontrivial differentials in the q_1^{-1}-BSS.

The main result is simple to prove as long as one has the correct picture in mind; otherwise, the proof may seem rather opaque. Figure I.1 on page 29 displays some of Christian Nassau's chart [19] for $E_2(\text{ASS-}S^0)$ when $p = 3$. His chart tells us about the object we are trying to calculate in a range by proposition 4.1.4 and the facts that $E_2(\text{ASS-}S^0) = \operatorname{Cotor}_P(Q(0))$ and

$$\Sigma_s \operatorname{Cotor}_P(Q(0)/q_0^\infty)/\, \mathbb{F}_p \langle q_0^t : t < 0\rangle = \operatorname{Cotor}_P(Q(0))/\, \mathbb{F}_p\, [q_0]. \tag{5.3.1}$$

A q_0-tower corresponds to a differential in the $Q(0)$-BSS. Labels at the top of towers are the sources of the corresponding Bockstein differentials; labels at the bottom of towers are the targets of the corresponding Bockstein differentials. We note that the part of figure I.1 in grey is not displayed in Nassau's charts and is deduced from the results of this section.

Recall from corollary 5.1.4 that $\operatorname{Cotor}_P(q_1^{-1}Q(1))$ is a polynomial algebra tensored with an exterior algebra. Thus, we have a convenient $\mathbb{F}_p$-basis for it, given by monomials in q_1, the h_i's and the b_i's. We need some notation to clarify matters.

Notation 5.3.2. Suppose given $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$ and $K = (k_1, \dots, k_r)$ such that $i_1 > \dots > i_r \geq 1$, $j_1 > \dots > j_s \geq 1$ and $k_a \geq 0$ for $a \in \{1, \dots, r\}$. We write

1. $b_I^K h_J$ for the monomial $b_{i_1}^{k_1} \cdots b_{i_r}^{k_r} h_{j_1} \cdots h_{j_s}$;

2. $N_{I,J,K}$ for $\sum_a k_a(1 - p^{[i_a+1]}) - \sum_c p^{[j_c-1]}$;

3. $J - 1$ for $(j_1, \dots, j_{s-1})$ if $s \geq 1$;

4. $K - 1$ for $(k_1, \dots, k_r - 1)$ if $r \geq 1$ and $k_r \geq 1$.

Notice that the indexing of a monomial in the h_i's and b_i's by I, J and K is unique once we impose the conditions $i_1 > \dots > i_r \geq 1$, $j_1 > \dots > j_s \geq 1$ and $k_a \geq 1$ for $a \in \{1, \dots, r\}$. Moreover, $\{[\![M]\!] b_I^K h_J\}$ gives a basis for $\operatorname{Cotor}_P(q_1^{-1}Q(1))$.

Here is the corollary to theorem 5.2.10 which completely describes all the nontrivial differentials in the q_1^{-1}-BSS.

Corollary 5.3.3. *Suppose given $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$ and $K = (k_1, \dots, k_r)$ such that $i_1 > \dots > i_r \geq 1$, $j_1 > \dots > j_s \geq 1$ and $k_a \geq 1$ for $a \in \{1, \dots, r\}$.*

Suppose $s \geq 1$, that either $r = 0$ or $r \geq 1$ and $i_r \geq j_s$, and that $p \nmid k \in \mathbb{Z}$. We have

$$d_{p^{[j_s]}} [\![kp^{j_s-1}]\!] \Big[[\![N_{I,J-1,K}]\!] b_I^K h_{J-1} \Big] \doteq [\![(k-1)p^{j_s-1}]\!] \Big[[\![N_{I,J,K}]\!] b_I^K h_J \Big]. \tag{5.3.4}$$

Suppose $r \geq 1$, that either $s = 0$ or $s \geq 1$ and $i_r < j_s$ and that $k \in \mathbb{Z}$. We have

$$d_{p^{i_r}-1} [\![kp^{i_r}]\!] \Big[[\![-p^{[i_r]}]\!] h_{i_r} \Big] \Big[[\![N_{I,J,K-1}]\!] b_I^{K-1} h_J \Big] \doteq [\![kp^{i_r}]\!] \Big[[\![N_{I,J,K}]\!] b_I^K h_J \Big]. \tag{5.3.5}$$

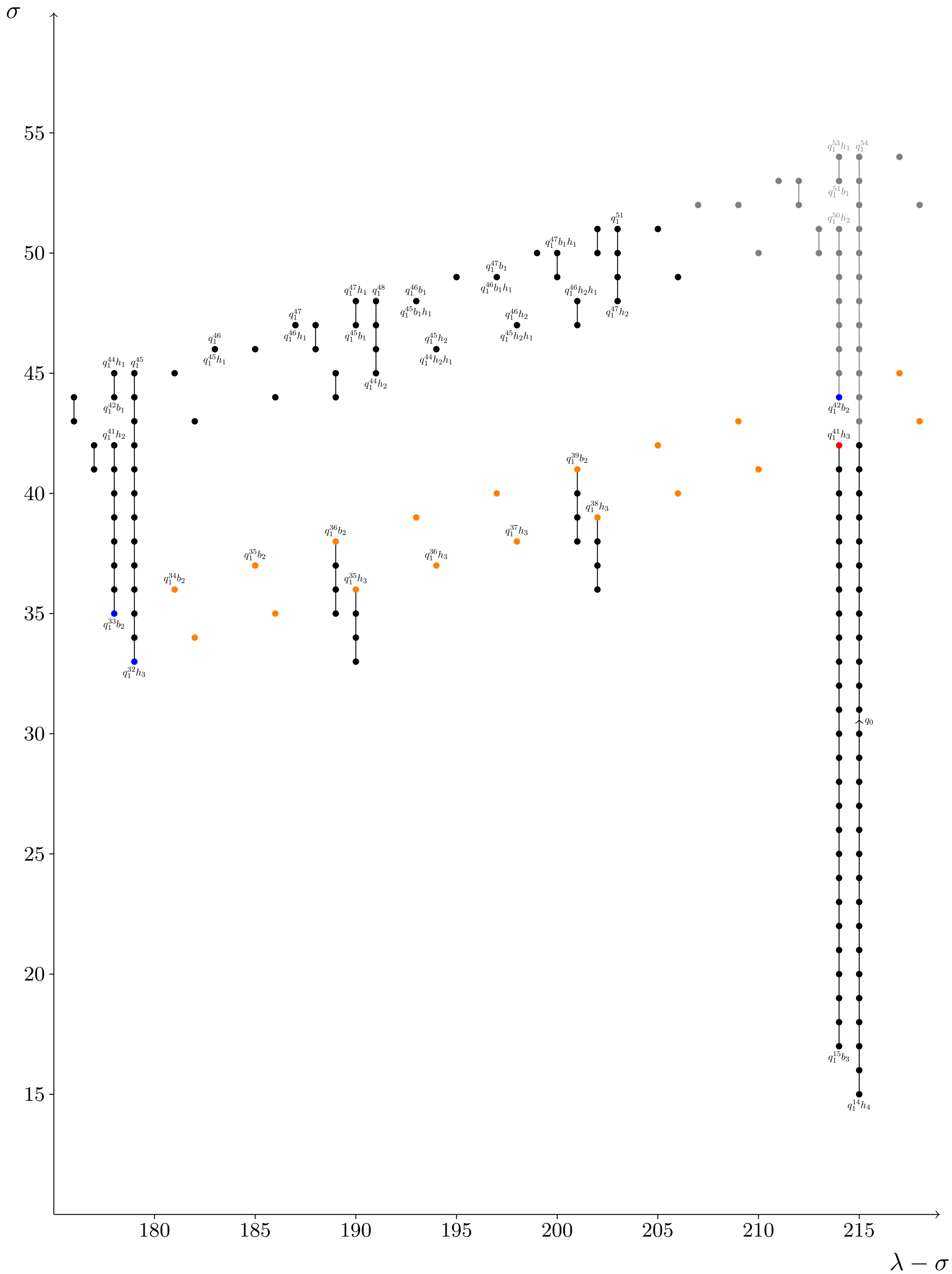

Figure I.1: A portion of $E_2^{\sigma,\lambda}(\text{ASS-}S^0)$ when $p = 3$

Proof. By theorem 5.2.10 and lemma 2.7 $[\![N_{I,J,K}]\!]b_I^K h_J$ is a permanent cycle. The first class of differentials is verified by observing that $d_{p^{[j_s]}}[\![kp^{j_s-1}]\!] \doteq [\![(k-1)p^{j_s-1}]\!][\![-p^{[j_s-1]}]\!]h_{j_s}$. The second class of differentials is verified by observing that

$$d_{p^{i_r}-1}[\![kp^{i_r}]\!] = 0 \text{ and } d_{p^{i_r}-1}[\![-p^{[i_r]}]\!]h_{i_r} \doteq [\![1-p^{[i_r+1]}]\!]b_{i_r}.$$

□

The content of the next lemma is that corollary 5.3.3 describes all of the nontrivial differentials in the q_1^{-1}-BSS.

Lemma 5.3.6. *The union*

$$\{1\} \cup \{x : x \textit{ is a source of one of the differentials in corollary 5.3.3}\}$$
$$\cup \{y : y \textit{ is a target of one of the differentials in corollary 5.3.3}\}$$

is a basis for $\mathrm{Cotor}_P(q_1^{-1}Q(1))$*. Moreover, the sources and targets of the differentials in corollary 5.3.3 are distinct and never equal to* 1*.*

Proof. We note that for any $M \neq 0$, $[\![M]\!]$ is the source of a differential like the one in (5.3.4).

Take I, J and K as in (5.3.4). We wish to show that $[\![M]\!]b_I^K h_J$ is the source or target of one of the differentials in corollary 5.3.3. There are three cases (the second case is empty if $j_s = 1$):

1. $M = (k-1)p^{j_s-1} + N_{I,J,K}$ for some $k \in \mathbb{Z}$ with $p \nmid k$.

2. $M = kp^{j_{s+1}-1} + N_{I,J,K}$ for some $k \in \mathbb{Z}$ with $p \nmid k$ and some $j_{s+1} \geq 1$ with $j_s > j_{s+1}$.

3. $M = (kp-1)p^{j_s-1} + N_{I,J,K}$ for some $k \in \mathbb{Z}$.

In the first case $[\![M]\!]b_I^K h_J$ is the target of the differential (5.3.4). In the second case, $[\![M]\!]b_I^K h_J$ is the source of a differential like the one in (5.3.4). In the third case, $[\![M]\!]b_I^K h_J$ is the source of a differential like the one in (5.3.5).

These cases are highlighted in figure I.1 when $p = 3$, $J = (3)$, and I and K are empty. This means that $N_{I,J,K} = -4$ and the three cases are

1. $M = 9(k-1) - 4$ for some $k \in \mathbb{Z}$ with $3 \nmid k$.

2. $M = 3^{j-1}k - 4$ for some $k \in \mathbb{Z}$ with $3 \nmid k$ and some j with $1 \leq j < 3$.

3. $M = 9(3k-1) - 4$ for some $k \in \mathbb{Z}$.

The first case is highlighted in blue when $k = 5$; the second case is highlighted in orange and we see both the cases $j = 1$ and $j = 2$ occurring; the last case is highlighted in red when $k = 2$.

Take I, J and K as in (5.3.5). We wish to show that $[\![M]\!]b_I^K h_J$ is the source or target of one of the differentials in corollary 5.3.3. There are two cases:

1. $M = kp^{i_r} + N_{I,J,K}$ for some $k \in \mathbb{Z}$.

2. $M = kp^{j_{s+1}-1} + N_{I,J,K}$ for some $k \in \mathbb{Z}$ with $p \nmid k$ and some $j_{s+1} \geq 1$ with $i_r \geq j_{s+1}$.

In the first case $[\![M]\!]b_I^K h_J$ is the target of the differential (5.3.5). In the second case, $[\![M]\!]b_I^K h_J$ is the source of a differential like the one in (5.3.4).

These cases are highlighted in figure I.1 when $p = 3$, $I = (2)$, $K = (1)$ and J is empty. This means that $N_{I,J,K} = -12$ and the two cases are

1. $M = 9k - 12$ for some $k \in \mathbb{Z}$ with $3 \nmid k$.

2. $M = 3^{j-1}k - 12$ for some $k \in \mathbb{Z}$ with $3 \nmid k$ and some j with $1 \leq j \leq 2$.

The first case is highlighted in blue when $k = 5$ and $k = 6$; the second case is highlighted in orange and we see both the cases $j = 1$ and $j = 2$ occurring; the last case is highlighted in red when $k = 2$.

Since the empty sequences I, J and K together with those satisfying the conditions in (5.3.4) or (5.3.5) make up all choices of I, J and K, and since $\{[\![M]\!]b_I^K h_J\}$ gives a basis for $\mathrm{Cotor}_P(q_1^{-1}Q(1))$ (corollary 5.1.4), we have proved the first claim.

Careful inspection of the previous argument shows that this also proves the second claim. □

5.4 Interpretting the q_1^{-1}-BSS

Finally, we address the fourth item of of our list and explain how our understanding of the q_1^{-1}-BSS differentials allows us to determine an $\mathbb{F}_p$-basis of $E_\infty^{*,*,*,*}(q_1^{-1}\text{-BSS})$. We use the following lemma.

Lemma 5.4.1. *Suppose we have an indexing set A and an $\mathbb{F}_p$-basis*

$$\{1\} \cup \{x_\alpha\}_{\alpha \in A} \cup \{y_\alpha\}_{\alpha \in A}$$

of $\mathrm{Cotor}_P(q_1^{-1}Q(1))$ *such that each x_α supports a differential $d_{r_\alpha} x_\alpha = y_\alpha$. Then we have an $\mathbb{F}_p$-basis of $E_\infty^{*,*,*,*}(q_1^{-1}$-BSS$)$ given by*

$$\{\langle 1 \rangle_v : v < 0\} \cup \{\langle x_\alpha \rangle_v : \alpha \in A,\ -r_\alpha \leq v < 0\}.$$

In the above statement we intend for 1, the x_α's and the y_α's to be distinct as in lemma 5.3.6. The notation $\langle - \rangle_v$ is used to denote the v-grading of an element.

Proof. Let $v < 0$. We see make some observations.

1. $E_1^{*,*,*,v} \cap \bigcup_{s<r} \mathrm{im}\, d_s$ has basis $\{y_\alpha : \alpha \in A,\ r_\alpha < r\}$.

2. $\{y_\alpha : \alpha \in A,\ r_\alpha = r\}$ is independent in $E_1^{*,*,*,v} / \left(E_1^{*,*,*,v} \cap \bigcup_{s<r} \mathrm{im}\, d_s\right)$.

3. $E_1^{*,*,*,v} \cap \bigcap_{s<r} \ker d_s$ has basis $\{1\} \cup \{x_\alpha : \alpha \in A,\ r_\alpha \geq \min\{r, -v\}\} \cup \{y_\alpha : \alpha \in A\}$.

4. $E_\infty^{*,*,*,v} = \left(E_1^{*,*,*,v} \cap \bigcap_s \ker d_s\right) / \left(E_1^{*,*,*,v} \cap \bigcup_s \mathrm{im}\, d_s\right)$ has basis $\{1\} \cup \{x_\alpha : \alpha \in A,\ r_\alpha \geq -v\}$.

We see that 1 is a basis element for $E_\infty^{*,*,*,v}$ for all $v < 0$ and that x_α is a basis element for $E_\infty^{*,*,*,v}$ as long as $-r_\alpha \leq v < 0$. This completes the proof. □

Corollary 5.4.2. *$E_\infty^{*,*,*,*}(q_1^{-1}$-BSS$)$ has basis*

$$\begin{aligned}
&\{\langle 1 \rangle_v : v < 0\} \\
\cup &\left\{ \left\langle [\![kp^{j_s - 1}]\!] \left[[\![N_{I,J-1,K}]\!] b_I^K h_{J-1} \right] \right\rangle_v \ :\ I, J, K, k \text{ satisfy the conditions in } (5.3.4),\ -p^{[j_s]} \leq v < 0 \right\} \\
\cup &\left\{ \left\langle [\![kp^{i_r}]\!] \left[[\![-p^{[i_r]}]\!] h_{i_r} \right] \left[[\![N_{I,J,K-1}]\!] b_I^{K-1} h_J \right] \right\rangle_v \ :\ I, J, K, k \text{ satisfy } (5.3.5),\ 1 - p^{i_r} \leq v < 0 \right\}.
\end{aligned}$$

Of course, this allows us to find a basis of $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$ if we wish.

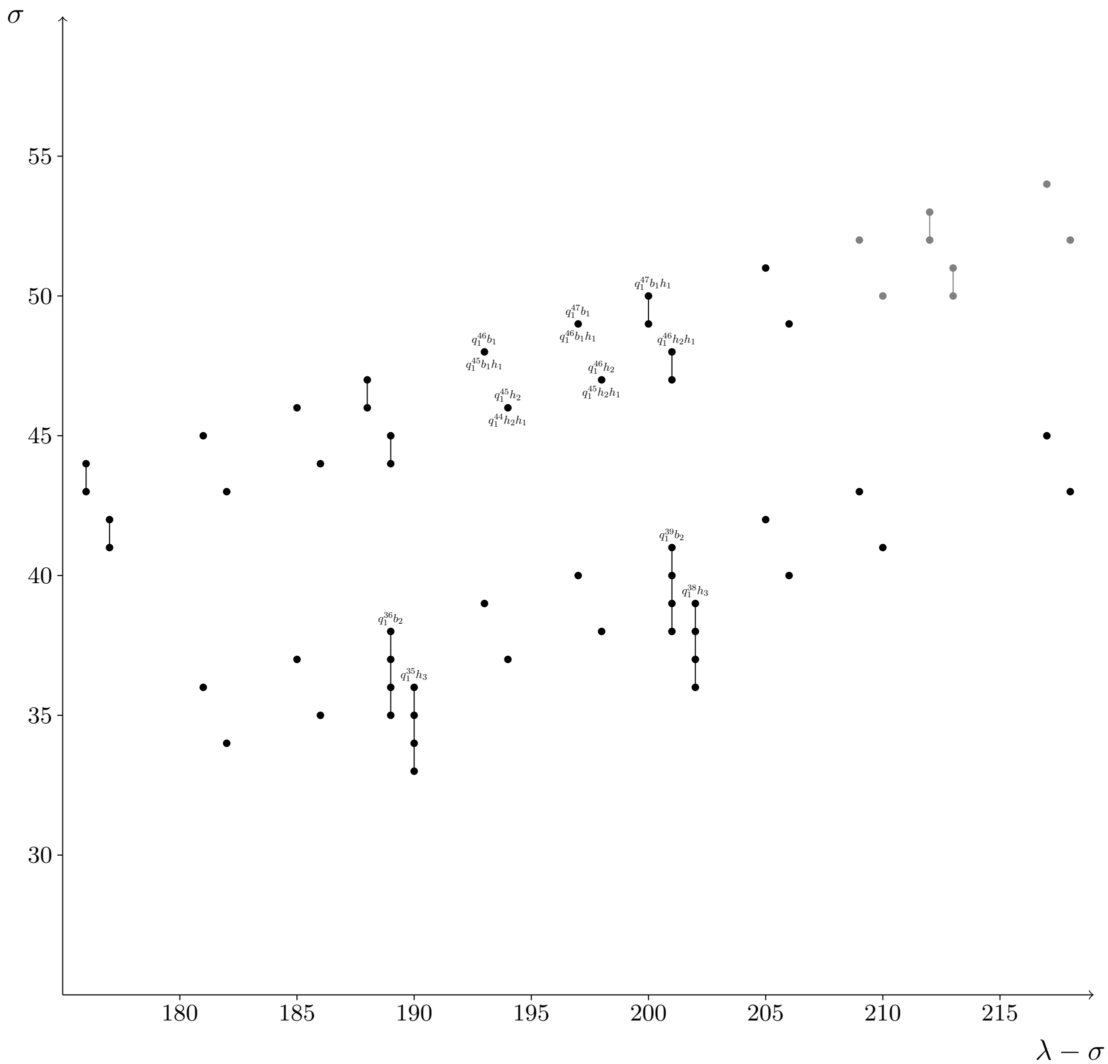

Figure I.2: A portion of $E_2^{\sigma,\lambda}(\text{ASS-}S^0)$ when $p = 3$

6 The combinatorics for computing E_3(LASS-∞)

Although setting up the localized Adams spectral sequence for the v_1-periodic sphere $v_1^{-1}S/p^{\infty}$ is delayed until section II.5 the computation of its E_3-page can essentially be completed here. It is in this sense that the algebra lies at the heart of our computation.

We return to figure I.1 which highlights other patterns too. In particular, after removing some of the towers we obtain figure I.2 and we see that the remaining towers come in pairs, arranged perfectly, so that there is a chance that they form an ayclic complex with respect to d_2. Moreover, the labelling at the top of the towers obeys a nice pattern with respect to this arrangement. If the differentials do what we hope then we have

$$q_1^{35}h_3 \longmapsto q_1^{36}b_2,\ q_1^{38}h_3 \longmapsto q_1^{39}b_2,\ q_1^{45}h_2 \longmapsto q_1^{46}b_1,\ q_1^{46}h_2 \longmapsto q_1^{47}b_1,\ q_1^{46}h_2h_1 \longmapsto q_1^{47}b_1h_1,$$

and in each case this comes from replacing an h_{i+1} by q_1b_i. This resembles the following result of

Miller concerning the *localized Adams spectral sequence for* $v_1^{-1}S/p$ (LASS-1).

Theorem 6.1 ([17, 4.8])**.** *We will see (II.5.2.4) that*

$$E_2^{\sigma,\lambda}(LASS\text{-}1) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \mathrm{Cotor}_P^{s,t,u}(q_1^{-1}Q(1))$$

and so $E_2^{*,*}(LASS\text{-}1)$ *has a filtration given by*

$$F^{\hat{s}}E_2^{\sigma,\lambda}(LASS\text{-}1) = \bigoplus_{\substack{s+t=\sigma,\ s\geq\hat{s} \\ u+t=\lambda}} \mathrm{Cotor}_P^{s,t,u}(q_1^{-1}Q(1)).$$

In the LASS-1 we have $d_2h_{i+1} \doteq q_1b_i$ *for* $i>0$*, up to higher filtration.*

This is precisely the result that we will use to compute $E_2(\mathrm{MAH})$. Although we delay setting up this spectral sequence until section II.9 we will essentially perform the computation of its E_2-page now. The next proposition shows that the towers lining up as they do in figure I.2 is not a fluke.

Definition 6.2. Define an operator D on $\mathrm{Cotor}_P(q_1^{-1}Q(1))$ by $D(q_1)=0=D(h_1)$, $D(h_{i+1})=q_1b_i$ for $i>0$, $D(b_i)=0$ for $i>0$ and the property that it is a derivation (recall that the h_i's are odd dimensional classes and the b_i's are even dimensional classes). We have $D^2=0$.

Proposition 6.3. *D induces an operation on $E_\infty^{*,*,*,*}(q_1^{-1})$. Fix, $i,j\geq 1$. Then D restricts to an operation on the subspaces with basis*

$$\{\langle 1\rangle_v : v<0\},$$

$$\left\{\left\langle [\![kp^{j_s-1}]\!]\left[[\![N_{I,J-1,K}]\!]b_I^K h_{J-1}\right]\right\rangle_v \ :\ I,J,K,k \textit{ satisfy } (5.3.4),\ j_s=j,\ -p^{[j]}\leq v<0\right\}$$

and

$$\left\{\left\langle [\![kp^{i_r}]\!]\left[[\![-p^{[i_r]}]\!]h_{i_r}\right]\left[[\![N_{I,J,K-1}]\!]b_I^{K-1}h_J\right]\right\rangle_v \ :\ I,J,K,k \textit{ satisfy } (5.3.5),\ i_r=i,\ 1-p^i\leq v<0\right\},$$

respectively. Moreover, the homology of D on each of these subcomplexes has basis

$$\{\langle 1\rangle_v : v<0\},$$

$$\left\{\left\langle [\![kp^{j-1}]\!]\right\rangle_v :\ p\nmid k\in\mathbb{Z},\ -p^{[j]}\leq v<0\right\} \quad \textit{and} \quad \left\{\left\langle [\![kp^{i}]\!]\left[[\![-p^{[i]}]\!]h_i\right]\right\rangle_v :\ k\in\mathbb{Z},\ 1-p^i\leq v<0\right\},$$

respectively.

Proof. The fact that we can set up the Mahowald spectral sequence (II.9.3) is the conceptual reason for why D induces an operation on $E_\infty^{*,*,*,*}(q_1^{-1})$. Presently, we need to show that $D : \bigcup_s \operatorname{im} d_s \longrightarrow \bigcup_s \operatorname{im} d_s$ and that the basis elements above are mapped to linear combinations of each other in an appropriate way.

We leave it to the reader to check that $D : \bigcup_s \operatorname{im} d_s \longrightarrow \bigcup_s \operatorname{im} d_s$ because this requires a similar argument to the one discussed in detail below. The reader will find that if we apply D to the inner expressions of (5.3.5) we obtain a valid formula. The same is true for (5.3.4) if $j_s=1$; when $j_s>1$

replacing h_{j_s} by $q_1 b_{j_s-1}$ creates a term which is the boundary of a differential like that in (5.3.5) (with the same k).

We concentrate on the basis elements listed above.

$D(1) = 0$ and so the claims concerning $\{\langle 1\rangle_v : v < 0\}$ are evident.

First fix $j \geq 1$ and consider

$$x = \left\langle [\![kp^{j_s-1}]\!]\left[[\![N_{I,J-1,K}]\!]b_I^K h_{J-1}\right]\right\rangle_v$$

where I, J, K and k satisfy 5.3.4, $j_s = j$, and $-p^{[j]} \leq v < 0$. If $s = 1$ then $D(x) = 0$ so suppose that $s > 1$ and let $c \in \{1, \ldots, s-1\}$. We wish to show that replacing h_{j_c} by $q_1 b_{j_c-1}$ in x gives an element x' of the same form as x. Let I', J', K' be obtained from I, J, K by imposing

1. $b_{I'}^{K'} h_{J'-1}$ is obtained from $b_I^K h_{J-1}$ by replacing h_{j_c} by b_{j_c-1};
2. $i'_1 > \ldots > i'_{r'} \geq 1$;
3. $s' = s - 1$ and $j'_1 > \ldots > j'_{s'} = j$;
4. $k_a \geq 1$ for all $a \in \{1, \ldots, r'\}$.

Notice that $j_c > j \geq 1$. Moreover, $r' \geq 1$, $i'_{r'} \geq j'_{s'}$ and if we let $k' = k + p^{j_c - j}$ then $p \nmid k' \in \mathbb{Z}$. We have just observed that I', J', K', and k' satisfy 5.3.4. Finally,

$$\begin{aligned} k'p^{j'_{s'}-1} + N_{I',J'-1,K'} &= (k + p^{j_c-j})p^{j-1} + [N_{I,J-1,K} + (1 - p^{[j_c]}) + p^{[j_c-1]}] \\ &= (kp^{j-1} + p^{j_c-1}) + [N_{I,J-1,K} + 1 - p^{j_c-1}] \\ &= [kp^{j_s-1} + N_{I,J-1,K}] + 1 \end{aligned}$$

so that replacing h_{j_c} by $q_1 b_{j_c-1}$ in x gives

$$x' = \left\langle [\![k'p^{j'_{s'}-1}]\!]\left[[\![N_{I',J'-1,K'}]\!]b_{I'}^{K'} h_{J'-1}\right]\right\rangle_v$$

an element of the same form as x. Since D is a derivation, this shows that D induces an operation on the second subspace of the proposition. The claim about the homology is true because

$$(E[h_i : i > j] \otimes \mathbb{F}_p[b_i : i \geq j] : dh_{i+1} = b_i)$$

has homology $\mathbb{F}_p$.

Secondly, consider

$$y = \left\langle [\![kp^{i_r}]\!]\left[[\![-p^{[i_r]}]\!]h_{i_r}\right]\left[[\![N_{I,J,K-1}]\!]b_I^{K-1}h_J\right]\right\rangle_v$$

where I, J, K and k satisfy 5.3.5, $i_r = i$, and $1 - p^i \leq v < 0$.

Firstly, we wish to show the term obtained from applying D to h_{i_r} is trivial. If $i_r = 1$ then $D(h_{i_r}) = 0$ so suppose that $i_r > 1$. Let I', J', K' be obtained from I, J, K by imposing

1. $b_{I'}^{K'} h_{J'} = b_{i_r-1} b_I^{K-1} h_J$;

2. $i'_1 > \ldots > i'_{r'} = i_r - 1$;

3. $J' = J$;

4. $k_a \geq 1$ for all $a \in \{1, \ldots, r'\}$.

Let $k' = kp$. Then I', J', K' and k' satisfy 5.3.5 and replacing h_{i_r} by $q_1 b_{i_r-1}$ in y gives

$$y' = \left\langle [\![kp^{i_r}]\!] \left[[\![1 - p^{[i_r]}]\!] b_{i_r-1} \right] \left[[\![N_{I,J,K-1}]\!] b_I^{K-1} h_J \right] \right\rangle_v = \left\langle [\![k' p^{i'_{r'}}]\!] \left[[\![N_{I',J',K'}]\!] b_{I'}^{K'} h_{J'} \right] \right\rangle_v = 0,$$

where the last eqality comes from 5.3.5. We deduce that when applying D the only terms of interest come from applying D to the $b_I^{K-1} h_J$ part of y.

If $s = 0$ then $D(y) = 0$ so suppose that $s > 0$ and let $c \in \{1, \ldots, s\}$. We wish to show that replacing h_{j_c} by $q_1 b_{j_c-1}$ in y gives an element y' of the same form as y. Let I', J', K' be obtained from I, J, K by imposing

1. $b_{I'}^{K'-1} h_{J'}$ is obtained from $b_I^{K-1} h_J$ by replacing h_{j_c} by b_{j_c-1} for some $c \in \{1, \ldots, s\}$;

2. $i'_1 > \ldots > i'_{r'} = i$;

3. $s' = s - 1$ and $j'_1 > \ldots > j'_{s'}$;

4. $k_a \geq 1$ for all $a \in \{1, \ldots, r'\}$.

Notice that $j_c > i \geq 1$ so that $k' = k + p^{j_c - i - 1} \in \mathbb{Z}$. I', J', K' and k' satisfy 5.3.5. Moreover,

$$\begin{aligned} k' p^{i'_{r'}} - p^{[i'_{r'}]} + N_{I',J',K'-1} &= (k + p^{j_c - i - 1}) p^i - p^{[i]} + [N_{I,J,K-1} + (1 - p^{[j_c]}) + p^{[j_c - 1]}] \\ &= (kp^i + p^{j_c - 1}) - p^{[i]} + [N_{I,J,K-1} + 1 - p^{j_c - 1}] \\ &= [kp^{i_r} - p^{[i_r]} + N_{I,J,K-1}] + 1 \end{aligned}$$

so that replacing h_{j_c} by $q_1 b_{j_c-1}$ in y gives

$$y' = \left\langle [\![k' p^{i'_{r'}}]\!] \left[[\![-p^{[i'_{r'}]}]\!] h_{i'_{r'}} \right] \left[[\![N_{I',J',K'-1}]\!] b_{I'}^{K'-1} h_{J'} \right] \right\rangle_v,$$

an element of the same form as y. Since D is a derivation, this shows that D induces an operation on the third subspace of the proposition. The claim about the homology is true because

$$(E[h_j : j > i] \otimes \mathbb{F}_p[b_j : j \geq i] : dh_{j+1} = b_j)$$

has homology $\mathbb{F}_p$. $\square$

7 Proof of proposition 5.2.4, the first class of differentials

7.1 The strategy

We prove that there exist cocyles $x_j \in \Omega(P; q_1^{-1}Q(0)/q_0^\infty)$ and $y_j \in \Omega(P; q_1^{-1}Q(1))$ such that in the diagram

$$\begin{tikzcd}
\Omega(P; q_1^{-1}Q(0)/q_0^\infty) & \Omega(P; q_1^{-1}Q(0)/q_0^\infty) \arrow[l, "q_0^{p^{[j]}-1}"'] \arrow[dr, dotted, "q_0 d(-/q_0)"] & \\
\Omega(P; q_1^{-1}Q(1)) \arrow[u] \arrow[d] & & \Omega(P; q_1^{-1}Q(1)) \arrow[d] \\
\Omega(P(1); \mathbb{F}_p[q_1, q_1^{-1}]) & & \Omega(P(1); \mathbb{F}_p[q_1, q_1^{-1}])
\end{tikzcd}$$

we have

$$\begin{tikzcd}
q_0^{-1}q_1^{p^{j-1}} & x_j \arrow[l, mapsto, "q_0^{p^{[j]}-1}"'] \arrow[dr, mapsto, "q_0 d(-/q_0)"] & \\
q_1^{p^{j-1}} \arrow[u, mapsto] \arrow[d, mapsto] & & y_j \arrow[d, mapsto] \\
q_1^{p^{j-1}} & & (-1)^{j-1}[\xi_j]q_1^{-p^{[j-1]}}.
\end{tikzcd}$$

Here $(-/q_0)$ denotes the function on $\Omega(P; q_1^{-1}Q(0)/q_0^\infty)$ with the following two properties:

1. x/q_0 is mapped to x under multiplication by q_0;
2. the denominators of the terms in x/q_0 have q_0 raised to a power greater than or equal to 2.

The author hopes it is clear that $q_0 d(-/q_0)$ denotes the function on cocycles of $\Omega(P; q_1^{-1}Q(0)/q_0^\infty)$, which appears when applying the snake lemma to the short exact sequence

$$0 \longrightarrow q_1^{-1}Q(1) \longrightarrow q_1^{-1}Q(0)/q_0^\infty \xrightarrow{q_0} q_1^{-1}Q(0)/q_0^\infty \longrightarrow 0.$$

We note that x_j determines y_j and that once we pass to homology the first diagram above becomes

$$\begin{tikzcd}
\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty) & \mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty) \arrow[l, "q_0^{p^{[j]}-1}"'] \arrow[dr, "\partial"] & \\
\mathrm{Cotor}_P(q_1^{-1}Q(1)) \arrow[u] \arrow[d, "\cong"] & & \mathrm{Cotor}_P(q_1^{-1}Q(1)) \arrow[d, "\cong"] \\
\mathrm{Cotor}_{P(1)}(\mathbb{F}_p[q_1, q_1^{-1}]) & & \mathrm{Cotor}_{P(1)}(\mathbb{F}_p[q_1, q_1^{-1}])
\end{tikzcd}$$

so that what we prove does, in fact, prove propostion 5.2.4.

7.2 $j = 1$ and $j = 2$

We proceed by induction. The $j = 1$ and $j = 2$ cases are

$$\begin{tikzcd}
x_1 = q_0^{-1} q_1 \arrow[dr, mapsto, "q_0 d(-/q_0)"] & \\
q_1 \arrow[u, mapsto] \arrow[d, mapsto] & y_1 = [\xi_1] \arrow[d, mapsto] \\
q_1 & [\xi_1]
\end{tikzcd}$$

and

$$\begin{tikzcd}
q_0^{-1} q_1^p & x_2 = q_0^{-p-1} q_1^p - q_0^{-1} q_1^{-1} q_2 \arrow[l, mapsto, "q_0^p"'] \arrow[dr, mapsto, "q_0 d(-/q_0)"] & \\
q_1^p \arrow[u, mapsto] \arrow[d, mapsto] & & y_2 = [\bar{\xi}_2] q_1^{-1} + [\xi_1] q_1^{-2} q_2 \arrow[d, mapsto] \\
q_1^p & & -[\xi_2] q_1^{-1}.
\end{tikzcd}$$

7.3 The inductive step

Suppose we have proved the case $j = n$, where $n \geq 1$. So we have cocycles x_n and y_n such that

$$\begin{tikzcd}
q_0^{-1} q_1^{p^{n-1}} & x_n \arrow[l, mapsto, "q_0^{p^{[n]}-1}"'] \arrow[dr, mapsto, "q_0 d(-/q_0)"] & \\
q_1^{p^{n-1}} \arrow[u, mapsto] \arrow[d, mapsto] & & y_n \arrow[d, mapsto] \\
q_1^{p^{n-1}} & & (-1)^{n-1} [\xi_n] q_1^{-p^{[n-1]}}.
\end{tikzcd}$$

Write $\tilde{P}^0 x_n$ and $\tilde{P}^0 y_n$ for the cochains in which we have raised every symbol to the p^{th} power. We prove in lemma 7.5.1 that $\tilde{P}^0 x_n$ is a cocycle. One can see directly that $\tilde{P}^0 y_n$ is a cocycle.

Since the map $\Omega(P; q_1^{-1} Q(1)) \to \Omega(P(1); \mathbb{F}_p[q_1, q_1^{-1}])$ is a homology isomorphism (theorem 5.1.2) and $\xi_n^p = 0$ in $P(1)$, we can write $\tilde{P}^0 y_n = d w_n$ for some $w_n \in \Omega(P; q_1^{-1} Q(1))$.

Lemmas 7.5.1 and 7.5.2 of subsection 7.5 imply that in the diagram

$$\begin{tikzcd}
\Omega(P; q_1^{-1}Q(0)/q_0^\infty) & \Omega(P; q_1^{-1}Q(0)/q_0^\infty) \arrow[l, "q_0^{p^{[n+1]}-2}"'] \arrow[dr, dotted, "q_0 d(-/q_0)"] & \\
\Omega(P; q_1^{-1}Q(1)) \arrow[u] \arrow[d] & & \Omega(P; q_1^{-1}Q(1)) \arrow[d] \\
\Omega(P(1); \mathbb{F}_p[q_1, q_1^{-1}]) & & \Omega(P(1); \mathbb{F}_p[q_1, q_1^{-1}])
\end{tikzcd}$$

we have

$$\begin{tikzcd}
q_0^{-1}q_1^{p^n} & \tilde{P}^0 x_n \arrow[l, mapsto, "q_0^{p^{[n+1]}-2}"'] \arrow[dr, mapsto, "q_0 d(-/q_0)"] & \\
q_1^{p^n} \arrow[u, mapsto] \arrow[d, mapsto] & & \tilde{P}^0 y_n = dw_n \arrow[d, mapsto] \\
q_1^{p^n} & & (-1)^{n-1}[\xi_n^p]q_1^{-p\cdot p^{[n-1]}} = 0.
\end{tikzcd}$$

We summarise some of the information in the following diagram.

$$\begin{tikzcd}
\Omega(P; q_1^{-1}Q(1)) \arrow[r] & \Omega(P; q_1^{-1}Q(0)/q_0^\infty) \arrow[r] & \Omega(P; q_1^{-1}Q(0)/q_0^\infty) \\
w_n \arrow[d, mapsto, "d"] & q_0^{-1}\tilde{P}^0 x_n \arrow[r, mapsto] \arrow[d, mapsto, "d"] & \tilde{P}^0 x_n \\
\tilde{P}^0 y_n \arrow[r, mapsto] & q_0^{-1}\tilde{P}^0 y_n &
\end{tikzcd}$$

Let $x_{n+1} = q_0^{-1}\tilde{P}^0 x_n - q_0^{-1} w_n$, a cocycle in $\Omega(P; q_1^{-1}Q(0)/q_0^\infty)$ and $y_{n+1} = q_0 d(x_{n+1}/q_0)$, a cocycle in $\Omega(P; q_1^{-1}Q(1))$. Then

$$\begin{tikzcd}
q_0^{-1}q_1^{p^n} & x_{n+1} \arrow[l, mapsto, "q_0^{p^{[n+1]}-1}"'] \arrow[dr, mapsto, "q_0 d(-/q_0)"] & \\
q_1^{p^n} \arrow[u, mapsto] \arrow[d, mapsto] & & y_{n+1} \arrow[d, mapsto] \\
q_1^{p^n} & & (-1)^n[\xi_{n+1}]q_1^{-p^{[n]}}?
\end{tikzcd}$$

and we are left with showing that y_{n+1} is mapped to $(-1)^n[\xi_{n+1}]q_1^{-p^{[n]}}$.

7.4 The image of y_{n+1} in $\Omega(P(1);\mathbb{F}_p[q_1,q_1^{-1}])$: completing the inductive step

$$\begin{tikzcd}
\Omega(P;q_1^{-1}Q(1)) \arrow[r] & \Omega(P;q_1^{-1}Q(0)/q_0^{\infty}) \arrow[r] & \Omega(P;q_1^{-1}Q(0)/q_0^{\infty}) \\
 & q_0^{-1}x_{n+1} = q_0^{-2}\tilde{P}^0 x_n - q_0^{-2}w_n \arrow[r, mapsto] \arrow[d, mapsto, "d"] & x_{n+1} \\
y_{n+1} \arrow[r, mapsto] & q_0^{-1}y_{n+1} &
\end{tikzcd}$$

We wish to show that the image of y_{n+1} in $\Omega(P(1);\mathbb{F}_p[q_1,q_1^{-1}])$ is $(-1)^n[\xi_{n+1}]q_1^{-p^{[n]}}$. We note that we can ignore all contributions from $d(q_0^{-2}\tilde{P}^0 x_n)$, for they involve ξ_j's raised to the p (to see this one can use (7.5.3)). Let

$$w_n' = w_n + (-1)^n q_1^{-p^{[n]}} q_{n+1} \in \Omega(P;q_1^{-1}Q(1))$$

so that

$$-q_0^{-2}w_n = (-1)^n q_0^{-2} q_1^{-p^{[n]}} q_{n+1} - q_0^{-2}w_n' \in \Omega(P;q_1^{-1}Q(0)/q_0^{\infty}).$$

We consider the contributions from the two terms in this expression separately.

7.4.1 $(-1)^n d(q_0^{-2}q_1^{-p^{[n]}}q_{n+1})$

Firstly, we consider the contribution from $(-1)^n d(q_0^{-2}q_1^{-p^{[n]}}q_{n+1})$. Recall definitions 1.5 and 1.7. We have a P-comodule map

$$M_2(p^{[n-1]}+1) \longrightarrow q_1^{-1}M_2 \subset q_1^{-1}Q(0)/q_0^{\infty}, \quad q_0^{-2}q_1^{p-1}q_{n+1} \longmapsto q_0^{-2}q_1^{-p^{[n]}}q_{n+1}.$$

Under $Q(0) \longrightarrow P\otimes Q(0)$

$$q_1^{p-1} \longmapsto \sum_{i+j=p-1} (-1)^i \xi_1^i \otimes q_0^i q_1^j \quad \text{and} \quad q_{n+1} \longmapsto \sum_{r+s=n+1} \xi_r^{p^s} \otimes q_s.$$

Under $q_0^{-1}Q(0) \longrightarrow P \otimes q_0^{-1}Q(0)$

$$q_0^{-2}q_1^{p-1}q_{n+1} \longmapsto \sum_{i+j=p-1}\ \sum_{r+s=n+1} (-1)^i \xi_1^i \xi_r^{p^s} \otimes q_0^{i-2}q_1^j q_s$$

so that $q_1^{-1}Q(0)/q_0^{\infty} \longrightarrow P\otimes q_1^{-1}Q(0)/q_0^{\infty}$ takes

$$(-1)^n q_0^{-2}q_1^{-p^{[n]}}q_{n+1} \longmapsto \sum_{\substack{i+j=p-1\\ i=0,1}}\ \sum_{r+s=n+1} (-1)^{i+n}\xi_1^i\xi_r^{p^s} \otimes q_0^{i-2}q_1^{j-p(p^{[n-1]}+1)}q_s.$$

We know that terms involving q_0^{-2} must eventually cancel in some way so we ignore these. Because we are concerned with an image in $\Omega(P(1);\mathbb{F}_p[q_1,q_1^{-1}])$ we ignore terms involving ξ_j's raised to a power greater than or equal to p and terms involving q_j's other than q_1 and q_0^{-1}. Since $n\geq 1$, we are left with the term corresponding to $s=0$, $r=n+1$, $i=0$ and $j=p-1$: it is

$$(-1)^n \xi_{n+1} \otimes q_0^{-1}q_1^{-p^{[n]}}.$$

7.4.2 $d(q_0^{-2}w_n')$

We have almost shown that the image of y_{n+1} in $\Omega(P(1);\mathbb{F}_p[q_1,q_1^{-1}])$ is $(-1)^n[\xi_{n+1}]q_1^{-p^{[n]}}$; we just need to show that $d(q_0^{-2}w_n')$ contributes nothing to the image of y_{n+1} in $\Omega(P(1);\mathbb{F}_p[q_1,q_1^{-1}])$.

First, we make a critical observation. Recall that

$$w_n' = w_n + (-1)^n q_1^{-p^{[n]}} q_{n+1} \in \Omega(P; q_1^{-1}Q(1))$$

and that $dw_n = \tilde{P}^0 y_n$.

Lemma 7.4.2.1. *The terms of*

$$dw_n' = \tilde{P}^0 y_n + (-1)^n \sum_{\substack{i+j=n+1 \\ i,j\geq 1}} [\xi_i^{p^j}]q_1^{-p^{[n]}} q_j \in \Omega(P; q_1^{-1}Q(1))$$

involve a q_j other than q_1 or a ξ_j raised to a power greater than or equal to p^2.

Proof. By the induction hypothesis

$$y_n = (-1)^{n-1}[\xi_n]q_1^{-p^{[n-1]}} + \dots$$

where the terms we have omitted involve a q_j other than q_1 or a ξ_j raised to a power greater than or equal to p. So

$$\tilde{P}^0 y_n = (-1)^{n-1}[\xi_n^p]q_1^{-p^{[n]}+1} + \dots$$

where the terms we have omitted involve a q_j other than q_1 or a ξ_j raised to a power greater than or equal to p^2. The term we have indicated cancels with the $j=1$ term of the summation in the lemma statement and this completes the proof. $\square$

We are now in a position to start work.

Suppose no power of q_1 worse (more negative) than q_1^{-lp} appears in w_n'. Making use of the map (see definitions 1.5 and 1.7)

$$\Omega(P; M_2(l)) \longrightarrow \Omega(P; q_1^{-1}M_2) \subset \Omega(P; q_1^{-1}Q(0)/q_0^{\infty}), \quad q_0^{-2}q_1^{lp}w_n' \longmapsto q_0^{-2}w_n'$$

it is sufficient to analyze $d(q_0^{-2}q_1^{lp}w_n')$: viewing $q_1^{lp}w_n'$ as lying in $\Omega(P;Q(0))$, we care about terms of $d(q_1^{lp}w_n')$ involving a single power of q_0. We have the following corollary to lemma 7.4.2.1.

Corollary 7.4.2.2. *The terms of $dq_1^{lp}w_n' = q_1^{lp}dw_n' \in \Omega(P;Q(1))$ involve a q_j other than q_1 or a ξ_j raised to a power greater than or equal to p^2.*

For $k_1,\dots,k_r \in \{1,2,\dots,p-1\}$ and $1 < n_1 < \dots < n_r$ we have

$$Q(1) \longrightarrow P \otimes Q(1), \quad q_{n_1}^{k_1}\cdots q_{n_r}^{k_r} \longmapsto \xi_{n_1-1}^{k_1 p}\cdots\xi_{n_r-1}^{k_r p} \otimes q_1^{\sum k_i} + \dots$$

where the terms we have omitted involve q_j's other than q_1. This is the only way that the indicated term can arise; there is no way to cancel it. Thus our critical observation (corollary 7.4.2.2) implies that any monomial apppearing in $q_1^{lp}w_n'$ must contain some q_k ($k>1$) raised to a power greater than or equal to p. We conclude that the contribution from $d(q_0^{-2}w_n')$ is zero in $\Omega(P(1);\mathbb{F}_p[q_1,q_1^{-1}])$.

This completes the inductive step and the proof of the proposition 5.2.4 modulo the lemmas of the next subsection.

7.5 Two lemmas

We are just left two prove the two lemmas used in subsection 7.3

Lemma 7.5.1. *Suppose that $x \in \Omega(P; q_1^{-1}Q(0)/q_0^\infty)$ is a cocycle. Then $\tilde{P}^0 x \in \Omega(P; q_1^{-1}Q(0)/q_0^\infty)$ is a cocyle. Moreover,*

$$q_0^{p^{[n]}-1}x = q_0^{-1}q_1^{p^{n-1}} \implies q_0^{p^{[n+1]}-2}\tilde{P}^0 x = q_0^{-1}q_1^{p^n}.$$

Proof. Suppose that x and $\tilde{P}^0 x$ involve negative powers of q_0 at worst q_0^{-k} and that x involves negative powers of q_1 at worst q_1^{-l}. Then we have a sequence of injections

$$\begin{array}{ccccccc}
\Omega(P; M_k(lp)) & \longrightarrow & \Omega(P; M_k(l)) & \longrightarrow & \Omega(P; q_1^{-1}M_k) & \longrightarrow & \Omega(P; q_1^{-1}Q(0)/q_0^\infty) \\
 & & q_1^{lp^{k-1}}x & \longmapsto & x & \longmapsto & x \\
q_1^{lp^k}\tilde{P}^0 x & & \longmapsto & & \tilde{P}^0 x & \longmapsto & \tilde{P}^0 x.
\end{array}$$

Since x is a cocycle in $\Omega(P; q_1^{-1}Q(0)/q_0^\infty)$, $q_1^{lp^{k-1}}x$ is a cocycle in $\Omega(P; M_k(l))$. Thus $q_1^{lp^k}\tilde{P}^0 x$ is a cocycle in $\Omega(P; M_k(lp))$ and we see that $\tilde{P}^0 x$ is a cocycle in $\Omega(P; q_1^{-1}Q(0)/q_0^\infty)$. Also,

$$q_0^{p^{[n]}-1}x = q_0^{-1}q_1^{p^{n-1}} \implies q_0^{p\cdot p^{[n]}-p}\tilde{P}^0 x = q_0^{-p}q_1^{p^n} \implies q_0^{p\cdot p^{[n]}-1}\tilde{P}^0 x = q_0^{-1}q_1^{p^n}$$

and since $p \cdot p^{[n]} - 1 = p^{[n+1]} - 2$ we're done. ∎

Lemma 7.5.2. *Suppose $x \in \Omega(P; q_1^{-1}Q(0)/q_0^\infty)$ is a cocycle and $q_0 d(x/q_0) = y \in \Omega(P; q_1^{-1}Q(1))$. Then $q_0 d(\tilde{P}^0 x/q_0) = \tilde{P}^0 y \in \Omega(P; q_1^{-1}Q(1))$.*

Proof. Suppose that x/q_0 and $\tilde{P}^0 x/q_0^p$ involve negative powers of q_0 at worst q_0^{-k} and that x/q_0 involves negative powers of q_1 at worst q_1^{-l}. Then we have a sequence of injections

$$\begin{array}{ccccccc}
\Omega(P; M_k(lp)) & \longrightarrow & \Omega(P; M_k(l)) & \longrightarrow & \Omega(P; q_1^{-1}M_k) & \longrightarrow & \Omega(P; q_1^{-1}Q(0)/q_0^\infty) \\
 & & q_1^{lp^{k-1}}x/q_0 & \longmapsto & x/q_0 & \longmapsto & x/q_0 \\
q_1^{lp^k}\tilde{P}^0 x/q_0^p & & \longmapsto & & \tilde{P}^0 x/q_0^p & \longmapsto & \tilde{P}^0 x/q_0^p.
\end{array}$$

We have

$$d(q_1^{lp^k}\tilde{P}^0 x/q_0^p) = \tilde{P}^0 d(q_1^{lp^{k-1}}x/q_0) \in \Omega(P; M_k(l)) \tag{7.5.3}$$

and so

$$d(\tilde{P}^0 x/q_0^p) = d(q_1^{lp^k}\tilde{P}^0 x/q_0^p)/q_1^{lp^k} = \tilde{P}^0\left[d(q_1^{lp^{k-1}}x/q_0)/q_1^{lp^{k-1}}\right] = \tilde{P}^0 d(x/q_0) \in \Omega(P; q_1^{-1}Q(0)/q_0^\infty).$$

We obtain

$$q_0 d(\tilde{P}^0 x/q_0) = q_0 d(q_0^{p-1}(\tilde{P}^0 x/q_0^p)) = q_0^p d(\tilde{P}^0 x/q_0^p) = \tilde{P}^0(q_0 d(x/q_0)) = \tilde{P}^0 y$$

where the penultimate equality comes from the preceding observation and this completes the proof. ∎

7.6 An illustration of the method

To illustrate the method notice that we have

$$x_1 = q_0^{-1}q_1, \qquad y_1 = [\xi_1], \qquad w_1 = q_1^{-1}q_2, \qquad w_1' = 0,$$

and

$$\begin{aligned} x_2 &= q_0^{-p-1}q_1^p - q_0^{-1}q_1^{-1}q_2, & y_2 &= [\bar{\xi}_2]q_1^{-1} + [\xi_1]q_1^{-2}q_2, \\ w_2 &= q_1^{-2p-1}q_2^{p+1} - q_1^{-p-1}q_3, & w_2' &= q_1^{-2p-1}q_2^{p+1}. \end{aligned}$$

8 Proof of proposition 5.2.6, the Kudo trangression

8.1 Notation

The reader should refer to [15, pages 75-76] for notation regarding twisting morphisms and twisted tensor products. We write τ for the universal twisting morphism instead of [ ].

8.2 The strategy

Suppose given a connected commutative Hopf algebra P and a commutative algebra Q over P and suppose that all nontrivial elements of P and Q have even degree. We will mimic theorem 3.1 of [14] to define a natural operation

$$\beta\tilde{P}^0 : \Omega^0(P;Q) \longrightarrow \Omega^1(P;Q).$$

Once this operation has been defined and we have observed its basic properties the proof of proposition 5.2.6 follows quickly.

8.3 Homotopy commutativity of $\Omega(P;Q)$ and Φ

The first step towards proving the existence of the operation $\beta\tilde{P}^0$ is to describe a map

$$\Phi : W \otimes \Omega(P;Q)^{\otimes p} \longrightarrow \Omega(P;Q),$$

which acts as the θ in [14, theorem 3.1]. This can be obtained by dualizing the construction in [14, lemma 11.3]. Conveniently, this has been written up in [5, lemma 2.3].

Let P be a commutative connected Hopf algebra and Q be a commutative algebra over P, i.e. a commutative algebra, which is also a P-comodule, whose structure map is multiplicative.

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] \arrow[d, "\psi_Q"', "i_0"] & 0 \arrow[r] \arrow[d, "i_1"] & 0 \arrow[r] \arrow[d, "i_2"] & \dots \\
0 \arrow[r] & P \otimes Q \arrow[r, "d"] \arrow[d, "\epsilon\otimes 1"', "r_0"] & P \otimes \overline{P} \otimes Q \arrow[r, "d"] \arrow[d, "r_1"] & P \otimes \overline{P} \otimes \overline{P} \otimes Q \arrow[r, "d"] \arrow[d, "r_2"] & \dots \\
0 \arrow[r] & Q \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Consider the diagram above. The top and bottom row are equal to the chain complex consisting of Q concentrated in cohomological degree zero and the middle row is the chain complex $P \otimes_\tau \Omega(P;Q)$.

We have the counit $\epsilon : P \to \mathbb{F}_p$ and the coaction $\psi_Q : Q \to P \otimes Q$. The definition of a P-comodule gives $1 - ri = 0$. We also have $1 - ir = dS + Sd$ where S is the contraction defined by

$$S(p_0[p_1|\dots|p_s]q) = \epsilon(p_0)p_1[p_2|\dots|p_s]q.$$

Let C_p denote the cyclic group of order p and let W be the standard $\mathbb{F}_p[C_p]$-free resolution of $\mathbb{F}_p$ (see [5, definition 2.2]). Following Brüner's account in [5, lemma 2.3], we can extend the multiplication map displayed at the top of the following diagram and construct Φ, a C_p-equivariant map of DG P-comodules (with $\Phi(W_i \otimes (P \otimes_\tau \Omega(P;Q))_j^{\otimes p}) = 0$ if $pi > (p-1)j$).

$$\begin{tikzcd}
Q^{\otimes p} \arrow[r] \arrow[d, "e_0 \otimes i^{\otimes p}"'] & Q \arrow[d, "i"] \\
W \otimes (P \otimes_\tau \Omega(P;Q))^{\otimes p} \arrow[r, "\Phi"] & P \otimes_\tau \Omega(P;Q)
\end{tikzcd}$$

Precisely, we make the following definition.

Definition 8.3.1.

$$\Phi : W \otimes (P \otimes_\tau \Omega(P;Q))^{\otimes p} \longrightarrow P \otimes_\tau \Omega(P;Q)$$

is the map obtained by applying [5, lemma 2.3] to the following set up:

1. $r = p$, $\rho = \langle (12\cdots p)\rangle = C_p$ and $\mathcal{V} = W$;
2. $(R, A) = (\mathbb{F}_p, P)$, $M = N = Q$ and $K = L = P \otimes_\tau \Omega(P;Q)$;
3. $f : M^{\otimes r} \longrightarrow N$ is the iterated multiplication $Q^{\otimes p} \longrightarrow Q$.

Let's recall the construction. Brüner defines

$$\Phi_{i,j} : W_i \otimes (P \otimes_\tau \Omega(P;Q))_j^{\otimes p} \longrightarrow (P \otimes_\tau \Omega(P;Q))_{j-i}$$

inductively. Other than a connectedness assumption we have not mentioned anything about the gradings on P and Q in this subsection; the gradings here are all (co)homological gradings.

As documented in [20, page 325, A1.2.15] there is a natural associative multiplication

$$(P \otimes_\tau \Omega(P;Q)) \otimes^\Delta (P \otimes_\tau \Omega(P;Q)) \longrightarrow P \otimes_\tau \Omega(P;Q)$$

$$p[p_1|\cdots|p_s]q \cdot p'[p'_1|\cdots p'_t]q' = \sum \pm pp'_{(0)}[p_1p'_{(1)}|\cdots|p_sp'_{(s)}|q_{(1)}p'_1|\cdots|q_{(t)}p'_t]q_{(t+1)}q'. \tag{8.3.2}$$

Here, $\sum p'_{(0)} \otimes\cdots\otimes p'_{(s)} \in P^{\otimes(s+1)}$ is the s-fold diagonal of $p' \in P$ and $\sum q_{(1)} \otimes\cdots\otimes q_{(t+1)} \in P^{\otimes t} \otimes Q$ is the t-fold diagonal of $q \in Q$. The sign is the usual one introduced when moving graded elements past one another. Also, $\otimes^\Delta$ denotes the internal tensor product in the category of P-comodules as in [15, page 74]; one checks directly that the multiplication above is a P-comodule map.

Iterating this multiplication gives a map

$$(P \otimes_\tau \Omega(P;Q))^{\otimes p} \longrightarrow P \otimes_\tau \Omega(P;Q)$$

which determines $\Phi_{0,*}$.

Suppose we have defined $\Phi_{i',j}$ for $i' < i$. Since $\Phi_{i,j} = 0$ for $j < i$ we may suppose that we have defined $\Phi_{i,j'}$ for $j' < j$. We define $\Phi_{i,j}$ using C_p-equivariance, the adjunction

$$P\text{-comodules} \rightleftarrows \mathbb{F}_p\text{-modules} \qquad f \longmapsto \tilde{f}$$

and the contracting homotopy

$$T = \sum_{i=1}^{p} (ir)^{i-1} \otimes S \otimes 1^{p-i}.$$

In particular, we define $\widetilde{\Phi_{i,j}}$ on $e_i \otimes x$ by

$$\widetilde{\Phi_{i,j}} = ([d\Phi_{i,j-1}]^{\sim} - [\Phi_{i-1,j-1}(d \otimes 1)]^{\sim})(1 \otimes T).$$

Our choice of Φ is natural in P and Q because we specified the multiplication determining $\Phi_{0,*}$ and the contracting homotopy T in a natural way.

Applying $\mathbb{F}_p \square_P (-)$ shows that Φ restricts to a natural C_p-equivariant DG homomorphism

$$\Phi : W \otimes \Omega(P;Q)^{\otimes p} \longrightarrow \Omega(P;Q).$$

8.4 Φ and primitives in Q

Assume P and Q are as in subsection 8.3.

Definition 8.4.1. Suppose that $x \in P \otimes_\tau \Omega(P;Q)$ and that $q \in Q$ is a P-comodule primitive. We write qx for $x \cdot 1[]q$.

Lemma 8.4.2. *Suppose that $q \in Q$ is P-comodule primitive with even degree. Then*

$$\Phi(e_i \otimes q^{i_1} x_1 \otimes \cdots q^{i_p} x_p) = q^{\sum_j i_j} \Phi(e_i \otimes x_1 \otimes \cdots x_p).$$

Proof. A special case of formula (8.3.2) gives

$$p'[p'_1|\cdots|p'_s]q' \cdot 1[]q = p'[p'_1|\cdots|p'_s]q'q.$$

Since $q \in Q$ is a P-comodule primitive with even degree we also obtain

$$1[]q \cdot p'[p'_1|\cdots p'_t]q' = p'[p'_1|\cdots|p'_t]q'q;$$

left and right multiplication by $1[]q$ agree. This observation proves the $i = 0$ case of the result since $\Phi_{0,*}(e_0 \otimes - \otimes \ldots \otimes -)$ is is equal to the map $(P \otimes \Omega(P;Q))^{\otimes p} \longrightarrow P \otimes \Omega(P;Q)$. We can now make use of the inductive formula

$$\widetilde{\Phi_{i,j}} = ([d\Phi_{i,j-1}]^{\sim} - [\Phi_{i-1,j-1}(d \otimes 1)]^{\sim})(1 \otimes T).$$

S, $\epsilon \otimes 1$ and ψ_Q commute with multiplication by q and so $1 \otimes T$ commutes with multiplication by $1 \otimes q^{i_1} \otimes \ldots \otimes q^{i_p}$. By an inductive hypothesis we can suppose $\Phi_{i,j-1}$ and $\Phi_{i-1,j-1}$ have the required property. It follows that $d\Phi_{i,j-1}$ and $\Phi_{i-1,j-1}(d \otimes 1)$ have the required property. The same is true of their adjoints and so the result holds for the adjoint of $\Phi_{i,j}$ and thus for $\Phi_{i,j}$ itself. $\square$

8.5 The operation $\beta\tilde{P}^0 : \Omega^0(P;Q) \longrightarrow \Omega^1(P;Q)$

In this subsection we define $\beta\tilde{P}^0 : \Omega^0(P;Q) \longrightarrow \Omega^1(P;Q)$ and note a couple of its properties. One should refer to the proof of [14, theorem 3.1]; this definition mimics that of $\beta P_0 : K_0 \longrightarrow K_{-1}$. In particular, we take $q = s = 0$ and the reader will note that we omit a $\nu(-1)$ in our definition.

Definition 8.5.1. Suppose that P is a connected commutative Hopf algebra, that Q is a commutative algebra over P and suppose that all nontrivial elements of P and Q have even degree.

[The assumption on degree is to ensure that no panic about signs ensues as a result of reading the discussion preceding [14, theorem 11.8].]

1. Let $a \in \Omega^0(P;Q)$ and let $b = da \in \Omega^1(P;Q)$.
2. We define $t_k \in \Omega(P;Q)^{\otimes p}$ for $0 < k < p$; in the following two formulae juxtaposition denotes tensor product. Write $p = 2m+1$ and define for $0 < k \leq m$
$$t_{2k} = (k-1)! \sum_I b^{i_1} a^2 b^{i_2} a^2 \cdots b^{i_k} a^2$$
summed over all k-tuples $I = (i_1, \ldots, i_k)$ such that $\sum_j i_j = p - 2k$. Define for $0 \leq k < m$
$$t_{2k+1} = k! \sum_I b^{i_1} a^2 \cdots b^{i_k} a^2 b^{i_{k+1}} a$$
summed over all $(k+1)$-tuples $I = (i_1, \ldots, i_{k+1})$ such that $\sum_j i_j = p - 2k - 1$.
3. Define $c \in W \otimes \Omega(P;Q)^{\otimes p}$ by
$$c = \sum_{k=1}^{m} (-1)^k \left[e_{p-2k-1} \otimes t_{2k} - e_{p-2k} \otimes t_{2k-1} \right].$$
4. $\beta\tilde{P}^0 a$ is defined to be Φc.

The sum defining c involves $t_1, \ldots, t_{2m}$. t_{2m} is given by $(m-1)! \sum_{i=0}^{m-1} a^{2i} b a^{2m-2i}$ and the terms in the sums defining $t_1, \ldots, t_{2m-1}$ all involve at least two b's.

We note that naturality of $\beta\tilde{P}^0$ follows from the naturality of Φ.

Lemma 8.5.2 ([14, 3.1(8)])**.** *With b and c as in definition 8.5.1 we have $dc = -e_{p-2} \otimes b^p$.*

Corollary 8.5.3. *With a and b as in definition 8.5.1 we have $d\beta\tilde{P}^0 a = -\Phi(e_{p-2} \otimes b^p)$.*

8.6 The proof

We turn to the proof of the proposition. First, we make the requisite definition.

Definition 8.6.1. Assume P and Q are as in definition 8.5.1. Given $b \in \Omega^1(P;Q)$, we define $\langle b \rangle^p$ to be $\Phi(e_{p-2} \otimes b^p)$. If $y \in \mathrm{Cotor}_P(Q)$ is represented by b, $\langle y \rangle^p$ is defined to be the class represented by $\langle b \rangle^p$.

$\langle y \rangle^p$ is well-defined in the previous definition; this fact is used in [14, definition 2.2]. We are now ready to prove the proposition.

Proof of proposition 5.2.6. Looking back to the proof of lemma 3.3.2 we see that we can choose $a, b \in \Omega(P; Q(1))$ together with lifts $a', \tilde{b} \in \Omega(P; Q(0))$ such that $a, b, \tilde{b}$ are cocycles, $da' = q_0^r \tilde{b}$, and a, b represent the x, y in the proposition statement.

By the remarks after definition 8.5.1 and the fact that $da' = q_0^r \tilde{b}$, $\beta \tilde{P}^0 a'$ is obtained by evaluating Φ on a sum in which the symbol $q_0^r \tilde{b}$ appears many times: one collection of terms contains precisely one $q_0^r \tilde{b}$ in each term and the others collections contain more than one $q_0^r \tilde{b}$ in each term. By lemma 8.4.2 $\beta \tilde{P}^0 a'$ is divisible by q_0^r and the image of $(\beta \tilde{P}^0 a')/q_0^r$ in $\Omega(P; Q(1))$ is a unit multiple of the image of $\Phi(e_0 \otimes t_{2m})/q_0^r$ in $\Omega(P; Q(1))$. This latter image is equal to

$$\gamma = (m-1)! \sum_{i=0}^{m-1} a^{2i} b a^{2m-2i}$$

which represents a unit multiple of $x^{p-1} y$. [In the formula above juxtaposition denotes multiplication.] Moreover, by corollary 8.5.3 and lemma 8.4.2, $d\beta \tilde{P}^0 a'$ is divisible by q_0^{pr} and $(d\beta \tilde{P}^0 a')/q_0^{pr}$ is equal to a unit multiple of $\Phi(e_{p-2} \otimes \tilde{b}^p) = \langle \tilde{b} \rangle^p$. The image of this element in $\Omega(P; Q(1))$ is $\langle b \rangle^p$.

We have shown that a unit multiple of $(\beta \tilde{P}^0 a')/q_0^r$ lifts γ. Moreover,

$$d\Big((\beta \tilde{P}^0 a')/q_0^r\Big) = q_0^{(p-1)r} \Big((d\beta \tilde{P}^0 a')/q_0^{pr}\Big) \doteq q_0^{(p-1)r} \langle \tilde{b} \rangle^p$$

and $\langle \tilde{b} \rangle^p$ lifts $\langle b \rangle^p$. Since γ represents a unit multiple of $x^{p-1} y$ and $\langle b \rangle^p$ respresents $\langle y \rangle^p$ we are done by definition 2.2 and definition 3.1.1. □

8.7 An auxilary calculation

We still need to prove lemma 5.2.7. We proceed by using the following lemma.

Lemma 8.7.1. *Let P be the primitively generated Hopf algebra $\mathbb{F}_p[\xi]/(\xi^p)$ where $|\xi|$ is even. Let h and b be classes in $\operatorname{Cotor}_P(\mathbb{F}_p)$ which are represented in ΩP by $[\xi]$ and*

$$\sum_{j=1}^{p-1} \frac{(-1)^j}{j} [\xi^j | \xi^{p-j}],$$

respectively. Then $\langle h \rangle^p \doteq b$.

Proof. This follows from remarks 6.9 and 11.1 of [14]. Beware of the different use of notation: our $\langle y \rangle^p$ is May's $\beta \tilde{P}^0 y$ and May defines $\langle y \rangle^p$ using the $\cup_1$-product associated to ΩP. □

The author thinks that the lemma above is true at the level of cochains. It is simple to check for $p = 3$ but for $p > 3$ the calculation become tedious.

Proof of lemma 5.2.7. Lemma 8.7.1 shows that $\langle h_i \rangle^p \doteq b_i$ in $\operatorname{Cotor}_{\mathbb{F}_p[\xi_i]/(\xi_i^p)}(\mathbb{F}_p)$. Since q_1 is primitive, definition 8.6.1 and lemma 8.4.2 show that

$$\langle [\![p^i - p^{[i-1]}]\!] h_i \rangle^p = [\![p \cdot [p^i - p^{[i-1]}]]\!] b_i$$

in $\operatorname{Cotor}_{\mathbb{F}_p[\xi_i]/(\xi_i^p)}(\mathbb{F}_p[q_1, q_1^{-1}])$. We have homomorphisms

$$\operatorname{Cotor}_P(Q(1)) \longrightarrow \operatorname{Cotor}_P(q_1^{-1} Q(1)) \longrightarrow \operatorname{Cotor}_{P(1)}(\mathbb{F}_p[q_1, q_1^{-1}]) \longleftarrow \operatorname{Cotor}_{\mathbb{F}_p[\xi_i]/(\xi_i^p)}(\mathbb{F}_p[q_1, q_1^{-1}]).$$

The third is induced by the inclusion $\mathbb{F}_p[\xi_i]/(\xi_i^p) \longrightarrow P(1)$. Theorem 5.1.2 tells us that the second is an isomorphism. Proposition 4.1.1 tells us the first is an isomorphism in a range; in lemma 5.2.8 we see that $[\![p^i - p^{[i-1]}h_i]\!]$ and $[\![p \cdot [p^i - p^{[i-1]}]]\!]b_i$ lie in this range and so have unique lifts to $\mathrm{Cotor}_P(Q(1))$. We are done by naturality of $\langle - \rangle^p$. $\square$

9 Some differentials in the $Q(0)$-BSS

In this section we make note of some differentials in the $Q(0)$-BSS and observe that they give nontrivial q_0-towers in $\mathrm{Cotor}_P(Q(0))$. These are of interest if we wish to analyze the differentials in the ASS-S^0 relevant to the image of J. This is not important for the main goal of the thesis but it may be useful for future work on the ASS-S^0.

Theorem 9.1. *For $i, j \geq 1$, $k > 1$, $[\![kp^{j-1} - p^{[j]}]\!]h_j$, $[\![kp^i - p^{[i]}]\!]h_i$, and $[\![kp^i + 1 - p^{[i+1]}]\!]b_i$, elements of $\mathrm{Cotor}_P(q_1^{-1}Q(1))$, have unique lifts to $\mathrm{Cotor}_P(Q(1))$. In addition, in the $Q(0)$-BSS we have, for $i, j \geq 1$ and $k > 1$, the following differentials:*

1. *$d_{p^{[j]}}[\![kp^{j-1}]\!] \doteq [\![kp^{j-1} - p^{[j]}]\!]h_j$ if $p \nmid k$;*
2. *$d_{p^i-1}[\![kp^i - p^{[i]}]\!]h_i \doteq [\![kp^i + 1 - p^{[i+1]}]\!]b_i$.*

Proof. The differentials hold in the q_1^{-1}-BSS by theorem 5.2.10 and lemma 3.5.4. It is enough by lemma 3.3.2 to prove that the differentials hold in the q_0^∞-BSS. As in the proof of lemma 5.2.8 we refer to proposition 4.1.1 and lemma 4.1.2. We note that

$$\begin{aligned}
[\![kp^{j-1}]\!] &\in E_1^{0,kp^{j-1}-p^{[j]}-1,2kp^{j-1}(p-1),-p^{[j]}-1}(q_1^{-1});\\
[\![kp^{j-1} - p^{[j]}]\!]h_j &\in E_1^{1,kp^{j-1}-p^{[j]}-1,2kp^{j-1}(p-1),-1}(q_1^{-1});\\
[\![kp^i - p^{[i]}]\!]h_i &\in E_1^{1,kp^i-p^{[i+1]},2kp^i(p-1),-p^i}(q_1^{-1});\\
[\![kp^i + 1 - p^{[i+1]}]\!]b_i &\in E_1^{2,kp^i-p^{[i+1]},2kp^i(p-1),-1}(q_1^{-1});
\end{aligned}$$

and so it is enough to show $2kp^{j-1}(p-1) < U(0) + 2(p^2-1)(kp^{j-1} - p^{[j]} + 1) - 2(p-1)$. The worst case is when $k = 2$ and in this case the inequality is equivalent to $(p+1)p^{[j]} < 2p^j + p$. $\square$

We rewrite the theorem using the more cumbersome notation $h_{j,0}$ and $b_{i,0}$ since we have cause to use the second grading in the proposition which follows.

Theorem 9.2. *In the $Q(0)$-BSS we have, for $i, j \geq 1$ and $k > 1$, the following differentials:*

1. *$d_{p^{[j]}}[\![kp^{j-1}]\!] \doteq [\![kp^{j-1} - p^{[j]}]\!]h_{j,0}$ if $p \nmid k$;*
2. *$d_{p^i-1}[\![kp^i - p^{[i]}]\!]h_{i,0} \doteq [\![kp^i + 1 - p^{[i+1]}]\!]b_{i,0}$.*

The following proposition takes care of the $k = 1$ case.

Proposition 9.3. *For $i \geq 1$, $[\![p^i - p^{[i]}]\!]h_{i,0}$ has a unique lift to $\mathrm{Cotor}_P(Q(1))$. Moreover, in the $Q(0)$-BSS we have, for $i, j \geq 1$, the following differentials:*

1. *$d_{p^{j-1}}[\![p^{j-1}]\!] \doteq h_{1,j-1}$;*

2. $d_{p^i-p^{[i]}}[\![p^i-p^{[i]}]\!]h_{i,0} \doteq b_{1,i-1}$.

Proof. $[\![p^i-p^{[i]}]\!]h_i \in \mathrm{Cotor}_P^{1,p^i-p^{[i]},2p^i(p-1)}(q_1^{-1}Q(1))$. Moreover, $p^i+(p+1)p^{[i]}+1 < (p+1)(p^i+1)$ which implies $2p^i(p-1) < U(0)+2(p^2-1)(p^i-p^{[i]}+1)-2(p-1)$ and so proposition 4.1.1 gives the first claim.

The first class of differentials follows from the following formulae in the cobar complex $\Omega(P;Q(0))$.

$$d\big(q_1^{p^{j-1}}\big) = \big[\xi_1^{p^{j-1}}\big]q_0^{p^{j-1}}$$

We turn to the second class of differentials. The proof is by an induction. Some of the ideas used are similar in flavour to those in section 7. Because this result will not be used in the remainder of the thesis we will not go through all the details.

The $i=1$ case is verified by checking the following formula in $\Omega(P;Q(0))$.

$$d\left[\sum_{j=1}^{p-1}\frac{(-1)^j}{j}[\xi_1^j]q_0^{j-1}q_1^{p-j}\right] = \sum_{j=1}^{p-1}\frac{\binom{p}{j}}{p}[\xi_1^j|\xi_1^{p-j}]q_0^{p-1}$$

Suppose that for some $n \geq 1$ we have $\tilde{x}$ and $\tilde{y}$ such that

$$\begin{tikzcd}
\Omega(P;Q(0)) \arrow[d] & \tilde{x} \arrow[r, mapsto, "d"] \arrow[d, mapsto] & q_0^{p^n-p^{[n]}}\tilde{y} & \tilde{y} \arrow[d, mapsto] \\
\Omega(P;Q(1)) \arrow[d] & x \arrow[d, mapsto] & & y = \sum_{j=1}^{p-1}\frac{\binom{p}{j}}{p}[\xi_1^{jp^{n-1}}|\xi_1^{(p-j)p^{n-1}}] \\
\Omega(P(1);\mathbb{F}_p[q_1]) & (-1)^n[\xi_n]q_1^{p^n-p^{[n]}} & &
\end{tikzcd}$$

We can raise all expressions to the p^{th} power to get

$$\begin{tikzcd}
\Omega(P;Q(0)) \arrow[d] & \tilde{z} \arrow[d, mapsto] & \tilde{P}^0\tilde{x} \arrow[r, mapsto, "d"] \arrow[d, mapsto] & q_0^{p^{n+1}-p\cdot p^{[n]}}\tilde{P}^0\tilde{y} & \tilde{P}^0\tilde{y} \arrow[d, mapsto] \\
\Omega(P;Q(1)) \arrow[d] & z & dz = \tilde{P}^0 x \arrow[d, mapsto] & & \tilde{P}^0 y = \sum_{j=1}^{p-1}\frac{\binom{p}{j}}{p}[\xi_1^{jp^n}|\xi_1^{(p-j)p^n}] \\
\Omega(P(1);\mathbb{F}_p[q_1]) & & 0 = (-1)^n[\xi_n^p]q_1^{p^{n+1}-p\cdot p^{[n]}} & &
\end{tikzcd}$$

We see that $\tilde{P}^0\tilde{x}-d\tilde{z}$ maps to zero in $\Omega(P;Q(1))$ and so it is divisible by q_0. Let $\tilde{w} = \frac{\tilde{P}^0\tilde{x}-d\tilde{z}}{q_0}$. Then

$$q_0 d\tilde{w} = dq_0\tilde{w} = d\tilde{P}^0\tilde{x} = q_0^{p^{n+1}-p\cdot p^{[n]}}\tilde{P}^0\tilde{y} \implies d\tilde{w} = q_0^{p^{n+1}-p^{[n+1]}}\tilde{P}^0\tilde{y}.$$

The induction is completed by showing that the image of $\tilde{w}$ in $\Omega(P(1);\mathbb{F}_p[q_1])$ is

$$(-1)^{n+1}[\xi_{n+1}]q_1^{p^{n+1}-p^{[n+1]}},$$

for then we have the following diagram.

$$
\begin{tikzcd}
\Omega(P;Q(0)) \arrow[d] & \tilde{w} \arrow[r, mapsto, "d"] \arrow[d, mapsto] & q_0^{p^{n+1}-p^{[n+1]}}\tilde{P}^0\tilde{y} & \tilde{P}^0\tilde{y} \arrow[d, mapsto] \\
\Omega(P;Q(1)) \arrow[d] & w \arrow[d, mapsto] & & \tilde{P}^0 y = \sum_{j=1}^{p-1}\frac{\binom{p}{j}}{p}[\xi_1^{jp^n}|\xi_1^{(p-j)p^n}] \\
\Omega(P(1);\mathbb{F}_p[q_1]) & (-1)^{n+1}[\xi_{n+1}]q_1^{p^{n+1}-p^{[n+1]}} & &
\end{tikzcd}
$$

$\tilde{P}^0\tilde{x}$ will not contribute to the image of $\tilde{w}$ in $\Omega(P(1);\mathbb{F}_p[q_1])$. We see that z must have a term equal to

$$(-1)^n q_1^{p^{n+1}-p^{[n+1]}} q_{n+1}$$

Thus $-d\tilde{z}$ has a term equal to $(-1)^{n+1}[\xi_{n+1}]q_0 q_1^{p^{n+1}-p^{[n+1]}}$. With care, one can complete the proof by mimicking the methods of I.7.4. $\square$

Corollary 9.4. *We have nonzero elements in* $E_\infty^{*,*,*,*}(q_0^{-1})$

$$
\begin{aligned}
&\{\langle 1\rangle_v : v<0\}\\
&\cup\left\{\left\langle [\![p^{j-1}]\!]\right\rangle_v :\ -p^{j-1}\le v<0\right\}\cup\left\{\left\langle [\![kp^{j-1}]\!]\right\rangle_v :\ p\nmid k>1,\ -p^{[j]}\le v<0,\right\}\\
&\cup\left\{\left\langle [\![p^i-p^{[i]}]\!]h_{i,0}\right\rangle_v :\ p^{[i]}-p^i\le v<0\right\}\cup\left\{\left\langle [\![kp^i-p^{[i]}]\!]h_{i,0}\right\rangle_v :\ k>1,\ 1-p^i\le v<0\right\},
\end{aligned}
$$

where the i and j indices range over all positive integers.

Proof. The previous results together with lemma 3.3.2 show that the elements are permanent cycles. Corollary 5.4.2 tells us that all the elements are nonzero in $E_\infty^{*,*,*,*}(q_1^{-1})$. $\square$

This result can be interpretted as saying that certain elements in $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$ are permanent cycles in the "chromatic spectral sequence" for $\mathrm{Cotor}_P(Q(0))$ (see [15, section 5]). The only boundaries on the 1-line are of the form q_0^t for $t<0$. Thus, the subgroup of permanent cycles in $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)/\ \mathbb{F}_p\ \langle q_0^t : t<0\rangle$ determines elements in $\mathrm{Cotor}_P(Q(0))$. In particular, by using the identification of (5.3.1) we obtain q_0-towers in $\mathrm{Cotor}_P(Q(0))$.

Corollary 9.5. *We have nonzero elements in* $E_\infty^{*,*,*,*}(Q(0))$

$$
\begin{aligned}
&\{\langle 1\rangle_v : v\ge 0\}\\
&\cup\left\{\left\langle h_{1,j-1}\right\rangle_v :\ 0\le v<p^{j-1}\right\}\cup\left\{\left\langle [\![kp^{j-1}-p^{[j]}]\!]h_{j,0}\right\rangle_v :\ p\nmid k>1,\ 0\le v<p^{[j]},\right\}\\
&\cup\left\{\left\langle b_{1,i-1}\right\rangle_v :\ 0\le v<p^i-p^{[i]}\right\}\cup\left\{\left\langle [\![kp^i+1-p^{[i+1]}]\!]b_{i,0}\right\rangle_v :\ k>1,\ 0\le v<p^i-1\right\},
\end{aligned}
$$

where the i and j indices range over all positive integers.

For the elements in $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$ corresponding to

$$\left\{\left\langle [\![p^{j-1}]\!] \right\rangle_v : \ -p^{[j]} \le v < -p^{j-1}\right\} \cup \left\{\left\langle [\![p^i - p^{[i]}]\!] h_{i,0} \right\rangle_v : \ 1 - p^i \le v < p^{[i]} - p^i\right\} \subset E_\infty^{*,*,*,*}(q_1^{-1})$$

we can ask about what differentials they support in the chromatic spectral sequence of [15]. We do not say anything more about this problem.

Chapter II

Adams spectral sequences

1 Main results and outline of chapter

Recall that the goal of this thesis is to compute the homotopy of the v_1-periodic sphere $v_1^{-1}S/p^\infty$ using classical Adams spectral sequence methods and to use this computation to obtain information about the classical mod p Adams spectral sequence for S^0. Our main result of this chapter is the following theorem.

Theorem 1.1. *There is a spectral sequence with E_2-page $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$ which converges to $\pi_*(v_1^{-1}S/p^\infty)$. We call this the localized Adams spectral sequence for the v_1-periodic sphere $v_1^{-1}S/p^\infty$ (LASS-∞).*

In order to obtain information about the classical mod p Adams spectral sequence for S^0 we need two propositions.

Proposition 1.2. *There is a spectral sequence with E_2-page $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$ which converges to $\pi_*(S/p^\infty)$, called the modified Adams spectral sequence for the Prüfer sphere S/p^∞ (MASS-∞).*

By construction there is a map of spectral sequences $E_*^{*,*}(MASS\text{-}\infty) \longrightarrow E_*^{*,*}(LASS\text{-}\infty)$.

Proposition 1.3. *Associated to the map $\Sigma^{-1}S/p^\infty \longrightarrow S^0$ is a map of spectral sequences*

$$E_*^{*,*}(MASS\text{-}\infty) \longrightarrow E_*^{*,*}(ASS\text{-}S^0).$$

At E_2-pages this map can be identified with the connecting homomorphism

$$\mathrm{Cotor}_P(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P(Q(0))$$

arising from the short exact sequences of P-comodules $0 \longrightarrow Q(0) \longrightarrow q_0^{-1}Q(0) \longrightarrow Q(0)/q_0^\infty \longrightarrow 0$ (see definition I.1.4).

The LASS-∞ has an involved construction and there are many intermediate spectral sequences to set up because the LASS-∞ is a direct limit of localized modified Adams spectral sequences. We note that this spectral sequence is set up in [13] but we choose to give a self-contained exposition.

We begin the chapter by recalling the construction of the classical Adams spectral sequence (ASS) for a p-complete spectrum. We proceed to show how this construction is altered to give a modified Adams spectral sequence for S/p^n (MASS-n). The work of Miller in [15] allows us to give convenient descriptions of $E_2(\text{ASS-}S^0)$ and $E_2(\text{MASS-}n)$. By taking a direct limit of the MASS-n's we obtain a modified Adams spectral sequence for the Prüfer sphere S/p^∞ (MASS-∞). We then repeat the story, localizing throughout, to construct the LASS-n and the LASS-∞.

There are four technical issues which we have to battle against. We wish to give the MASS-n a multiplicative structure. In order to set up the LASS-n we need to show that a particular element in the MASS-n is a permanent cycle. We must show that our spectral sequences converge. We must construct various maps between our spectral sequences.

Giving the MASS-n a multiplicative structure leads us to consider pointset level constructions and these are addressed in chapter III, section 2. Section 1 of chapter III addresses the permanent cycle issue.

The convergence problems associated to a localized Adams spectral sequence are considered in [12, theorem 2.13]. The convergence problems associated to a modified Adams spectral sequence are considered in [23, theorem 3.6]. Since we localize *modified* Adams spectral sequences the result of [12] does not apply. We make explicit use of [23, 3.6] but we address all other convergence issues in a self-contained manner in section 7. For this we need some results on vanishing lines which are proved in section 6.

The hardest map of spectral sequences we need to construct would be given to us immediately by a general geometric boundary. The cobar construction was useful for us in chapter I and we imitate the algebra by using canonical resolutions for our Adams spectral sequences. This leads to a pretty way of constructing the requisite map and we do not try and prove the most general result.

In the final part of the chapter we set up two more spectral sequences which are needed to calculate $E_3(\text{LASS-}\infty)$ and we tie up all the loose ends in the main computation.

2 The classical Adams spectral sequence

In this section we recall the construction of the classical Adams spectral sequences along with some of its properties. This section may be omitted as it contains nothing new for the expert.

2.1 The main result

Before stating the main result of the section we recall some notation.

Recall that P is the polynomial algebra on generators $\{\xi_n : n \geq 1\}$ where $|\xi_n| = (0, 2(p^n-1))$ and that P is a Hopf algebra when equipped with the Milnor diagonal

$$P \longrightarrow P \otimes P, \quad \xi_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i, \quad (\xi_0 = 1).$$

Definition 2.1.1. Let E be the exterior Hopf algebra on primitive generators $\{\tau_n : n \geq 0\}$ where $|\tau_n| = (0, 2p^n - 1)$. E is a Hopf algebra over P, i.e. we have a multiplicative map

$$E \longrightarrow P \otimes E, \quad \tau_n \longmapsto \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i$$

which makes E a coalgebra over P.

Since $p > 2$ the semi-tensor product Hopf algebra $A = E\widetilde{\otimes}P$ is the dual of Steenrod algebra and A-comodules are the same as E-comodules over P [15, pages 78, 75]. The reader should refer to [15, pages 75-76] for notation regarding twisting morphisms and twisted tensor products. We have a twisting morphism $\theta : E \to Q(0)$ which takes $1 \longmapsto 0$, $\tau_n \longmapsto q_n$, and $\tau_{n_1} \cdots \tau_{n_r} \longmapsto 0$ when $r > 1$. We write τ for the universal twisting morphism instead of [ ].

The next proposition contains all the properties that we wish to recall about the classical Adams spectral sequence.

Proposition 2.1.2. *Let Y be a p-complete bounded below spectrum. There is a spectral sequence, the Adams spectral sequence for Y, with E_1-page $\Omega(A; H_*(Y))$ and E_2-page $\mathrm{Cotor}_P(Q(0) \otimes_\theta H_*(Y))$. Moreover, it converges to $\pi_*(Y)$ and is functorial in Y.*

Suppose Y' is another p-complete bounded below spectrum. Then there is a pairing

$$E_*^{*,*}(ASS\text{-}Y) \otimes E_*^{*,*}(ASS\text{-}Y') \longrightarrow E_*^{*,*}(ASS\text{-}(Y \wedge Y'))$$

converging to the smash product $\pi_*(Y) \otimes \pi_*(Y') \longrightarrow \pi_*(Y \wedge Y')$ which, at the E_1-page, agrees with the multiplication

$$\Omega(A; H_*(Y)) \otimes \Omega(A; H_*(Y')) \longrightarrow \Omega(A; H_*(Y) \otimes^\Delta H_*(Y')) = \Omega(A; H_*(Y \wedge Y'))$$

and at the E_2-page agrees with the multiplication

$$\mathrm{Cotor}_P(Q(0) \otimes_\theta H_*(Y)) \otimes \mathrm{Cotor}_P(Q(0) \otimes_\theta H_*(Y')) \longrightarrow \mathrm{Cotor}_P(Q(0) \otimes_\theta H_*(Y \wedge Y')).$$

This result is well-known and, of course, the spectral sequence is originally due to Adams. We take convergence for granted but we recall how the other properties are verified. In this section we address all the claims of the propostion except those concerning the E_2-page; they will be verified in section 4. We hope that recalling these arguments will help illuminate our later arguments.

2.2 Setting up the ASS-Y

Notation 2.2.1. Recall that S^0 denotes the sphere spectrum completed at p. We write S for the sphere spectrum, H for the Eilenberg-MacLane spectrum of type $\mathbb{F}_p$ (a ring spectrum), $\eta : S \longrightarrow H$ for the unit, $\overline{H}$ for the homotopy fiber of η and $\mu : H \wedge H \longrightarrow H$ for the multiplication.

Let Y be the p-completion of a bounded below spectrum. Applying $\pi_*(-)$ to the *H-canonical tower* for Y

$$\begin{tikzcd}
Y \arrow[d, "\eta\wedge Y"'] & \overline{H} \wedge Y \arrow[l] \arrow[d] & \cdots \arrow[l] & \overline{H}^s \wedge Y \arrow[l] \arrow[d] & \overline{H}^{s+1} \wedge Y \arrow[l] \arrow[d] & \cdots \arrow[l] \\
H \wedge Y \arrow[ur, dotted] & H \wedge \overline{H} \wedge Y \arrow[ur, dotted] & & H \wedge \overline{H}^s \wedge Y \arrow[ur, dotted] & H \wedge \overline{H}^{s+1} \wedge Y \arrow[ur, dotted] &
\end{tikzcd}$$

gives an exact couple.

Definition 2.2.2. The spectral sequence arising from this exact couple is called the *classical Adams spectral sequence for Y* (ASS-Y). It has E_1-page

$$E_1^{s,t}(\text{ASS-}Y) = \begin{cases} \pi_{t-s}(H \wedge \overline{H}^{\wedge s} \wedge Y) = \pi_t(H \wedge (\Sigma\overline{H})^{\wedge s} \wedge Y) & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases}$$

and d_r has degree $(r, r-1)$. The spectral sequence converges to $\pi_*(Y)$ and the filtration degree is given by s. In particular, we have an identification

$$E_\infty^{s,t}(\text{ASS-}Y) = F^s\pi_{t-s}(Y)/F^{s+1}\pi_{t-s}(Y)$$

where $F^s\pi_*(Y) = \mathrm{im}(\pi_*(\overline{H}^{\wedge s} \wedge Y) \longrightarrow \pi_*(Y))$. The identification is given by lifting an element of $F^s\pi_*(Y)$ to $\pi_*(\overline{H}^{\wedge s} \wedge Y)$ and mapping this lift down to $\pi_*(H \wedge \overline{H}^{\wedge s} \wedge Y)$ to give a permanent cycle.

2.3 Relation of E_1(ASS-Y) to the cobar construction

Throughout this subsection Y will denote a spectrum; one can set up the ASS for any spectrum Y but it may not converge. Our goal is to identify $E_1^{\bullet,*}(\text{ASS-}Y)$ with $\Omega(A; H_*(Y))$. Firstly, we need to fix notation and some identifications.

Notation 2.3.1. In this subsection and the beginning of the next we will write $\overline{H}$ for the homotopy **co**fiber of η so that we have a cofibration sequence

$$S \xrightarrow{\eta} H \xrightarrow{p} \overline{H}$$

and a split cofibration sequence

$$\begin{tikzcd} H \arrow[r, "H\wedge\eta"] & H \wedge H \arrow[l, bend right, "\mu"'] \arrow[r, "H\wedge p"] & H \wedge \overline{H}. \arrow[l, bend left, "s"] \end{tikzcd}$$

[Because $H^s(H) = 0$ for $s < 0$, the μ and s are actually determined uniquely by the properties that $\mu \circ (H \wedge \eta) = 1$ and $(H \wedge p) \circ s = 1$.]

Lemma 2.3.2. *Write A, $\overline{A}$, $\iota : \overline{A} \longrightarrow A$ and $H_*(Y)$ for $\pi_*(H\wedge H)$, $\pi_*(H\wedge\overline{H})$, $\pi_*(s)$ and $\pi_*(H\wedge Y)$. We have isomorphisms*

$$a_{s,i} : A \otimes A^{\otimes(s-i)} \otimes \overline{A}^{\otimes i} \otimes H_*(Y) \longrightarrow H_*(H \wedge H^{\wedge(s-i)} \wedge \overline{H}^{\wedge i} \wedge Y)$$

for all s and i with $0 \le i \le s$. They can be chosen so that the following diagrams commute.

$$\begin{tikzcd} A \otimes A^{\otimes(s-i)} \otimes \overline{A}^{\otimes i} \otimes H_*(Y) \arrow[r, "a_{s,i}"] \arrow[d, "A\otimes A^{\otimes(s-i)}\otimes\iota\otimes\overline{A}^{\otimes(i-1)}\otimes H_*(Y)"'] & H_*(H \wedge H^{\wedge(s-i)} \wedge \overline{H}^{\wedge i} \wedge Y) \arrow[d, "H_*(H^{\wedge(s-i)}\wedge s\wedge\overline{H}^{\wedge(i-1)}\wedge Y)"] \\ A \otimes A^{\otimes(s-i+1)} \otimes \overline{A}^{\otimes(i-1)} \otimes H_*(Y) \arrow[r, "a_{s,i-1}"] & H_*(H \wedge H^{\wedge(s-i+1)} \wedge \overline{H}^{\wedge(i-1)} \wedge Y) \end{tikzcd}$$

In particular, $a_{s,i}$ can be described as the composite

$$\begin{tikzcd} A \otimes A^{\otimes(s-i)} \otimes \overline{A}^{\otimes i} \otimes H_*(Y) \arrow[r, "A\otimes A^{\otimes(s-i)}\otimes\iota^{\otimes i}\otimes H_*(Y)"] & A \otimes A^{\otimes s} \otimes H_*(Y) \arrow[dl, "a_{s,0}"] \\ H_*(H \wedge H^{\wedge s} \wedge Y) \arrow[r, "H_*(H\wedge H^{\wedge(s-i)}\wedge p^{\wedge i}\wedge Y)"'] & H_*(H \wedge H^{\wedge(s-i)} \wedge \overline{H}^{\wedge i} \wedge Y). \end{tikzcd}$$

Proof. We define the maps by induction on i. Let $a_{s,0} : A \otimes A^{\otimes s} \otimes H_*(Y) \longrightarrow H_*(H \wedge H^{\wedge s} \wedge Y)$ be defined by

$$f_0 \otimes f_1 \otimes \ldots \otimes f_s \otimes g \longmapsto H_*(\mu \wedge \mu^s \wedge Y)(f_0 \wedge f_1 \wedge \ldots \wedge f_s \wedge g).$$

It is a familiar result that this is an isomorphism. For the inductive step we note that the cofibration sequence

$$H \wedge \overline{H} \xrightarrow{s} H \wedge H \xrightarrow{\mu} H$$

gives the short exact sequences in the following diagram (we have written j for $\pi_*(\mu)$).

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
A \otimes A^{\otimes(s-i)} \otimes \overline{A} \otimes \overline{A}^{\otimes(i-1)} \otimes H_*(Y) \arrow[r, dotted] \arrow[d, "1\otimes 1\otimes\iota\otimes 1\otimes 1"'] & H_*(H^{\wedge(s-i)} \wedge (H \wedge \overline{H}) \wedge \overline{H}^{\wedge(i-1)} \wedge Y) \arrow[d, "H_*(1\wedge s\wedge 1\wedge 1)"] \\
A \otimes A^{\otimes(s-i)} \otimes A \otimes \overline{A}^{\otimes(i-1)} \otimes H_*(Y) \arrow[r, "a_{s,i-1}"] \arrow[d, "1\otimes 1\otimes j\otimes 1\otimes 1"'] & H_*(H^{\wedge(s-i)} \wedge (H \wedge H) \wedge \overline{H}^{\wedge(i-1)} \wedge Y) \arrow[d, "H_*(1\wedge\mu\wedge 1\wedge 1)"] \\
A \otimes A^{\otimes(s-i)} \otimes \mathbb{F}_p \otimes \overline{A}^{\otimes(i-1)} \otimes H_*(Y) \arrow[r, "a_{s-1,i-1}"] \arrow[d] & H_*(H^{\wedge(s-i)} \wedge H \wedge \overline{H}^{\wedge(i-1)} \wedge Y) \arrow[d] \\
0 & 0
\end{tikzcd}$$

One checks that the bottom square commutes so we can take $a_{s,i}$ to be the induced dashed arrow. $\square$

It is well-known that A is the dual of the Steenrod algebra (and so the notation of the lemma is consistent with the notation used earlier). $a_{0,0} : A \otimes H_*(Y) \longrightarrow H_*(H \wedge Y)$ and the A-comodule structure map of $H_*(Y)$ is given by $a_{0,0}^{-1} \circ H_*(\eta \wedge 1)$. $a_{s,i}$ is an isomorphism of A-comodules as long as we give the domain the extended comodule structure.

Notation 2.3.3. Write $\mathscr{S}$ for the stable homotopy category.

Now we go about defining three cochain complexes in $\mathscr{S}$. The second complex's relationship with $E_1^{\bullet,*}(\text{ASS-}Y)$ is clear and the third complex is constructed so that it realises $A \otimes_\tau \Omega(A; H_*(Y))$. The first complex is used as an intermediate object and the letters N and D are chosen with the terminology 'normalized' and 'degenerate' in mind. The reader may need to recall the definition of H-injective from [17].

Definition 2.3.4. Let $C^\bullet$ be the cochain complex in $\mathscr{S}$ with $C^s = H^{\wedge(s+1)} \wedge Y$ and $d : C^{s-1} \longrightarrow C^s$ given by

$$\left(\sum_{i=0}^{s} (-1)^i \Big[H^{\wedge i} \wedge \eta \wedge H^{\wedge(s-i)} \Big] \right) \wedge Y.$$

Using the isomorphisms, $a_{s,0}$, we can identify $H_*(C^\bullet)$ with $C(A, A, H_*(Y))^\bullet$, the unreduced two-sided cobar construction. This cochain complex has s^{th} term $A \otimes A^{\otimes s} \otimes H_*(Y)$ and a differential given by an alternating sum of coaction maps.

Definition 2.3.5. Let $N^\bullet$ be the cochain complex in $\mathscr{S}$ with $N^s = H \wedge \overline{H}^{\wedge s} \wedge Y$ and $d : N^{s-1} \longrightarrow N^s$ given by

$$\left(H \wedge \overline{H}^{\wedge(s-1)} \xrightarrow{\quad p\wedge\overline{H}^{\wedge(s-1)}\quad} \overline{H}^{\wedge s} \xrightarrow{\quad \eta\wedge\overline{H}^{\wedge s}\quad} H \wedge \overline{H}^{\wedge s}\right) \wedge Y.$$

Let $r : C^\bullet \longrightarrow N^\bullet$ be the map $H \wedge p^{\wedge\bullet} \wedge Y$. We have $E_1^{\bullet,*}(\text{ASS-}Y) = \pi_*(N^\bullet)$.

Definition 2.3.6. Let $D^s = H \wedge \overline{H}^{\wedge s} \wedge Y$. Using the isomorphisms, $a_{s,s}$, we can identify $H_*(D^s)$ with $(A \otimes_\tau \Omega(A; H_*(Y)))_s$. Since each D^s is H-injective we can define a map $d : D^{s-1} \longrightarrow D^s$ by requiring that it induce the coboundary map $d : (A \otimes_\tau \Omega(A; H_*(Y)))_{s-1} \longrightarrow (A \otimes_\tau \Omega(A; H_*(Y)))_s$. Similarly, we can define a map $i : D^\bullet \longrightarrow C^\bullet$ by requiring that it induce the inclusion

$$A \otimes \overline{A}^{\otimes s} \otimes H_*(Y) \longrightarrow A \otimes A^{\otimes s} \otimes H_*(Y)$$

at the s^{th} level. By lemma 2.3.2 we can describe this map explcitly as

$$\left((H^{\wedge(s-1)}\wedge s)\circ(H^{\wedge(s-2)}\wedge s\wedge\overline{H})\circ\ldots\circ(H^{\wedge(s-i)}\wedge s\wedge\overline{H}^{\wedge(i-1)})\circ\ldots\circ(H\wedge s\wedge\overline{H}^{\wedge(s-2)})\circ(s\wedge\overline{H}^{\wedge(s-1)})\right)\wedge Y.$$

We wish to identify $N^\bullet$ and $D^\bullet$ since the first arises in our construction of the ASS-Y, whereas the second gives rise to the cobar construction $\Omega(A; H_*(Y))$. By the explicit descriptions above we have the following result.

Lemma 2.3.7. *The composite* $D^\bullet \xrightarrow{\ i\ } C^\bullet \xrightarrow{\ r\ } N^\bullet$ *is the identity.*

Corollary 2.3.8. *Using the isomorphisms, $a_{s,s}$, we can identify $H_*(N^\bullet)$ with $A \otimes_\tau \Omega(A; H_*(Y)$. Thus we can identify $E_1^{\bullet,*}(\text{ASS-}Y) = \pi_*(N^\bullet) = \mathbb{F}_p\square_A H_*(N^\bullet)$ with $\Omega(A; H_*(Y))$.*

$\Omega(A; H_*(Y))$ is bigraded since we have a cohomological grading s and a 'total' grading t coming from the gradings on A and $H_*(Y)$. In the indentification above $E_1^{s,t}(\text{ASS-Y}) = \Omega^{s,t}(A; H_*(Y))$.

We note that the differentials in $C^\bullet$ and $N^\bullet$ and the maps i and r are all obtained by smashing various maps with Y.

2.4 Multiplicativity of the ASS-S^0

$\pi_*(S^0)$ and $\Omega(A)$ are rings and the ASS-S^0 is multiplicative. In this subsection we recall why the ASS-S^0 is multiplicative and how we can identify $E_1^{\bullet,*}(\text{ASS-}S^0) = E_1^{\bullet,*}(\text{ASS-}S)$ with $\Omega(A)$ as rings.

In short, the spectral sequence is given a multiplicative structure using the argument in theorem IV.4.4 of [5] which makes use of a map of towers. To make sure we obtain the desired structure on the E_1-page we take care when constructing the underlying map on augmented chain complexes. For an explanation of the terminology just used see III.2.

First, we adapt some of the notation from the last subsection to this subsection.

Notation 2.4.1. Let $C^\bullet$ be the augmented cochain complex in $\mathscr{S}$ with $C^s = H^{\wedge(s+1)}$ and $d : C^{s-1} \longrightarrow C^s$ given by

$$\sum_{i=0}^{s}(-1)^i\left[H^{\wedge i} \wedge \eta \wedge H^{\wedge(s-i)}\right].$$

Let $N^\bullet$ be the augmented cochain complex in $\mathscr{S}$ with $N^{-1} = S$ and $N^s = H \wedge \overline{H}^{\wedge s}$ for $s \geq 0$, $d : N^{-1} \longrightarrow N^0$ given by $\eta : S \longrightarrow H$ and $d : N^{s-1} \longrightarrow N^s$ given by

$$H \wedge \overline{H}^{\wedge(s-1)} \xrightarrow{p \wedge \overline{H}^{\wedge(s-1)}} \overline{H}^{\wedge s} \xrightarrow{\eta \wedge \overline{H}^{\wedge s}} H \wedge \overline{H}^{\wedge s}$$

for $s > 0$. Let $i : N^\bullet \to C^\bullet$ be the unique map of chain complexes with the property that it induces the inclusion

$$A \otimes \overline{A}^{\otimes s} \longrightarrow A \otimes A^{\otimes s}$$

at the s^{th} level. Let $r : C^\bullet \longrightarrow N^\bullet$ be the map of chain complexes $H \wedge p^{\wedge \bullet}$. Note that both maps respect the augmentations and recall that the composite ri is the identity.

We wish to define multiplications $m_N : N^\bullet \wedge N^\bullet \longrightarrow N^\bullet$ and $m_C : C^\bullet \wedge C^\bullet \longrightarrow C^\bullet$ so that the multiplication on the cobar complex ΩA is realised by m_N and the following diagram commutes.

$$\begin{tikzcd}
N^\bullet \wedge N^\bullet \arrow[r, "m_N"] \arrow[d, "i \wedge i"'] & N^\bullet \arrow[d, "i"] \\
C^\bullet \wedge C^\bullet \arrow[r, "m_C"] & C^\bullet
\end{tikzcd} \tag{2.4.2}$$

Definition 2.4.3. Using the Künneth isomorphism together with the isomorphisms above we can identify $H_*(N^\bullet \wedge N^\bullet)$ with $(A \otimes_\tau \Omega A) \otimes^\Delta (A \otimes_\tau \Omega A)$. Analogous to that which is defined in [15], there is a multiplication $(A \otimes_\tau \Omega A) \otimes^\Delta (A \otimes_\tau \Omega A) \longrightarrow A \otimes_\tau \Omega A$. Since everything in sight is H-injective this defines a map $m_N : N^\bullet \wedge N^\bullet \longrightarrow N^\bullet$.

Definition 2.4.4. $m_C : C^\bullet \wedge C^\bullet \longrightarrow C^\bullet$ is given by $H^{\wedge s} \wedge \mu \wedge H^{\wedge t} : C^s \wedge C^t \longrightarrow C^{s+t}$.

On homology m_C induces a map extending the multiplication $(A \otimes_\tau \Omega A) \otimes^\Delta (A \otimes_\tau \Omega A) \longrightarrow A \otimes_\tau \Omega A$ and so (2.4.2) commutes. Thus, we can describe m_N as the composite

$$N^\bullet \wedge N^\bullet \xrightarrow{i \wedge i} C^\bullet \wedge C^\bullet \xrightarrow{m_C} C^\bullet \xrightarrow{r} N^\bullet.$$

This map is compatible with the multiplication $S \wedge S \longrightarrow S$ (the map on the augmentations) and we claim that it extends to a map on the level of towers to which we can apply the argument of [5].

Notation 2.4.5. We resort back to writing $\overline{H}$ for the homotopy fiber of η.

Definition 2.4.6. Let (X, I) be the H-canonical tower for S

$$\begin{tikzcd}
S \arrow[d] & \overline{H} \arrow[l] \arrow[d] & \dots \arrow[l] & \overline{H}^s \arrow[l] \arrow[d] & \overline{H}^{s+1} \arrow[l] \arrow[d] & \dots \arrow[l] \\
H \arrow[ur, dotted] & H \wedge \overline{H} \arrow[ur, dotted] & & H \wedge \overline{H}^s \arrow[ur, dotted] & H \wedge \overline{H}^{s+1} \arrow[ur, dotted] &
\end{tikzcd}$$

so that $X \in \mathscr{S}^{\mathbb{Z}}$ is the sequence given by

$$X_s = \begin{cases} S & \text{if } s \leq 0 \\ \overline{H}^{\wedge s} & \text{if } s \geq 0, \end{cases}$$

where $X_{s+1} \longrightarrow X_s$ is

$$\overline{H}^{\wedge(s+1)} \xrightarrow{(\overline{H} \longrightarrow S) \wedge \overline{H}^{\wedge s}} \overline{H}^s$$

for $s \geq 0$ and the identity on S otherwise.

In III.2 we will discuss precisely what we mean by the augmented chain complex associated to a tower and what we mean by extending a map of augmented chain complexes to a map of towers. We will also discuss in depth the smash product of two towers.

By an obstruction theory argument utilizing the technology of [17] the map m_N can be extended to a map of towers $(X, I) \wedge (X, I) \to (X, I)$ and the argument in theorem IV.4.4 of [5] gives us what we need to obtain a multiplicative structure on the ASS-S. One sees directly from the definition that we have $E_1^{\bullet,*}(\text{ASS-}S) = \pi_*(N^\bullet) = \mathbb{F}_p \Box_A H_*(N^\bullet) = \Omega A$ as rings.

Smashing with $Y \wedge Y'$, where Y and Y' are any two spectra, and applying the same argument gives a pairing $E_*^{*,*}(\text{ASS-}Y) \otimes E_*^{*,*}(\text{ASS-}Y') \longrightarrow E_*^{*,*}(\text{ASS-}(Y \wedge Y'))$ which converges to the smash product $\pi_*(Y) \otimes \pi_*(Y') \longrightarrow \pi_*(Y \wedge Y')$. At the E_1-page this agrees with the multiplication

$$\Omega(A; H_*(Y)) \otimes \Omega(A; H_*(Y')) \longrightarrow \Omega(A; H_*(Y \wedge Y')) = \Omega(A; H_*(Y) \otimes^\Delta H_*(Y')). \tag{2.4.7}$$

Miller claims this at the bottom of page 76 in [15].

3 A modified Adams spectral sequence for S/p^n

3.1 The main result

Recall that S/p^n denotes the mod p^n Moore spectrum. The first step towards computing E_2(ASS-S/p) is to understand the A-comodule $H_*(S/p)$. This is straightforward. It is given by $E[\tau_0]$ which has a nontrivial action under the Bockstein β. However, for $n > 1$ the A-comodule structure of $H_*(S/p^n)$ is trivial; when calculating the E_2-page for S/p^n we obtain two copies of the E_2-page for the sphere. In [15] Miller identifies the E_2-pages for the sphere and the Moore spectrum as $\mathrm{Cotor}_P(Q(0))$ and $\mathrm{Cotor}_P(Q(1))$, respectively. We would like a spectral sequence converging to $\pi_*(S/p^n)$, with E_2-page $\mathrm{Cotor}_P(Q(0)/q_0^n)$. This is a more convenient E_2-page because multiplication by q_0^n is zero. It is also clearer, based on our algebraic work, how to obtain a localized spectral sequence from this one. We need a modified Adams spectral sequence for S/p^n.

The next proposition is the main result of this section. Before stating it, we need to introduce some notation, which is explained more thoroughly in definition 3.3.1.

We have a DG algebra over A called $M(n)$. As an $\mathbb{F}_p$-vector space $M(n) = \bigoplus_{i=0}^{n-1} E[\tau_{0,i}]$; $M(n)$ has differential defined by $d1_i = 0$, $d\tau_{0,i} = 1_{i+1}$ (with the convention that $1_i = 0 = \tau_{0,i}$ for $i \geq n$).

Proposition 3.1.1. *There is a spectral sequence called the modified Adams spectral sequence for S/p^n with E_1-page $\Omega(A; M(n))$ and E_2-page $\mathrm{Cotor}_P(Q(0)/q_0^n)$. Moreover, it converges to $\pi_*(S/p^n)$ and there is a pairing*

$$E_*^{*,*}(\mathit{MASS}\text{-}n) \otimes E_*^{*,*}(\mathit{MASS}\text{-}n) \longrightarrow E_*^{*,*}(\mathit{MASS}\text{-}n)$$

converging to the multiplication $\pi_*(S/p^n) \otimes \pi_*(S/p^n) \to \pi_*(S/p^n)$ which, at the E_1-page, agrees with the multiplication on $\Omega(A; M(n))$ and, at the E_2-page, with the multiplication on $\mathrm{Cotor}_P(Q(0)/q_0^n)$.

This result is probably well-known to the experts. However, the multiplicative structure does not appear to be well-documented in the literature. In this section we address all the claims of the proposition except those concerning the E_2-page; they will be verified in section 4.

3.2 Setting up the MASS-n

Definition 3.2.1. Let $(Y(n), J(n))$ be the tower

$$\begin{tikzcd}
S/p^n \arrow[d] & S/p^{n-1} \arrow[l, "p"'] \arrow[d] & \dots \arrow[l, "p"'] & S/p \arrow[l, "p"'] \arrow[d] & * \arrow[l] \arrow[d] & \dots \arrow[l] \\
S/p \arrow[ur, dotted] & S/p \arrow[ur, dotted] & & S/p \arrow[ur, dotted] & *
\end{tikzcd}$$

so that $Y(n) \in \mathscr{S}^{\mathbb{Z}}$ is the sequence given by

$$Y(n)_s = \begin{cases} S/p^n & \text{if } s \leq 0 \\ S/p^{n-s} & \text{if } 0 \leq s < n \\ * & \text{if } s \geq n, \end{cases}$$

where $Y(n)_{n-s+1} \longrightarrow Y(n)_{n-s}$ is $p : S/p^{s-1} \longrightarrow S/p^s$ for $0 < s \leq n$ and the identity on S/p^n or $*$ otherwise.

Recall definition 2.4.6. Applying $\pi_*(-)$ to the tower $(Z(n), K(n)) = (X, I) \wedge (Y(n), J(n))$ gives an exact couple.

Definition 3.2.2. The spectral sequence arising from this exact couple is called the *modified Adams spectral sequence for* S/p^n (MASS-n). It has E_1-page

$$E_1^{s,t}(\text{MASS-}n) = \pi_{t-s}(K(n)_s) = \pi_t(\Sigma^s K(n)_s)$$

and d_r has degree $(r, r-1)$. The spectral sequence converges to $\pi_*(S/p^n)$ and the filtration degree is given by s. In particular, we have an identification

$$E_\infty^{s,t}(\text{MASS-}n) = F^s\pi_{t-s}(S/p^n)/F^{s+1}\pi_{t-s}(S/p^n)$$

where $F^s\pi_*(S/p^n) = \mathrm{im}(\pi_*(Z(n)_s) \longrightarrow \pi_*(S/p^n))$. The identification is given by lifting an element of $F^s\pi_*(S/p^n)$ to $\pi_*(Z(n)_s)$ and mapping this lift down to $\pi_*(K(n)_s)$ to give a permanent cycle.

The reader might be unfamiliar with modified Adams spectral sequences and have doubts about the convergence of the MASS-n. In fact, convergence is proved in [23, theorem 3.6]. We give more details in subsection 7.3.

3.3 Relation of E_1(MASS-n) to the cobar construction

Our goal is to identify $E_1^{\bullet,*}(\text{MASS-}n)$ with $\Omega(A; M(n))$ as DG vector spaces, where $M(n)$ is defined below.

Definition 3.3.1. We describe a DG algebra over A which we call $M(n)$. As an $\mathbb{F}_p$-vector space we have

$$M(n) = \bigoplus_{i=0}^{n-1} E[\tau_{0,i}].$$

We grade $M(n)$ by $|1_i| = (i,i)$ and $|\tau_{0,i}| = (i, i+1)$. We have the convention that $1_i = 0 = \tau_{0,i}$ for $i \geq n$. We equip $M(n)$ with a with differential $d1_i = 0$, $d\tau_{0,i} = 1_{i+1}$ and a multiplication

$$1_i \otimes 1_j \longmapsto 1_{i+j}, \qquad \tau_{0,i} \otimes 1_j \longmapsto \tau_{i+j,0}, \qquad 1_i \otimes \tau_{0,j} \longmapsto \tau_{i+j,0}, \qquad \tau_{0,i} \otimes \tau_{0,j} \longmapsto 0.$$

We make it into an A-comodule via the following structure map. We note that A only has a single grading and the structure map is a graded map using the second grading on $M(n)$.

$$1_i \longmapsto 1 \otimes 1_i, \qquad \tau_{0,i} \longmapsto 1 \otimes \tau_{0,i} + \tau_0 \otimes 1_i.$$

We note that as a cochain complex in A-comodules $M(n)$ is obtained by applying $H_*(-)$ to $\Sigma^\bullet J(n)_\bullet$: the first grading is the cohomological grading, the second is the grading from $H_*(-)$.

Notice that we can consider $M(\infty)$. This object will only be important momentarily and it will *not* be used to construct the MASS-∞. To avoid any potential confusion we will call this object M. Notice that $M(n)$ is a quotient of M.

All the work in identifying $E_1^{\bullet,*}(\text{MASS-}n)$ was done in corollary 2.3.8. As $\mathbb{F}_p$-vector spaces we have

$$\begin{aligned} E_1^{s,t}(\text{MASS-}n) &= \pi_t(\Sigma^s K(n)_s) = \pi_t\Big(\bigvee_{i+j=s} \Sigma^i I_i \wedge \Sigma^j J(n)_j\Big) \\ &= \bigoplus_{i+j=s} \pi_t(\Sigma^i I_i \wedge \Sigma^j J(n)_j) = \bigoplus_{i+j=s} \Omega^{i,t}(A; H_*(\Sigma^j J(n)_j)) = \bigoplus_{i+j=s} \Omega^{i,t}(A; M(n)_{j,*}). \end{aligned}$$

Because the isomorphism $a_{s,s}$ of lemma 2.3.2 is natural in Y, proposition III.2.4.5 together with the definition of $\Omega(A; M(n))$ tells us that this is, in fact, an identification of cochain complexes $E_1^{\bullet,*}(\text{MASS-}n) = \Omega(A; M(n))$.

3.4 Multiplicativity of the MASS-n

$\pi_*(S/p^n)$ and $\Omega(A; M(n))$ are rings. In this subsection we show that the MASS-n is multiplicative and identify $E_1^{\bullet,*}(\text{MASS-}n)$ with $\Omega(A; M(n))$ as rings. Our strategy is as follows.

First, we note that $M(n)$ has the structure of an algebra. We show that the topology follows suit and that we can give the augmented cochain complex $\Sigma^\bullet J(n)_\bullet$ a multiplication by using the multiplication on S/p. By a technical pointset level construction we extend this multiplication to towers. Finally, we smash together the multiplications on the towers (X, I) and $(Y(n), J(n))$. Here are most of the details.

Notation 3.4.1. Let $\beta : S/p \longrightarrow \Sigma S/p$ denote the Bockstein map.

Lemma 3.4.2. *$[S/p, S/p] = \mathbb{Z}/p\langle 1\rangle$ and $[S/p, \Sigma S/p] = \mathbb{Z}/p\langle\beta\rangle$.*

Proof. This comes down to computing π_0 and π_{-1} of the endomorphism spectrum $\text{End}(S/p)$. Using Spanier-Whitehead duality we have $\text{End}(S/p) = \Sigma^{-1}S/p \wedge S/p = \Sigma^{-1}S/p \vee S/p$ and so the result follows from the fact that $\pi_0(S/p) = \mathbb{Z}/p$, $\pi_{-1}(S/p) = \pi_1(S/p) = 0$ and that 1 and β are nontrivial. $\square$

Lemma 3.4.3. *The following diagram commutes*

$$\begin{tikzcd}
S/p \wedge S/p \arrow[r, "(\beta\wedge S/p,\, S/p\wedge\beta)"] \arrow[d, "\mu"'] & (\Sigma S/p \wedge S/p) \vee (S/p \wedge \Sigma S/p) \arrow[d, "\Sigma(\mu,\mu)"] \\
S/p \arrow[r, "\beta"] & \Sigma S/p.
\end{tikzcd}$$

where $\mu : S/p \wedge S/p \longrightarrow S/p$ is the multiplication on the ring spectrum S/p.

Proof. Since $S/p \wedge S/p = S/p \vee \Sigma S/p$ it is enough to restrict to each factor. We are then comparing maps in $[S/p, \Sigma S/p]$ and $[\Sigma S/p, \Sigma S/p] = [S/p, S/p]$. Since 1 and β are homologically nontrivial, it is enough, by the previous lemma, to check that the diagram commutes after applying homology:

$$\begin{tikzcd}
E[\tau_0] \otimes^{\Delta} E[\tau_0] \arrow[r, "((\tau_0\mapsto 1)\otimes E[\tau_0],\, E[\tau_0]\otimes(\tau_0\mapsto 1))"] \arrow[d, "\mu"'] & (\Sigma E[\tau_0] \otimes E[\tau_0]) \oplus (E[\tau_0] \otimes \Sigma E[\tau_0]) \arrow[d, "\Sigma(\mu,\mu)"] \\
E[\tau_0] \arrow[r, "\tau_0\mapsto 1"] & \Sigma E[\tau_0].
\end{tikzcd}$$

Here $E[\tau_0]$ is the sub-Hopf algebra of A generated by τ_0; it is a subalgebra in A-comodules. □

Corollary 3.4.4. *We obtain a map $m_n : \Sigma^{\bullet} J(n)_{\bullet} \wedge \Sigma^{\bullet} J(n)_{\bullet} \to \Sigma^{\bullet} J(n)_{\bullet}$ by using the multiplication of S/p or the zero map on each factor. Applying $H_*(-)$ returns the multiplication $M(n) \otimes^{\Delta} M(n) \to M(n)$.*

Recall that the map m_N of definition 2.4.3 extends to a map of towers $(X, I) \wedge (X, I) \longrightarrow (X, I)$ and this enabled us to make the ASS-S multiplicative. We'd like to extend the map m_n to a map of towers. The obstruction theory seems hard if one attacks it directly. Instead, we apply the obstruction theory to the S/p-canonical tower for S.

Definition 3.4.5. Let (Y, J) be the S/p-canonical tower for S

$$\begin{tikzcd}
S \arrow[d] & S \arrow[l, "p"'] \arrow[d] & \dots \arrow[l] & S \arrow[l, "p"'] \arrow[d] & S \arrow[l, "p"'] \arrow[d] & \dots \arrow[l] \\
S/p \arrow[ur, dotted] & S/p \arrow[ur, dotted] & & S/p \arrow[ur, dotted] & S/p \arrow[ur, dotted] &
\end{tikzcd}$$

so that $Y \in \mathscr{S}^{\mathbb{Z}}$ is the sequence given by $Y_s = S$, where $Y_{s+1} \longrightarrow Y_s$ is multiplication by p for $s \geq 0$ and the identity otherwise.

Using Miller's technology [17] we have another corollary to lemma 3.4.3.

Corollary 3.4.6. *We obtain a map $m_S : \Sigma^{\bullet} J_{\bullet} \wedge \Sigma^{\bullet} J_{\bullet} \to \Sigma^{\bullet} J_{\bullet}$ by using the multiplication of S/p on each factor. It is compatible with the multiplication $S \wedge S \longrightarrow S$ (i.e. it respects the augmentation) and we can extend it to a map of towers $(Y, J) \wedge (Y, J) \longrightarrow (Y, J)$. Moreover, $H_*(m_S)$ gives the multiplication $M \otimes^{\Delta} M \longrightarrow M$ (see definition 3.3.1).*

Just like we can quotient M to give $M(n)$ we wish to 'quotient' the map $(Y, J) \wedge (Y, J) \longrightarrow (Y, J)$ by the part of the tower from the n^{th} position onwards to obtain a map of towers $(Y(n), J(n)) \wedge (Y(n), J(n)) \to (Y(n), J(n))$. This requires the pointset model for $\mathscr{S}$, **Spec** which is discussed in III.2.2.

Lemma 3.4.7. *m_n extends to a map of towers $(Y(n), J(n)) \wedge (Y(n), J(n)) \longrightarrow (Y(n), J(n))$.*

Proof. The idea of the proof is straightforward: we strictify the map of towers in corollary 3.4.6 up to the $(2n-1)^{\text{th}}$ position and collapse from the n^{th} position onwards. However, the proof is messy. In order to avoid having to delve into any true pointset level discussion of spectra we use a Quillen adjunction with spaces and work there. The reader who wishes to understand all the details should look ahead to subsections III.2.2 and III.2.3 for all the relevant notation. The proof is completed in subsection III.2.5. □

We now have a map of towers $(Z(n), K(n)) \wedge (Z(n), K(n)) \longrightarrow (Z(n), K(n))$ given by

$$\Big[(X,I)\wedge(Y(n),J(n))\Big] \wedge \Big[(X,I)\wedge(Y(n),J(n))\Big] \xrightarrow{\cong} \Big[(X,I)\wedge(X,I)\Big] \wedge \Big[(Y(n),J(n))\wedge(Y(n),J(n))\Big] \longrightarrow (X,I)\wedge(Y(n),J(n)).$$

Proposition III.2.4.7 is devoted to the construction of a map in this way and Bruner's argument ([5, IV.4.4]) gives us a multiplicative structure on the MASS-n. Using the definition of the multiplication on $\Omega(A; M(n))$, the observation of 2.4.7 and the property verified in III.2.4.7 one can see directly from Bruner's definition that we have $E_1^{\bullet,*}(\text{MASS-}n) = \Omega(A; M(n))$ as rings.

4 E_2-pages

In this section we complete the proofs of proposition 2.1.2 and proposition 3.1.1 by recalling the homological algebra needed to identify the E_2-pages.

4.1 $E_2(\textbf{ASS-}S^0) = \text{Cotor}_P(Q(0))$

In [15] Miller identified the $E_2(\text{ASS-}S^0)$ as $\text{Cotor}_P(Q(0))$. We begin this section by recalling how this identification is obtained.

The twisting homomorphism $\theta : E \longrightarrow Q(0)$ defined before propostion 2.1.2 gives a map of DG P-comodules $\Omega E \longrightarrow Q(0)$. Miller defines a map $\Omega A \longrightarrow \Omega(P; \Omega E)$ ([15, proposition 1.2]) and he proves the following theorem.

Theorem 4.1.1. *The composite $\Omega A \longrightarrow \Omega(P; \Omega E) \longrightarrow \Omega(P; Q(0))$ is a homology isomorphism.*

Let's be precise about gradings.

1. ΩA has a cohomological grading σ and a 'total' grading λ coming from the grading on A.

2. $\Omega(P; \Omega E)$ has an external cohomological grading w, an internal cohomological grading x coming from the cohomological grading on ΩE, a 'total' grading z coming from the second gradings on E and P, and one checks that these are respected by the multiplication.

3. $\Omega(P; Q(0))$ has a cohomological grading s and gradings t and u coming from the fact that P and $Q(0)$ are bigraded.

We see directly from the formula in [15, page 77] that

$$(\Omega A)_{1,\lambda} \longrightarrow \Omega(P; \Omega E)_{1,0,\lambda} \oplus \Omega(P; \Omega E)_{0,1,\lambda} \quad \Longrightarrow \quad (\Omega A)_{\sigma,\lambda} \longrightarrow \bigoplus_{\substack{w+x=\sigma \\ z=\lambda}} \Omega(P; \Omega E)_{w,x,z}.$$

We also see that $\Omega(P;\Omega E)_{w,x,z} \longrightarrow \Omega(P;Q(0))_{s,t,u}$, where $s = w$, $t = x$, and $u = z - x$; the minus x comes from the fact that $|q_n|_u = |\tau_n|_u - 1$. Thus

$$(\Omega A)_{\sigma,\lambda} \longrightarrow \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \Omega(P;Q(0))_{s,t,u}.$$

These gradings persist to Cotor and so we have the following result.

Proposition 4.1.2.

$$E_2^{\sigma,\lambda}(\text{ASS-}S^0) = \mathrm{Cotor}_A^{\sigma,\lambda}(\mathbb{F}_p) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \mathrm{Cotor}_P^{s,t,u}(Q(0)).$$

Note the bigrading (σ,λ) that we have introduced for the ASS-S^0 above. We will continue to use this bigrading for all of our Adams spectral sequences. This will help avoid confusion with the s, t and u of our Bockstein spectral sequences.

4.2 E_2(MASS-n) = $\mathrm{Cotor}_P(Q(0)/q_0^n)$

We wish to perform an analogous calculation for the MASS-n. The starting point is the observation that Miller proves a stronger result than that of theorem 4.1.1.

Theorem 4.2.1 ([15, page 80])**.** *For any A-comodule M which is bounded below the composite*

$$\Omega(A;M) \longrightarrow \Omega(P;\Omega E \otimes_\tau M) \longrightarrow \Omega(P;Q(0) \otimes_\theta M)$$

is a homology isomorphism.

Thus, to identify the E_2-page of the MASS-n we need to identify $\mathrm{Cotor}_P(Q(0) \otimes_\theta M(n))$.

Lemma 4.2.2. *We have a homology isomorphism $\Omega(P;Q(0) \otimes_\theta M(n)) \longrightarrow \Omega(P;Q(0)/q_0^n)$. Morever, this is a map of DG algebras.*

Proof. A short calculation in $Q(0) \otimes_\theta M(n)$ shows that

$$d(q \otimes 1_i) = 0 \quad \text{and} \quad d(q \otimes \tau_{0,i}) = q_0 q \otimes 1_i - q \otimes 1_{i+1}.$$

[A sign might be wrong here but the end result will still be the same.] Define a map

$$Q(0) \otimes_\theta M(n) \longrightarrow Q(0)/q_0^n$$

by $q \otimes 1_i \longmapsto q_0^i q$ and $q \otimes \tau_{0,i} \longmapsto 0$. This is a map of DG algebras over P, where the target has a trivial differential. In addition, it is a homology isomorphism and so the Eilenberg-Moore spectral sequence completes the proof. □

We keep track of the gradings in the composite

$$\Omega(A;M(n)) \longrightarrow \Omega(P;\Omega E \otimes_\tau M(n)) \longrightarrow \Omega(P;Q(0)/q_0^n).$$

1. $\Omega(A; M(n))$ has an external cohomological grading i, an internal 'cohomological' grading j, which comes from the first grading on $M(n)$ and a 'total' grading λ coming from the grading on A and the second grading on $M(n)$. These are preserved by the multiplication.

2. $\Omega(P; \Omega E \otimes_\tau M(n))$ has an external cohomological grading w, a middle cohomological grading x coming from the cohomological grading on ΩE, an internal 'cohomological' grading y coming from the first grading on $M(n)$, a 'total' grading z coming from the second gradings on E, P and $M(n)$, and one checks that these are respected by the multiplication.

3. $\Omega(P; Q(0)/q_0^n)$ has a cohomological grading s and gradings t and u coming from the fact that P and $Q(0)/q_0^n$ are bigraded.

As before, we see directly from the formula in [15, page 77] that

$$\Omega(A; M(n))_{1,j,\lambda} \longrightarrow \Omega(P; \Omega E \otimes_\tau M(n))_{1,0,j,\lambda} \oplus \Omega(P; \Omega E \otimes_\tau M(n))_{0,1,j,\lambda}$$

$$\Longrightarrow \qquad \Omega(A; M(n))_{i,j,\lambda} \longrightarrow \bigoplus_{\substack{w+x=i \\ y=j,\ z=\lambda}} \Omega(P; \Omega E \otimes_\tau M(n))_{w,x,y,z}.$$

We also see that $\Omega(P; \Omega E \otimes_\tau M(n))_{w,x,y,z} \longrightarrow \Omega(P; Q(0)/q_0^n)_{s,t,u}$, where

$$s = w, \quad t = x + y \quad \text{and} \quad u = z - (x + y);$$

the minus $(x + y)$ comes from the fact that $|q_n|_u = |\tau_n|_u - 1$ and that $|q_0^i|_u = |1_i|_u - i$. Thus

$$\Omega(A; M(n))_{i,j,\lambda} \longrightarrow \bigoplus_{\substack{s+t=i+j \\ u+t=\lambda}} \Omega(P; Q(0)/q_0^n)_{s,t,u}$$

and we obtain the following result.

Proposition 4.2.3. *We have a homology isomorphism*

$$E_1^{\sigma,\lambda}(\text{MASS-}n) = \bigoplus_{i+j=\sigma} \Omega(A; M(n))_{i,j,\lambda} \longrightarrow \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \Omega(P; Q(0)/q_0^n)_{s,t,u}$$

and the gradings σ, λ, s, t and u persist to homology so that

$$E_2^{\sigma,\lambda}(\text{MASS-}n) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \text{Cotor}_P^{s,t,u}(k, Q(0)/q_0^n).$$

We have now completed the proof of proposition 2.1.2 and proposition 3.1.1.

5 The journey towards setting up the LASS-∞

5.1 The reindexed MASS-n and setting up the MASS-∞

This subsection begins the work required to prove the following proposition.

Proposition 5.1.1. *There is a spectral sequence called the modified Adams spectral seqence for the Prüfer sphere S/p^∞ (MASS-∞). It has E_2-page* $\mathrm{Cotor}_P(Q(0)/q_0^\infty)$ *and converges to* $\pi_*(S/p^\infty)$.

Our first step in the construction of this spectral sequence is a reindexing procedure.

Definition 5.1.2. Let $(Y(n)', J(n)')$ be a shifted version of $(Y(n), J(n))$ so that $Y(n)' \in \mathscr{S}^{\mathbb{Z}}$ is the sequence given by

$$Y(n)'_s = \begin{cases} S/p^n & \text{if } s \leq -n \\ S/p^{-s} & \text{if } -n \leq s < 0 \\ * & \text{if } s \geq 0. \end{cases}$$

Applying $\pi_*(-)$ to the tower $(Z(n)', K(n)') = (X, I) \wedge (Y(n)', J(n)')$ gives an exact couple.

Definition 5.1.3. The spectral sequence arising from this exact couple is called the *reindexed modified Adams spectral sequence for S/p^n* (MASS-n'). It has E_1-page

$$E_1^{\sigma,\lambda}(\text{MASS-}n') = \pi_\lambda(\Sigma^\sigma K(n)'_\sigma) = \pi_{\lambda+n}(\Sigma^{\sigma+n} K(n)_{\sigma+n}) = E_1^{\sigma+n,\lambda+n}(\text{MASS-}n)$$

and hence, by proposition 4.2.3, E_2-page

$$E_2^{\sigma,\lambda}(\text{MASS-}n') = \bigoplus_{\substack{s+(t-n)=\sigma \\ u+(t-n)=\lambda}} \mathrm{Cotor}_P^{s,t,u}(Q(0)/q_0^n) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \mathrm{Cotor}_P^{s,t,u}(k, M_n)$$

(see definition 1.5 in chapter I). d_r has degree $(r, r-1)$, the spectral sequence converges to $\pi_*(S/p^n)$ and the filtration degree is given by σ. In particular, we have an identification

$$E_\infty^{\sigma,\lambda}(\text{MASS-}n') = F^\sigma \pi_{\lambda-\sigma}(S/p^n)/F^{\sigma+1}\pi_{\lambda-\sigma}(S/p^n)$$

where $F^\sigma\pi_*(S/p^n) = \mathrm{im}(\pi_*(Z(n)'_\sigma) \longrightarrow \pi_*(S/p^n))$. The identification is given by lifting an element of $F^\sigma\pi_*(S/p^n)$ to $\pi_*(Z(n)'_\sigma)$ and mapping this lift down to $\pi_*(K(n)'_\sigma)$ to give a permanent cycle.

We have a map of towers $(Y(n)', J(n)') \longrightarrow (Y(n+1)', J(n+1)')$. Here, $Y(n)'_s \longrightarrow Y(n+1)'_s$ is the identity for $s \geq -n$ and $p : S/p^n \longrightarrow S/p^{n+1}$ for $s < -n$. This map of towers gives rise to an induced map of spectral sequences $E_*^{*,*}(\text{MASS-}n') \longrightarrow E_*^{*,*}(\text{MASS-}(n+1)')$. One checks using the map in the proof of lemma 4.2.2 that the map on E_2-pages, $\mathrm{Cotor}_P(M_n) \longrightarrow \mathrm{Cotor}_P(M_{n+1})$ is induced by the inclusion $M_n \longrightarrow M_{n+1}$.

Since taking filtered colimits is exact, we can take the colimit of the diagram

$$E_*^{*,*}(\text{MASS-}1') \longrightarrow E_*^{*,*}(\text{MASS-}2') \longrightarrow \ldots \longrightarrow E_*^{*,*}(\text{MASS-}n') \longrightarrow E_*^{*,*}(\text{MASS-}(n+1)') \longrightarrow \ldots$$

to obtain a spectral sequence.

Definition 5.1.4. The spectral sequence constructed above is called the *modified Adams spectral sequence for the Prüfer sphere S/p^∞* (MASS-∞). It has E_2-page

$$E_2^{\sigma,\lambda}(\text{MASS-}\infty) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \mathrm{Cotor}_P^{s,t,u}(k, Q(0)/q_0^\infty)$$

and d_r has degree $(r, r-1)$.

It is nonobvious that the spectral sequence above converges to $\pi_*(S/p^\infty)$. In fact, it is not even clear what we should mean by $E_\infty(\text{MASS-}\infty)$ since it is not obtained by the procedure in section 2 of chapter I.

We explain, in a little more detail, the construction of the MASS-∞. We have natural identifications $E_{r+1}(\text{MASS-}n') = H(E_r(\text{MASS-}n'), d_r)$. We define $E_r(\text{MASS-}\infty)$ to be $\text{colim}_n E_r(\text{MASS-}n')$ and then we have natural identifications

$$\begin{aligned} E_{r+1}(\text{MASS-}\infty) &= \text{colim}_n E_{r+1}(\text{MASS-}n') \\ &= \text{colim}_n H(E_r(\text{MASS-}n'), d_r) \\ &= H(\text{colim}_n E_r(\text{MASS-}n'), d_r) \\ &= H(E_r(\text{MASS-}\infty), d_r). \end{aligned}$$

This justifies us calling the MASS-∞ a spectral sequence.

The vanishing lines for the MASS-n' (lemma 6.5) provide us with maps $E_r^{\sigma,\lambda}(\text{MASS-}n') \longrightarrow E_{r+1}^{\sigma,\lambda}(\text{MASS-}n')$ for large r; how large r is required to be depends on (σ, λ) but not on n. By corollary 6.6, n is allowed to be ∞. Thus we may define $E_\infty^{\sigma,\lambda}(\text{MASS-}\infty)$ to be $\text{colim}_{r>>0} E_r^{\sigma,\lambda}(\text{MASS-}\infty)$ and we obtain

$$\begin{aligned} E_\infty^{\sigma,\lambda}(\text{MASS-}\infty) &= \text{colim}_{r>>0} E_r^{\sigma,\lambda}(\text{MASS-}\infty) \\ &= \text{colim}_{r>>0}\text{colim}_n E_r^{\sigma,\lambda}(\text{MASS-}n') \\ &= \text{colim}_n\text{colim}_{r>>0} E_r^{\sigma,\lambda}(\text{MASS-}n') \\ &= \text{colim}_n E_\infty^{\sigma,\lambda}(\text{MASS-}n'). \end{aligned}$$

We leave the issue of convergence until subsection 7.3.

5.2 The localized Adams spectral sequence for $v_1^{-1}S/p^n$

We now proceed to localize the MASS-n. Our main result is the following proposition.

Proposition 5.2.1. *There is a spectral sequence, which we call the localized Adams spectral sequence for $v_1^{-1}S/p^n$ with E_2-page* $\text{Cotor}_P(q_1^{-1}Q(0)/q_0^n)$*. It converges to $\pi_*(v_1^{-1}S/p^n)$ and there is a pairing*

$$E_*^{*,*}(LASS\text{-}n) \otimes E_*^{*,*}(LASS\text{-}n) \longrightarrow E_*^{*,*}(LASS\text{-}n)$$

converging to the multiplication $\pi_*(v_1^{-1}S/p^n)\otimes\pi_*(v_1^{-1}S/p^n) \longrightarrow \pi_*(v_1^{-1}S/p^n)$ which, at the E_2-page, agrees with the multiplication on $\text{Cotor}_P(q_1^{-1}Q(0)/q_0^n)$.

This section sets up the spectral sequence. Convergence is left until subsection 7.4. We do not need the full multiplicative structure, only the fact that d_2 is a derivation. This follows quickly from the construction and the fact that the MASS-n is multiplicative. We omit other details concerning the multiplicative structure although they are not hard to verify.

In order to localize the MASS-$(n+1)$, we would like to find a permanent cycle detecting the map

$$v_1^{p^n} : S/p^{n+1} \longrightarrow \Sigma^{-p^n q} S/p^{n+1}$$

constructed by Crabb and Knapp in [6, proposition 1.1]. We could not achieve this “on the nose” and so we make use of the periodicity theorem of [10] to prove the following lemma.

Lemma 5.2.2. *Suppose that a map $f : S/p^{n+1} \to \Sigma^{-p^n q} S/p^{n+1}$ induces an isomorphism on K-theory. Then there exists an $i \in \mathbb{N}$ such that $f^i = (v_1^{p^n})^i$.*

Proof. By [10, corollary 3.7] it is enough to show that f is a v_1-self map (see [10, definition 8]). Since $K(0)_*(S/p^{n+1}) = 0$ and $K(1)_*(f)$ is an isomorphism we are just left with showing that $K(m)_*(f)$ is nilpotent for $m > 1$. This occurs for degree reasons: for $m \geq 1$, $K(m)_*(S/p^{n+1}) = K(m)_* \oplus \Sigma K(m)_*$ and $K(m)_* = \mathbb{F}_p[v_m, v_m^{-1}]$, while $p^n q/|v_m| \in \mathbb{Z}$ if and only if $m = 1$. □

The following theorem is our main technical result.

Theorem 5.2.3. *The element $q_1^{p^n} \in \mathrm{Cotor}_P(Q(0)/q_0^{n+1})$ is a permanent cycle in the MASS-$(n+1)$ detecting an element $\alpha_{p^n} : S^{p^n q} \longrightarrow S/p^{n+1}$ such that*

$$\Sigma^{p^n q} S/p^{n+1} \xrightarrow{\alpha_{p^n} \wedge S/p^{n+1}} S/p^{n+1} \wedge S/p^{n+1} \xrightarrow{\mu} S/p^{n+1}$$

induces an isomorphism on K-theory.

Proof. The proof is long and carried out in III.1. □

Multiplication by $q_1^{p^{n-1}}$ defines a map of spectral sequences. Since taking filtered colimits is exact, we can take the colimit of the diagram

$$E_*^{*,*}(\text{MASS-}n) \xrightarrow{q_1^{p^{n-1}}} E_*^{*,*}(\text{MASS-}n) \xrightarrow{q_1^{p^{n-1}}} E_*^{*,*}(\text{MASS-}n) \xrightarrow{q_1^{p^{n-1}}} \dots$$

to obtain a spectral sequence. [In the above diagram we are not precise about gradings.]

Definition 5.2.4. The spectral sequence constructed above is called the *localized Adams spectral sequence for $v_1^{-1}S/p^n$* (LASS-n). It has E_2-page

$$E_2^{\sigma,\lambda}(\text{LASS-}n) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \mathrm{Cotor}_P^{s,t,u}(k, q_1^{-1}Q(0)/q_0^n)$$

and d_r has degree $(r, r-1)$.

It is nonobvious that the spectral sequence above converges to $\pi_*(v_1^{-1}S/p^n)$ and we leave this verification until subsection 7.4. An identical discussion to the one following the construction of the MASS-∞ explains what we mean by $E_\infty(\text{LASS-}n)$.

5.3 The reindexed LASS-n and the LASS-∞

We are finally ready to set up the LASS-∞ and we begin the work required to prove the following theorem.

Theorem 5.3.1. *There is a spectral sequence with E_2-page $\mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^\infty)$ which converges to $\pi_*(v_1^{-1}S/p^\infty)$. We call this the localized Adams spectral sequence for the v_1-periodic sphere $v_1^{-1}S/p^\infty$ (LASS-∞).*

As with the MASS-∞, the first step in the construction is a reindexing procedure. We can take the colimit of the diagram

$$E_*^{*,*}(\text{MASS-}n') \xrightarrow{q_1^{p^{n-1}}} E_*^{*,*}(\text{MASS-}n') \xrightarrow{q_1^{p^{n-1}}} E_*^{*,*}(\text{MASS-}n') \xrightarrow{q_1^{p^{n-1}}} \dots$$

to obtain a spectral sequence.

Definition 5.3.2. The spectral sequence constructed above is called the *reindexed localized Adams spectral sequence for* $v_1^{-1}S/p^n$ (LASS-n'). It has E_2-page

$$E_2^{\sigma,\lambda}(\text{LASS-}n') = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \operatorname{Cotor}_P^{s,t,u}(k, q_1^{-1}M_n)$$

and d_r has degree $(r, r-1)$.

The following diagram commutes at the level of E_2-pages, where the vertical maps are those used in the construction of the MASS-∞. By induction on the page, repeatedly taking homology, we see that this square is a commutative diagram of spectral sequences.

$$\begin{tikzcd}
E_*^{*,*}(\text{MASS-}n') \arrow[r, "q_1^{p^n}"] \arrow[d] & E_*^{*,*}(\text{MASS-}n') \arrow[d] \\
E_*^{*,*}(\text{MASS-}(n+1)') \arrow[r, "q_1^{p^n}"] & E_*^{*,*}(\text{MASS-}(n+1)')
\end{tikzcd}$$

Thus the maps of spectral sequences

$$E_*^{*,*}(\text{MASS-}1') \longrightarrow E_*^{*,*}(\text{MASS-}2') \longrightarrow \dots \longrightarrow E_*^{*,*}(\text{MASS-}n') \longrightarrow E_*^{*,*}(\text{MASS-}(n+1)') \longrightarrow \dots$$

induce maps of spectral sequences

$$E_*^{*,*}(\text{LASS-}1') \longrightarrow E_*^{*,*}(\text{LASS-}2') \longrightarrow \dots \longrightarrow E_*^{*,*}(\text{LASS-}n') \longrightarrow E_*^{*,*}(\text{LASS-}(n+1)') \longrightarrow \dots$$

Taking the colimit of the last diagram gives a spectral sequence.

Definition 5.3.3. The spectral sequence just constructed is called the *localized Adams spectral sequence for the* v_1*-periodic sphere* $v_1^{-1}S/p^\infty$ (LASS-∞). It has E_2-page

$$E_2^{\sigma,\lambda}(\text{LASS-}\infty) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \operatorname{Cotor}_P^{s,t,u}(k, q_1^{-1}Q(0)/q_0^\infty)$$

and d_r has degree $(r, r-1)$.

It is nonobvious that the spectral sequence above converges to $\pi_*(v_1^{-1}S/p^\infty)$ and we leave this verification until subsection 7.4. An identical discussion to the one following the construction of the MASS-∞ explains what we mean by $E_\infty(\text{LASS-}\infty)$.

6 Vanishing Lines

Some results on vanishing lines are essential for proving convergence of our spectral sequences and the technical theorem 5.2.3.

Definition 6.1. For $s \in \mathbb{N} \cup \{0\}$ let $U(2s) = pqs$ and $U(2s+1) = pqs + q$ and write $U(-1) = \infty$.

In [15] Miller uses the following result of Adams.

Lemma 6.2. $\mathrm{Cotor}_P^{s,t,u}(k, Q(1)) = 0$ *when* $u < U(s) + 2(p-1)t$.

Corollary 6.3. $E_2^{\sigma,\lambda}(\mathit{MASS}\text{-}1) = 0$ *when* $\lambda < (2p-1)\sigma - 1$.

Proof. By the $n = 1$ case of proposition 4.2.3 it is enough to observe that $(2p-1)s - 1 \leq U(s)$. □

Since q_1 has (σ, λ) bigrading $(1, 2p-1)$ we obtain the following corollary.

Corollary 6.4. $E_2^{\sigma,\lambda}(\mathit{LASS}\text{-}1) = 0$ *when* $\lambda < (2p-1)\sigma - 1$.

The main results on vanishing lines which we need are given by the following lemma and its corollaries.

Lemma 6.5. $E_2^{\sigma,\lambda}(\mathit{MASS}\text{-}n') = 0$ *when* $\lambda < (2p-1)\sigma + (2p-3)$.

Proof. We proceed by induction on n.

Corollary 6.3 gives $E_2^{\sigma,\lambda}(\text{MASS-1}) = 0$ when $\lambda < (2p-1)\sigma - 1$ and so $E_2^{\sigma,\lambda}(\text{MASS-1}') = 0$ when $\lambda + 1 < (2p-1)(\sigma+1) - 1$ giving the base case.

The short exact sequence of P-comodules $0 \longrightarrow M_1 \longrightarrow M_{n+1} \xrightarrow{q_0} M_n \longrightarrow 0$ gives a long exact sequence some of which is displayed below.

$$\mathrm{Cotor}_P^{s,t,u}(k, M_1) \longrightarrow \mathrm{Cotor}_P^{s,t,u}(k, M_{n+1}) \xrightarrow{q_0} \mathrm{Cotor}_P^{s,t+1,u}(k, M_n)$$

By taking direct sums over appropriate indexings we obtain a long exact sequence

$$\ldots \longrightarrow E_2^{\sigma,\lambda}(\text{MASS-1}') \longrightarrow E_2^{\sigma,\lambda}(\text{MASS-}(n+1)') \longrightarrow E_2^{\sigma+1,\lambda+1}(\text{MASS-}n') \longrightarrow \ldots.$$

We conclude that $E_2^{\sigma,\lambda}(\text{MASS-}(n+1)')$ is zero provided that $E_2^{\sigma,\lambda}(\text{MASS-1}')$ and $E_2^{\sigma+1,\lambda+1}(\text{MASS-}n')$ are zero. Since $\lambda < (2p-1)\sigma + (2p-3)$ implies $\lambda + 1 < (2p-1)(\sigma+1) + (2p-3)$ the inductive step is complete. □

Corollary 6.6. $E_2^{\sigma,\lambda}(\mathit{MASS}\text{-}\infty) = 0$ *when* $\lambda < (2p-1)\sigma + (2p-3)$.

Corollary 6.7. $E_2^{\sigma,\lambda}(\mathit{LASS}\text{-}n') = 0$ *when* $\lambda < (2p-1)\sigma + (2p-3)$.

Corollary 6.8. $E_2^{\sigma,\lambda}(\mathit{LASS}\text{-}\infty) = 0$ *when* $\lambda < (2p-1)\sigma + (2p-3)$.

7 Convergence Issues

7.1 What is convergence?

In this section we prove that all of the spectral sequences we use converge.

A spectral sequence converges if we can recover the graded abelian group we are trying to calculate from the E_∞-page of the spectral sequence. The author is aware that the most recommended account addressing these type of issues is [4]. However, the previously undocumented convergence problems arising in this thesis are easily tackled without such a reference.

In each of the spectral sequences of this thesis we have a graded abelian group A which we are trying to calculate and the recovery procedure can be viewed as having three steps.

1. Define a filtration of A

$$A \supset \ldots \supset F^{s-1}A \supset F^sA \supset F^{s+1}A \supset \ldots \supset 0, \quad s \in \mathbb{Z}$$

   together with an idenfication of the associated graded object $E_\infty^s = F^sA/F^{s+1}A$.

2. Resolve extension problems. Depending on circumstance this will give either F^sA for each s or A/F^sA for each s.

3. Recover A. Depending on circumstance this will either be via an isomorphism $\mathrm{colim}_s F^sA \longrightarrow A$ or an isomorphism $A \longrightarrow \lim_s A/F^sA$.

There are three cases which arise for us. We highlight how each affects the procedure above.

1. Each case is determined by the way in which the filtration behaves.
   (a) $F^0A = A$ and $\bigcap F^sA = 0$.
   (b) $F^0A = 0$ and $\bigcup F^sA = A$.
   (c) $\bigcup F^sA = A$; if we keep track of the grading of A we have $E_\infty^{s,t} = F^sA_{t-s}/F^{s+1}A_{t-s}$ and for each u there exists an s such that $F^sA_u = 0$.

2. The way in which we would go about resolving extension problems varies according to which case we are in.
   (a) $A/F^0A = 0$. Suppose that we know A/F^sA where $s \geq 0$. (1) gives us $F^sA/F^{s+1}A$ and so resolving an extension problem gives $A/F^{s+1}A$. By induction we know A/F^sA for all s.
   (b) $F^0A = 0$. Suppose that we know $F^{s+1}A$ where $s < 0$. (1) gives us $F^sA/F^{s+1}A$ and so resolving an extension problem gives F^sA. By induction we know F^sA for all s.
   (c) This is similar to (2b). Fix u. Then there exists an s_0 with $F^{s_0}A_u = 0$. Suppose that we know $F^{s+1}A_u$ where $s < s_0$. (1) gives us $F^sA_u/F^{s+1}A_u$ and so resolving an extension problem gives F^sA_u. By induction we know F^sA_u for all s. We can now vary u.

3. In case (a) we need an isomorphism $A \longrightarrow \lim_s A/F^sA$. In cases (b) and (c) we have an isomorphism $\mathrm{colim}_s F^sA \longrightarrow A$.

When we say that our spectral sequences converge we ignore whether or not we can resolve the extension problems to carry out stage (2). This is paralleled by the fact that, when making such a statement, we ignore whether or not we can calculate the differentials in the spectral sequence. The point is that theoretically, both of these computations are possible even if they are extremely difficult in practice. Thus, the important statements in convergence, for us, are given in stage (1) and (3) of our recovery procedure.

Definition 7.1.1. Suppose given a graded abelian group A and a spectral sequence E_*^*. Suppose that A is filtered, that we have an identification $E_\infty^s = F^sA/F^{s+1}A$ and that one of the following conditions holds.

1. $F^0A = A$, $\bigcap F^sA = 0$ and the natural map $A \longrightarrow \lim A/F^sA$ is an isomorphism.
2. $F^0A = 0$ and $\bigcup F^sA = A$.
3. $\bigcup F^sA = A$; if we keep track of the grading of A we have $E_\infty^{s,t} = F^sA_{t-s}/F^{s+1}A_{t-s}$ and for each u there exists an s such that $F^sA_u = 0$.

Then the spectral sequence is said to converge.

The ASS-Y converges in the sense of case 1 and this shows why one should expect to need a completeness condition.

We now go about proving that each of our spectral sequences converges.

7.2 Algebraic spectral sequences

In section I.3 we set up the $Q(0)$-BSS (I.3.1.1), the q_0^∞-BSS (I.3.2.2) and the q_1^{-1}-BSS (I.3.4.1). We made claims about the convergence of these spectral sequences and it is only now that we address them. The reason for the delay is that it is actually a rather easy fact that these spectral sequences converge and we did not wish to clutter the exposition in chapter I.

Proposition 7.2.1. *The $Q(0)$-BSS, the q_0^∞-BSS and the q_1^{-1}-BSS converge in the sense of definition 7.1.1.*

Proof. The relevant filtrations are given in I.3.1.1, I.3.2.2 and I.3.4.1 as are the identifications $E_\infty^v = F^v/F^{v+1}$.

For the $Q(0)$-BSS we are in case 1. We have

$$F^0\,\mathrm{Cotor}_P(Q(0)) = \mathrm{Cotor}_P(Q(0)) \text{ and } F^{t+1}\,\mathrm{Cotor}_P^{s,t,u}(Q(0)) = 0$$

and so the requisite conditions hold.

For the q_0^∞-BSS and the q_1^{-1}-BSS we are in case 2. Let $N(t,u) = \max\{0, \lceil u/q - t\rceil\}$, which is the maximum possible power of q_0 in the denominator of element with bigrading (t,u). We note, although it is not required, that we have

$$F^{-N(t,u)}\,\mathrm{Cotor}_P^{s,t,u}(Q(0)/q_0^\infty) = \mathrm{Cotor}_P^{s,t,u}(Q(0)/q_0^\infty)$$

and

$$F^{-N(t,u)}\,\mathrm{Cotor}_P^{s,t,u}(q_1^{-1}Q(0)/q_0^\infty) = \mathrm{Cotor}_P^{s,t,u}(q_1^{-1}Q(0)/q_0^\infty).$$

$\square$

7.3 The MASS-n and MASS-∞

In this subsection we recall why the MASS-n converges and then prove that the MASS-∞ converges.

Proposition 7.3.1. *The MASS-n (definition 3.2.2) converges.*

Proof. We are in case 1 of definition 7.1.1. We need to check the following conditions.

1. The map $F^s\pi_*(S/p^n)/F^{s+1}\pi_*(S/p^n) \longrightarrow E_\infty^{s,*}(\text{MASS-}n)$ constructed in 3.2.2 is an isomorphism.

2. $\bigcap_s F^s\pi_*(S/p^n) = 0$ and the natural map $\pi_*(S/p^n) \longrightarrow \lim_s \pi_*(S/p^n)/F^s\pi_*(S/p^n)$ is an isomorphism.

Dualizing the tower $(Y(n), J(n))$ of definition 3.2.1 gives (up to a desuspension) the following tower.

$$\begin{tikzcd}
S/p^n \arrow[r] & S/p^{n-1} \arrow[r] \arrow[dl, dotted] & \dots \arrow[r] & S/p \arrow[r] & * \arrow[r] \arrow[dl, dotted] & \dots \\
S/p \arrow[u] & S/p \arrow[u] & & S/p \arrow[u] & * \arrow[u] &
\end{tikzcd}$$

We appeal to theorem 3.6 of [23]; with the notation of that paper we have $S/p^n = F(DS/p^n, S)$ and the MASS-n is obtained by using the tower $D(Y(n), J(n))$ in the source and the tower (X, I) (2.4.6) in the target. The result is applicable because each S/p^s has such good properties: S/p^s is finite and because $p^s : S/p^s \longrightarrow S/p^s$ is zero, proposition 1.2(a) of [23] tells us S/p^s is p-adically complete and p-adically cocomplete. The connectivity hypothesis is not strictly satisfied but this is not a problem; it is satisfied once we suspend the source variable and this does not affect convergence.

This result certainly gives 1 and the first part of 2. The vanishing line of lemma 6.5 implies a vanishing line for the MASS-n. Combining these facts we see that for each u there exists an s such that $F^s\pi_u(S/p^n) = 0$ and so we conclude that the natural map $\pi_*(S/p^n) \longrightarrow \lim_s \pi_*(S/p^n)/F^s\pi_*(S/p^n)$ is an isomorphism. $\square$

Corollary 7.3.2. *The MASS-n' (definition 5.1.3) converges.*

Proof. We are in case 3 of definition 7.1.1. The previous argument completes the proof. $\square$

Proposition 7.3.3. *The MASS-∞ (definition 5.1.4) converges to $\pi_*(S/p^\infty)$.*

Proof. We draw the following diagram in which each row and column is exact and where the last nontrivial map in the short exact sequence is described in definition 5.1.3. Notice that we are using the filtration associated to the MASS-n', not the MASS-n.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & F^{\sigma+1}\pi_{\lambda-\sigma}(S/p^n) \arrow[r] \arrow[d] & F^{\sigma}\pi_{\lambda-\sigma}(S/p^n) \arrow[r] \arrow[d] & E_\infty^{\sigma,\lambda}(\text{MASS-}n') \arrow[r] & 0 \\
 & \pi_{\lambda-\sigma}(S/p^n) \arrow[r, "="] & \pi_{\lambda-\sigma}(S/p^n) & &
\end{tikzcd}$$

Recall from the discussion after definition 5.1.4 that $E_\infty^{\sigma,\lambda}(\text{MASS-}\infty) = \text{colim}_n E_\infty^{\sigma,\lambda}(\text{MASS-}n')$ and so taking filtered colimits gives the following diagram in which the rows and columns remain exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \text{colim}_n F^{\sigma+1}\pi_{\lambda-\sigma}(S/p^n) \arrow[r] \arrow[d] & \text{colim}_n F^{\sigma}\pi_{\lambda-\sigma}(S/p^n) \arrow[r] \arrow[d] & E_\infty^{\sigma,\lambda}(\text{MASS-}\infty) \arrow[r] & 0 \\
 & \pi_{\lambda-\sigma}(S/p^\infty) \arrow[r, "="] & \pi_{\lambda-\sigma}(S/p^\infty) & &
\end{tikzcd}$$

Defining $F^\sigma\pi_*(S/p^\infty) = \text{im}(\text{colim}_n F^\sigma\pi_*(S/p^n) \longrightarrow \pi_*(S/p^\infty))$ the short exact sequence above gives an identification

$$E_\infty^{\sigma,\lambda}(\text{MASS-}\infty) = F^\sigma\pi_{\lambda-\sigma}(S/p^\infty)/F^{\sigma+1}\pi_{\lambda-\sigma}(S/p^\infty)$$

and we see that

$$\begin{aligned}
\bigcup_\sigma F^\sigma\pi_*(S/p^\infty) &= \text{im}(\text{colim}_\sigma\text{colim}_n F^\sigma\pi_*(S/p^n) \longrightarrow \pi_*(S/p^\infty)) \\
&= \text{im}(\text{colim}_n\text{colim}_\sigma F^\sigma\pi_*(S/p^n) \longrightarrow \pi_*(S/p^\infty)) \\
&= \text{im}(\text{colim}_n \pi_*(S/p^n) \longrightarrow \pi_*(S/p^\infty)) \\
&= \pi_*(S/p^\infty).
\end{aligned}$$

Lemma 6.5, together with convergence of the MASS-n' shows that $F^\sigma\pi_{\lambda-\sigma}(S/p^n) = 0$ for $\sigma > K$ when we let

$$K = \frac{(\lambda-\sigma)-(2p-3)}{2(p-1)}.$$

K is indepenedent of n and so $F^\sigma\pi_{\lambda-\sigma}(S/p^\infty) = 0$ for $\sigma > K$. All of this gives convergence as in case 3 of definition 7.1.1. ☐

7.4 The LASS-n and LASS-∞

An almost identical argument to that for the MASS-∞ shows that the LASS-n converges. However, some preliminary observations are in order and they justify why we took so much care when verifying the multiplicative structure of the MASS-n.

Proposition 7.4.1. *The LASS-$(n+1)$ (definition 5.2.4) converges.*

Proof. We start with some preliminary observations. Recall from theorem 5.2.3 that we have an element $\alpha_{p^n} : S^{p^n q} \longrightarrow S/p^{n+1}$ detected by $q_1^{p^n}$ in the MASS-$(n+1)$. Moreover,

$$f_n : \Sigma^{p^n q}S/p^{n+1} \xrightarrow{\alpha_{p^n}\wedge S/p^{n+1}} S/p^{n+1}\wedge S/p^{n+1} \xrightarrow{\mu} S/p^{n+1}$$

induces an isomorphism on K-theory and the periodicity theorem tells us, via lemma 5.2.2, that there exists an $i \in \mathbb{N}$ such that $(f_n)^i = (v_1^{p^n})^i$. We deduce the following identity.

$$\begin{aligned}
v_1^{-1}S/p^{n+1} &= \text{hocolim}(S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-p^n q}S/p^{n+1} \xrightarrow{v_1^{p^n}} \Sigma^{-2p^n q}S/p^{n+1} \longrightarrow \dots) \\
&= \text{hocolim}(S/p^{n+1} \xrightarrow{f_n} \Sigma^{-p^n q}S/p^{n+1} \xrightarrow{f_n} \Sigma^{-2p^n q}S/p^{n+1} \longrightarrow \dots)
\end{aligned}$$

By construction the induced homomorphism $(f_n)_* : \pi_*(S/p^{n+1}) \longrightarrow \pi_*(S/p^{n+1})$ is multiplication by $\alpha_{p^n} \in \pi_*(S/p^{n+1})$ so that

$$\begin{aligned}
\pi_*(v_1^{-1}S/p^{n+1}) &= \operatorname{colim}(\pi_*(S/p^{n+1}) \xrightarrow{(f_n)_*} \pi_*(S/p^{n+1}) \xrightarrow{(f_n)_*} \pi_*(S/p^{n+1}) \longrightarrow \ldots) \\
&= \operatorname{colim}(\pi_*(S/p^{n+1}) \xrightarrow{\cdot\alpha_{p^n}} \pi_*(S/p^{n+1}) \xrightarrow{\cdot\alpha_{p^n}} \pi_*(S/p^{n+1}) \longrightarrow \ldots) \\
&= \alpha_{p^n}^{-1}\pi_*(S/p^{n+1}).
\end{aligned}$$

The point in this observation is that it allows us to use the multiplicative structure of the MASS-n to localize the spectral sequence as opposed to constructing maps of towers. Maps of towers are constructed in [12, section 2.3] but the situation there is easier: to construct a map of Adams resolutions only requires one map; to construct a map of 'modified Adams resolutions' appears to be much harder.

For each k let $[\sigma, k] = \sigma + p^n k$ and $[\lambda, k] = \lambda + p^n(q+1)k$; we draw the following diagram in which each row and column is exact and where the last nontrivial map in the short exact sequence is described in definition 3.2.2. Notice that we are using the filtration associated to the MASS-$(n+1)$, not the MASS-$(n+1)'$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & F^{[\sigma,k]+1}\pi_{[\lambda,k]-[\sigma,k]}(S/p^{n+1}) \arrow[r] \arrow[d] & F^{[\sigma,k]}\pi_{[\lambda,k]-[\sigma,k]}(S/p^{n+1}) \arrow[r] \arrow[d] & E_\infty^{[\sigma,k],[\lambda,k]}(\text{MASS-}(n+1)) \arrow[r] & 0 \\
 & \pi_{[\lambda,k]-[\sigma,k]}(S/p^{n+1}) \arrow[r, "="] & \pi_{[\lambda,k]-[\sigma,k]}(S/p^{n+1}) & &
\end{tikzcd}$$

Multiplication by α_{p^n} defines maps between the $F^{[\sigma,k]}\pi_*(S/p^{n+1})$ as k varies. Since multiplication by α_{p^n} is seen as multiplication by $q_1^{p^n}$ on the E_∞-page we may take filtered colimits over k to give the following diagram in which the rows and columns remain exact and the middle row is part of a short exact sequence.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\operatorname{colim}_k F^{[\sigma,k]+1}\pi_{[\lambda,k]-[\sigma,k]}(S/p^{n+1}) \arrow[r] \arrow[d] & \operatorname{colim}_k F^{[\sigma,k]}\pi_{[\lambda,k]-[\sigma,k]}(S/p^{n+1}) \arrow[r] \arrow[d] & E_\infty^{\sigma,\lambda}(\text{LASS-}(n+1)) \\
\pi_{\lambda-\sigma}(v_1^{-1}S/p^{n+1}) \arrow[r, "="] & \pi_{\lambda-\sigma}(v_1^{-1}S/p^{n+1}) &
\end{tikzcd}$$

Defining $F^\sigma\pi_*(v_1^{-1}S/p^{n+1}) = \operatorname{im}(\operatorname{colim}_k F^{[\sigma,k]}\pi_*(S/p^{n+1}) \longrightarrow \pi_*(v_1^{-1}S/p^{n+1}))$ the short exact sequence above gives an identification

$$E_\infty^{\sigma,\lambda}(\text{LASS-}(n+1)) = F^\sigma\pi_{\lambda-\sigma}(v_1^{-1}S/p^{n+1})/F^{\sigma+1}\pi_{\lambda-\sigma}(v_1^{-1}S/p^{n+1})$$

and we see that

$$\begin{aligned}
\bigcup_\sigma F^\sigma \pi_*(v_1^{-1}S/p^{n+1}) &= \mathrm{im}(\mathrm{colim}_\sigma \mathrm{colim}_k F^{[\sigma,k]}\pi_*(S/p^{n+1}) \longrightarrow \pi_*(v_1^{-1}S/p^{n+1})) \\
&= \mathrm{im}(\mathrm{colim}_k \mathrm{colim}_\sigma F^{[\sigma,k]}\pi_*(S/p^{n+1}) \longrightarrow \pi_*(v_1^{-1}S/p^{n+1})) \\
&= \mathrm{im}(\mathrm{colim}_k \pi_*(S/p^{n+1}) \longrightarrow \pi_*(v_1^{-1}S/p^{n+1})) \\
&= \pi_*(v_1^{-1}S/p^{n+1}).
\end{aligned}$$

To verify case 3 of definition 7.1.1 we just need to check that for each u we can find a σ so that $F^\sigma\pi_u(v_1^{-1}S/p^{n+1})$ vanishes. The vanishing line of lemma 6.5 implies a vanishing line for the MASS-$(n+1)$ and so convergence of the MASS-$(n+1)$ shows that for each u we can systematically find a σ with $F^\sigma\pi_u(S/p^{n+1}) = 0$. Since multiplication by $q_1^{p^n}$ acts parallel to the vanishing line we have $F^\sigma\pi_u(v_1^{-1}S/p^{n+1}) = 0$, too. □

Corollary 7.4.2. *The LASS-n' (definition 5.3.2) converges.*

Finally, we are in a position to complete the proof of theorem 5.3.1.

Proposition 7.4.3. *The LASS-∞ (definition 5.3.3) converges.*

Proof. The proof is almost identical to that for the MASS-∞ although there is an extra subtlety. First, we indicate the changes to the proof for the MASS-∞. The two diagrams are replaced by

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & F^{\sigma+1}\pi_{\lambda-\sigma}(v_1^{-1}S/p^n) \arrow[r] \arrow[d] & F^{\sigma}\pi_{\lambda-\sigma}(v_1^{-1}S/p^n) \arrow[r] \arrow[d] & E_\infty^{\sigma,\lambda}(\text{LASS-}n') \arrow[r] & 0 \\
 & \pi_{\lambda-\sigma}(v_1^{-1}S/p^n) \arrow[r, "="] & \pi_{\lambda-\sigma}(v_1^{-1}S/p^n) & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{colim}_n F^{\sigma+1}\pi_{\lambda-\sigma}(v_1^{-1}S/p^n) \arrow[r] \arrow[d] & \mathrm{colim}_n F^{\sigma}\pi_{\lambda-\sigma}(v_1^{-1}S/p^n) \arrow[r] \arrow[d] & E_\infty^{\sigma,\lambda}(\text{LASS-}\infty) \arrow[r] & 0 \\
 & \pi_{\lambda-\sigma}(v_1^{-1}S/p^\infty) \arrow[r, "="] & \pi_{\lambda-\sigma}(v_1^{-1}S/p^\infty) & &
\end{tikzcd}$$

and we define $F^\sigma\pi_*(v_1^{-1}S/p^\infty) = \mathrm{im}(\mathrm{colim}_n F^\sigma\pi_*(v_1^{-1}S/p^n) \longrightarrow \pi_*(v_1^{-1}S/p^\infty))$. We use corollary 6.7 instead of lemma 6.5 and convergence of the LASS-n' instead of convergence of the MASS-n'. We use exactly the same inequality.

The remaining issue is to show how we can pass from the first diagram to the second diagram. This comes down to constructing a map of diagrams as we let n vary in the first.

Lemma 5.2.2 tells us that there exists an i such that the top and bottom rectangles in the following diagram commute; the v_1-maps are chosen as in [6, 1.1] so that each square commutes.

$$\begin{tikzcd}
S/p^n \arrow[r, "(f_{n-1})^p"] \arrow[d, "="] & \Sigma^{-p^n q}S/p^n \arrow[r] & \dots \arrow[r] & \Sigma^{-(i-1)p^n q}S/p^n \arrow[r, "(f_{n-1})^p"] & \Sigma^{-ip^n q}S/p^n \arrow[d, "="] \\
S/p^n \arrow[r, "\left(v_1^{p^{n-1}}\right)^p"] \arrow[d, "p"] & \Sigma^{-p^n q}S/p^n \arrow[r] \arrow[d, "p"] & \dots \arrow[r] & \Sigma^{-(i-1)p^n q}S/p^n \arrow[r, "\left(v_1^{p^{n-1}}\right)^p"] \arrow[d, "p"] & \Sigma^{-ip^n q}S/p^n \arrow[d, "p"] \\
S/p^{n+1} \arrow[r, "v_1^{p^n}"] \arrow[d, "="] & \Sigma^{-p^n q}S/p^n \arrow[r] & \dots \arrow[r] & \Sigma^{-(i-1)p^n q}S/p^{n+1} \arrow[r, "v_1^{p^n}"] & \Sigma^{-ip^n q}S/p^{n+1} \arrow[d, "="] \\
S/p^{n+1} \arrow[r, "f_n"] & \Sigma^{-p^n q}S/p^n \arrow[r] & \dots \arrow[r] & \Sigma^{-(i-1)p^n q}S/p^{n+1} \arrow[r, "f_n"] & \Sigma^{-ip^n q}S/p^{n+1}
\end{tikzcd}$$

The diagram above implies commutativity of the following diagram and so we deduce from a cofinality argument that the maps $p : F^\sigma\pi_*(S/p^n) \longrightarrow F^\sigma\pi_*(S/p^{n+1})$ induce maps $p : F^\sigma\pi_*(v_1^{-1}S/p^n) \longrightarrow F^\sigma\pi_*(v_1^{-1}S/p^{n+1})$.

$$\begin{tikzcd}
F^\sigma\pi_*(S/p^n) \arrow[r, "\cdot(\alpha_{p^{n-1}})^{ip}"] \arrow[d, "p"] & F^{\sigma+ip^n}\pi_*(S/p^n) \arrow[d, "p"] \\
F^\sigma\pi_*(S/p^{n+1}) \arrow[r, "\cdot(\alpha_{p^n})^i"] & F^{\sigma+ip^n}\pi_*(S/p^{n+1})
\end{tikzcd}$$

We saw in the construction of the LASS-∞ that the maps of spectral sequences

$$E_*^{*,*}(\text{MASS-}1') \longrightarrow E_*^{*,*}(\text{MASS-}2') \longrightarrow \dots \longrightarrow E_*^{*,*}(\text{MASS-}n') \longrightarrow E_*^{*,*}(\text{MASS-}(n+1)') \longrightarrow \dots$$

induce maps of spectral sequences

$$E_*^{*,*}(\text{LASS-}1') \longrightarrow E_*^{*,*}(\text{LASS-}2') \longrightarrow \dots \longrightarrow E_*^{*,*}(\text{LASS-}n') \longrightarrow E_*^{*,*}(\text{LASS-}(n+1)') \longrightarrow \dots$$

These constructions are compatible and so this completes the proof. $\square$

8 Maps of spectral sequences

We have set up a number of spectral sequences now and have seen some maps between them. For instance, we have the maps $E_*^{*,*}(\text{MASS-}n') \longrightarrow E_*^{*,*}(\text{MASS-}(n+1)')$ used in the construction of the MASS-∞. Inspecting the construction of the LASS-n and LASS-∞ we find maps of spectral sequences $E_*^{*,*}(\text{MASS-}n) \longrightarrow E_*^{*,*}(\text{LASS-}n)$ and $E_*^{*,*}(\text{MASS-}\infty) \longrightarrow E_*^{*,*}(\text{LASS-}\infty)$.

In order to obtain information about the ASS-S^0 we need a map of spectral sequences

$$\Sigma_\sigma E_*^{*,*}(\text{MASS-}\infty) \longrightarrow E_*^{*,*}(\text{ASS-}S^0).$$

This is also crucial for proving theorem 5.2.3.

In order to calculate the LASS-∞ we use the filtration of the E_2-page given by the q_1^{-1}-BSS (I.3.4.1). We need to know that the maps used to construct the q_0^∞-BSS (those given by applying $\text{Cotor}_P(-)$ to I.3.2.1) and q_1^{-1}-BSS come from maps of topological spectral sequences.

This section sets up all the maps of spectral sequences that we need and identifies their effect algebraically at the E_2-page.

8.1 $\Sigma^{-1}S/p^\infty \longrightarrow S^0$

The hardest map of spectral sequences to identify at the level of E_2-pages is given by the following proposition.

Proposition 8.1.1. *Associated to the map $\Sigma^{-1}S/p^n \longrightarrow S^0$ is a map of spectral sequences*

$$\Sigma_\sigma E_*^{*,*}(MASS\text{-}n') \longrightarrow E_*^{*,*}(ASS\text{-}S^0).$$

At E_2-pages this map can be identified, up to a sign, with the connecting homomorphism

$$\partial : \mathrm{Cotor}_P(M_n) \longrightarrow \mathrm{Cotor}_P(Q(0))$$

arising from the short exact sequences of P-comodules $0 \longrightarrow Q(0) \longrightarrow Q(0)\langle q_0^{-k}\rangle \longrightarrow M_k \longrightarrow 0$ (see definition I.1.5).

From the construction, which we address shortly, we immediately obtain the following corollary.

Corollary 8.1.2. *Associated to the map $\Sigma^{-1}S/p^\infty \longrightarrow S^0$ is a map of spectral sequences*

$$\Sigma_\sigma E_*^{*,*}(MASS\text{-}\infty) \longrightarrow E_*^{*,*}(ASS\text{-}S^0).$$

At E_2-pages this map can be identified, up to a sign, with the connecting homomorphism

$$\partial : \mathrm{Cotor}_P(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P(Q(0))$$

arising from the short exact sequences of P-comodules $0 \longrightarrow Q(0) \longrightarrow q_0^{-1}Q(0) \longrightarrow Q(0)/q_0^\infty \longrightarrow 0$ (see definition I.1.4).

The connecting homomorphism of the previous corollary is an isomorphism in a large range and the MASS-∞ is isomorphic to the ASS-S^0 in this range. More precisely, because $\mathrm{Cotor}_P(q_0^{-1}Q(0)) = \mathbb{F}_p[q_0, q_0^{-1}]$ we have

$$\Sigma_s \mathrm{Cotor}_P(Q(0)/q_0^\infty)/\, \mathbb{F}_p\, \langle q_0^t : t < 0\rangle = \mathrm{Cotor}_P(Q(0))/\, \mathbb{F}_p\, [q_0] \tag{8.1.3}$$

and the spectral sequences are isomorphic in the range $\lambda - \sigma > 0$. This corresponds to the fact, obtained using the cofibration sequence $S_{(p)} \longrightarrow H\mathbb{Q} \longrightarrow S/p^\infty$, that

$$\Sigma^{-1}\pi_*(S/p^\infty)/\pi_0(S/p^\infty) = \pi_*(S^0)/\pi_0(S^0).$$

Using these observations together with proposition 4.2.6, we find that the LASS-∞ tells us a lot about the ASS-S^0. We can be more precise once we compute the LASS-∞.

We turn to the proof of proposition 8.1.1. First we need to construct the map of towers which gives rise to the map of spectral sequences. We introduce the relevant notation.

Definition 8.1.4. Write (C, L) for the tower in which the sequence $C \in \mathscr{S}^{\mathbb{Z}}$ is given by

$$C_s = \begin{cases} S & \text{if } s \leq 0 \\ * & \text{if } s > 0 \end{cases}$$

and all nontrivial structure maps are the identity.

We have a map of towers $(Y(n)', J(n)') \longrightarrow (C, L)$ of nonzero degree. More precisely, we have compatible maps $\Sigma^{-1}Y(n)'_{s-1} \longrightarrow C_s$ and $\Sigma^{-1}J(n)'_{s-1} \longrightarrow L_s$: the map $\Sigma^{-1}S/p^s \longrightarrow S$ is given by composing $\Sigma^{-1}S/p^s \longrightarrow \Sigma^{-1}S/p^\infty$ with the connecting map obtained from the cofibration sequence

$$S \longrightarrow p^{-1}S \longrightarrow S/p^\infty.$$

Smashing with (X, I) (2.4.6) induces map of spectral sequences $\Sigma_\sigma E_*^{*,*}(\text{MASS-}n') \to E_*^{*,*}(\text{ASS-}S^0)$. Moreover, these maps are compatible as we vary n and so we obtain a map

$$\Sigma_\sigma E_*^{*,*}(\text{MASS-}\infty) \longrightarrow E_*^{*,*}(\text{ASS-}S^0).$$

Definition 8.1.5. The maps of spectral sequences

$$\Sigma_\sigma E_*^{*,*}(\text{MASS-}n') \longrightarrow E_*^{*,*}(\text{ASS-}S^0), \qquad \Sigma_\sigma E_*^{*,*}(\text{MASS-}\infty) \longrightarrow E_*^{*,*}(\text{ASS-}S^0)$$

just constructed are the maps of spectral sequences associated to the maps $\Sigma^{-1}S/p^n \longrightarrow S^0$ and $\Sigma^{-1}S/p^\infty \longrightarrow S^0$, respectively.

Proof of proposition 8.1.1. Consider the following diagram of cochain complexes.

$$\begin{tikzcd}
-n-1 & -n & -n+1 & & & -1 & 0 & 1 \\
* \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \dots \arrow[r] & * \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, "="'] & * \arrow[d] \\
* \arrow[r] \arrow[d] & S/p \arrow[r, "\beta"] \arrow[d, "="'] & S/p \arrow[r] \arrow[d, "="'] & \dots \arrow[r] & S/p \arrow[r, "\beta"] \arrow[d, "="'] & S/p \arrow[r] \arrow[d, "="'] & S \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & S/p \arrow[r, "\beta"] & S/p \arrow[r] & \dots \arrow[r] & S/p \arrow[r, "\beta"] & S/p \arrow[r] & * \arrow[r] & *
\end{tikzcd}$$

We have omitted suspensions: each spectrum lying in 'cohomological' grading s should be suspended s times. The map $\Sigma^{-1}S/p \to S$ is the same one that we used before. We note that the first row of the diagram is $\Sigma^\bullet L_\bullet$ and the last row of the diagram is $\Sigma^\bullet J(n)'_\bullet$. We call the middle row $\Sigma^\bullet L(n)'_\bullet$. Applying $H_*(-)$ gives the following diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{F}_p \arrow[r] \arrow[d, "="'] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & E[\tau_0] \arrow[r, "\beta"] \arrow[d, "="'] & E[\tau_0] \arrow[r] \arrow[d, "="'] & \dots \arrow[r] & E[\tau_0] \arrow[r, "\beta"] \arrow[d, "="'] & E[\tau_0] \arrow[r] \arrow[d, "="'] & \mathbb{F}_p \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & E[\tau_0] \arrow[r, "\beta"] & E[\tau_0] \arrow[r] & \dots \arrow[r] & E[\tau_0] \arrow[r, "\beta"] & E[\tau_0] \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

This diagram is a short exact sequence of DG A-comodules and we can apply the snake lemma to the short exact sequence

$$0 \longrightarrow \Omega(A; H_*(\Sigma^\bullet L_\bullet)) \longrightarrow \Omega(A; H_*(\Sigma^\bullet L(n)'_\bullet)) \longrightarrow \Omega(A; H_*(\Sigma^\bullet J(n)'_\bullet)) \longrightarrow 0$$

to give a connecting homomorphism. In fact, we can perform the 'lift, apply coboundary, pullback' procedure of the snake lemma geometrically:

$$\Big[(\Sigma^\bullet I_\bullet)\wedge(\Sigma^\bullet J(n)'_\bullet)\Big]_\sigma \longrightarrow \Big[(\Sigma^\bullet I_\bullet)\wedge(\Sigma^\bullet L(n)'_\bullet)\Big]_\sigma \xrightarrow{d} \Big[(\Sigma^\bullet I_\bullet)\wedge(\Sigma^\bullet L(n)'_\bullet)\Big]_{\sigma+1} \longrightarrow \Big[(\Sigma^\bullet I_\bullet)\wedge(\Sigma^\bullet L_\bullet)\Big]_{\sigma+1}$$

The first and last map are 'inclusion' and 'collapse' maps, respectively. Of course, these are not maps of cochain complexes but they correspond to the 'extend by zero' and 'project' maps at the level of the cobar construction.

The map $(Z(n)', K(n)') = (X, I) \wedge (Y(n)', J(n)') \to (X, I) \wedge (C, L) = (X, I)$, which induces the map of spectral sequences has an associated map of cochain complexes. This is precisely the map above and so we deduce that at the level of E_2-pages the map of spectral sequences is given by

$$\partial : \mathrm{Cotor}_A(H_*(\Sigma^\bullet J(n)'_\bullet)) \longrightarrow \mathrm{Cotor}_A(H_*(\Sigma^\bullet L_\bullet)).$$

We have a commutative diagram in which the vertical maps are homology isomorphisms. The maps are constructed in the same way as in lemma 4.2.2.

$$\begin{tikzcd}
0 \arrow[r] & Q(0) \otimes_\theta H_*(\Sigma^\bullet L_\bullet) \arrow[r] \arrow[d] & Q(0) \otimes_\theta H_*(\Sigma^\bullet L'_\bullet) \arrow[r] \arrow[d] & Q(0) \otimes_\theta H_*(\Sigma^\bullet J(n)'_\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q(0) \arrow[r] & Q(0)\langle q_0^{-k}\rangle \arrow[r] & M_n \arrow[r] & 0
\end{tikzcd}$$

Thus, using the map in theorem 4.2.1 we obtain a commuting diagram in which the vertical maps are homology isomorphisms

$$\begin{tikzcd}
0 \arrow[r] & \Omega(A; H_*(\Sigma^\bullet L_\bullet)) \arrow[r] \arrow[d] & \Omega(A; H_*(\Sigma^\bullet L'_\bullet)) \arrow[r] \arrow[d] & \Omega(A; H_*(\Sigma^\bullet J(n)'_\bullet)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega(P; Q(0)) \arrow[r] & \Omega(P; Q(0)\langle q_0^{-k}\rangle) \arrow[r] & \Omega(P; M_n) \arrow[r] & 0
\end{tikzcd}$$

and the connecting homomorphism above is isomorphic to the connecting homomorphism in the proposition statement, completing the proof.

Note that in the preceding argument we omitted some details regarding signs. More signs than usual appear because $(Y(n)', J(n)') \to (C, L)$ has nonzero degree. □

8.2 $S/p \longrightarrow S/p^\infty \longrightarrow S/p^\infty$

Next, we need to identify the maps of spectral sequences induced by the maps in the cofibration sequence $S/p \to S/p^\infty \to S/p^\infty$ and show that at the E_2-pages they give the maps used to construct the q_0^∞-BSS.

First, we use corollary 8.1.2 to identify the map induced by the connecting homomorphism.

Corollary 8.2.1. *Associated to the map $\Sigma^{-1}S/p^\infty \longrightarrow S/p$ is a map of spectral sequences*

$$\Sigma_\sigma E_*^{*,*}(MASS\text{-}\infty) \longrightarrow E_*^{*,*}(MASS\text{-}1).$$

At E_2-pages this map can be identified with the connecting homomorphism

$$\partial : \mathrm{Cotor}_P(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P(Q(1))$$

arising from the short exact sequences of P-comodules $0 \longrightarrow Q(1) \longrightarrow Q(0)/q_0^\infty \xrightarrow{q_0} Q(0)/q_0^\infty \longrightarrow 0$ (see I.3.2.1).

Proof. We have the following map of cofibration sequences.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & p^{-1}S \arrow[r] \arrow[d] & S/p^\infty \arrow[d, "="] \\ S/p \arrow[r] & S/p^\infty \arrow[r] & S/p^\infty \end{tikzcd}$$

We have an analogous map between short exact sequences of P-comodules.

$$\begin{tikzcd} 0 \arrow[r] & Q(0) \arrow[r] \arrow[d] & q_0^{-1}Q(0) \arrow[r] \arrow[d, "/q_0"] & Q(0)/q_0^\infty \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & Q(1) \arrow[r] & Q(0)/q_0^\infty \arrow[r, "q_0"] & Q(0)/q_0^\infty \arrow[r] & 0 \end{tikzcd}$$

This shows that $\Sigma^{-1}S/p^\infty \longrightarrow S/p$ factors as $\Sigma^{-1}S/p^\infty \longrightarrow S^0 \longrightarrow S/p$. Similarly, the connecting homomorphism $\partial : \mathrm{Cotor}_P(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P(Q(1))$ factors as

$$\mathrm{Cotor}_P(Q(0)/q_0^\infty) \xrightarrow{\partial} \mathrm{Cotor}_P(Q(0)) \longrightarrow \mathrm{Cotor}_P(Q(1)).$$

The result follows by composing the map in corollary 8.1.2 with the map induced by $S^0 \longrightarrow S/p$, which can be identified at E_2-pages with $\mathrm{Cotor}_P(Q(0)) \longrightarrow \mathrm{Cotor}_P(Q(1))$. ☐

The other maps are identified by the following proposition.

Proposition 8.2.2. *Associated to the maps in the cofibration sequence $S/p \longrightarrow S/p^\infty \xrightarrow{p} S/p^\infty$ are maps of spectral sequences*

$$E_*^{*,*}(MASS\text{-}1) \longrightarrow \Sigma_{\sigma,\lambda} E_*^{*,*}(MASS\text{-}\infty) \longrightarrow E_*^{*,*}(MASS\text{-}\infty).$$

At E_2-pages these maps can be identified with the maps

$$\mathrm{Cotor}_P(Q(1)) \longrightarrow \mathrm{Cotor}_P(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P(Q(0)/q_0^\infty)$$

arising from the short exact sequences of P-comodules $0 \longrightarrow Q(1) \longrightarrow Q(0)/q_0^\infty \xrightarrow{q_0} Q(0)/q_0^\infty \longrightarrow 0$ (see I.3.2.1).

Proof. The first map is the composite $\Sigma^{-1}_{\sigma,\lambda}E^{*,*}_*(\text{MASS-1}) = E^{*,*}_*(\text{MASS-1}') \to E^{*,*}_*(\text{MASS-}\infty)$. We turn to the second map.

The maps $S/p^{s+1} \longrightarrow S/p^s$ for $s \geq 0$ give maps $(Y(n+1), J(n+1)) \longrightarrow (Y(n), J(n))$. Reindexing, we obtain maps $(Y(n+1)', J(n+1)') \longrightarrow (Y(n)', J(n)')$ of nonzero degree. These induce maps of spectral sequences

$$\Sigma_{\sigma,\lambda}E^{*,*}_*(\text{MASS-}(n+1)') \longrightarrow E^{*,*}_*(\text{MASS-}n')$$

The second map is given by taking a colimit of these maps. The E_2-page identifications are obtained using the same arguments as in lemma 4.2.2. □

We summarizing the previous two results in the following proposition.

Proposition 8.2.3. *Associated to the cofibration sequence $S/p \longrightarrow S/p^\infty \xrightarrow{p} S/p^\infty$ we have maps of spectral sequences. At E_2-pages these maps can be identified with the long exact sequence used to construct the q_0^∞-BSS (definition I.3.2.2).*

We also have the requisite localized version of this result.

Proposition 8.2.4. *Associated to the cofibration sequence $v_1^{-1}S/p \longrightarrow v_1^{-1}S/p^\infty \xrightarrow{p} v_1^{-1}S/p^\infty$ we have maps of spectral sequences. At E_2-pages these maps can be identified with the long exact sequence used to construct the q_1^{-1}-BSS (definition I.3.4.1).*

Proof. The first map is the composite $\Sigma^{-1}_{\sigma,\lambda}E^{*,*}_*(\text{LASS-1}) = E^{*,*}_*(\text{LASS-1}') \longrightarrow E^{*,*}_*(\text{LASS-}\infty)$.

For the second map we note that the maps $\Sigma_{\sigma,\lambda}E^{*,*}_*(\text{MASS-}(n+1)') \longrightarrow E^{*,*}_*(\text{MASS-}n')$ induce maps $\Sigma_{\sigma,\lambda}E^{*,*}_*(\text{LASS-}(n+1)') \longrightarrow E^{*,*}_*(\text{LASS-}n')$ and we take colimits.

For the connecting homomorphism we note that the map $\Sigma_\sigma E^{*,*}_*(\text{MASS-}\infty) \to E^{*,*}_*(\text{MASS-1})$ can be constructed in one shot. We have a map of towers $(Y(n)', J(n)') \to (Y(1), J(1))$ of nonzero degree. More precisely, we have compatible maps $\Sigma^{-1}Y(n)'_{s-1} \to Y(1)_s$ and $\Sigma^{-1}J(n)'_{s-1} \to J(1)_s$: the map $\Sigma^{-1}S/p^s \to S/p$ is given by composing $\Sigma^{-1}S/p^s \to \Sigma^{-1}S/p^\infty$ with the connecting map obtained from the cofibration sequence $S/p \longrightarrow S/p^\infty \longrightarrow S/p^\infty$. These induce maps of spectral sequences $\Sigma_\sigma E^{*,*}_*(\text{MASS-}n') \to E^{*,*}_*(\text{MASS-1})$; these maps are compatible as we vary n and so we obtain a map $\Sigma_\sigma E^{*,*}_*(\text{MASS-}\infty) \to E^{*,*}_*(\text{MASS-1})$. Moreover, the maps $\Sigma_\sigma E^{*,*}_*(\text{MASS-}n') \to E^{*,*}_*(\text{MASS-1})$ induce maps of spectral sequences $\Sigma_\sigma E^{*,*}_*(\text{LASS-}n') \to E^{*,*}_*(\text{LASS-1})$; these maps are compatible as we vary n and so we obtain a map $\Sigma_\sigma E^{*,*}_*(\text{LASS-}\infty) \to E^{*,*}_*(\text{LASS-1})$.

That the maps are as claimed on E_2-pages follows from the construction and proposition 8.2.3. □

9 Finishing up the computation

In this section we compute the LASS-∞. We have already addressed the necessary combinatorics for the calculation in I.6. We need to introduce two more spectral sequences so that we can see the value in proposition 6.3.

9.1 Setting up the q_0-filtration spectral sequence

The heart of the calculation is computing $E_3(\text{LASS-}\infty) = H(E_2(\text{LASS-}\infty), d_2)$. $E_2(\text{LASS-}\infty)$ has the filtration arising from the q_1^{-1}-BSS and this is respected by d_2. There is an associated spectral sequence, which we proceed to set up.

Definition 9.1.1. For $v \le 0$ let $F_B^v E_2^{*,*}(\text{LASS-}\infty) = \ker(q_0^{-v} : E_2^{*,*}(\text{LASS-}\infty) \to E_2^{*,*}(\text{LASS-}\infty))$. For $v > 0$ let $F_B^v E_2^{*,*}(\text{LASS-}\infty) = 0$.

$q_0 \in \mathrm{Cotor}_P(q_1^{-1}Q(0)/q_0^n)$ is a permanent cycle in the LASS-n. Because d_2 is a derivation we see that multiplication by q_0 commutes with d_2. Thus, the same is true in the LASS-n' and the LASS-∞. We summarize in the following lemma.

Lemma 9.1.2. *$d_2 : E_2^{*,*}(LASS\text{-}\infty) \longrightarrow E_2^{*,*}(LASS\text{-}\infty)$ respects the filtration of definition 9.1.1. In particular, we have an induced map*

$$\overline{d_2} : F_B^v E_2^{*,*}(LASS\text{-}\infty)/F_B^{v+1}E_2^{*,*}(LASS\text{-}\infty) \longrightarrow F_B^v E_2^{*,*}(LASS\text{-}\infty)/F_B^{v+1}E_2^{*,*}(LASS\text{-}\infty).$$

Applying $H^{*,*}(-)$ to the short exact sequence

$$0 \longrightarrow F_B^{v+1}E_2^{*,*}(\text{LASS-}\infty) \longrightarrow F_B^v E_2^{*,*}(\text{LASS-}\infty) \longrightarrow F_B^v E_2^{*,*}(\text{LASS-}\infty)/F_B^{v+1}E_2^{*,*}(\text{LASS-}\infty) \longrightarrow 0$$

gives a long exact sequence and intertwining all of these long exact sequences gives an exact couple.

Definition 9.1.3. The spectral sequence arising from this exact couple is called the *q_0-filtration spectral sequence* (q_0-FILT). It has E_1-page

$$E_1^{\sigma,\lambda,v}(q_0\text{-FILT}) = H^{\sigma,\lambda}(F_B^v E_2^{*,*}(\text{LASS-}\infty)/F_B^{v+1}E_2^{*,*}(\text{LASS-}\infty), \overline{d_2})$$

and d_r has degree $(2,1,r)$. As a notational device we define the E_0-page (recall I.3.4.1 and I.4.2.1).

$$E_0^{\sigma,\lambda,v}(q_0\text{-FILT}) = F_B^v E_2^{\sigma,\lambda}(\text{LASS-}\infty)/F_B^{v+1}E_2^{\sigma,\lambda}(\text{LASS-}\infty) = E_\infty^{\sigma,\lambda,v}(\text{bi-}q_1^{-1}) = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} E_\infty^{s,t,u,v}(q_1^{-1})$$

The spectral sequence converges to $E_3^{*,*}(\text{LASS-}\infty)$; the filtration degree is given by v. In particular, we have an identification

$$E_\infty^{\sigma,\lambda,v}(q_0\text{-FILT}) = F^v E_3^{\sigma,\lambda}(\text{LASS-}\infty)/F^{v+1}E_3^{\sigma,\lambda}(\text{LASS-}\infty)$$

where $F^v E_3^{*,*}(\text{LASS-}\infty) = \mathrm{im}(H^{*,*}(F_B^v E_2^{*,*}(\text{LASS-}\infty), d_2) \to H^{*,*}(E_2^{*,*}(\text{LASS-}\infty), d_2))$. The identification is given by lifting an element of $F^v E_3^{*,*}(\text{LASS-}\infty)$ to $H^{*,*}(F_B^v E_2^{*,*}(\text{LASS-}\infty), d_2)$ and mapping this down to $E_1^{*,*,v}(q_0\text{-FILT})$ to give a permanent cycle.

We note that convergence is given by case 2 of definition 7.1.1, using convergence of the q_1^{-1}-BSS.

9.2 The Mahowald Filtration

To make the next subsection clearer we recap, in this subsection, some of the work of Miller. In [17] he computes the v_1-periodic homotopy of the Moore spectrum S/p using the LASS-1 (notice that the MASS-1 is simply the ASS-S/p which is why there is no mention of modified Adams spectral sequences). In proposition 4.2.3 we saw that the E_2-page of the MASS-1 can be given three gradings; we can filter it using the s grading. Miller observes that, by constructing this filtration geometrically, we can show that the d_2's in the MASS-1 interact nicely with this additional structure. We explain in a little more detail.

Definition 9.2.1. All of the spectral sequences of sections 2 through 5 have E_2-pages of the form

$$E_2^{\sigma,\lambda} = \bigoplus_{\substack{s+t=\sigma \\ u+t=\lambda}} \mathrm{Cotor}_P^{s,t,u}(N)$$

for some bigraded P-comodule N. In each case, the *Mahowald filtration* is given by

$$F_M^{\hat{s}} E_2^{\sigma,\lambda} = \bigoplus_{\substack{s+t=\sigma,\ s\geq\hat{s} \\ u+t=\lambda}} E_2^{s,t,u}.$$

Our nomenclature follows Miller in [17] who named this filtration in honour of Mahowald who made use of a related filtration in [11]. Miller constructed this filtration geometrically in the case of the ASS-Y for a "(BP,H)-primary" spectrum Y, in particular, when Y is S^0 or S/p.

Definition 9.2.2. Let (W,G) be the BP-canonical tower for S

$$\begin{tikzcd}
S \arrow[d] & \overline{BP} \arrow[l] \arrow[d] & \cdots \arrow[l] & \overline{BP}^s \arrow[l] \arrow[d] & \overline{BP}^{s+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
BP \arrow[ur, dotted] & BP\wedge\overline{BP} \arrow[ur, dotted] & & BP\wedge\overline{BP}^s \arrow[ur, dotted] & BP\wedge\overline{BP}^{s+1} \arrow[ur, dotted] &
\end{tikzcd}$$

so that $W \in \mathscr{S}^{\mathbb{Z}}$ is the sequence given by

$$W_s = \begin{cases} S & \text{if } s \leq 0 \\ \overline{BP}^{\wedge s} & \text{if } s \geq 0, \end{cases}$$

where $W_{s+1} \longrightarrow W_s$ is

$$\overline{BP}^{\wedge(s+1)} \xrightarrow{(\overline{BP}\longrightarrow S)\wedge\overline{BP}^{\wedge s}} \overline{BP}^s$$

for $s \geq 0$ and the identity on S otherwise.

Recall the definition of (X,I), 2.4.6. The Mahowald filtration is constructed geometrically using the observation that $(W,G)\wedge(X,I)$ is an H-Adams towers for S and truncating (W,G) to a tower for $\overline{BP}^s$ gives a filtration of $(W,G)\wedge(X,I)$. On Adams E_2-pages this gives the Mahowald filtration (see remark 5.3, (5.5), (5.10) and remark 8.15 of [17]). We could surely apply the same reasoning to the MASS-n since [23, lemma 3.5] tells us that we do not have to use the canonical resolution (X,I) in its construction; we're free to use $(W,G)\wedge(X,I)$ instead.

The key result, which Miller proves, is the following theorem (a restated version of I.6.1).

Theorem 9.2.3 ([17, 4.8])**.** *In the LASS-1 we have $d_2 : F_M^s E_2^{*,*}(LASS\text{-}1) \longrightarrow F_M^{s+1} E_2^{*,*}(LASS\text{-}1)$ and $d_2 h_{i+1} \doteq q_1 b_i$ for $i > 0$, up to higher filtration.*

The proof proceeds in two stages. Firstly, Miller checks the claim concerning the filtration and that, up to higher filtration, differentials in the LASS-1 are determined by differentials in another spectral sequence. Then he computes the relevant differentials in the other spectral sequence. We believe that the first part of the argument holds for the LASS-n and the LASS-∞, in particular that d_2 increases Mahowald filtration by one. In fact, we have a different proof of [17, theorem 6.1] that is easily generalized. We do not give it here since it is not necessary. However, it does put the mind at rest, knowing that underpinning the algebra is geometry.

9.3 Setting up the Mahowald spectal sequence

The bulk of the work in computing the q_0-FILT spectral sequence is determining the E_1-page. For this we introduce our final spectral sequence. Its set up uses the fact that the maps in the exact couple giving rise to the the q_1^{-1}-BSS come from maps of topological spectral sequences and that $E_0(q_0\text{-FILT})$ has a Mahowald filtration.

Lemma 9.3.1. *We know from definition I.4.2.1 and proposition 4.2.3 that $E_\infty^{*,*,*}(bi\text{-}q_1^{-1})$ is a subquotient of $\bigoplus_{v<0} E_2^{*,*}(LASS\text{-}1)_v$.*

$d_2 : E_2^{*,*}(LASS\text{-}1) \to E_2^{*,*}(LASS\text{-}1)$ induces a map $\overline{d_2} : E_\infty^{*,*,v}(bi\text{-}q_1^{-1}) \to E_\infty^{*,*,v}(bi\text{-}q_1^{-1})$ such that the identification

$$(E_\infty^{\sigma,\lambda,v}(bi\text{-}q_1^{-1}), \overline{d_2}) = (F_B^v E_2^{\sigma,\lambda}(LASS\text{-}\infty)/F_B^{v+1} E_2^{\sigma,\lambda}(LASS\text{-}\infty), \overline{d_2})$$

is an idenfication of complexes (recall lemma 9.1.2).

Proof. This follows immediately from the fact that the bi-q_1^{-1}-BSS is set up using the exact couple

$$\begin{tikzcd}
E_2^{\sigma-v+r-1,\lambda-v+r-1}(\text{LASS-}\infty) & \dots \arrow[l, "q_0"'] & E_2^{\sigma-v,\lambda-v}(\text{LASS-}\infty) \arrow[l, "q_0"'] \arrow[drr, dotted, "\partial"] & & E_2^{\sigma-v-1,\lambda-v-1}(\text{LASS-}\infty) \arrow[ll] \\
E_2^{\sigma-v+r,\lambda-v+r}(\text{LASS-}1)_{v-r} \arrow[u] & & & & E_2^{\sigma-v,\lambda-v}(\text{LASS-}1)_v \arrow[u]
\end{tikzcd}$$

and each of the maps comes from a map of spectral sequences (8.2.4). ☐

Corollary 9.3.2. $E_1^{\sigma,\lambda,v}(q_0\text{-}FILT) = H^{\sigma,\lambda}(E_\infty^{*,*,v}(bi\text{-}q_1^{-1}), \overline{d_2})$.

Since the bi-q_1^{-1}-BSS is obtained from the q_1^{-1}-BSS by collapsing one of the gradings, $E_\infty(\text{bi-}q_1^{-1})$ has a Mahowald filtration.

Definition 9.3.3. The *Mahowald filtration* on $E_\infty^{*,*,*}(\text{bi-}q_1^{-1})$ is induced from the Mahowald filtration on $E_1^{*,*}(\text{bi-}q_1^{-1}) = \bigoplus_{v<0} E_2^{*,*}(\text{LASS-}1)_v$:

$$\begin{aligned}
F_M^{\hat{s}} E_\infty^{\sigma,\lambda,v}(\text{bi-}q_1^{-1}) &= \bigoplus_{\substack{s+t=\sigma,\ s\geq\hat{s} \\ u+t=\lambda}} E_\infty^{s,t,u,v}(q_1^{-1}) \\
&= \bigoplus_{\substack{s+t=\sigma,\ s\geq\hat{s} \\ u+t=\lambda}} \left(E_1^{s,t,u,v} \cap \bigcap_r \ker d_r \right) / \left(E_1^{s,t,u,v} \cap \bigcup_r \operatorname{im} d_r \right).
\end{aligned}$$

Theorem 9.2.3 and the proof of lemma 9.3.1 give the following result.

Proposition 9.3.4. $\overline{d_2} : E_\infty^{*,*,v}(bi\text{-}q_1^{-1}) \longrightarrow E_\infty^{*,*,v}(bi\text{-}q_1^{-1})$ *induces a map*

$$\overline{d_2} : F_M^s E_\infty^{*,*,v}(bi\text{-}q_1^{-1}) \longrightarrow F_M^{s+1} E_\infty^{*,*,v}(bi\text{-}q_1^{-1}).$$

Applying $H^{*,*}(-)$ to the short exact sequence

$$0 \longrightarrow F_M^{s+1} E_\infty^{*,*,v}(\text{bi-}q_1^{-1}) \longrightarrow F_M^s E_\infty^{*,*,v}(\text{bi-}q_1^{-1}) \longrightarrow F_M^s E_\infty^{*,*,v}(\text{bi-}q_1^{-1})/F_M^{s+1} E_\infty^{*,*,v}(\text{bi-}q_1^{-1}) \longrightarrow 0$$

gives a long exact sequence and intertwining all of these long exact sequences gives an exact couple.

Definition 9.3.5. The spectral sequence arising from this exact couple is called the *Mahowald spectral sequence* (MAHSS). It has E_1-page

$$E_1^{s,\sigma,\lambda,v}(\mathrm{MAH}) = F_M^s E_\infty^{\sigma,\lambda,v}(\text{bi-}q_1^{-1})/F_M^{s+1}E_\infty^{\sigma,\lambda,v}(\text{bi-}q_1^{-1}) = E_\infty^{s,\sigma-s,(\lambda-\sigma)+s,v}(q_1^{-1})$$

and d_r has degree $(r,2,1,0)$. The spectral sequence converges to $E_1(q_0\text{-FILT})$ and the filtration degree is given by s. In particular, we have an identification

$$E_\infty^{s,\sigma,\lambda,v}(\mathrm{MAH}) = F^s E_1^{\sigma,\lambda,v}(q_0\text{-FILT})/F^{s+1}E_1^{\sigma,\lambda,v}(q_0\text{-FILT})$$

where

$$F^s E_1^{*,*,*}(q_0\text{-FILT}) = \mathrm{im}(H^{*,*,*}(F_M^s E_\infty^{*,*,*}(\text{bi-}q_1^{-1}),\overline{d_2}) \longrightarrow H^{*,*,*}(E_\infty^{*,*,*}(\text{bi-}q_1^{-1}),\overline{d_2})).$$

The identification is given by lifting an element of $F^s E_1^{*,*,*}(q_0\text{-FILT})$ to $H^{*,*,*}(F_M^s E_\infty^{*,*,*}(\text{bi-}q_1^{-1}),\overline{d_2})$ and mapping this down to $E_1^{s,*,*,*}(\mathrm{MAH})$ to give a permanent cycle.

Convergence of this spectral sequence is given by case 1 of definition 7.1.1, although we need the corollary of the following lemma.

Lemma 9.3.6. *For each (σ,λ) there are only finitely many s such that $\mathrm{Cotor}_P^{s,\sigma-s,(\lambda-\sigma)+s}(q_1^{-1}Q(1))$ is nonzero.*

Proof. Recall from I.5.1.4 that $\mathrm{Cotor}_P(q_1^{-1}Q(1)) = \mathbb{F}_p[q_1,q_1^{-1}]\otimes E[h_i : i>0]\otimes \mathbb{F}_p[b_i : i>0]$, using the notation of I.5.2.2. Certainly s needs to be non-negative so assume $s\geq 0$ throughout.

Working in the $(\lambda-\sigma,\sigma)$ grading in which we plot our Adams spectral sequences, we find that the line from the origin to any of the h_i's or b_i's except h_1 has slope strictly less than $1/(2p-2)$, in particular, slope less than or equal to $1/(p^2-p-1)$. Also, with the exception of h_1, each has $(\lambda-\sigma)$ grading greater than or equal to $2(p^2-p-1)$. Since

$$\frac{2(p^2-p-1)}{2p-2} - \frac{2(p^2-p-1)}{p^2-p-1} = \frac{p(p-3)+1}{p-1} > \frac{1}{p},$$

all except h_1 have vertical distance greater than $1/p$ to the vanishing line of corollary 6.4.

If an element in $\mathrm{Cotor}_P(q_1^{-1}Q(1))$ has grading (s,t,u), its monomials contain precisely s symbols in the set $\{h_i, b_i : i>0\}$. Since $h_1^2=0$, and multiplication by q_1 acts parallel to the vanishing line, we see that if a nonzero element has grading $(s+1,t,u)$ then it has vertical distance is greater than s/p to the vanishing line. The proof follows quickly. □

Corollary 9.3.7. *For each (σ,λ,v) there are only finitely many s such that $E_\infty^{s,\sigma-s,(\lambda-\sigma)+s,v}(q_1^{-1})$ is nonzero.*

9.4 The MAHSS, the q_0-FILT and the LASS-∞

We are finally ready to compute the LASS-∞ (5.3.3).

Miller's theorem (9.2.3) tells us that $(E_1^{\bullet,*,*,v}(\mathrm{MAH}),d_1)$ is determined by definition I.6.2 and propostion I.6.3. We obtain the following results.

Theorem 9.4.1. *$E_2^{*,*,*,*}(MAH)$ has $\mathbb{F}_p$-basis*

$$\{\langle 1\rangle_v : v<0\} \cup \left\{ \left\langle [\![kp^{j-1}]\!] \right\rangle_v : \ p \nmid k \in \mathbb{Z},\ j \geq 1,\ -p^{[j]} \leq v < 0 \right\}$$
$$\cup \left\{ \left\langle [\![kp^{i}]\!] \left[[\![-p^{[i]}]\!] h_i \right] \right\rangle_v : \ k \in \mathbb{Z},\ i \geq 1,\ 1-p^i \leq v < 0 \right\}.$$

Degree considerations give the following corollary.

Corollary 9.4.2. $E_\infty^{*,*,*,*}(MAH) = E_2^{*,*,*,*}(MAH)$.

Proof. $[\![n]\!] \in \mathrm{Cotor}_P^{0,n,2n(p-1)}(q_1^{-1}Q(1))$ and so for the appropriate n and v (given by theorem 9.4.1) we have

$$\left\langle [\![n]\!] \right\rangle_v \in E_2^{0,n+v,(2p-1)n+v,v}(\mathrm{MAH}).$$

$[\![n]\!] h_i \in \mathrm{Cotor}_P^{1,n,2[n+p^{[i]}](p-1)}(q_1^{-1}Q(1))$ and so for the appropriate n and v (given by theorem 9.4.1) we have

$$\left\langle [\![n]\!] h_i \right\rangle_v \in E_2^{1,(n+v)+1,2[n+p^{[i]}](p-1)+(n+v),v}(\mathrm{MAH}).$$

Write q for $2(p-1)$ as usual, and consider the topological dimension $\lambda-\sigma$ of these classes. In the first case we have $\lambda-\sigma = nq$; in the second case we have $\lambda-\sigma = [n+p^{[i]}]q-1$. Using just this data we know that there can only be differentials from elements in the first class and there can only be differentials to elements in the second class.

We have

$$d_r \left\langle [\![n]\!] \right\rangle_v \in E_r^{r,n+v+2,(2p-1)n+v+1,v}(\mathrm{MAH}).$$

and the vanishing line of corollary 6.8 tells us that

$$F_B^{v+2} E_2^{n+v+2,(2p-1)n+v+1}(\text{LASS-}\infty) = E_2^{n+v+2,(2p-1)n+v+1}(\text{LASS-}\infty).$$

Thus $E_\infty^{n+v+2,(2p-1)n+v+1,v}(\text{bi-}q_1^{-1}) = 0$ and so $E_r^{r,n+v+2,(2p-1)n+v+1,v}(\mathrm{MAH}) = 0$. We deduce that there cannot be any more nontrivial differentials, which completes the proof. $\square$

Corollary 9.4.3. *$E_1^{*,*,*}(q_0\text{-}FILT)$ has an $\mathbb{F}_p$-basis, which we write abusively as*

$$\{\langle 1\rangle_v : v<0\} \cup \left\{ \left\langle [\![kp^{j-1}]\!] \right\rangle_v : \ p \nmid k \in \mathbb{Z},\ j \geq 1,\ -p^{[j]} \leq v < 0 \right\}$$
$$\cup \left\{ \left\langle [\![kp^{i}]\!] \left[[\![-p^{[i]}]\!] h_i \right] \right\rangle_v : \ k \in \mathbb{Z},\ i \geq 1,\ 1-p^i \leq v < 0 \right\}.$$

This gives an almost perfect upper bound on the size of $E_3^{*,*}(\text{LASS-}\infty)$ and because $\pi_*(v_1^{-1}S/p^\infty)$ is already known (see [21]), we can deduce the rest of the spectral sequence. For completeness we note the following results.

Corollary 9.4.4. *With the notation of corollary 9.4.3, $E_\infty^{*,*,*}(q_0\text{-}FILT)$ has an $\mathbb{F}_p$-basis given by*

$$\{\langle 1\rangle_v : v<0\} \cup \left\{ \left\langle [\![kp^{j-1}]\!] \right\rangle_v : \ p \nmid k \in \mathbb{Z},\ j \geq 1,\ -p^{[j-1]}-1 \leq v < 0 \right\}$$
$$\cup \left\{ \left\langle [\![kp^{i+1}]\!] \left[[\![-p^{[i]}]\!] h_i \right] \right\rangle_v : \ k \in \mathbb{Z},\ i \geq 1,\ 1-p^i \leq v < 0 \right\}.$$

Corollary 9.4.5. *$E_3^{*,*}(LASS\text{-}\infty)$ has an $\mathbb{F}_p$-basis, which we write abusively as*

$$\{\langle 1\rangle_v : v<0\} \cup \left\{ \left\langle [\![kp^{j-1}]\!] \right\rangle_v : \ p\nmid k\in\mathbb{Z},\ j\geq 1,\ -p^{[j-1]}-1\leq v<0 \right\}$$
$$\cup \left\{ \left\langle [\![kp^{i+1}]\!] \left[[\![-p^{[i]}]\!] h_i \right] \right\rangle_v : \ k\in\mathbb{Z},\ i\geq 1,\ 1-p^i\leq v<0 \right\}.$$

Corollary 9.4.6. *With the notation of corollary 9.4.5, and the convention that for $r>j$,*

$$-p^{[j-r]}-r=-j,$$

$E_{r+2}^{*,*}(LASS\text{-}\infty)$ has an $\mathbb{F}_p$-basis given by

$$\{\langle 1\rangle_v : v<0\} \cup \left\{ \left\langle [\![kp^{j-1}]\!] \right\rangle_v : \ p\nmid k\in\mathbb{Z},\ j\geq 1,\ -p^{[j-r]}-r\leq v<0 \right\}$$
$$\cup \left\{ \left\langle [\![kp^{i+r}]\!] \left[[\![-p^{[i]}]\!] h_i \right] \right\rangle_v : \ k\in\mathbb{Z},\ i\geq 1,\ 1-p^i\leq v<0 \right\}$$

whenever $r\geq 1$.

Corollary 9.4.7. *With the notation of corollary 9.4.5, $E_\infty^{*,*}(LASS\text{-}\infty)$ has an $\mathbb{F}_p$-basis given by*

$$\{\langle 1\rangle_v : v<0\} \cup \left\{ \left\langle [\![kp^{j-1}]\!] \right\rangle_v : \ p\nmid k\in\mathbb{Z},\ j\geq 1,\ -j\leq v<0 \right\}$$
$$\cup \left\{ \left\langle [\![-p^{[i]}]\!] h_i \right\rangle_v : \ i\geq 1,\ 1-p^i\leq v<0 \right\}.$$

These results are hard to digest if one has not been staring at Christian Nassau's charts [19] for three months. Recall, figure I.1, which displays some of his chart for $E_2(\text{ASS-}S^0)$ when $p=3$. This tells us about $E_2(\text{LASS-}\infty)$ in a range by (8.1.3) and proposition 4.1.4. We obtained figure I.2 by removing some of the towers in figure I.1 and the complement of figure I.2 in figure I.1 gives $E_2(\text{MAH})=E_1(q_0\text{-FILT})$ in the plotted range (up to the regrading coming from the Σ_σ in corollary 8.1.2). Figure II.1 displays the plot of $E_1(q_0\text{-FILT})$ with the gradings fixed. It also highlights the differentials which occur in the LASS-∞. The d_2's occur as differentials in the q_0-FILT: at what point they show up in the spectral sequence depends on the difference in the size of the towers they map between. d_3's and d_4's are also displayed in the picture. They are all forced from knowledge of $\pi_*(v_1^{-1}S/p^\infty)$. We highlight the permanent cycles in blue.

9.5 The LASS-∞ and the ASS-S^0

We finally address the information that the LASS-∞ gives us about the ASS-S^0. Proposition 4.2.6 together with lemma III.1.5.2 give the following result.

Corollary 9.5.1. *The localization map $E_3^{\sigma,\lambda}(MASS\text{-}\infty)\longrightarrow E_3^{\sigma,\lambda}(LASS\text{-}\infty)$ is*

1. *a surjection if $\lambda<p(p-1)(\sigma+1)-2$;*
2. *an isomorphism if $\lambda-1<p(p-1)(\sigma-1)-2$.*

Using (8.1.3) we obtain the following.

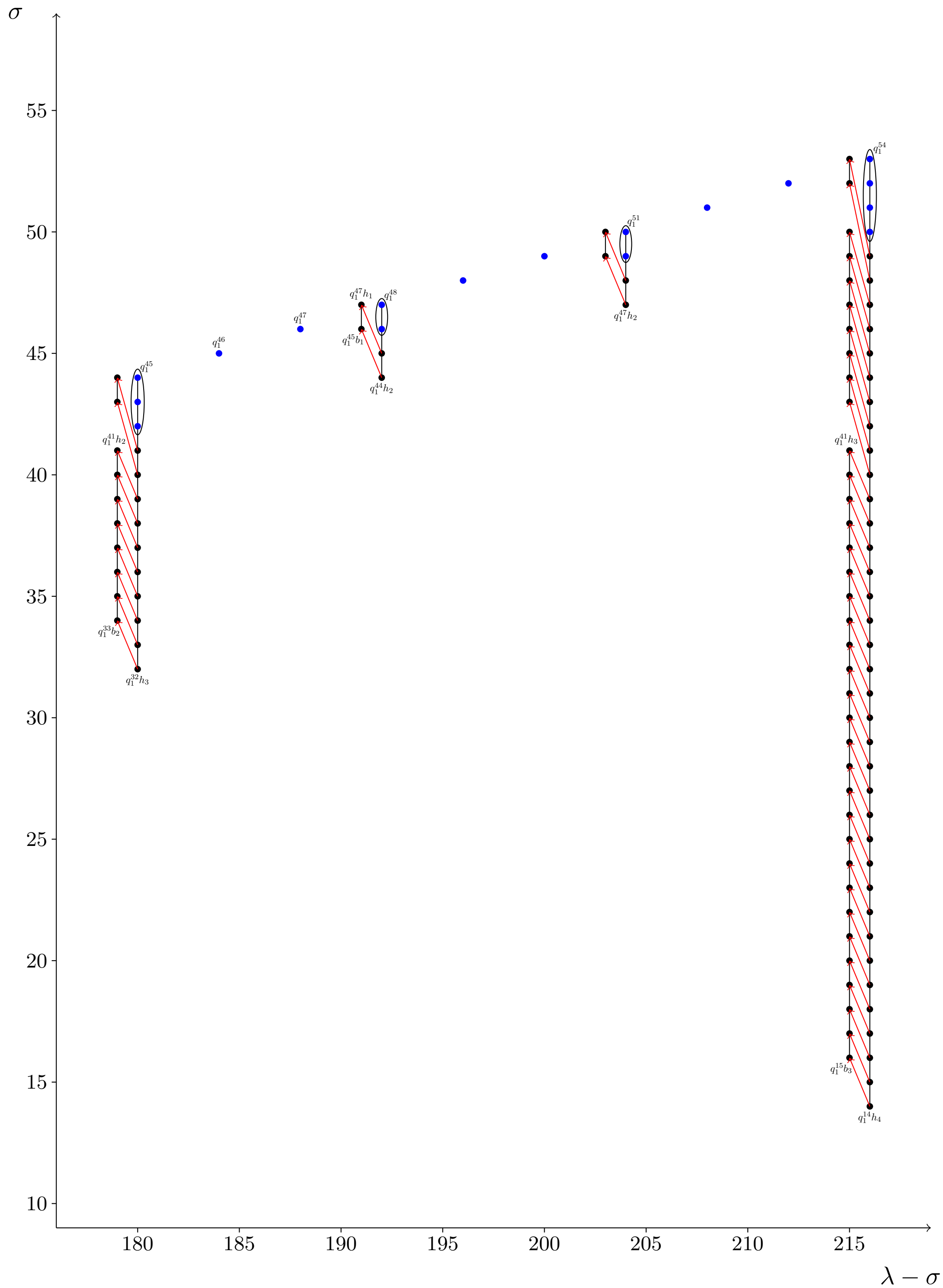

Figure II.1: A portion of $E_*^{\sigma,\lambda}(\text{LASS-}\infty)$ when $p = 3$

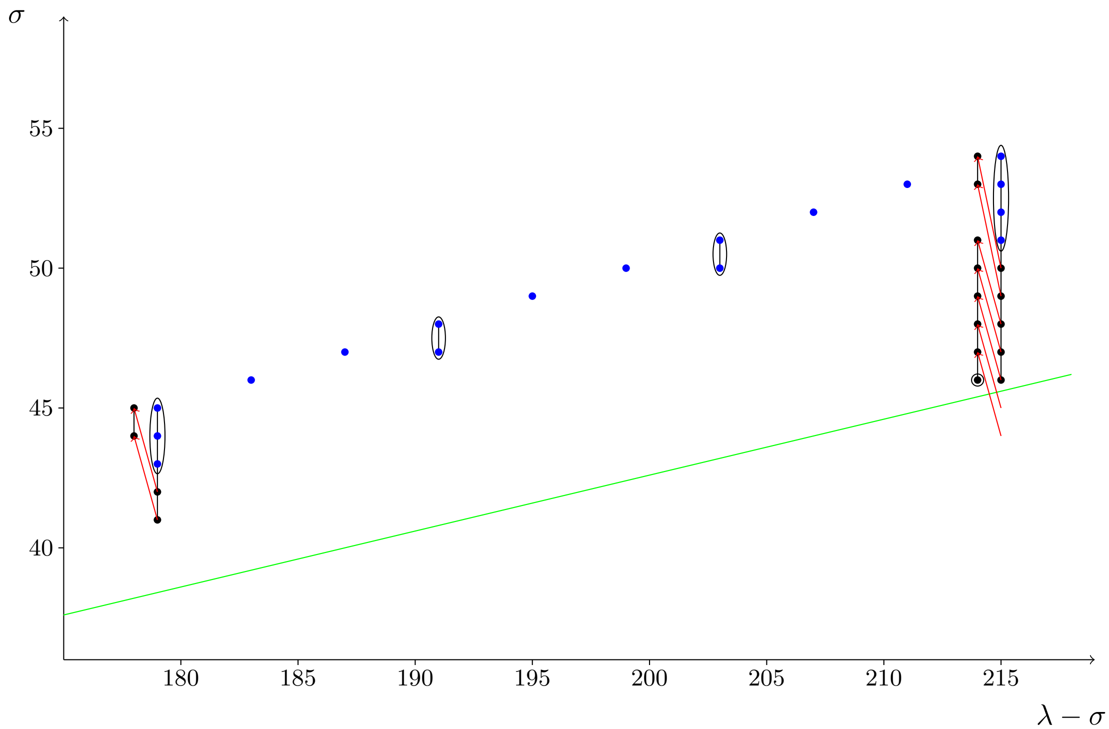

Figure II.2: A portion of $E_*^{\sigma,\lambda}(\text{ASS-}S^0)$ when $p = 3$

Corollary 9.5.2. *$E_3^{\sigma,\lambda}(ASS\text{-}S^0) = E_3^{\sigma-1,\lambda}(LASS\text{-}\infty)$ if $\lambda < p(p-1)(\sigma-2) - 1$ and $\lambda - \sigma > 0$.*

Writing down exactly what else can be deduced from these corollaries is tricky. Rather than writing opaque looking statements we draw the picture and discuss what we can say about it.

Corollary 9.5.2 tells us that figure II.2 displays $E_3(\text{ASS-}S^0)$ faithfully above the green line. First, we identify the permanent cycles.

Theorem 5.2.3 tells us that $q_1^{p^n} \in \text{Cotor}_P(Q(0)/q_0^{n+1})$ is a permanent cycle in the MASS-$(n+1)$. Since the MASS-$(n+1)$ is multiplicative the same is true for powers of this element. Using the maps of spectral sequences $E_*^{*,*}(\text{MASS-}(n+1)) \longrightarrow E_*^{*,*}(\text{MASS-}\infty) \longrightarrow E_*^{*,*}(\text{ASS-}S^0)$ we obtain permanent cycles in the ASS-S^0. Their q_0-multiples are displayed as the blue dots in figure II.2.

Now we discuss differentials in figure II.2. The elements directly below the blue dots cannot be permanent cycles since otherwise, the corresponding elements in the LASS-∞ would be permanent cycles, too. Similarly, the elements to the left of the towers containing the blue dots cannot be hit before they are hit in the LASS-∞. This allows us to conclude that the towers containing the blue dots support differentials which obey exactly the same pattern as in the LASS-∞.

By the surjectivity statement of corollary 9.5.1 we can deduce the existence of a few more differentials such as those in figure II.2 whose sources are not drawn. However, we cannot see an easy argument for why the circled element is the target of a differential. As we move further out in the $(\lambda - \sigma)$-direction our green line will intersect towers supporting longer differentials and we will have towers of questionable elements like this one. We believe that these questionable elements are always hit by a differential like that in the LASS-∞ and we have already proved a useful result in this direction (I.9.5). Rather than writing down the weaker statements that we can deduce from the results above, we make the following conjecture and postpone its proof for future work.

Conjecture: *For all $r \geq 2$, $E_r^{\sigma,\lambda}(ASS\text{-}S^0) = E_r^{\sigma-1,\lambda}(LASS\text{-}\infty)$ if $\sigma < \lambda < p(p-1)(\sigma-2)-1$.*

In the case of the circled element in the figure, our result in this direction together with Nassau's charts give an argument for why it *is* the target of a differential. But in a less specific case, there is still the possibility that a questionable element like this is a permanent cycle detecting a nontrivial homotopy class. Although the following example does not give a counterexample to our conjecture (the elements used lie below the green line and we are considering d_2's) it does illustrate how this sort of phenomenon might occur.

1. Let $x \in E_2^{-1,2p(p-1)-1}$(LASS-∞) be an element corresponding to
$$\langle q_1^p \rangle_{-1-p} \in E_\infty^{0,-1,2p(p-1),-1-p}(q_1^{-1}).$$

2. Let $y \in E_2^{1,2p(p-1)}$(LASS-∞) be an element corresponding to
$$\langle q_1^{p-1} h_1 \rangle_{1-p} \in E_\infty^{1,0,2p(p-1),1-p}(q_1^{-1}).$$

3. $E_2^{-1,2p(p-1)-1}(\text{MASS-}\infty) = E_2^{0,2p(p-1)-1}(\text{ASS-}S^0) = 0.$

4. We can take y to be the image of the element in $E_2^{1,2p(p-1)}$(MASS-∞) mapping to $b_{1,0}$.

$$\begin{tikzcd} E_2^{2,2p(p-1)}(\text{ASS-}S^0) & E_2^{1,2p(p-1)}(\text{MASS-}\infty) \arrow[l] \arrow[r] & E_2^{1,2p(p-1)}(\text{LASS-}\infty) \\ b_{1,0} & \bullet \arrow[l, mapsto] \arrow[r, mapsto] & y \end{tikzcd}$$

5. We find that $d_2 x = y$ in the LASS-∞ whereas $b_{1,0}$ is a permanent cycle detecting a nonzero homotopy class $\beta_1 \in \pi_{2p(p-1)-2}(S^0)$, the first non-trivial element in the cokernel of the J-homomorphism.

This example reminds us of Ravenel's work in [22].

Chapter III

Appendix

1 A permanent cycle in the MASS-n

1.1 Strategy

This section is devoted to a proof of theorem II.5.2.3.

Firstly, we wish to show that the element $q_1^{p^n} \in \mathrm{Cotor}_P(Q(0)/q_0^{n+1}) = E_2(\text{MASS-}(n+1))$ is a permanent cycle. The idea of the proof follows from the observation that if this is true, then there is a related permanent cycle in the MASS-∞ and thus, in the ASS-S^0. This is because we have a map of spectral sequences from the MASS-$(n+1)'$ to the MASS-∞ and because in subsection II.8.1 we construct a map of spectral sequences from the MASS-∞ to the ASS-S^0. We prove that the corresponding element is a permanent cycle in the ASS-S^0 (theorem 1.4.1), deduce the same for the MASS-∞, and then use an injectivity argument (lemma 1.5.3) to prove the result for the MASS-$(n+1)$.

In order to prove the statement for the ASS-S^0 we construct a corresponding homotopy class. To do this requires a thorough analysis of stunted projective spaces and this is performed in subsections 1.2 and 1.3. Once we have proven the permanent cycle statement we need to verify the K-theory statement and this is done in subsection 1.6.

1.2 Some classes in the (co)homology of stunted projective spaces

We make extensive use of stunted projective spaces. Firstly, let's recall the cohomology of $B\Sigma_p$.

Proposition 1.2.1 ([1, 2.1])**.** *Let $i : C_p \longrightarrow \Sigma_p$ be the inclusion of a Sylow subgroup. $H^*(BC_p) = E[x]\otimes\mathbb{F}_p[y]$ where $|x| = 1$, $|y| = 2$ and $\beta x = y$ and $H^*(B\Sigma_p) = E[x_{q-1}]\otimes\mathbb{F}_p[y_q]$ where $(Bi)^*(x_{q-1}) = xy^{p-2}$ and $(Bi)^*(y_q) = y^{p-1}$.*

Proposition 2.7 of [5], first proved by Adams in [1], says that there is a CW spectrum L with one cell in each nonnegative dimension congruent to 0 or -1 modulo q, such that $L \simeq (\Sigma^\infty B\Sigma_p)_{(p)}$. Denote the skeletal fitration by a superscript in parentheses. Then we make the following definition.

Definition 1.2.2. Write B for the spectrum of [5, 2.7]. For $n \geq 0$ let $B^n = B^{(nq)}$ and for $1 \leq n \leq m$ let $B_n^m = B^m/B^{n-1}$. For $n > m$ let $B_n^m = *$.

Notation 1.2.3. For $j > 0$ write e^j for $x_{q-1}y_q^{j-1} \in \widetilde{H}^{jq-1}(B\Sigma_p) = H^{jq-1}(B)$ and write e_j for the class in $H_{jq-1}(B)$ dual to e^j.

To construct the relevant permanent cycle in the ASS-S^0 we make use of a homotopy class in $\pi_{p^n q-1}(B_{p^n-n}^{p^n})$. Firstly, we analyze the algebraic picture and identify the corresponding A-comodule primitive.

Proposition 1.2.4. *For each $n \geq 0$, $e_{p^n} \in H_{p^n q-1}(B)$ is an A-comodule primitive.*

Proof. The result is obvious when $n = 0$ so assume from now on that $n > 0$.

Since the (co)homology of B is concentrated in dimensions which are 0 or -1 congruent to q, the dual result is that $P^i e^j = 0$ whenever $i, j > 0$ and $i + j = p^n$.

Let $i : C_p \longrightarrow \Sigma_p$ be the inclusion of a Sylow subgroup. Since $(Bi)^*$ is injective it is enough to show that the equation is true after applying $(Bi)^*$. Writing this out explicitly, we must show that

$$P^i(xy^{(p-1)j-1}) = 0 \text{ whenever } i, j > 0 \text{ and } i + j = p^n.$$

Write P for the total reduced p-th power. Then we have $P(x) = x$ and $P(y) = y + y^p = y(1 + y^{p-1})$. Suppose that $i, j > 0$ and that $i + j = p^n$. Then

$$P(xy^{(p-1)j-1}) = xy^{(p-1)j-1}(1 + y^{p-1})^{(p-1)j-1} = xy^{(p-1)j-1} \sum_{k=0}^{(p-1)j-1} \binom{(p-1)j-1}{k} y^{(p-1)k}$$

which gives

$$P^i(xy^{(p-1)j-1}) = \binom{(p-1)j-1}{i} xy^{(p-1)p^n-1}$$

as long as $i \leq (p-1)j - 1$ and $P^i(xy^{(p-1)j-1}) = 0$ otherwise. We just need to show that

$$p \,\Big|\, \binom{(p-1)(p^n-i)-1}{i}$$

whenever $0 < i \leq (p-1)(p^n - i) - 1$. The largest value of i for which we have $i \leq (p-1)(p^n-i)-1$ is $(p-1)p^{n-1} - 1$ so write $i = sp^k$ for $0 \leq k < n$ and $s \not\equiv 0 \pmod p$. Let $m = (p-1)(p^n - i) - 1$ so that we are interested in $\binom{m}{i}$. $m - i \equiv -1 \pmod{p^{k+1}}$ and so when we add $m - i$ to i in base p there is a carry. An elementary fact about binomial coefficients completes the proof. $\square$

The relevant topological result is given by the following proposition.

Proposition 1.2.5. *For each $n \geq 0$, $e_{p^n} \in H_{p^n q-1}(B_{p^n-n}^{p^n})$ is in the image of the Hurewicz homomorphism.*

Proof. Setting $\epsilon = 0$, $i = n + 1$, $j = p^n - n - 1$ and $k = iq - 1$ in [5, V.2.9(v)] shows that

$$Z = B^{(p^n q-1)}/B^{((p^n-n-1)q-1)}$$

has reductive top cell and we have an 'include-collapse' map $Z \longrightarrow B_{p^n-n}^{p^n}$. $\square$

In constructing the relevant permanent cycle in the ASS-S^0 we make use of the transfer. First, we analyze it algebraically.

Definition 1.2.6. Write $t : B \longrightarrow S^0$ for the transfer map of [1, 2.3(i)] and let C be the cofiber of $\Sigma^{-1}t$.

Notation 1.2.7. We have a cofibration sequence $S^{-1} \longrightarrow C \longrightarrow B$. Abuse notation and write e^j and e_j for the elements in $H^*(C)$ and $H_*(C)$ which correspond to the elements of the same name in $H^*(B)$ and $H_*(B)$. Write U and u for the dual classes in $H^*(C)$ and $H_*(C)$ corresponding to generators of $H^{-1}(S^{-1})$ and $H_{-1}(S^{-1})$.

Proposition 1.2.8. *Let $n \geq 0$. Then $e_{p^n} \in H_{p^n q-1}(C)$ is mapped to $1 \otimes e_{p^n} \dot{+} \xi_1^{p^n} \otimes u$ under the A-coaction map.*

Proof. First, let's introduce some notation which will be useful for the proof. Write Sq_p^{kq} and Sq_p^{kq+1} for P^k and βP^k, respectively. Recall that the Steenrod algebra A^* has a $\mathbb{F}_p$-vector space basis given by admissable monomials

$$\mathcal{B} = \{\mathrm{Sq}_p^{i_1} \cdots \mathrm{Sq}_p^{i_r} : i_j \geq p i_{j+1},\ i_j \equiv 0 \text{ or } 1 \pmod{p}\}.$$

We claim that $\mathrm{Sq}_p^{p^n q} U \doteq e^{p^n}$, and that $bU = 0$ for any $b \in \mathcal{B}$ of length greater than 1. Here, length greater than one means that $r > 1$ and $i_r > 0$.

By proposition 1.2.4 we know that e_{p^n} is mapped, under the coaction map, to $1 \otimes e_{p^n} + a \otimes u$ for some $a \in A$. If we can prove the claim above then we will deduce that $a \doteq \xi_1^{p^n}$.

Take an element $b = \mathrm{Sq}_p^{i_1} \cdots \mathrm{Sq}_p^{i_r} \in \mathcal{B}$ of length greater than one and let $k = \lfloor i_{r-1}/q \rfloor$. We have

$$i_{r-1} \geq p i_r \implies i_{r-1}/p \geq i_r \implies k \geq i_r \implies 2k > i_r - 1.$$

Since $|\mathrm{Sq}_p^{i_r} U| = i_r - 1$ and $\mathrm{Sq}_p^{i_r} U$ comes from the cohomology of a space we deduce that $P^k \mathrm{Sq}_p^{i_r} U = 0$. Now either $i_{r-1} = kq$ or $kq+1$ so that $\mathrm{Sq}_p^{i_{r-1}} = P^k$ or βP^k. Thus $\mathrm{Sq}_p^{i_{r-1}} \mathrm{Sq}_p^{i_r} U = 0$ and $bU = 0$ as required for the second part of the claim.

To prove that $P^{p^n} U \doteq e^{p^n}$ it is enough to show that $\beta P^{p^n} U \doteq \beta e^{p^n}$. Notice that $|\beta e^1| = q$ and so

$$P^{p^{n-1}q/2} \cdots P^{pq/2} P^{q/2} \beta e^1 = (\beta e^1)^{p^n} = \beta e^{p^n}.$$

The Kahn-Priddy theorem ([1, 2.3]) tells us that the map $t : B \longrightarrow S^0$ is surjective in homotopy. The ASS shows that there is a unique nontrivial class $\pi_{q-1}(S^0)$ (up to unit) detected by $h_{1,0}$ and by cellular approximation this is the composite

$$S^{q-1} \xrightarrow{i} B \xrightarrow{t} S^0.$$

We conclude that $P^1 U \doteq e^1$ and so it is enough to prove that

$$P^{p^{n-1}q/2} \cdots P^{pq/2} P^{q/2} \beta P^1 U \doteq \beta P^{p^n} U.$$

We induct on n, the result being trivial for $n = 0$. Suppose it is proven for some $n \geq 0$. Then we have

$$\begin{aligned} P^{p^n q/2} P^{p^{n-1}q/2} \cdots P^{pq/2} P^{q/2} \beta P^1 U &\doteq P^{p^n q/2} \beta P^{p^n} U \\ &\doteq (\beta P^{p^n + p^n q/2} + \text{elements of } \mathcal{B} \text{ of length greater than } 1)U \\ &= \beta P^{p^{n+1}} U, \end{aligned}$$

which completes the inductive step and the proof of the proposition. ∎

1.3 Maps between stunted projective spaces

We now proceed to construct maps between stunted projective spaces whilst analyzing their Adams filtration. For our purposes the following spectra are more convenient than those in the H-canonical Adams tower.

Definition 1.3.1. For $1 \leq n \leq m \leq \infty$ define $B_n^m\langle 1\rangle$ by the following cofibration sequence.

$$B_n^m\langle 1\rangle \longrightarrow B_n^m \xrightarrow{(e^n,\ldots,e^m)} \bigvee_{i=n}^m \Sigma^{iq-1}H$$

The following diagram commutes

$$\begin{tikzcd} B^{n-1} \arrow[r, "i"] \arrow[d] & B^m \arrow[r, "j"] \arrow[d] & B_n^m \arrow[d] \\ \bigvee_{i=1}^{n-1} \Sigma^{iq-1}H \arrow[r] & \bigvee_{i=1}^{m} \Sigma^{iq-1}H \arrow[r] & \bigvee_{i=n}^{m} \Sigma^{iq-1}H \end{tikzcd}$$

and so we obtain cofibration sequences

$$B^{n-1}\langle 1\rangle \xrightarrow{i} B^m\langle 1\rangle \xrightarrow{j} B_n^m\langle 1\rangle.$$

The following proposition might appear long and technical but the proof is, in fact, very straightforward.

Proposition 1.3.2. *For each $n \in \mathbb{N}$ there exists a unique map $f : B^n \longrightarrow B^{n-1}$ such that the left diagram commutes. Moreover, the centre diagram commutes so that the right diagram commutes.*

$$\begin{tikzcd} B^n \arrow[d, "f"'] \arrow[dr, "p"] & \\ B^{n-1} \arrow[r, "i"] & B^n \end{tikzcd} \qquad \begin{tikzcd} B^n \arrow[r, "i"] \arrow[dr, "p"'] & B^{n+1} \arrow[d, "f"] \\ & B^n \end{tikzcd} \qquad \begin{tikzcd} B^n \arrow[r, "i"] \arrow[d, "f"'] & B^{n+1} \arrow[d, "f"] \\ B^{n-1} \arrow[r, "i"] & B^n \end{tikzcd}$$

For $1 \leq n \leq m$ the filler for the diagram

$$\begin{tikzcd} B^n \arrow[r, "i"] \arrow[d, "f"'] & B^{m+1} \arrow[r, "j"] \arrow[d, "f"'] & B_{n+1}^{m+1} \arrow[d, dotted] \\ B^{n-1} \arrow[r, "i"] & B^m \arrow[r, "j"] & B_n^m \end{tikzcd}$$

is unique and we call it f. For $1 \leq n \leq m$ the filler for the diagram

$$\begin{tikzcd} B^{n-1} \arrow[r, "i"] \arrow[d, "p"'] & B^{m} \arrow[r, "j"] \arrow[d, "p"'] & B_{n}^{m} \arrow[d, dotted] \\ B^{n-1} \arrow[r, "i"] & B^m \arrow[r, "j"] & B_n^m \end{tikzcd}$$

is unique and so equal to p. Thus, the following diagrams commute for the appropriate vaules of m and n.

$$\begin{tikzcd} B^{m+1}_{n+1} \arrow[d, "f"'] \arrow[dr, "p"] & \\ B^m_n \arrow[r, "ij"] & B^{m+1}_{n+1} \end{tikzcd} \qquad \begin{tikzcd} B^m_n \arrow[r, "ij"] \arrow[dr, "p"'] & B^{m+1}_{n+1} \arrow[d, "f"] \\ & B^m_n \end{tikzcd} \qquad \begin{tikzcd} B^m_{n+1} \arrow[r, "p"] \arrow[d, "i"'] & B^m_{n+1} \\ B^{m+1}_{n+1} \arrow[r, "f"] & B^m_n \arrow[u, "j"'] \end{tikzcd} \qquad \begin{tikzcd} B^{m+1}_{n+1} \arrow[r, "f"] & B^m_n \arrow[d, "i"] \\ B^{m+1}_n \arrow[u, "j"] \arrow[r, "p"] & B^{m+1}_n \end{tikzcd}$$

For each $n \in \mathbb{N}$ there exists a unique map $g : B^n \longrightarrow B^{n-1}\langle 1\rangle$ such that the left diagram commutes. Moreover, the right diagram commutes.

$$\begin{tikzcd} & B^n \arrow[dl, "g"'] \arrow[d, "f"] \\ B^{n-1}\langle 1\rangle \arrow[r] & B^{n-1} \end{tikzcd} \qquad \begin{tikzcd} B^n \arrow[r, "i"] \arrow[d, "g"'] & B^{n+1} \arrow[d, "g"] \\ B^{n-1}\langle 1\rangle \arrow[r, "i"] & B^n\langle 1\rangle \end{tikzcd}$$

For $1 \le n \le m$ the filler for the diagram

$$\begin{tikzcd} B^n \arrow[r, "i"] \arrow[d, "g"'] & B^{m+1} \arrow[r, "j"] \arrow[d, "g"'] & B^{m+1}_{n+1} \arrow[d, dotted] \\ B^{n-1}\langle 1\rangle \arrow[r, "i"] & B^m\langle 1\rangle \arrow[r, "j"] & B^m_n\langle 1\rangle \end{tikzcd}$$

is unique and we call it g. For $1 \le n \le m$ the following diagram commutes.

$$\begin{tikzcd} & B^{m+1}_{n+1} \arrow[dl, "g"'] \arrow[d, "f"] \\ B^m_n\langle 1\rangle \arrow[r] & B^m_n \end{tikzcd}$$

Before the proving the proposition we make a preliminary calculation.

Lemma 1.3.3. *For $m, n \ge 1$ $[\Sigma B^{n-1}, B^m_n] = 0$, $[\Sigma B^n, B^m_n] = 0$, $[\Sigma B^n, B^m_n\langle 1\rangle] = 0$.*

Proof. The results are all obvious if $m < n$ so suppose that $m \ge n$.

The first follows from cellular approximation; the third does too, although we will give a different proof.

Cellular approximation gives $[\Sigma B^n, B^m_n] = [\Sigma B^n_n, B^n_n] = [\Sigma S/p, S/p]$. We have an exact sequence

$$\pi_2(S/p) \longrightarrow [\Sigma S/p, S/p] \longrightarrow \pi_1(S/p)$$

and $\pi_1(S/p) = \pi_2(S/p) = 0$, which gives the second identification. Since $[\Sigma B^n, \bigvee_{i=n}^m \Sigma^{iq-2} H] = 0$, $[\Sigma B^n, B^m_n\langle 1\rangle] \longrightarrow [\Sigma B^n, B^m_n]$ is injective and this completes the proof. $\square$

Proof of proposition 1.3.2. f exists because the composite $B^n \xrightarrow{p} B^n \longrightarrow B^n_n = \Sigma^{nq-1} S/p$ is null. f is unique because $[B^n, \Sigma^{-1} B^n_n] = 0$.

Since $[B^n, \Sigma^{-1} B^{n+1}_{n+1}] = 0$ the map $i_* : [B^n, B^n] \longrightarrow [B^n, B^{n+1}]$ is injective and so commutativity of the following diagram gives commutativity of the second diagram of the proposition.

$$\begin{tikzcd}
B^n \arrow[r, "i"] \arrow[dd, "p"'] & B^{n+1} \arrow[d, "f"] \arrow[dd, bend right=60, "p"'] \\
 & B^n \arrow[d, "i"] \\
B^n \arrow[r, "i"] & B^{n+1}
\end{tikzcd}$$

Uniqueness of the first and second fillers of the proposition is given by the facts $[\Sigma B^n, B^m_n] = 0$ and $[\Sigma B^{n-1}, B^m_n] = 0$, respectively. The deductions that each of the four diagrams commute are similar. We'll need the fourth diagram so we show this in detail. We have a commuting diagram.

$$\begin{tikzcd}
B^{n-1} \arrow[r] \arrow[d, "i"] & B^{m+1} \arrow[r] \arrow[d, "="] & B^{m+1}_n \arrow[d, "j"] \\
B^n \arrow[r] \arrow[d, "f"] & B^{m+1} \arrow[r] \arrow[d, "f"] & B^{m+1}_{n+1} \arrow[d, "f"] \\
B^{n-1} \arrow[r] \arrow[d, "="] & B^m \arrow[r] \arrow[d, "i"] & B^m_n \arrow[d, "i"] \\
B^{n-1} \arrow[r] & B^{m+1} \arrow[r] & B^{m+1}_n
\end{tikzcd}$$

The vertical composites in the first two columns are p and so the third is too.

We turn to the existence of g. We have $[B^n, \Sigma^{nq-2} H] = 0$ and so the map $[B^n, \bigvee_{i=1}^{n-1} \Sigma^{iq-1} H] \longrightarrow [B^n, \bigvee_{i=1}^{n} \Sigma^{iq-1} H]$ is injective. Thus, the following diagram proves the existence of g.

$$\begin{tikzcd}
 & B^n \arrow[d, "f"'] \arrow[rrr, bend left, "0"] & & & \\
B^{n-1}\langle 1 \rangle \arrow[r] & B^{n-1} \arrow[r] \arrow[rr, "i"'] & \bigvee_{i=1}^{n-1} \Sigma^{iq-1} H \arrow[rr] & B^n \arrow[r] & \bigvee_{i=1}^{n} \Sigma^{iq-1} H
\end{tikzcd}$$

Uniqueness of g is given by the fact that $[B^n, \bigvee_{i=1}^{n-1} \Sigma^{iq-2} H] = 0$.

Since $[B^n, \bigvee_{i=1}^{n} \Sigma^{iq-2} H] = 0$ the map $[B^n, B^n\langle 1 \rangle] \longrightarrow [B^n, B^n]$ is injective and so commutativity of the following diagram gives commutativity of the second diagram involving g in the proposition.

$$\begin{tikzcd}
B^n \arrow[dd, "g"'] \arrow[ddr, "f"] \arrow[drr, "i"] & & & \\
& & B^{n+1} \arrow[r, "g"] \arrow[ddr, "f"'] & B^n\langle 1\rangle \arrow[dd] \\
B^{n-1}\langle 1\rangle \arrow[r] \arrow[drr, "i"'] & B^{n-1} \arrow[drr, "i"] & & \\
& & B^n\langle 1\rangle \arrow[r] & B^n
\end{tikzcd}$$

The final filler is unique because $[\Sigma B^n, B_n^m\langle 1\rangle] = 0$. The final diagram commutes because we have the following commutative diagram and a uniqueness condition on f as a filler.

$$\begin{tikzcd}
B^n \arrow[r, "i"] \arrow[d, "g"] \arrow[dd, bend right=60, "f"'] & B^{m+1} \arrow[d, "g"] \arrow[dd, bend left=60, "f"] \\
B^{n-1}\langle 1\rangle \arrow[r, "i"] \arrow[d] & B^m\langle 1\rangle \arrow[d] \\
B^{n-1} \arrow[r, "i"] & B^m
\end{tikzcd}$$

□

The $B_n^m\langle 1\rangle$ are useful because they simultaneously appear in an H-Adams resolution for B_n^m and allow the statement and proof of the previous proposition to be so simple.

Lemma 1.3.4. *A map to B_n^m can be factored through $B_n^m\langle 1\rangle$ if and only if it can be factored through $\overline{H} \wedge B_n^m$*

Proof. $H^*(B_n^m; \mathbb{F}_p)$ is free over $E[\beta]$ with basis $e^n, \ldots, e^m$. This basis allowed us to construct the top map in the following diagram.

$$\begin{tikzcd}
B_n^m \arrow[r, "{(e^n,\ldots,e^m)}"] \arrow[d] & \bigvee_n^m \Sigma^{iq-1} H \arrow[d, "{\bigvee_n^m \Sigma^{iq-1}(1,\beta)}"] \\
H \wedge B_n^m \arrow[r, "\simeq"] & \bigvee_n^m \left(\Sigma^{iq-1} H \vee \Sigma^{iq} H\right)
\end{tikzcd}$$

We have a map $(1, \beta) : H \longrightarrow H \wedge \Sigma H$ which is used to construct the map on the right. Since the target of this map is an H-module we obtain the bottom map and one can check that this is an equivalence. Thus we obtain the map of cofibration sequences displayed at the top of the following diagram.

$$\begin{tikzcd}
B_n^m\langle 1\rangle \arrow[r] \arrow[d] & B_n^m \arrow[r, "{(e^n,\dots,e^m)}"] \arrow[d, "="] & \bigvee_n^m \Sigma^{iq-1}H \arrow[d, "\simeq\circ\left[\bigvee_n^m \Sigma^{iq-1}(1,\beta)\right]"] \\
\overline{H}\wedge B_n^m \arrow[r] \arrow[d] & B_n^m \arrow[r] \arrow[d, "="] & H\wedge B_n^m \arrow[d, "\left[\bigvee_n^m \Sigma^{iq-1}(1,*)\right]\circ\simeq"] \\
B_n^m\langle 1\rangle \arrow[r] & B_n^m \arrow[r, "{(e^n,\dots,e^m)}"] & \bigvee_n^m \Sigma^{iq-1}H
\end{tikzcd}$$

The bottom right square is checked to commute and so we obtain the map of cofibration sequences displayed at the bottom. This diagrams shows that a map to B_n^m can be factored through $B_n^m\langle 1\rangle$ if and only if it can be factoring through $\overline{H}\wedge B_n^m$; this is clear if one uses the more general theory of Adams resolutions disussed in [17].

[One sees that $(e_n,\dots,e_m)$ is an $H_*(-;\mathbb{F}_p)$-isomorphism in dimensions which are strictly less than $(n+1)q-1$ so $B_n^m\langle 1\rangle$ is $((n+1)q-3)$-connected and hence nq-connected.] □

The purpose of proposition 1.3.2 now comes to light. It allow us to prove the following lemma.

Lemma 1.3.5. *The maps $f : B_{n+1}^{m+1}\longrightarrow B_n^m$ are compatible and have Adams filtration one.*

Proof. The last diagram of proposition 1.3.2 together with lemma 1.3.4 tells us that they have Adams filtration one. Proposition 1.3.2 also gives us the following commutative diagram.

$$\begin{tikzcd}
B^1 \arrow[r] \arrow[d, "f"'] & B^2 \arrow[r] \arrow[d, "f"'] & \dots \arrow[r] & B^n \arrow[r, "i"] \arrow[d, "f"] & B^{n+1} \arrow[r] \arrow[d, "f"] & \dots \\
* \arrow[r] & B^1 \arrow[r] & \dots \arrow[r] & B^{n-1} \arrow[r, "i"] & B^n \arrow[r] & \dots
\end{tikzcd}$$

This is commutative in the homotopy category. For concreteness suppose that we a have pointset level model for this diagram in which each $i : B^{n-1}\longrightarrow B^n$ is a cofibration between cofibrant spectra (see 2.2 for details). The 'homotopy extension property' says that we can make any of the squares strictly commute at the cost of changing the right map to a homotopic one. By proceeding inductively, starting with the left most square, we can assume that the f's are chosen so that each square strictly commutes.

Each $f : B_{n+1}^{m+1}\longrightarrow B_n^m$ can be obtained by taking strict cofibers of the appropriate diagram. It is now clear that the $f : B_{n+1}^{m+1}\to B_n^m$ are compatible. □

1.4 A permanent cycle in the ASS-S^0

We are now ready to construct the relevant homotopy class in the ASS-S^0.

Theorem 1.4.1. *The element $q_0^{p^n-n-1}h_{1,n}\doteq\{[\tau_0]^{p^n-n-1}[\xi_1^{p^n}]\}\in E_2^{p^n-n,p^n(q+1)-n-1}(ASS\text{-}S^0)$ is a permanent cycle.*

Proof. Firstly, we'll construct the homotopy class that $q_0^{p^n-n-1}h_{1,n}$ detects. By propositon 1.2.5 we have a map $i : S^{p^nq-1} \to B^{p^n}_{p^n-n}$ which is nontrivial homology. We take $\overline{\alpha}_{p^n}$ to be $t \circ f^{p^n-n-1} \circ i$ as displayed in the diagram below.

$$\begin{tikzcd}
S^{p^nq-1} \arrow[dr, "i"'] \arrow[rrrrrrd, bend left=10, "\overline{\alpha}_{p^n}"] & & & & & & \\
& B^{p^n}_{p^n-n} \arrow[r, "f"] & B^{p^n-1}_{p^n-n-1} \arrow[r] & \dots \arrow[r] & B^{n+2}_2 \arrow[r, "f"] & B^{n+1}_1 \arrow[r] \arrow[d, "i"'] & S^0 \\
& B^{p^n}_1 \arrow[u, "j"] \arrow[rrrr, "p^{p^n-n-1}"] & & & & B^{p^n}_1 \arrow[d, "i"'] & \\
& & & & & B \arrow[ruu, bend right, "t"'] &
\end{tikzcd}$$

We look at the maps induced on E_2-pages.

$t_* : E_2(B) \to E_2(S^0)$ is described by the geometric boundary theorem. The cofibration sequence $S^{-1} \to C \to B$ induces a short exact sequence of A-comodules. The boundary map obtained by applying $\mathrm{Cotor}_A(-)$ is the map induced by t.

$$\begin{tikzcd}
0 \arrow[r] & \Omega(A; H_*(S^{-1})) \arrow[r] & \Omega(A; H_*(C)) \arrow[r] & \Omega(A; H_*(B)) \arrow[r] & 0 \\
& & {[\tau_0]^{p^n-n-1}e_{p^n}} \arrow[r, mapsto] \arrow[d, mapsto] & {[\tau_0]^{p^n-n-1}e_{p^n}} & \\
& {[\tau_0]^{p^n-n-1}[\xi_1^{p^n}]} \arrow[r, mapsto] & {[\tau_0]^{p^n-n-1}[\xi_1^{p^n}]u} & &
\end{tikzcd}$$

Using propostion 1.2.4 and propostion 1.2.8 we see that $t_*(q_0^{p^n-n-1} \cdot e_{p^n}) = q_0^{p^n-n-1}h_{1,n}$.

The maps labelled by i and j are all nontrivial on homology and so are easily described on E_2-pages. The following diagram almost completes the proof.

$$\begin{tikzcd}
e_{p^n} \arrow[r, mapsto, "f_*"] & ? \arrow[r, mapsto] & \dots \arrow[r, mapsto] & ? \arrow[r, mapsto, "f_*"] & ? \arrow[r, mapsto] \arrow[d, mapsto, "i"'] & q_0^{p^n-n-1}h_{1,n} \\
e_{p^n} \arrow[u, mapsto, "j"] \arrow[rrrr, mapsto, "p^{p^n-n-1}"] & & & & q_0^{p^n-n-1} \cdot e_{p^n} \arrow[d, mapsto, "i"'] & \\
& & & & q_0^{p^n-n-1} \cdot e_{p^n} \arrow[ruu, mapsto, bend right, "t"'] &
\end{tikzcd}$$

There is a subtlety, however. A map of filtration degree k only gives a well-defined map on E_{k+1} pages. This means means that, as we have drawn the diagram above, it is not completely obvious that the rectangle commutes. This is easily resolved. We can break the rectangle up into $(p^n-n-1)^2$ squares. We have demonstrated this for the case when $p = 5$ and $n = 1$ below. Each square involves two maps of Adams filtration zero in the vertical direction and two maps of Adams filtration one in the horizontal direction. Each square commutes by proposition 1.3.2 and lemma 1.3.5 and the maps induced on E_2-pages are well-defined. This completes the proof.

$$\begin{tikzcd}
B^5_4 \arrow[r, "f"] & B^4_3 \arrow[r, "f"] & B^3_2 \arrow[r, "f"] & B^2_1 \arrow[d, "i"] \\
B^5_3 \arrow[u, "j"] \arrow[r, "f"] & B^4_2 \arrow[u, "j"] \arrow[r, "f"] & B^3_1 \arrow[u, "j"] \arrow[r, "5"] \arrow[d, "i"] & B^3_1 \arrow[d, "i"] \\
B^5_2 \arrow[u, "j"] \arrow[r, "f"] & B^4_1 \arrow[u, "j"] \arrow[r, "5"] \arrow[d, "i"] & B^4_1 \arrow[r, "5"] \arrow[d, "i"] & B^4_1 \arrow[d, "i"] \\
B^5_1 \arrow[u, "j"] \arrow[r, "5"] & B^5_1 \arrow[r, "5"] & B^5_1 \arrow[r, "5"] & B^5_1
\end{tikzcd}
\qquad
\begin{tikzcd}
e_5 \arrow[r, mapsto] & ? \arrow[r, mapsto] & ? \arrow[r, mapsto] & ? \arrow[d, mapsto] \\
e_5 \arrow[u, mapsto] \arrow[r, mapsto] & ? \arrow[u, mapsto] \arrow[r, mapsto] & ? \arrow[u, mapsto] \arrow[r, mapsto] \arrow[d, mapsto] & ? \arrow[d, mapsto] \\
e_5 \arrow[u, mapsto] \arrow[r, mapsto] & ? \arrow[u, mapsto] \arrow[r, mapsto] \arrow[d, mapsto] & ? \arrow[r, mapsto] \arrow[d, mapsto] & ? \arrow[d, mapsto] \\
e_5 \arrow[u, mapsto] \arrow[r, mapsto] & q_0 \cdot e_5 \arrow[r, mapsto] & q_0^2 \cdot e_5 \arrow[r, mapsto] & q_0^3 \cdot e_5
\end{tikzcd}$$

□

This gives a permanent cylce in the MASS-∞.

Corollary 1.4.2. *The element $q_1^{p^n}/q_0^{n+1} \in \mathrm{Cotor}_P^{0,p^n-n-1,p^nq}(Q(0)/q_0^\infty)$ is a permanent cycle in the* MASS-∞*.*

Proof. The map $\Sigma^{-1}S/p^\infty \longrightarrow S^0$ induces a map of spectral sequences E_2(MASS-∞) $\longrightarrow E_2$(ASS-S^0). Corollary II.8.1.2 tells us that the map on E_2-pages can be identified, up to a sign, with the map

$$\partial : \mathrm{Cotor}_P(Q(0)/q_0^\infty) \longrightarrow \mathrm{Cotor}_P(Q(0))$$

induced by the short exact sequence $0 \longrightarrow Q(0) \longrightarrow q_0^{-1}Q(0) \longrightarrow Q(0)/q_0^\infty \longrightarrow 0$.

Since $\partial(q_1^{p^n}/q_0^{n+1}) = q_0^{p^n-n-1}h_{1,n}$ and the map of spectral sequences is an isomorphism in the region of interest theorem 1.4.1 gives the result. □

1.5 A permanent cycle in the MASS-n

In this subsection we perform an injectivity argument to prove that $q_1^{p^n}$ is a permanent cycle in the MASS-$(n+1)$. The next three lemmas are the required technical lemmas.

Lemma 1.5.1. *Write $(\overline{\sigma}, \overline{\lambda})$ for the bigrading of $q_1^{p^{n-1}}/q_0^n \in E_2(MASS\text{-}n')$ and consider the map*

$$E_2(MASS\text{-}n') \longrightarrow E_2(MASS\text{-}\infty)$$

induced by $S/p^n \longrightarrow S/p^\infty$. It is

1. *injective in bigrading $(\overline{\sigma} + r, \overline{\lambda} + r - 1)$ for $r \geq 2$.*

2. *surjective in bigrading $(\overline{\sigma} + s, \overline{\lambda} + s)$ for $s \geq 1$.*

Proof. We have a short exact sequence of P-comodules

$$0 \longrightarrow M_n \longrightarrow Q(0)/q_0^\infty \xrightarrow{q_0^n} Q(0)/q_0^\infty \longrightarrow 0$$

which induces the long exact sequence

$$E_2^{\sigma+n-1,\lambda+n}(\text{MASS-}\infty) \xrightarrow{\partial} E_2^{\sigma,\lambda}(\text{MASS-}n') \longrightarrow E_2^{\sigma,\lambda}(\text{MASS-}\infty) \xrightarrow{q_0^n} E_2^{\sigma+n,\lambda+n}(\text{MASS-}\infty).$$

The map $S/p^n \longrightarrow S/p^\infty$ induces a map of spectral sequences. On E_2-pages this is precisely the middle map above. [The other maps are induced from maps of spectra, though we shall not need this.]

Recall corollary II.6.6 which says that $E_2^{\sigma,\lambda}(\text{MASS-}\infty) = 0$ when $\lambda < (2p-1)\sigma + (2p-3)$ and thus when $\lambda < (2p-1)\sigma - (2p-3)$. We deduce that $E_2^{\sigma,\lambda}(\text{MASS-}n') \longrightarrow E_2^{\sigma,\lambda}(\text{MASS-}\infty)$ is

1. injective when $\lambda + n < (2p-1)(\sigma + n - 1) - (2p-3)$;
2. surjective when $\lambda + n < (2p-1)(\sigma + n) - (2p-3)$.

Thus, we need to check that

1. $(\overline{\lambda} + r - 1) + n < (2p-1)((\overline{\sigma} + r) + n - 1) - (2p-3)$ for $r \geq 2$;
2. $(\overline{\lambda} + s) + n < (2p-1)((\overline{\sigma} + s) + n) - (2p-3)$ for $s \geq 1$.

We see that these statements are equivalent by setting $s = r - 1$. We also see that it is enough to check the $s = 1$ case:

$$\overline{\lambda} + n + 1 < (2p-1)(\overline{\sigma} + n + 1) - (2p-3).$$

$q_1^{p^{n-1}}/q_0^n$ lies in bigrading $(\overline{\sigma}, \overline{\lambda}) = (p^{n-1} - n, (2p-1)p^{n-1} - n)$ and so the result follows.

[One might wonder why we made the inequality tighter and our lives harder. Originally I made a sign error and so this was the bound I had to go on. Luckily it didn't make a difference, otherwise I might have rejected this idea erroneously.] □

Lemma 1.5.2. *Suppose we have a commutative diagram of abelian groups in which the rows are exact.*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[rr] & & X' & & \\
& & Y \arrow[ul, "f"] \arrow[rr] & & Y' \arrow[ul, "f'"'] \arrow[r] & 0
\end{tikzcd}$$

Then the induced maps $\operatorname{coker} f \to \operatorname{coker} f'$ *and* $\ker f \to \ker f'$ *are injective and surjective, respectively.*

Proof. An elementary diagram chase. □

Lemma 1.5.3. *Write $(\overline{\sigma}, \overline{\lambda})$ for the bigrading of $q_1^{p^{n-1}}/q_0^n \in E_2(MASS\text{-}n')$ and consider the map*

$$E_*(MASS\text{-}n') \longrightarrow E_*(MASS\text{-}\infty)$$

induced by $S/p^n \longrightarrow S/p^\infty$. Let $r \geq 2$. Then on the r-th page the map is

1. *injective in bigrading $(\overline{\sigma} + \widetilde{r}, \overline{\lambda} + \widetilde{r} - 1)$ for $\widetilde{r} \geq r$.*
2. *surjective in bigrading $(\overline{\sigma} + s, \overline{\lambda} + s)$ for $s \geq 1$.*

Proof. Lemma 1.5.1 is the base case for an induction on the page number starting at the second page. Using the following diagram together with lemma 1.5.2 we obtain the inductive step [taking subgroups does not affect injectivity and quotienting does not affect surjectivity].

$$\begin{tikzcd}
0 \arrow[r] & E_r^{\overline{\sigma}+r+s,\overline{\lambda}+r+s-1}(\text{MASS-}n') \arrow[r] & E_r^{\overline{\sigma}+r+s,\overline{\lambda}+r+s-1}(\text{MASS-}\infty) & \\
& E_r^{\overline{\sigma}+s,\overline{\lambda}+s}(\text{MASS-}n') \arrow[r] \arrow[u, "d_r"] & E_r^{\overline{\sigma}+s,\overline{\lambda}+s}(\text{MASS-}\infty) \arrow[r] \arrow[u, "d_r"'] & 0
\end{tikzcd}$$

□

We are finally ready to show that $q_1^{p^n}$ is a permanent cycle. We work in the reindexed spectral sequence.

Theorem 1.5.4. *The element* $q_1^{p^n}/q_0^{n+1} \in \mathrm{Cotor}_P^{0,p^n-n-1,p^nq}(M_{n+1})$ *is a permanent cycle in the* MASS-$(n+1)'$.

Proof. Write $(\overline{\sigma}, \overline{\lambda})$ for the bigrading of $q_1^{p^n}/q_0^{n+1} \in E_2(\text{MASS-}(n+1)')$. Lemma 1.5.3 tells us that

$$E_r^{\overline{\sigma}+r,\overline{\lambda}+r-1}(\text{MASS-}(n+1)') \longrightarrow E_r^{\overline{\sigma}+r,\overline{\lambda}+r-1}(\text{MASS-}\infty)$$

induced by $S/p^{n+1} \longrightarrow S/p^\infty$ is injective for $r \geq 2$. This completes the proof by corollary 1.4.2. □

Since the MASS-$(n+1)'$ is just the reindexed MASS-$(n+1)$ we obtain the following corollary.

Corollary 1.5.5. *The element* $q_1^{p^n} \in \mathrm{Cotor}_P^{0,p^n,p^nq}(Q(0)/q_0^{n+1})$ *is a permanent cycle in the* MASS-$(n+1)$.

1.6 K-theory

Finally, we need to address the claim concerning K-theory. First, we look at the maps between the stunted projective spaces.

Lemma 1.6.1. *The maps* $f : B_{n+1}^{m+1} \longrightarrow B_n^m$ *induce an isomorphism on* K*-theory.*

Proof. Applying K_{-1} to the diagram on the left gives the diagram on the right and we deduce that $K_{-1}(f : B^n \longrightarrow B^{n-1})$ is surjective because the image of $p : \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^n$ has size p^{n-1}.

$$\begin{tikzcd}
B^n \arrow[d, "f"'] \arrow[dr, "p"] & \\
B^{n-1} \arrow[r, "i"] & B^n
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{Z}/p^n \arrow[d] \arrow[dr, "p"] & \\
\mathbb{Z}/p^{n-1} \arrow[r] & \mathbb{Z}/p^n
\end{tikzcd}$$

Applying K_{-1} to the diagram

$$\begin{tikzcd}
B^n \arrow[r, "i"] \arrow[d, "f"'] & B^{m+1} \arrow[r, "j"] \arrow[d, "f"'] & B_{n+1}^{m+1} \arrow[d, "f"'] \\
B^{n-1} \arrow[r, "i"] & B^m \arrow[r, "j"] & B_n^m
\end{tikzcd}$$

gives

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p^n \arrow[r] \arrow[d] & \mathbb{Z}/p^{m+1} \arrow[r] \arrow[d] & \mathbb{Z}/p^{m-n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/p^{n-1} \arrow[r] & \mathbb{Z}/p^m \arrow[r] & \mathbb{Z}/p^{m-n+1} \arrow[r] & 0.
\end{tikzcd}$$

We deduce that $K_{-1}(f : B^{m+1}_{n+1} \longrightarrow B^m_n)$ is surjective and so it is an isomorphism. $K_0(B^m_n) = 0$ and so by duality each $K^*(f)$ is an isomorphism. $\square$

The following result of Miller and Snaith gives us the information we need about the transfer.

Theorem 1.6.2 ([18])**.** *The transfer $t : B \longrightarrow S^0$ induces an isomorphism on K-theory.*

The following proposition almost completes the proof of theorem 5.2.3.

Proposition 1.6.3. *The composite $\overline{\alpha}_{p^n}$ used in theorem 1.4.1 induces a map $\tilde{\alpha}_{p^n} : \Sigma^{p^n q} S/p^{n+1} \longrightarrow S/p^{n+1}$ with the property that its desuspension fits into the following commutative diagram. Moreover, this map induces an isomorphism on K-theory.*

$$\begin{tikzcd}
S^{p^n q-1} \arrow[d, "p^{n+1}"'] & & & & & & S^0 \\
S^{p^n q-1} \arrow[r, "i"] \arrow[d] \arrow[rrrrrr, bend left, "\overline{\alpha}_{p^n}"] & B^{p^n}_{p^n-n} \arrow[r, "f"] & B^{p^n-1}_{p^n-n-1} \arrow[r] & \cdots \arrow[r] & B^{n+2}_2 \arrow[r, "f"] & B^{n+1}_1 \arrow[r, "t"] \arrow[dr] & S^0 \arrow[u, "p^{n+1}"'] \\
\Sigma^{p^n q-1} S/p^{n+1} \arrow[ur] \arrow[rrrrrr, "\Sigma^{-1}\tilde{\alpha}_{p^n}"] & & & & & & \Sigma^{-1} S/p^{n+1} \arrow[u]
\end{tikzcd}$$

Proof. Multiplication by p^{n+1} is zero on $B^{p^n}_{p^n-n}$ and B^{n+1}_1 since they are built up from $(n+1)$ Moore spectra S/p on which multiplication by p is zero. Thus we obtain the two angled maps. We take the map $\tilde{\alpha}_{p^n} : \Sigma^{p^n q} S/p^{n+1} \longrightarrow S/p^{n+1}$ to be the suspension of the obvious composite.

By lemma 1.6.1 we are just left to show that the angled maps induce isomorphisms on K-theory.

For the first angled map, the Atiyah-Hirzebruch SS gives us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & K_{-1}(\Sigma^{p^n q-1} S/p^{n+1}) \arrow[r] \arrow[d] & H_{p^n q-1}(\Sigma^{p^n q-1} S/p^{n+1}; \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_{-1}(B^{p^n-1}_{p^n-n}) \arrow[r] & K_{-1}(B^{p^n}_{p^n-n}) \arrow[r] & H_{p^n q-1}(B^{p^n}_{p^n-n}; \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

which takes the following form.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/p^{n+1} \arrow[r] \arrow[d] & \mathbb{Z}/p^{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/p^n \arrow[r] & \mathbb{Z}/p^{n+1} \arrow[r] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

Since the map on the right is surjective we deduce that the middle map is an isomorphism. Again, $K_0(\Sigma^{p^n q-1} S/p^{n+1})$ and $K_0(B^{p^n}_{p^n-n})$ are zero and so we have an isomorphism on K^*. For the other map we apply K^0 to the left diagram to obtain the right diagram. The top map is surjective and the right map is an isomorphism. Thus the left map is surjective and hence an isomorphism. Since $K^{-1}(B_1^{n+1})$ and $K^{-1}(\Sigma^{-1}S/p^{n+1})$ are zero, we're done.

$$\begin{tikzcd} B_1^{n+1} \arrow[r] \arrow[d] & B \arrow[d, "t"] \\ \Sigma^{-1}S/p^{n+1} \arrow[r] & S^0 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}/p^{n+1} & \mathbb{Z}_p \arrow[l] \\ \mathbb{Z}/p^{n+1} \arrow[u] & \mathbb{Z}_p \arrow[l] \arrow[u] \end{tikzcd}$$

□

However, the proof is still not complete. If $q_1^{p^n}$ detects an element $\alpha_{p^n} : S^{p^n q} \longrightarrow S/p^{n+1}$, we do not know that $\mu \circ (\alpha_{p^n} \wedge S/p^{n+1}) = \tilde{\alpha}_{p^n}$. To address this problem we use the following lemma together with the vanishing line of the MASS-∞ (corollary II.6.6).

Lemma 1.6.4. *Suppose that $\gamma : \Sigma^{p^n q} S/p^{n+1} \longrightarrow S/p^{n+1}$ induces an isomorphism on K-theory. Then the composite*

$$S^{p^n q} \longrightarrow \Sigma^{p^n q} S/p^{n+1} \xrightarrow{\gamma} S/p^{n+1} \longrightarrow S^1$$

has order p^{n+1}.

Proof. First, recall that $\mathrm{Cotor}^{0,p^n q}_{BP_*BP}(BP_*, BP_*)$ is zero and that $\mathrm{Cotor}^{1,p^n q}_{BP_*BP}(BP_*, BP_*)$ is $\mathbb{Z}/p^{n+1}$, so the boundary map associated to the short exact sequence $0 \longrightarrow BP_* \longrightarrow BP_* \longrightarrow BP_*/p^{n+1} \longrightarrow 0$ of BP_*BP-comodules

$$\begin{aligned} \partial : \mathrm{Cotor}^{0,p^n q}_{BP_*BP}(BP_*/p^{n+1}, BP_*) &\longrightarrow \mathrm{Cotor}^{1,p^n q}_{BP_*BP}(BP_*, BP_*) \\ v_1^{p^n} &\longmapsto dv_1^{p^n}/p^{n+1} \end{aligned}$$

is an isomorphism. Moreover, $v_1^{p^n}$ generates $\mathrm{Cotor}^{0,p^n q}_{BP_*BP}(BP_*/p^{n+1}, BP_*)$.

$BP_*(\gamma)$ is multiplication by an element $P(v_1, v_2, \ldots) \in BP_*/p^{n+1}$ and the hypothesis on γ tells us that the coefficient of $v_1^{p^n}$ is a unit e in $\mathbb{Z}/p^{n+1}$. Since this element is primitive we conclude that $P = ev_1^{p^n}$. Because the composite of interest is detected by $\partial(P)$ in the Adams-Novikov spectral sequence we know that it has order at least p^{n+1}. Since the class factors through S/p^{n+1} it has order precisely p^{n+1}. □

We are finally ready to prove the theorem.

Proof of theorem 5.2.3. Proposition 1.6.3 gives us the following commuting diagram and tells us that $\tilde{\alpha}_{p^n}$ induces an isomorphism on K-theory.

$$\begin{tikzcd} S^{p^n q} \arrow[r, "\Sigma\overline{\alpha}_{p^n}"] \arrow[d] & S^1 \\ \Sigma^{p^n q} S/p^{n+1} \arrow[r, "\tilde{\alpha}_{p^n}"] & S/p^{n+1} \arrow[u] \end{tikzcd}$$

Corollary 1.5.5 tells us that $q_1^{p^n}$ is a permanent cycle so it detects some $\alpha_{p^n} : S^{p^n q} \longrightarrow S/p^{n+1}$ and upon mapping to S^1 we obtain $\Sigma\overline{\alpha}_{p^n}$ up to elements of higher Adams filtration. Summarizing, we have the commuting diagram below.

$$\begin{tikzcd}
S^{p^n q} \arrow[rr, "\Sigma\overline{\alpha}_{p^n} + \text{elements of higher Adams filtration}"] \arrow[d] \arrow[drr, "\alpha_{p^n}"] & & S^1 \\
\Sigma^{p^n q} S/p^{n+1} \arrow[rr, "\mu\circ(\alpha_{p^n} \wedge S/p^{n+1})"] & & S/p^{n+1} \arrow[u]
\end{tikzcd}$$

Corollary II.6.6, together with the fact that

$$\Sigma_s \operatorname{Cotor}_P(Q(0)/q_0^\infty)/\, \mathbb{F}_p \langle q_0^t : t < 0\rangle = \operatorname{Cotor}_P(Q(0))/\, \mathbb{F}_p\, [q_0]$$

and convergence of the ASS-S^0 tell us that elements of higher Adams filtration have order strictly less than p^{n+1}. Using the fact that the stable homotopy category is additive we obtain the following diagram by taking the difference of the previous diagrams.

$$\begin{tikzcd}
S^{p^n q} \arrow[rr, "\text{elements of order strictly less than } p^{n+1}"] \arrow[d] & & S^1 \\
\Sigma^{p^n q} S/p^{n+1} \arrow[rr, "\mu\circ(\alpha_{p^n} \wedge S/p^{n+1}) - \tilde{\alpha}_{p^n}"] & & S/p^{n+1} \arrow[u]
\end{tikzcd}$$

If $\mu \circ (\alpha_{p^n} \wedge S/p^{n+1}) - \tilde{\alpha}_{p^n}$ induced an isomorphism on K-theory then the composite along the top would have order p^{n+1} (lemma 1.6.4) and so, by the contrapositive, $\mu \circ (\alpha_{p^n} \wedge S/p^{n+1}) - \tilde{\alpha}_{p^n}$ does not induce an isomorphism on K-theory; it must induce multiplication by some number divisible by p. An isomorphism of $\mathbb{Z}/p^{n+1}$ plus a homomorphism divisible by p is an isomorphism. Since $\tilde{\alpha}_{p^n}$ induces an isomorphism on K-theory so does

$$\mu \circ (\alpha_{p^n} \wedge S/p^{n+1}) = \tilde{\alpha}_{p^n} + (\mu \circ (\alpha_{p^n} \wedge S/p^{n+1}) - \tilde{\alpha}_{p^n}).$$

□

2 Pointset level constructions

In the proof of lemma II.3.4.7 we require working at the pointset level. The final stage in making the MASS-n (definition II.3.2.2) a multiplicative spectral sequence requires smashing maps of towers together. This section deals with some of the technicalities that arise and we discuss some of the questions that they lead one to ask.

2.1 Motivation

To aid the following discussion we immediately introduce some terminology.

Notation 2.1.1. Let $\mathscr{S}$ denote the stable homotopy category and write $\mathbb{Z}$ for the category with the integers as objects and hom-sets determined by $|\mathbb{Z}(n,m)| = 1$ if $n \geq m$ and $|\mathbb{Z}(n,m)| = 0$

otherwise. Write $\mathbb{Z}_{\geq 0}$ for the full subcategory of $\mathbb{Z}$ with the non-negative integers as objects and write $\mathrm{Ch}(\mathscr{S})$ for the category of non-negative cochain complexes in $\mathscr{S}$; an object of this category is a diagram

$$C^0 \xrightarrow{d} C^1 \longrightarrow \dots \longrightarrow C^s \xrightarrow{d} C^{s+1} \longrightarrow \dots$$

in $\mathscr{S}$ with $d^2 = 0$.

Definition 2.1.2. An object of the diagram category $\mathscr{S}^{\mathbb{Z}_{\geq 0}}$ will be called a *sequence* in $\mathscr{S}$. A system of interlocking cofibration sequences

$$\begin{tikzcd} \dots & X_{s-1} \arrow[l] \arrow[d] & X_s \arrow[l] \arrow[d] & X_{s+1} \arrow[l] \arrow[d] & \dots \arrow[l] \\ & I_{s-1} \arrow[ur, dotted] & I_s \arrow[ur, dotted] & I_{s+1} \arrow[ur, dotted] & \end{tikzcd}$$

in $\mathscr{S}$ will be called a *tower* and we will use the notation (X, I) for it. A *map of towers* $(X, I) \longrightarrow (Y, J)$ is a compatible collection of maps $\{X_s \longrightarrow Y_s\} \cup \{I_s \longrightarrow J_s\}$. An *augmentation* $X \longrightarrow C^\bullet$ of a cochain complex $C^\bullet \in \mathrm{Ch}(\mathscr{S})$ is a map of cochain complexes from

$$X \longrightarrow * \longrightarrow \dots \longrightarrow * \longrightarrow * \longrightarrow \dots$$

to $C^\bullet$.

Many of the spectral sequences in chapter II are obtained by applying $\pi_*(-)$ to a tower (see, for example, II.2.2.2); a map of towers induces a map of spectral sequences. A tower (X, I) gives a sequence $X_\bullet \in \mathscr{S}^{\mathbb{Z}_{\geq 0}}$ and an augmented cochain complex $X_0 \longrightarrow \Sigma^\bullet I_\bullet$. A map of towers gives a map of sequences and a map of augmented cochain complexes and applying $\pi_*(-)$ to the map of cochain complexes describes the corresponding map of spectral sequences at the E_1-page.

Often we have a spectral sequence associated to a tower (X, I), the E_1-page has the structure of an algebra and we have a map $X_0 \wedge X_0 \longrightarrow X_0$. The argument of Bruner in theorem IV.4.4 of [5] shows that we can make our spectral sequence multiplicative in a way compatible with the E_1-page multiplication and the $\pi_*(X_0)$ multiplication by realizing this structure geometrically. From towers (X, I) and (Y, J) he constructs a third tower (Z, K):

- $(Z, K) = (X, I) \wedge (Y, J)$ is given by $Z_s = \bigcup_{i+j=s} X_i \wedge Y_j$; $K_s = \bigvee_{i+j=s} I_i \wedge J_j$;
- the augmented cochain complex $Z_0 \longrightarrow \Sigma^\bullet K_\bullet$ is the 'tensor product' of the augmented cochain complexes $X_0 \longrightarrow \Sigma^\bullet I_\bullet$ and $Y_0 \longrightarrow \Sigma^\bullet J_\bullet$.

The definition of (Z, K) makes sense when one uses Adams' CW-spectra and takes X_{s+1} and Y_{s+1} to be subcomplexes of X_s and Y_s, respectively, but with other more sophisticated models of spectra around this seems like a slightly unsatifying definition. This gives our first motivation for a pointset level discussion even if it is a purely aesthetic one.

We return to the problem of putting a multiplicative structure on a spectral sequence. Given the property of the construction above, which is listed in the second bullet point, our approach is to realize the algebra structure on the E_1-page as a map of cochain complexes $\Sigma^\bullet I_\bullet \wedge \Sigma^\bullet I_\bullet \longrightarrow \Sigma^\bullet I_\bullet$. Taking into consideration the map $X_0 \wedge X_0 \longrightarrow X_0$, this should be a map of augmented cochain complexes; we then attempt to extend this to a map of towers $(X, I) \wedge (X, I) \longrightarrow (X, I)$. Provided that all this is possible, the result referred to above ([5, IV.4.4]) gives the required structure.

In one instance (II.3.2.2) our tower is decomposable as $(X, I) \wedge (Y, J)$ (in fact, it is defined this way). In order to define a multiplication $(X, I) \wedge (Y, J) \wedge (X, I) \wedge (Y, J) \to (X, I) \wedge (Y, J)$ we wish to smash together two multiplications $(X, I) \wedge (X, I) \to (X, I)$ and $(Y, J) \wedge (Y, J) \to (Y, J)$. Ideally we would like to make the construction above functorial although we need less than this. Suppose we have maps of augmented cochain complexes

$$(X_0 \to \Sigma^\bullet I_\bullet) \longrightarrow (X_0' \to \Sigma^\bullet I_\bullet'), \qquad (Y_0 \to \Sigma^\bullet J_\bullet) \longrightarrow (Y_0' \to \Sigma^\bullet J_\bullet')$$

and that these can be extended to maps of towers $(X, I) \longrightarrow (X', I')$ and $(Y, J) \longrightarrow (Y', J')$. We wish to be able to construct a map of towers

$$(X, I) \wedge (Y, J) \longrightarrow (X', I') \wedge (Y', J'),$$

which agrees with the map

$$\Big[(X_0 \to \Sigma^\bullet I_\bullet) \longrightarrow (X_0' \to \Sigma^\bullet I_\bullet')\Big] \wedge \Big[(Y_0 \to \Sigma^\bullet J_\bullet) \longrightarrow (Y_0' \to \Sigma^\bullet J_\bullet')\Big]$$

at the level of the associated augmented cochain complexes; this is the real motivation for our foray into the pointset world.

Finally, as mentioned above, lemma II.3.4.7 gives us motivation for addressing these issues.

The section begins by recalling the properties of the category of spectra **Spec** that we use. We introduce a symmetric monoidal product on the category of sequences in any pointed simplicial monoidal model category and show how we can deform it to be homotopical using a telescope construction. This allows us to recover the classical construction documented by Bruner in [5]. It also allows us to identify why one should not expect a 'smashing together sequences in the homotopy category' functor. We prove a lemma which shows how we can sensibly smash together two maps of sequences in the homotopy category. Then we address the additional structure a tower gives us. We check that the cofibers of a tower behave as expected under the construction of [5] and show that maps of towers can be smashed together in a way that respects the underlying augmented chain complexes. Finally, we give the proof of lemma II.3.4.7.

2.2 S-modules

We use S-modules as our model for spectra. The main reason we find them convenient is that all objects are fibrant in the standard model structure.

Notation 2.2.1. We write **sSet** and **Top** for the categories of simplicial sets and compactly generated spaces, respectively. We write $\mathbf{sSet}_*$ and $\mathbf{Top}_*$ for their based analogues.

We recall that **sSet**, **Top**, $\mathbf{sSet}_*$ and $\mathbf{Top}_*$ are symmetric monoidal categories with respect to $\times, \times, \wedge$ and $\wedge$, respectively and that we have strong monoidal Quillen adjunctions

$$\begin{tikzcd}
\mathbf{sSet} \arrow[r, shift left, "(-)_+"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "|-|"] \arrow[d, phantom, "\vdash"] & \mathbf{sSet}_* \arrow[l, shift left, "U"] \arrow[d, shift left, "|-|"] \arrow[d, phantom, "\vdash"] \\
\mathbf{Top} \arrow[u, shift left, "\mathrm{Sing}_\bullet"] \arrow[r, shift left, "(-)_+"] \arrow[r, phantom, "\perp"] & \mathbf{Top}_* \arrow[l, shift left, "U"] \arrow[u, shift left, "\mathrm{Sing}_\bullet"]
\end{tikzcd}$$

Notation 2.2.2. Let **Spec** be the category of S-modules of [9]. We refer to an object of **Spec** as a *spectrum* rather than an S-module.

We recall that **Spec** has the following properties:

- **Spec** is a closed symmetric monoidal category with respect to the smash product $\wedge_S$, which we write as $\wedge$. The unit for the smash product is the *sphere spectrum* S.
- **Spec** is enriched, tensored and cotensored over the category $\mathbf{Top}_*$. For $X \in \mathbf{Spec}$, $Y \in \mathbf{Top}_*$ we write $X \wedge Y$ for the tensor object in **Spec**.
- **Spec** is a $\mathbf{Top}_*$-model category. From the strong monoidal Quillen adjunctions above it has the structure of a $\mathcal{V}$-model category where $\mathcal{V}$ is any one of **sSet**, **Top**, $\mathbf{sSet}_*$ and $\mathbf{Top}_*$. In particular, there is a cofibrantly generated pointed proper simplicial monoidal model structure on **Spec**. Every object of **Spec** is fibrant in this model structure.
- The sphere spectrum S is not cofibrant and so we fix a cofibrant replacement $S_c \xrightarrow{\sim} S$. The functor $S_c \wedge (-) : \mathbf{Top}_* \longrightarrow \mathbf{Spec}$ is a left Quillen functor modelling $\Sigma^\infty : \mathrm{Ho}(\mathbf{Top}_*) \longrightarrow \mathscr{S}$.

2.3 The telescope construction and smashing sequences together

This subsection looks at the homotopical properties of sequences (see definition 2.1.2). We do not discuss towers or the associated cochain complexes; that discussion is left for the next subsection.

For the necessary homotopical language we refer the reader to [24], a lovely reference, which is where the author learned this material.

We start off by introducing some notation which is used for the remainder of this section.

Notation 2.3.1. We write $\mathcal{M}$ for any pointed simplicial monoidal model category in which every object is fibrant and which comes equipped with a cofibrant replacement functor. We write

$$q : Q \overset{\sim}{\Longrightarrow} 1$$

for the natural weak equivalence from the cofibrant replacement functor to the identity and $\mathcal{M}_Q$ for the full subcategory of cofibrant objects. If $X \in \mathcal{M}$, $K \in \mathbf{sSet}_*$ and $L \in \mathbf{sSet}$ we write $X \wedge K$ for the tensor object with respect to the tensoring over $\mathbf{sSet}_*$ and $X \otimes L$ for the tensor object with respect to the tensoring over **sSet**. These tensorings are related by the formula $X \otimes L = X \wedge L_+$. Given $X, Y \in \mathcal{M}$, we write $\underline{\mathcal{M}}(X, Y)$ for the simplicial set of maps from X to Y, $\mathcal{M}(X, Y)$ for the underlying set of maps $\mathbf{sSet}(\Delta^0, \underline{\mathcal{M}}(X, Y))$, and $X \wedge Y$ for the symmetric monoidal product of X and Y.

For example $\mathcal{M}$ could denote either $\mathbf{Top}_*$ or **Spec**; the cofibrant replacement functors can be constructed using the small object argument, or in $\mathbf{Top}_*$ we could take it to be $|\mathbf{Sing}_\bullet(-)|$.

Notation 2.3.2. Write $\mathbb{N}$ for the category $\mathbb{Z}_{\geq 0}$. [This goes against my own personal convention that $0 \notin \mathbb{N}$ but makes the notation less cumbersome.]

Definition 2.3.3. As usual, $\mathcal{M}^{\mathbb{N}}$ denotes the diagram category, the category of *sequences* in $\mathcal{M}$. It is a homotopical category when given levelwise weak equivalences. It is symmetric monoidal with

$$(X \otimes Y)_s = \operatorname*{colim}_{\substack{i+j\geq s \\ 0\leq i,j\leq s}} X_i \wedge Y_j.$$

[The indexing category in this colimit is a full subcategory of $\mathbb{Z} \times \mathbb{Z}$ and the notation only indicates the objects.]

Notation 2.3.4.

- We write $\mathcal{M}_Q^{\mathbb{N}}$ for $(\mathcal{M}_Q)^{\mathbb{N}}$, the full subcategory of $\mathcal{M}^{\mathbb{N}}$ whose objects are levelwise cofibrant.
- Write $(\mathcal{M}^{\mathbb{N}})_Q$ for the full subcategory of $\mathcal{M}_Q^{\mathbb{N}}$ whose objects have cofibrations as structure maps (i.e. each map $X_s \longleftarrow X_{s+1}$ is a cofibration).

$\mathcal{M}_Q^{\mathbb{N}}$ and $(\mathcal{M}^{\mathbb{N}})_Q$ are homotopical categories ([24, definition 2.1.1]) with levelwise weak equivalences.

Now we wish to show that $(\mathcal{M}^{\mathbb{N}})_Q$ is symmetric monoidal with respect to $\otimes$, as defined above, and that $\otimes : (\mathcal{M}^{\mathbb{N}})_Q \times (\mathcal{M}^{\mathbb{N}})_Q \longrightarrow (\mathcal{M}^{\mathbb{N}})_Q$ is homotopical ([24, page 15]).

Lemma 2.3.5. *Suppose X and Y are objects of $(\mathcal{M}^{\mathbb{N}})_Q$; then so is $X \otimes Y$. Suppose, in addition, that $X \to X'$ and $Y \to Y'$ are weak equivalences; then $X \otimes Y \longrightarrow X' \otimes Y'$ is a weak equivalence.*

Proof. Write $\mathcal{E}_s$ for the indexing category in the colimit defining $(X \otimes Y)_s$, i.e. the full subcategory of $\mathbb{Z} \times \mathbb{Z}$ with objects

$$\{(i,j) : i + j \geq s,\ 0 \leq i,j \leq s\}.$$

$\mathcal{E}_s$ is a directed Reedy category (we can take $2s - i - j$ as the degree function). Giving the diagram category $\mathcal{M}^{\mathcal{E}_s}$ the Reedy model structure (which, by directedness, is equal to the projective model structure), the colimit and constant diagram functors form a Quillen adjunction. The pushout-product axiom for $\mathcal{M}$ allows one to check that the latching maps for the diagram $F : \mathcal{E}_s \to \mathcal{M}$, $(i,j) \longmapsto X_i \wedge Y_j$ are cofibrations. Thus F is Reedy cofibrant implying that $(X \otimes Y)_s$ is cofibrant. The pushout-product axiom also implies that the map of diagrams $F \implies F'$, where $F'(i,j) = X'_i \wedge Y'_j$ is a levelwise weak equivalence. Thus $(X \otimes Y) \to (X' \otimes Y')$ is a (levelwise) weak equivalence. It remains to show $(X \otimes Y)_{s+1} \to (X \otimes Y)_s$ is a cofibration. We let $\mathcal{E}'_s$ be the full subcategory of $\mathbb{Z} \times \mathbb{Z}$ with objects

$$\{(i,j) : i + j \geq s,\ 0 \leq i,j \leq s+1\}.$$

The inclusion $\mathcal{E}_{s+1} \xrightarrow{j} \mathcal{E}'_s$ gives rise to an adjunction $\mathrm{sk} : \mathcal{M}^{\mathcal{E}_{s+1}} \rightleftarrows \mathcal{M}^{\mathcal{E}'_s} : j^*$. Write F for the functor $\mathcal{E}'_s \longrightarrow \mathcal{M}$ with $F(i,j) = X_i \wedge Y_j$, an extension of the F above. The map $(X \otimes Y)_{s+1} \to (X \otimes Y)_s$ can be described as $\mathrm{colim}_{\mathcal{E}_{s+1}} j^* F \to \mathrm{colim}_{\mathcal{E}'_s} F$. Using the adjunction above we see this is the same as $\mathrm{colim}_{\mathcal{E}'_s}(\mathrm{sk} \circ j^*)F \longrightarrow \mathrm{colim}_{\mathcal{E}'_s} F$. Now $(\mathrm{sk} \circ j^*)F \implies F$ is a Reedy cofibration, because the relative latching maps are either the identity or latching maps or F, completing the proof. $\square$

(Q,q) is a left deformation $\mathcal{M} \to \mathcal{M}_Q$ ([24, definition 2.2.1]) and it induces $(Q^{\mathbb{N}}, q^{\mathbb{N}})$, a left deformation $\mathcal{M}^{\mathbb{N}} \longrightarrow \mathcal{M}_Q^{\mathbb{N}}$. We wish to define (Tel, t), a left deformation $\mathcal{M}_Q^{\mathbb{N}} \longrightarrow (\mathcal{M}^{\mathbb{N}})_Q$, which we call the *telescope functor*.

Definition 2.3.6. Suppose given $X \in \mathcal{M}^{\mathbb{N}}$. Label the structure maps $f_s : X_{s+1} \to X_s$. We define

$\mathrm{Tel}(X) \in \mathcal{M}^{\mathbb{N}}$ levelwise: $\mathrm{Tel}(X)_s$ is the colimit of the following diagram

$$\begin{tikzcd}
X_s \otimes \Delta^0 & X_{s+1} \otimes \Delta^0 \arrow[l, "f_s \otimes \Delta^0"'] \arrow[r, "X_{s+1}\otimes[0]"] & X_{s+1} \otimes \Delta^1 \\
X_{s+1} \otimes \Delta^0 \arrow[urr, "X_{s+1}\otimes[1]"] & X_{s+2} \otimes \Delta^0 \arrow[l, "f_{s+1} \otimes \Delta^0"'] \arrow[r, "X_{s+2}\otimes[0]"] & X_{s+2} \otimes \Delta^1 \\
X_{s+2} \otimes \Delta^0 \arrow[urr, "X_{s+2}\otimes[1]"] & \dots \arrow[l] & \dots
\end{tikzcd}$$

We define a natural transformation $t : \mathrm{Tel} \implies 1$ levelwise: the map $t_s : \mathrm{Tel}(X)_s \longrightarrow X_s$ is induced by the maps $\Delta^1 \longrightarrow \Delta^0$ and $f_s \cdots f_{s+r} : X_{s+r+1} \longrightarrow X_s$, $r \geq 0$.

Lemma 2.3.7. *The functor Tel* $: \mathcal{M}^{\mathbb{N}} \longrightarrow \mathcal{M}^{\mathbb{N}}$ *restricts to a functor Tel* $: \mathcal{M}^{\mathbb{N}}_Q \longrightarrow (\mathcal{M}^{\mathbb{N}})_Q$*. The natural transformation* t : *Tel* $\implies 1_{\mathcal{M}^{\mathbb{N}}}$ *restricts to a natural weak equivalence on* $\mathcal{M}^{\mathbb{N}}_Q$*.*

Proof. This makes use of the fact that $\otimes$ is a left Quillen bifunctor, that (acyclic) cofibrations are stable under cobase change and transfinite composition, and the 2-of-3 property.

Let $X \in \mathcal{M}^{\mathbb{N}}_Q$. One should be able to prove that $\mathrm{Tel}(X)_s$ is levelwise cofibrant by staring at the following diagrams the second of which suggests an inductive argument. That each structure map is a cofibration follows from the fact that $\mathrm{Tel}(X)_{s+1} \to \mathrm{Tel}(X)_s$ is obtained from $X_{s+1} \to \mathrm{Cyl}(f_s)$ by cobase change. The diagrams below also show that $X_s \to \mathrm{Tel}(X)_s$ is a weak equivalence. Postcomposition with t_s gives the identity so we are done by the 2-of-3 property.

$$\begin{tikzcd}
X_{s+1} \otimes \Delta^0 \arrow[r, hook, "1^{\mathrm{st}} \text{ factor}"] \arrow[rr, hook, bend left, "\sim"] \arrow[d, "f_s \otimes \Delta^0"'] & X_{s+1} \otimes (\Delta^0 \cup \Delta^0) \arrow[r, hook, "X_{s+1}\otimes([0]\cup[1])"] \arrow[d, "(f_s\otimes\Delta^0)\cup(X_{s+1}\otimes\Delta^0)"] & X_{s+1} \otimes \Delta^1 \arrow[d] \\
X_s \otimes \Delta^0 \arrow[r, hook, "1^{\mathrm{st}} \text{ factor}"] \arrow[rr, hook, bend right, "\sim"'] & (X_s \otimes \Delta^0) \cup (X_{s+1} \otimes \Delta^0) \arrow[r, hook] & \mathrm{Cyl}(f_s)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_{s+r+1} \arrow[d, hook] & \\
 & X_{s+r} \arrow[r, hook, "\sim"] \arrow[d, hook] & \mathrm{Cyl}(f_{s+r}) \arrow[d, hook] & \\
X_s \arrow[r, hook, "\sim"] & \mathrm{Cyl}(f_s) \cup \cdots \cup \mathrm{Cyl}(f_{s+r-1}) \arrow[r, hook, "\sim"] & \mathrm{Cyl}(f_s) \cup \cdots \cup \mathrm{Cyl}(f_{s+r}) \arrow[r, hook, "\sim"] & \mathrm{Tel}(X)_s
\end{tikzcd}$$

$\square$

$$
\begin{tikzcd}
\mathcal{M}^{\mathbb{N}} \times \mathcal{M}^{\mathbb{N}} \arrow[rrrrr, "\otimes"] \arrow[d, "="'] & & \arrow[d, phantom, "\Uparrow"] & & & \mathcal{M}^{\mathbb{N}} \arrow[d, "="] \\
\mathcal{M}^{\mathbb{N}} \times \mathcal{M}^{\mathbb{N}} \arrow[r, "Q^{\mathbb{N}} \times Q^{\mathbb{N}}"] \arrow[d] & \mathcal{M}^{\mathbb{N}}_Q \times \mathcal{M}^{\mathbb{N}}_Q \arrow[r, "\mathrm{Tel} \times \mathrm{Tel}"] \arrow[d] & (\mathcal{M}^{\mathbb{N}})_Q \times (\mathcal{M}^{\mathbb{N}})_Q \arrow[r, "\otimes"] \arrow[d] & (\mathcal{M}^{\mathbb{N}})_Q \arrow[r] \arrow[d] & \mathcal{M}^{\mathbb{N}}_Q \arrow[r] \arrow[d] & \mathcal{M}^{\mathbb{N}} \arrow[d] \\
\mathrm{Ho}(\mathcal{M}^{\mathbb{N}}) \times \mathrm{Ho}(\mathcal{M}^{\mathbb{N}}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Ho}(\mathcal{M}^{\mathbb{N}}_Q) \times \mathrm{Ho}(\mathcal{M}^{\mathbb{N}}_Q) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Ho}((\mathcal{M}^{\mathbb{N}})_Q) \times \mathrm{Ho}((\mathcal{M}^{\mathbb{N}})_Q) \arrow[r] & \mathrm{Ho}((\mathcal{M}^{\mathbb{N}})_Q) \arrow[r, "\simeq"] & \mathrm{Ho}(\mathcal{M}^{\mathbb{N}}_Q) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Ho}(\mathcal{M}^{\mathbb{N}}) \arrow[d] \\
\mathrm{Ho}(\mathcal{M})^{\mathbb{N}} \times \mathrm{Ho}(\mathcal{M})^{\mathbb{N}} \arrow[r, "\simeq"] & \mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}} \times \mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}} \arrow[rrr, dashed] & & & \mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}} \arrow[r, "\simeq"] & \mathrm{Ho}(\mathcal{M})^{\mathbb{N}}
\end{tikzcd}
$$

Figure III.1: Deriving the monoidal structure on $\mathcal{M}^{\mathbb{N}}$.

We are now in a position to derive the monoidal structure on $\mathcal{M}^{\mathbb{N}}$. To summarize what we have achieved so far and to aid the following discussion we draw the large diagram, figure III.1 on page 29. This diagram needs some explanation. Along the second row, we have the deformations just constructed, the monoidal product and the inclusions of the various full subcategories we have defined. This composite admits a natural transformation to the monoidal product displayed on the top row; it is given by $q^{\mathbb{N}}(t_{Q^{\mathbb{N}}}) \otimes q^{\mathbb{N}}(t_{Q^{\mathbb{N}}})$. Because $\mathrm{Ho}(\mathcal{N} \times \mathcal{N}) = \mathrm{Ho}(\mathcal{N}) \times \mathrm{Ho}(\mathcal{N})$ and each functor in the second row is homotopical, we obtain the third row; the deformations and inclusions of subcategories induce equivalences on the homotopy categories. The information described so far gives a natural transformation

$$\begin{tikzcd} \otimes : \mathcal{M}^{\mathbb{N}} \times \mathcal{M}^{\mathbb{N}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & \mathcal{M}^{\mathbb{N}} \arrow[d] \\ \mathrm{Ho}(\mathcal{M}^{\mathbb{N}}) \times \mathrm{Ho}(\mathcal{M}^{\mathbb{N}}) \arrow[r] & \mathrm{Ho}(\mathcal{M}^{\mathbb{N}}) \end{tikzcd}$$

which shows that the third row is the left derived functor for $\otimes$ ([24, theorem 2.2.8]). The functor $\mathcal{M}^{\mathbb{N}} \to \mathrm{Ho}(\mathcal{M})^{\mathbb{N}}$ induces a functor $\mathrm{Ho}(\mathcal{M}^{\mathbb{N}}) \to \mathrm{Ho}(\mathcal{M})^{\mathbb{N}}$. Because every object of $\mathcal{M}$ is fibrant, $\mathcal{M}_Q \to \mathrm{Ho}(\mathcal{M}_Q)$ is full ([24, theorem 10.5.1]). Thus $\mathcal{M}_Q^{\mathbb{N}} \to \mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}}$, and hence the induced functor $\mathrm{Ho}(\mathcal{M}_Q^{\mathbb{N}}) \to \mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}}$, is surjective on objects.

One might hope to find a functor where the dashed arrow appears making the diagram commute. Provided one did this successfully the composite along the bottom would be the 'smashing together sequences in the homotopy category' functor. However, the author thinks that such a functor does not exist. On the other hand, it is clear what such a functor should do on objects:

- suppose X, Y are objects in $\mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}}$;
- lift them to objects $\widetilde{X}, \widetilde{Y} \in \mathrm{Ho}(\mathcal{M}_Q^{\mathbb{N}})$;
- mapping to the right and down into $\mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}}$ recovers, in modern language, the construction discussed in the motivating subsection, documented by Bruner in [5].

The issue here is that two different choices for a lift of X might not be isomorphic; the construction here depends on the choices of $\widetilde{X}$ and $\widetilde{Y}$. The situation is even worse once one considers morphisms since the functor $\mathrm{Ho}(\mathcal{M}_Q^{\mathbb{N}}) \to \mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}}$ is not obviously full. Regardless of this state of affairs we will call the object constructed above $X \wedge Y$. Of course, using $Q^{\mathbb{N}}$, we can also define $X \wedge Y$ for sequences, X and Y, in $\mathrm{Ho}(\mathcal{M})$.

In summary, the diagram below shows that we should expect to have to get our hands a little dirty and so we get to work.

The following lemma is stated imprecisely although the construction used in the proof is useful. We see its value in the next subsection (proposition 2.4.7), once we consider towers (2.1.2) as well as sequences.

Lemma 2.3.8. *Suppose X, X', Y, $Y' \in (\mathcal{M}^{\mathbb{N}})_Q$ and that we have morphisms $X \to X'$ and $Y \to Y'$ in $\mathrm{Ho}(\mathcal{M})^{\mathbb{N}}$. Then we can construct a morphism $X \otimes Y \to X' \otimes Y'$ in $\mathrm{Ho}(\mathcal{M})^{\mathbb{N}}$ in a sensible way.*

Proof. Since all objects of $\mathcal{M}$ are fibrant we may use theorem 10.5.1 of [24] to view the morphisms $X \longrightarrow X'$ and $Y \longrightarrow Y'$ as homotopy commutative diagrams

$$\begin{tikzcd}
X_0 & X_1 \arrow[l] & \dots \arrow[l] & X_s \arrow[l] \arrow[d, "f_s^X"] & X_{s+1} \arrow[l] \arrow[d] & \dots \arrow[l] \\
X'_0 & X'_1 \arrow[l] & \dots \arrow[l] & X'_s \arrow[l] & X'_{s+1} \arrow[l] & \dots \arrow[l]
\arrow[from=1-1, to=2-1] \arrow[from=1-2, to=2-2]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_0 & Y_1 \arrow[l] & \dots \arrow[l] & Y_s \arrow[l] \arrow[d, "f_s^Y"] & Y_{s+1} \arrow[l] \arrow[d] & \dots \arrow[l] \\
Y'_0 & Y'_1 \arrow[l] & \dots \arrow[l] & Y'_s \arrow[l] & Y'_{s+1} \arrow[l] & \dots \arrow[l]
\arrow[from=1-1, to=2-1] \arrow[from=1-2, to=2-2]
\end{tikzcd}$$

Recall the full subcategories of $\mathbb{Z} \times \mathbb{Z}$ used in the proof of lemma 2.3.5: $\mathcal{E}_s$ and $\mathcal{E}'_s$ have objects

$$\{(i,j) : i+j \geq s,\ 0 \leq i,j \leq s\} \text{ and } \{(i,j) : i+j \geq s,\ 0 \leq i,j \leq s+1\},$$

respectively. We make note of the inclusions $\mathcal{E}_s \xrightarrow{i} \mathcal{E}'_s \xleftarrow{j} \mathcal{E}_{s+1}$. Let $F, F' : \mathcal{E}'_s \longrightarrow \mathcal{M}$ be defined by $F(i,j) = X_i \wedge Y_j$ and $F'(i,j) = X'_i \wedge Y'_j$. To construct the requisite morphism we construct the following commuting diagram in $\mathcal{M}$; we note that the top line is $(X \otimes Y)_s = (X \otimes Y)_s \longleftarrow (X \otimes Y)_{s+1}$ and the bottom line is the same with X and Y replaced by X' and Y'.

$$\begin{tikzcd}
\operatorname{colim}_{\mathcal{E}_s} Fi \arrow[rr, "="] & & \operatorname{colim}_{\mathcal{E}'_s} F & & \operatorname{colim}_{\mathcal{E}_{s+1}} Fj \arrow[ll] \\
B(*, \mathcal{E}_s, Fi) \arrow[u, "\sim"] \arrow[rr, "\sim"] \arrow[d, "="'] & & B(*, \mathcal{E}'_s, F) \arrow[u, "\sim"] \arrow[d, "="'] & & B(*, \mathcal{E}_{s+1}, Fj) \arrow[ll] \arrow[u, "\sim"] \arrow[d, "="'] \\
\operatorname{colim}^{N(-/\mathcal{E}_s)} Fi \arrow[r, "="] \arrow[d] & \operatorname{colim}^{N(-/i)} F \arrow[r] & \operatorname{colim}^{N(-/\mathcal{E}'_s)} F \arrow[d] & \operatorname{colim}^{N(-/j)} F \arrow[l] & \operatorname{colim}^{N(-/\mathcal{E}_{s+1})} Fj \arrow[l, "="'] \arrow[d] \\
\operatorname{colim}_{\mathcal{E}_s} F'i \arrow[rr, "="] & & \operatorname{colim}_{\mathcal{E}'_s} F' & & \operatorname{colim}_{\mathcal{E}_{s+1}} F'j \arrow[ll]
\end{tikzcd}$$

We refer to [24] for notation: 4.2.3 explains the objects in the second row; 7.4.1 explains the objects in the third row.

The point is that the middle two rows are standard models for the homotopy colimit, which is studied at length in Riehl's book, [24]. The description in the second row is useful for us later. The description in the third row is useful because of a universal property we employ shortly.

In proving lemma 2.3.5 we showed that the colimits in the top line actually compute a homotopy colimit. Thus we obtain the weak equivalences from the second row to the top row. The equalities in the third row come from 8.1.5 (as used in 8.1.8) of [24]. The weak equivalence in the second row comes from the 2-of-3 property or by 8.5.6 of [24].

We are left with the problem of constructing the map from the third row to the final row. We construct $\operatorname{colim}^{N(-/\mathcal{E}'_s)} F \longrightarrow \operatorname{colim}_{\mathcal{E}'_s} F'$ first. Applying the underlying sets functor to the formula in definition 7.4.1 of [24] gives

$$\mathcal{M}(\operatorname{colim}^{N(-/\mathcal{E}'_s)} F, \operatorname{colim}_{\mathcal{E}'_s} F') \cong \mathbf{sSet}^{\mathbb{N}^{\mathrm{op}}}(N(-/\mathcal{E}'_s), \underline{\mathcal{M}}(F-, \operatorname{colim}_{\mathcal{E}'_s} F')).$$

So we go about inductively constructing a natural transformation $N(-/\mathcal{E}'_s) \implies \underline{\mathcal{M}}(F-, \operatorname{colim}_{\mathcal{E}'_s} F')$.

For elements $e = (i,j) \in \mathcal{E}'_s$ with $i+j = s$ we have $N(e/\mathcal{E}'_s) = \Delta^0$ so defining a map $N(e/\mathcal{E}'_s) \longrightarrow \underline{\mathcal{M}}(Fe, \operatorname{colim}_{\mathcal{E}'_s} F'))$ is the same as specifying a morphism $Fe \longrightarrow \operatorname{colim}_{\mathcal{E}'_s} F'$. We take $Fe \to F'e \to \operatorname{colim}_{\mathcal{E}'_s} F'$.

To proceed further we need to fix choices of homotopies for our original diagram. Let

$$h_i^X : \Delta^1 \longrightarrow \underline{\mathcal{M}}(X_{i+1}, X'_s) \text{ and } h_j^Y : \Delta^1 \longrightarrow \underline{\mathcal{M}}(Y_{j+1}, Y'_s)$$

be homotopies from the Γ composite to the $\daleth$ composite. Whenever $(i', j') \in \mathcal{E}'_s$ we have maps $\overline{\wedge}$

$$\underline{\mathcal{M}}(X_i,X'_{i'})\times\underline{\mathcal{M}}(Y_j,Y'_{j'}) \xrightarrow{\wedge} \underline{\mathcal{M}}(X_i\wedge Y_j,X'_{i'}\wedge Y'_{j'}) \xrightarrow{=} \underline{\mathcal{M}}(F(i,j),F'(i',j')) \longrightarrow \underline{\mathcal{M}}(F(i,j),\mathrm{colim}_{\mathcal{E}'_s}F')$$

and these give maps

$$\Delta^0 \times \Delta^1 \xrightarrow{f_i^X \times h_{j-1}^Y} \underline{\mathcal{M}}(X_i, X'_i) \times \underline{\mathcal{M}}(Y_j, Y'_{j-1}) \xrightarrow{\overline{\wedge}} \underline{\mathcal{M}}(F(i,j), \mathrm{colim}_{\mathcal{E}'_s}F'),$$

$$\Delta^1 \times \Delta^0 \xrightarrow{h_{i-1}^X \times f_j^Y} \underline{\mathcal{M}}(X_i, X'_{i-1}) \times \underline{\mathcal{M}}(Y_j, Y'_j) \xrightarrow{\overline{\wedge}} \underline{\mathcal{M}}(F(i,j), \mathrm{colim}_{\mathcal{E}'_s}F'),$$

$$\Delta^1 \times \Delta^1 \xrightarrow{h_{i-1}^X \times h_{j-1}^Y} \underline{\mathcal{M}}(X_i, X'_{i-1}) \times \underline{\mathcal{M}}(Y_j, Y'_{j-1}) \xrightarrow{\overline{\wedge}} \underline{\mathcal{M}}(F(i,j), \mathrm{colim}_{\mathcal{E}'_s}F')$$

for $(i, j-1) \in \mathcal{E}'_s$, $(i-1, j) \in \mathcal{E}'_s$ and $(i-1, j-1) \in \mathcal{E}'_s$, respectively.

Suppose $e = (i, j) \in \mathcal{E}'_s$ with $i + j > s$. We wish to define the bottom map in the following diagram and if $e' = (i', j') \in \mathcal{E}'_s$ with $i' + j' = i + j - 1$, we must define the map in such a way that the whole diagram commutes.

$$\begin{tikzcd}
N(e'/\mathcal{E}'_s) \arrow[r] \arrow[d] & \underline{\mathcal{M}}(Fe', \mathrm{colim}_{\mathcal{E}'_s}F) \arrow[d] \\
N(e/\mathcal{E}'_s) \arrow[r] & \underline{\mathcal{M}}(Fe, \mathrm{colim}_{\mathcal{E}'_s}F)
\end{tikzcd}$$

Generically, we have two distinct choices of e' and we call them e' and e''. In this case $N(e/\mathcal{E}'_s)$ is the colimit of a diagram

$$\begin{tikzcd}
 & \Delta^0 \times \Delta^1 \arrow[dl] \arrow[dr, "{[0]\times\Delta^1}"] & & \Delta^1 \times \Delta^0 \arrow[dl, "{\Delta^1\times[0]}"'] \arrow[dr] & \\
N(e'/\mathcal{E}'_s) & & \Delta^1 \times \Delta^1 & & N(e''/\mathcal{E}'_s).
\end{tikzcd}$$

In general, there might only be one choice of e' but in either case we only have to define a map on one of $\Delta^0 \times \Delta^1$, $\Delta^1 \times \Delta^0$ or $\Delta^1 \times \Delta^1$ in a way compatible with what is already defined. The maps above achieve this.

Our map $\mathrm{colim}^{N(-/\mathcal{E}'_s)}F \longrightarrow \mathrm{colim}_{\mathcal{E}'_s}F'$ determines maps

$$\mathrm{colim}^{N(-/\mathcal{E}_s)}Fi \longrightarrow \mathrm{colim}_{\mathcal{E}_s}F'i, \qquad \mathrm{colim}^{N(-/\mathcal{E}_{s+1})}Fj \longrightarrow \mathrm{colim}_{\mathcal{E}'_s}F'.$$

We check that the second map factors through the map $\mathrm{colim}_{\mathcal{E}_{s+1}}F'j \longrightarrow \mathrm{colim}_{\mathcal{E}'_s}F'$ to give

$$\mathrm{colim}^{N(-/\mathcal{E}_{s+1})}Fj \longrightarrow \mathrm{colim}_{\mathcal{E}_{s+1}}F'j.$$

Moreover, one can see that we could construct these maps directly, using the procedure above and so the following morphisms in the homotopy category define a map $X \otimes Y \longrightarrow X' \otimes Y'$ in $\mathrm{Ho}(\mathcal{M})^{\mathbb{N}}$.

$$(X \otimes Y)_s = \mathrm{colim}_{\mathcal{E}_s}Fi \xleftarrow{\sim} B(*, \mathcal{E}_s, Fi) = \mathrm{colim}^{N(-/\mathcal{E}_s)}Fi \longrightarrow \mathrm{colim}_{\mathcal{E}_s}F'i = (X' \otimes Y')_s$$

□

2.4 Smashing towers together

In this subsection $\mathcal{M}$ continues to denote a pointed simplicial monoidal model category in which every object is fibrant and which comes equipped with a cofibrant replacement functor; $\mathbb{N}$ continues to denote $\mathbb{Z}_{\geq 0}$.

In the last section we constructed an object $X \wedge Y \in \mathrm{Ho}(\mathcal{M})^{\mathbb{N}}$ from elements $X, Y \in \mathrm{Ho}(\mathcal{M})^{\mathbb{N}}$. The first objective of this section is to show that the induced construction on the homotopy cofibers of our sequences is the expected one. There are two results that we wish to prove:

- if I and J denote the *sequences consisting of homotopy cofibers* of X and Y, respectively, then the sequence K consisting of homotopy cofibers of $X \wedge Y$ is given by $K_s = \bigvee_{i+j=s} I_i \wedge J_j$;
- the augmented cochain complex $(X \wedge Y)_0 \longrightarrow \Sigma^\bullet K_\bullet$ is the 'tensor product' of the augmented cochain complexes $X_0 \longrightarrow \Sigma^\bullet I_\bullet$ and $Y_0 \longrightarrow \Sigma^\bullet J_\bullet$.

We use a formal argument to analyze what happens to strict cofibers under the monoidal product $\otimes$; we begin by observing that we have an adjunction

$$i : \mathcal{M}^{\mathbb{N}} \rightleftarrows \mathcal{M}^{\mathbb{Z}} : r$$

$i(X)_s = X_s$ for $s \geq 0$ and $i(X)_s = X_0$ for $s < 0$, i.e. i extends a sequence to be constant in negative degrees. $r(X)_s = X_s$ for $s \geq 0$, i.e. r truncates a sequence at 0. Because ri is the identity functor we may view $\mathcal{M}^{\mathbb{N}}$ as a full subcategory of $\mathcal{M}^{\mathbb{Z}}$. Since the category $\mathbb{Z}$ is symmetric monoidal and $\mathcal{M}$ is closed symmetric monoidal, we obtain a closed symmetric monoidal structure on $\mathcal{M}^{\mathbb{Z}}$, the Day convolution, which we denote by $\otimes$ because it extends the monoidal structure on $\mathcal{M}^{\mathbb{N}}$.

We have a functor $(+1) : \mathbb{Z} \to \mathbb{Z}$ and a unique natural tansformation $\alpha : (+1) \implies \mathrm{id}$. Thus, for any $X \in \mathcal{M}^{\mathbb{Z}}$ we obtain a morphism $X\alpha : X(+1) \to X$ in $\mathcal{M}^{\mathbb{Z}}$. Since $\mathcal{M}$ is pointed, $\mathcal{M}^{\mathbb{Z}}$ is pointed and we can form the pushout diagram on the left.

$$\begin{tikzcd} X(+1) \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & I \end{tikzcd} \qquad \begin{tikzcd} X_{s+1} \arrow[r] \arrow[d] & * \arrow[d] \\ X_s \arrow[r] & I_s \end{tikzcd}$$

Pushouts in a functor category are calculated pointwise and so I_s is determined by the pushout diagram on the right. One can check that the morphism $I(+1) \longrightarrow I$ is 0, i.e. it factors through $*$.

Definition 2.4.1. I is said to be the *sequence consisting of strict cofibers* of X.

The following proposition contains the formal argument referred to above.

Proposition 2.4.2. *Let $X, Y \in \mathcal{M}^{\mathbb{Z}}$ and let I, J be the corresponding sequences consisting of strict cofibers. Then the sequence of strict cofibers of $X \otimes Y$ is given by $I \otimes J$ and*

$$(I \otimes J)_s = \bigvee_{i+j=s} I_i \wedge J_j.$$

Proof. Since the symmetric monoidal product is closed, $\otimes$ preserves colimits in each variable. We also note the canonical identifications $X(+1) \otimes Y = (X \otimes Y)(+1) = X \otimes Y(+1)$. These observations allow us to draw the following commuting diagram in which each of the asterisked squares is a pushout.

$$\begin{tikzcd}
* \arrow[d] \arrow[dr, phantom, "*"] & X(+1)\otimes Y \arrow[l] \arrow[r, equal] \arrow[d] & (X\otimes Y)(+1) \arrow[r, equal] \arrow[d] & X\otimes Y(+1) \arrow[r] \arrow[d] \arrow[dr, phantom, "*"] & * \arrow[d] \\
I\otimes Y \arrow[d, equal] & X\otimes Y \arrow[l] \arrow[r, equal] & X\otimes Y \arrow[r, equal] & X\otimes Y \arrow[r] & X\otimes J \arrow[d, equal] \\
I\otimes Y \arrow[r, equal] & I\otimes Y \arrow[r] & I\otimes J & X\otimes J \arrow[l] \arrow[r, equal] & X\otimes J \\
I(+1)\otimes Y \arrow[r, equal] \arrow[u, "0"] & I\otimes Y(+1) \arrow[r] \arrow[u] \arrow[ur, phantom, "*"] & * \arrow[u] & X(+1)\otimes J \arrow[l] \arrow[r, equal] \arrow[u] \arrow[ul, phantom, "*"] & X\otimes J(+1) \arrow[u, "0"']
\end{tikzcd}$$

The zero morphisms imply that the morphisms $I\otimes Y \longrightarrow I\otimes J \longleftarrow X\otimes J$ are isomorphisms and so the top left and top right pushout squares are in fact isomorphic to the following pushout square.

$$\begin{tikzcd}
(X\otimes Y)(+1) \arrow[r] \arrow[d] & * \arrow[d] \\
X\otimes Y \arrow[r] & I\otimes J
\end{tikzcd}$$

Direct computation yields $(I\otimes J)_s = \bigvee_{i+j=s} I_i \wedge J_j$ and so we are done. □

Since the monoidal structure on $\mathcal{M}^{\mathbb{Z}}$ extends that on $\mathcal{M}^{\mathbb{N}}$ we immediately obtain the following corollary.

Corollary 2.4.3. *Let $X, Y \in \mathcal{M}^{\mathbb{N}}$ and let I, J be corresponding sequences consisting of strict cofibers. Then the sequence of strict cofibers of $X\otimes Y$ is given by $I\otimes J$ and*

$$(I\otimes J)_s = \bigvee_{i+j=s} I_i \wedge J_j.$$

The following corollary addresses the first bullet point above.

Corollary 2.4.4. *Let $X, Y \in Ho(\mathcal{M})^{\mathbb{N}}$ and extend them to towers (X, I) and (Y, J) by taking homotopy cofibers. Extending $Z = X\wedge Y$ to a tower (Z, K) we have*

$$K_s = \bigvee_{i+j=s} I_i \wedge J_j.$$

Proof. Let $\widetilde{X}, \widetilde{Y} \in \mathrm{Ho}(\mathcal{M}_Q^{\mathbb{N}})$ lift $Q^{\mathbb{N}}X, Q^{\mathbb{N}}Y \in \mathrm{Ho}(\mathcal{M}_Q)^{\mathbb{N}}$.

Models for I and J are given by the strict cofibers of $\mathrm{Tel}(\widetilde{X})$ and $\mathrm{Tel}(\widetilde{Y})$ and $X\wedge Y = \mathrm{Tel}(\widetilde{X})\otimes \mathrm{Tel}(\widetilde{Y}) \in (\mathcal{M}^{\mathbb{N}})_Q$, whose strict cofibers give models for the homotopy cofibers. The result follows from corollary 2.4.3. □

For the second bullet point to make sense we need $\mathrm{Ho}(\mathcal{M})$ to be additive and so we take $\mathcal{M}$ to be **Spec**.

Proposition 2.4.5. *Let $X, Y \in \mathscr{S}^{\mathbb{N}}$ and extend them to towers (X, I) and (Y, J) by taking homotopy cofibers. Extend $Z = X \wedge Y$ to a tower (Z, K). The augmented cochain complex $Z_0 \longrightarrow \Sigma^\bullet K_\bullet$ is the 'tensor product' of the augmented cochain complexes $X_0 \longrightarrow \Sigma^\bullet I_\bullet$ and $Y_0 \longrightarrow \Sigma^\bullet J_\bullet$.*

In the course of the proof we need the following definition.

Definition 2.4.6. Given a sequence $X \in \mathcal{M}^{\mathbb{N}}$, write $X^{(s)}$ for the sequence

$$X_i^{(s)} = \begin{cases} X_i & \text{if } i \geq s \\ X_s & \text{if } i \leq s. \end{cases}$$

If $s \leq t$ define $X^{(s,t)}$ by the pushout square

$$\begin{tikzcd}
X^{(t+1)} \arrow[r] \arrow[d] & * \arrow[d] \\
X^{(s)} \arrow[r] & X^{(s,t)}.
\end{tikzcd}$$

Proof of proposition 2.4.5. As in corollary 2.4.4, after making various replacements we can work on the level of strict cofibers so suppose that $X, Y \in (\mathbf{Spec}^{\mathbb{N}})_Q$ and let I, J be the corresponding sequences consisting of strict cofibers. We consider the following diagram

$$\begin{tikzcd}
 & X \otimes Y & \\
X^{(s,s+1)} \otimes Y^{(t,t)} & X^{(s)} \otimes Y^{(t)} \arrow[l] \arrow[r] \arrow[u] & X^{(s,s)} \otimes Y^{(t,t+1)}
\end{tikzcd}$$

The maps induced on the corresponding sequences consisting of strict cofibers can be described at the $s+t$ and $s+t+1$ levels as follows.

$$\begin{tikzcd}
 & \bigvee_{i+j=s+t} I_i \wedge J_j & \\
I_s \wedge J_t & I_s \wedge J_t \arrow[l, "="'] \arrow[r, "="] \arrow[u] & I_s \wedge J_t
\end{tikzcd}$$

$$\begin{tikzcd}
 & \bigvee_{i+j=s+t+1} I_i \wedge J_j & \\
I_{s+1} \wedge J_t & (I_{s+1} \wedge J_t) \vee (I_s \wedge J_{t+1}) \arrow[l] \arrow[r] \arrow[u] & I_s \wedge J_{t+1}
\end{tikzcd}$$

We have a natural construction of the connecting map in the stable homotopy category which is given on the poinset level by the analogue of the space construction $W/A \xleftarrow{\sim} W \cup CA \longrightarrow \Sigma W$. The natural map $U \vee V \longrightarrow U \times V$ induces an isomorphism in the stable homotopy category and accounting for signs introduced by swapping suspension coordinates, we see that the following diagram commutes in the stable homotopy category, completing the proof.

$$
\begin{tikzcd}
\Sigma^{s+t} K_{s+t} \arrow[rr, "d_K^{s+t}"] & & \Sigma^{s+t+1} K_{s+t+1} \\
\Sigma^{s+t}(I_s \wedge J_t) \arrow[u] & & \Sigma^{s+t+1}\Big((I_{s+1} \wedge J_t) \vee (I_s \wedge J_{t+1}) \Big) \arrow[u] \\
(\Sigma^s I_s) \wedge (\Sigma^t J_t) \arrow[u, "\cong"] \arrow[r, "(d_I^s \wedge 1, (-1)^s \wedge d_J^t)"] & \Big((\Sigma^{s+1} I_{s+1}) \wedge (\Sigma^t J_t) \Big) \vee \Big((\Sigma^s I_s) \wedge (\Sigma^{t+1} J_{t+1}) \Big) \arrow[ur, "\cong"'] &
\end{tikzcd}
$$

□

The reader might be concerned by the signs appearing in the proof above and so we attempt to clarrify matters. Suppose that we have a cofibration sequence $A \longrightarrow B \longrightarrow C$ in spaces. We can model this sequence as a map $f : A \longrightarrow B$ together with the inclusion map to the cone on f and in this model the connecting map $C \longrightarrow \Sigma A$ is constructed by 'collapsing out' B. We note that the cone coordinate corresponds to the suspension coordinate and we use this observation to make a convention: the cone coordinate should always correspond to the outermost suspension coordinate (when other suspensions are lurking around). To summarise these ideas in the context of $\mathscr{S}$ we use the language of triagulated categories and *distinguished triangles.* If

$$A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \Sigma A$$

is a distinguished triangle then so are

$$\Sigma A \xrightarrow{-\Sigma i} \Sigma B \xrightarrow{-\Sigma j} \Sigma C \xrightarrow{-\Sigma k} \Sigma^2 A \qquad \text{and} \qquad \Sigma A \xrightarrow{\Sigma i} \Sigma B \xrightarrow{\Sigma j} \Sigma C \xrightarrow{-\Sigma k} \Sigma^2 A.$$

The sign in front of Σk appears because otherwise the cone coordinate would correspond to the inner suspension coordinate. In the proof above $d_I^s \wedge 1$ uses the outer suspension coordinate (by convention) but $1 \wedge d_J^t$ uses a suspension coordinate s places in and so we have to introduce a $(-1)^s$.

We can make lemma 2.3.8 of the previous subsection, which concerned sequences, more precise once we consider towers.

Proposition 2.4.7. *Suppose X, X', Y, $Y' \in (\boldsymbol{Spec}^{\mathbb{N}})_Q$ and that we extend their images in $\mathscr{S}^{\mathbb{N}}$ to towers (X, I), (X', I'), (Y, J), (Y', J') by taking homotopy cofibers. Suppose we have maps of towers*

$$(X, I) \longrightarrow (X', I') \quad \textit{and} \quad (Y, J) \longrightarrow (Y', J').$$

Extend the images of $Z = X \otimes Y$ and $Z' = X' \otimes Y'$ in $\mathscr{S}^{\mathbb{N}}$ to towers (Z, K) and (Z', K').

Then we can construct a map of towers $(Z, K) \to (Z', K')$, such that the map on augmented cochain complexes $(Z_0 \to \Sigma^\bullet K_\bullet) \longrightarrow (Z_0' \to \Sigma^\bullet K_\bullet')$ is the tensor product

$$\Big[(X_0 \to \Sigma^\bullet I_\bullet) \longrightarrow (X_0' \to \Sigma^\bullet I_\bullet')\Big] \wedge \Big[(Y_0 \to \Sigma^\bullet J_\bullet) \longrightarrow (Y_0' \to \Sigma^\bullet J_\bullet')\Big].$$

Proof. The maps of towers restrict to maps of sequences (morphisms in $\mathscr{S}^{\mathbb{N}}$). We are then free to apply lemma 2.3.8 to obtain a map $Z \longrightarrow Z'$ in $\mathscr{S}^{\mathbb{N}}$.

We need to be more precise. In the proof of lemma 2.3.8 a choice of homotopy had to be made for each square; in the current scenario, the map on cofibers determines which homotopy we should use. Making this choice allows one to check that we obtain a map of towers $(Z, K) \longrightarrow (Z', K')$ by defining the maps on cofibers

$$K_s = \bigvee_{i+j=s} I_i \wedge J_j \longrightarrow \bigvee_{i+j=s} I'_i \wedge J'_j = K'_s$$

to be the ones given by smashing together the maps $I_i \longrightarrow I'_i$ and $J_j \longrightarrow J'_j$.

Regarding this last point we omit some details. However, we note that this is where the second row in the diagram of lemma 2.3.8 becomes useful. An argument like the one in the proof of 5.2.1 in [24] shows that the map $B(*, \mathcal{E}_{s+1}, Fj) \longrightarrow B(*, \mathcal{E}'_s, F)$ is a cofibration between cofibrant objects. Thus the diagram induced by taking the strict cofiber of the map from the right column to the middle column has its top vertical map a weak equivalence; this diagram defines a map in the homotopy category, the induced map on homotopy cofibers. The collection of all such maps on cofibers gives a map of towers. One needs to check this coincides with the map described above and this is where we leave some details to the reader. The key point is that the cofiber of $B(*, \mathcal{E}_{s+1}, Fj) \longrightarrow B(*, \mathcal{E}'_s, F)$ receives a weak equivalence from a wedge of a smash product of cones, a fattening of

$$\bigvee_{i+j=s} I_i \wedge J_j.$$

□

By using $Q^{\mathbb{N}}$, lifting, and applying the telescope functor Tel we may assume, in the statement of propostion 2.4.7, that X, X', Y, $Y' \in \mathscr{S}^{\mathbb{N}}$. In fact, by fixing choices for lifts (using the axiom of choice) we can obtain a functor $\wedge : \mathcal{TO} \times \mathcal{TO} \longrightarrow \mathcal{TO}$, where $\mathcal{TO}$ denotes the category of towers in $\mathscr{S}$. As remarked in the motivational subsection this is more than we need; we don't elaborate further and we conclude our abstract pointset discussions.

We note that the results above can be applied to *bounded below* sequences and towers, sequences and towers indexed by $\mathbb{Z}$, which become constant below some $S \in \mathbb{Z}$.

Finally, we also note that the results of the discussion above are well-known to experts in the field. For instance, these sort of issues are addressed in section 3 of [23].

2.5 Quotienting towers using a pointset model

Proof of lemma II.3.4.7. The idea is straightforward: we find a strict model for the map of towers in corollary II.3.4.6 up to the $(2n-1)^{\text{th}}$ position and collapse from the n^{th} position onwards. In order to avoid having to delve into any true pointset level discussion of spectra we use the Quillen adjunction with spaces which we black-boxed in section 2.2.

Firstly, we construct a pointset model for ΣY:

$$\Sigma S \xleftarrow{p} \Sigma S \xleftarrow{p} \Sigma S \xleftarrow{p} \cdots.$$

We recall that the material of subsection 2.3 applies to $\mathbf{Top}_*$ so that we have a symmetric monoidal product $\otimes$ on sequences in $\mathbf{Top}_*$ (definition 2.3.3) and a telescope functor (definition 2.3.6). View

S^1 as $\{z \in \mathbb{C} : |z| = 1\}$, let

$$\widehat{Y} = (S^1 \xleftarrow{z \longmapsto z^p} S^1 \xleftarrow{z \longmapsto z^p} S^1 \xleftarrow{z \longmapsto z^p} \ldots) \in \mathbf{Top}_*^{\mathbb{Z}_{\geq 0}}$$

(note that this is levelwise cofibrant) and let $\widetilde{Y} = \mathrm{Tel}(\widehat{Y})$.

$$S_c \wedge \widetilde{Y} = S_c \wedge \mathrm{Tel}(\widehat{Y}) = \mathrm{Tel}(S_c \wedge \widehat{Y}) \in \mathbf{Spec}^{\mathbb{Z}_{\geq 0}}$$

is a model for ΣY. A model for $(\Sigma Y) \wedge (\Sigma Y)$ is given by $(S_c \wedge \widetilde{Y}) \otimes (S_c \wedge \widetilde{Y})$, which has s^{th} term given by

$$\operatorname*{colim}_{\substack{i+j \geq s \\ 0 \leq i,j \leq s}} (S_c \wedge \widetilde{Y}_i) \wedge (S_c \wedge \widetilde{Y}_j) = \operatorname*{colim}_{\substack{i+j \geq s \\ 0 \leq i,j \leq s}} (S_c \wedge S_c) \wedge (\widetilde{Y}_i \wedge \widetilde{Y}_j) = (S_c \wedge S_c) \wedge (\widetilde{Y} \otimes \widetilde{Y})_s$$

and the weak equivalence $S_c \wedge S_c \longrightarrow S \wedge S_c = S_c$ gives a weak equivalence $(S_c \wedge S_c) \wedge (\widetilde{Y} \otimes \widetilde{Y}) \longrightarrow S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})$.

The map of towers $(Y, J) \wedge (Y, J) \longrightarrow (Y, J)$ restricts to a map of sequences and and suspending twice gives us a map of sequences $S_c \wedge (\widetilde{Y} \otimes \widetilde{Y}) \longrightarrow S_c \wedge (\Sigma \widetilde{Y})$ in the stable homotopy category. Since everything in sight is bifibrant we have a diagram of maps which commutes up to homotopy.

$$\begin{tikzcd}
S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})_0 \arrow[d] & S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})_1 \arrow[l] \arrow[d] & \ldots \arrow[l] & S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})_s \arrow[l] \arrow[d] & S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})_{s+1} \arrow[l] \arrow[d] & \ldots \arrow[l] \\
S_c \wedge \Sigma \widetilde{Y}_0 & S_c \wedge \Sigma \widetilde{Y}_1 \arrow[l] & \ldots \arrow[l] & S_c \wedge \Sigma \widetilde{Y}_s \arrow[l] & S_c \wedge \Sigma \widetilde{Y}_{s+1} \arrow[l] & \ldots \arrow[l]
\end{tikzcd}$$

The 'homotopy extension property' says we can make any of the squares strictly commute at the cost of changing the left map to a homotopic one. Thus, by starting at the $(2n-1)^{\text{th}}$ position, we may suppose that we have a strictly commutative diagram consisting of the top two rows.

$$\begin{tikzcd}
S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})_0 \arrow[d] & S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})_1 \arrow[l] \arrow[d] & \ldots \arrow[l] & S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})_{2n-2} \arrow[l] \arrow[d] & S_c \wedge (\widetilde{Y} \otimes \widetilde{Y})_{2n-1} \arrow[l] \arrow[d] \\
S_c \wedge \Sigma \widetilde{Y}_0 \arrow[d] & S_c \wedge \Sigma \widetilde{Y}_1 \arrow[l] \arrow[d] & \ldots \arrow[l] & S_c \wedge \Sigma \widetilde{Y}_{2n-2} \arrow[l] \arrow[d] & S_c \wedge \Sigma \widetilde{Y}_{2n-1} \arrow[l] \arrow[d] \\
S_c \wedge \Sigma(\widetilde{Y}_0/\widetilde{Y}_n) & S_c \wedge \Sigma(\widetilde{Y}_1/\widetilde{Y}_n) \arrow[l] & \ldots \arrow[l] & S_c \wedge \Sigma(\widetilde{Y}_{2n-2}/\widetilde{Y}_{\max\{n,2n-2\}}) \arrow[l] & S_c \wedge \Sigma(*) = * \arrow[l]
\end{tikzcd}$$

Applying $S_c \wedge \Sigma(-)$ to the map, which collapses out $\widetilde{Y}_n$ we obtain the map down to the bottom row. Since $(\widetilde{Y}_0 \wedge \widetilde{Y}_n) \cup (\widetilde{Y}_n \wedge \widetilde{Y}_0) \subset (\widetilde{Y} \otimes \widetilde{Y})_n$ we see that $S_c \wedge ((\widetilde{Y}_0 \wedge \widetilde{Y}_n) \cup (\widetilde{Y}_n \wedge \widetilde{Y}_0))$ is mapped to $*$. Using the fact that $S_c \wedge (-)$ preserves colimits, or by using the tensor adjunction, arguing in spaces and using the tensor adjunction again, we obtain a map

$$S_c \wedge (\widetilde{Y}(n) \otimes \widetilde{Y}(n)) \longrightarrow S_c \wedge (\Sigma \widetilde{Y}(n))$$

where $\widetilde{Y}(n) = \widetilde{Y}/\widetilde{Y}^{(n)}$ (see definition 2.4.6). Since $S_c \wedge \widetilde{Y}(n)$ gives a model for $\Sigma Y(n)$

$$\Sigma S/p^n \xleftarrow{p} \Sigma S/p^{n-1} \xleftarrow{p} \ldots \xleftarrow{p} \Sigma S/p \longleftarrow * \longleftarrow \ldots$$

and $S_c \wedge (\widetilde{Y}(n) \otimes \widetilde{Y}(n))$ gives a model for $\Sigma Y(n) \wedge \Sigma Y(n)$ we obtain a map $Y(n) \wedge Y(n) \longrightarrow Y(n)$ by desuspending. One should check that the induced map on cofibers is what we want it to be. This is the case because one way to construct an induced map on cofibers is by a strictification process like the one used above (actually, we are going the opposite direction; the induced map tells us which homotopy we should extend when strictifying) and because the multiplication in the cochain complex $\Sigma^\bullet J(n)_\bullet$ is induced by the multiplication on the cochain complex $\Sigma^\bullet J_\bullet$ by collapsing. $\square$

2.6 A pointset level construction of S/p^∞

We use maps between Moore spectra extensively and in the bulk of the text we did not feel it was necessary to mention how such maps are constructed; they are well known to the expert. Many of the maps can be constructed using fillers for a distinguished triangle but this can cause concern due to the nonuniquness of such fillers. Here, we note that our Prüfer sphere has a good pointset model and the construction should make it clear that we could be very precise about all of the maps we use at the pointset level, if necessary.

Definition 2.6.1. Write $p : S^1 \longrightarrow S^1$ for the map $z \longmapsto z^p$. Then $p^{-1}S^1 \in \mathbf{Top}_*$ is the colimit of the following diagram

$$\begin{tikzcd}
S^1 \otimes \Delta^1 & S^1 \otimes \Delta^0 \arrow[l, "S^1\otimes[1]"'] \arrow[r, "p\otimes\Delta^0"] & S^1 \otimes \Delta^0 \arrow[dll, "S^1\otimes[0]"'] \\
S^1 \otimes \Delta^1 & S^1 \otimes \Delta^0 \arrow[l, "S^1\otimes[1]"'] \arrow[r, "p\otimes\Delta^0"] & S^1 \otimes \Delta^0 \arrow[dll, "S^1\otimes[0]"'] \\
S^1 \otimes \Delta^1 & \cdots \arrow[l] & \cdots
\end{tikzcd}$$

We have a map $S^1 \longrightarrow p^{-1}S^1$ given by

$$S^1 \otimes \Delta^0 \xrightarrow{S^1\otimes[0]} S^1 \otimes \Delta^1 \longrightarrow p^{-1}S^1$$

where the last map includes the first term in the colimit.

$S^1/p^\infty \in \mathbf{Top}_*$ is the strict cofiber of the map $S^1 \longrightarrow p^{-1}S^1$. We have a cofibration sequence in $\mathrm{Ho}(\mathbf{Top}_*)$: $S^1 \longrightarrow p^{-1}S^1 \longrightarrow S^1/p^\infty$. Applying $S_c \wedge (-)$ and desuspending gives a cofibration sequence in $\mathscr{S}$:

$$S \longrightarrow p^{-1}S \longrightarrow S/p^\infty.$$

S/p^∞ is called the *Prüfer sphere.*

We see that we can obtain the cofibration sequence above by taking homotopy colimits of the rows in the following diagram (the above definition explains how to obtain a good pointset model for such a diagram even though this is not strictly necessary).

$$\begin{tikzcd}
S \arrow[r, "="] \arrow[d, "="] & S \arrow[r, "="] \arrow[d, "p"] & S \arrow[r, "="] \arrow[d, "p^2"] & S \arrow[r, "="] \arrow[d, "p^3"] & \dots \\
S \arrow[r, "p"] \arrow[d] & S \arrow[r, "p"] \arrow[d] & S \arrow[r, "p"] \arrow[d] & S \arrow[r, "p"] \arrow[d] & \dots \\
* \arrow[r] & S/p \arrow[r, "p"] & S/p^2 \arrow[r, "p"] & S/p^3 \arrow[r, "p"] & \dots
\end{tikzcd}$$

Chapter IV

Miscellaneous results

This chapter contains results that will not appear in the thesis. This chapter is not proof-read and is probably only readable by me!

1 Relations between Adams spectral sequences (Miller's theorem)

In this section we give a different proof of [17, theorem 6.1].

We have a map a ring spectra $BP \longrightarrow H$ and the canonical BP-resolution for the mod p Moore spectrum X satisfies the hypothesis required for the May type SS. Thus we have a diagram of spectral sequences

$$\begin{tikzcd}
R^s_{BP}R^{t-s}_H\pi_u(X) \arrow[r, Rightarrow, "\mathrm{MAH}", "s"'] \arrow[d, Rightarrow, "\mathrm{MAY}"', "t"] & R^t_H\pi_u(X) \arrow[d, Rightarrow, "t"', "H\text{-ASS}"] \\
R^s_{BP}\pi_{u+s-t}(X) \arrow[r, Rightarrow, "BP\text{-ASS}", "s"'] & \pi_{u-t}(X)
\end{tikzcd}$$

Suppose given $z \in F^s_{\mathrm{MAH}}R^t_H\pi_u(X)$ detected in the Mahowald spectral sequence by $a \in R^s_{BP}R^{t-s}_H\pi_u(X)$; then $d_2^{\mathrm{MAY}}a \in R^{s+1}_{BP}R^{t-s+1}_H\pi_{u+1}(X)$.

Theorem 1.1 (Haynes Miller). *$d_2^H z \in F^{s+1}_{\mathrm{MAH}}R^{t+2}_H\pi_{u+1}(X)$ and is detected by $\pm d_2^{\mathrm{MAY}}a$.*

Proof. Let

$$\begin{tikzcd}
X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & X_s \arrow[l] \arrow[d] & X_{s+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
I_0 \arrow[ur, dotted] & I_1 \arrow[ur, dotted] & I_2 \arrow[ur, dotted] & & I_s \arrow[ur, dotted] & I_{s+1} \arrow[ur, dotted] &
\end{tikzcd}$$

be the canonical BP-resolution for X and let

$$\begin{tikzcd}
Y_0 \arrow[d] & Y_1 \arrow[l] \arrow[d] & Y_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & Y_s \arrow[l] \arrow[d] & Y_{s+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
J_0 \arrow[ur, dotted] & J_1 \arrow[ur, dotted] & J_2 \arrow[ur, dotted] & & J_s \arrow[ur, dotted] & J_{s+1} \arrow[ur, dotted] &
\end{tikzcd}$$

be any H-resolution for S^0. Then we have

$$
\begin{tikzcd}
R^t_H\pi_u(X_0) & R^{t-s}_H\pi_{u-s}(X_s) \arrow[l] \arrow[d] & & z & \tilde{z} \arrow[l, mapsto] \arrow[d, mapsto] \\
 & R^{t-s}_H\pi_{u-s}(I_s) & & & a'
\end{tikzcd}
$$

and $a = [a'] \in R^s_{BP}R^{t-s}_H\pi_u(X)$.

Since $X_{s+1} \longrightarrow X_s$ is BP-null, it is H-null and we obtain

$$
\begin{tikzcd}
X_0 \wedge Y_0 \\
X_0 \wedge Y_1 \arrow[u] & X_1 \wedge Y_0 \arrow[l] \\
X_0 \wedge Y_2 \arrow[u] & X_1 \wedge Y_1 \arrow[l] \arrow[u] & X_2 \wedge Y_0 \arrow[l] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
X_0 \wedge Y_t \arrow[u] & X_1 \wedge Y_{t-1} \arrow[l] \arrow[u] & X_2 \wedge Y_{t-2} \arrow[l] \arrow[u] & \cdots \arrow[l] & X_{s-1} \wedge Y_{t-s+1} \arrow[l] & X_s \wedge Y_{t-s} \arrow[l] & X_{s+1} \wedge Y_{t-s-1} \arrow[l] \\
X_0 \wedge Y_{t+1} \arrow[u] & X_1 \wedge Y_t \arrow[l] \arrow[u] & X_2 \wedge Y_{t-1} \arrow[l] \arrow[u] & \cdots \arrow[l] & X_{s-1} \wedge Y_{t-s+2} \arrow[l] \arrow[u] & X_s \wedge Y_{t-s+1} \arrow[l] \arrow[u] & X_{s+1} \wedge Y_{t-s} \arrow[l] \arrow[u] \\
X_0 \wedge Y_{t+2} \arrow[u] & X_1 \wedge Y_{t+1} \arrow[l] \arrow[u] & X_2 \wedge Y_t \arrow[l] \arrow[u] & \cdots \arrow[l] & X_{s-1} \wedge Y_{t-s+3} \arrow[l] \arrow[u] & X_s \wedge Y_{t-s+2} \arrow[l] \arrow[u] & X_{s+1} \wedge Y_{t-s+1} \arrow[l] \arrow[u]
\end{tikzcd}
$$

By picking a representative $\tilde{z}' \in \pi_{u-t}(X_s \wedge J_{t-s})$ for $\tilde{z}$ we determine representatives $z' \in \pi_{u-t}(X_0 \wedge J_t)$ and $a'' \in \pi_{u-t}(I_s \wedge J_{t-s})$ for z and a', respectively: all are cycles relative to the differential on J and $\tilde{z}'$ is mapped to z' under the maps induced by taking cofibers vertically in the above diagram.

$$
\begin{tikzcd}
\pi_{u-t}(X_0 \wedge J_t) & \pi_{u-t}(X_s \wedge J_{t-s}) \arrow[l] \arrow[d] & & z' & \tilde{z}' \arrow[l, mapsto] \arrow[d, mapsto] \\
 & \pi_{u-t}(I_s \wedge J_{t-s}) & & & a''
\end{tikzcd}
$$

We now smash together the cofiber sequences $X_{s+1} \longrightarrow X_s \longrightarrow I_s$ and $Y_{t-s} \longrightarrow J_{t-s} \longrightarrow \Sigma Y_{t-s+1}$

to obtain a Verdier system.

$$
\begin{tikzcd}
X_{s+1} \wedge Y_{t-s+1} \arrow[d] & X_{s+1} \wedge J_{t-s} \arrow[l, dotted] \arrow[d] & X_{s+1} \wedge Y_{t-s} \arrow[l] \arrow[d] \\
X_s \wedge Y_{t-s+1} \arrow[d] & X_s \wedge J_{t-s} \arrow[l, dotted] \arrow[d] & X_s \wedge Y_{t-s} \arrow[l] \arrow[d] \\
I_s \wedge Y_{t-s+1} & I_s \wedge J_{t-s} \arrow[l, dotted] & I_s \wedge Y_{t-s} \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\tilde{x} \arrow[d, mapsto] \arrow[rr, mapsto, bend left, "\tilde{\partial}"] & & \bullet \\
x \arrow[d, mapsto] & \tilde{z}' \arrow[l, mapsto] \arrow[d, mapsto] & \\
0 & a'' \arrow[l, mapsto] & \bullet \arrow[l, mapsto] \arrow[u, mapsto, bend right, "\partial"']
\end{tikzcd}
$$

$\tilde{z}'$ determines an element x and a'' is mapped to 0 because

1. it is a cycle relative to the differential on J;

2. the H-ASS for I_s collapses at E_2;

3. there is no convergence issue since all spectra in sight are p-complete.

Thus we can choose $\tilde{x}$ and the $\bullet$'s compatibly (up to a sign which we will ignore from now on). We redraw some of this information as follows where $(d_2^{\mathrm{MAY}} a)''$ represents an element representing $d_2^{\mathrm{MAY}} a \in R_{BP}^{s+1} R_H^{t-s+1} \pi_{u+1}(X)$ and $d_2 \tilde{z}'$ is just the element it has to be.

$$
\begin{tikzcd}
X_s \wedge J_{t-s} \arrow[d] & & & \\
I_s \wedge J_{t-s} & I_s \wedge Y_{t-s} \arrow[l] \arrow[d, dotted] & & \\
& X_{s+1} \wedge Y_{t-s} \arrow[d] & X_{s+1} \wedge Y_{t-s+1} \arrow[l] \arrow[r] \arrow[d] & X_{s+1} \wedge J_{t-s+1} \arrow[d] \\
& I_{s+1} \wedge Y_{t-s} & I_{s+1} \wedge Y_{t-s+1} \arrow[l] \arrow[r] & I_{s+1} \wedge J_{t-s+1}
\end{tikzcd}
$$

$$\begin{tikzcd}
\tilde{z}' \arrow[d, mapsto] & & \\
a'' & & \bullet \arrow[ll, mapsto] \arrow[d, mapsto, "\partial"'] & & \\
& & \bullet \arrow[d, mapsto] & \tilde{x} \arrow[l, mapsto, "\tilde{\partial}"'] \arrow[r, mapsto] \arrow[d, mapsto] & d_2\tilde{z}' \arrow[d, mapsto] \\
& & \bullet & \bullet \arrow[l, mapsto] \arrow[r, mapsto] & (d_2^{\mathrm{MAY}}a)''
\end{tikzcd}$$

We also have the following commutative diagram.

$$\begin{tikzcd}
& X_s \wedge Y_{t-s+1} & X_s \wedge Y_{t-s+2} \arrow[l] \arrow[d] \\
X_s \wedge J_{t-s} \arrow[ur, dotted] & X_{s+1} \wedge Y_{t-s+1} \arrow[u] \arrow[ur] \arrow[d] & X_s \wedge J_{t-s+2} \\
& X_{s+1} \wedge J_{t-s+1} \arrow[ur] &
\end{tikzcd}$$

Thus $\tilde{x}$ determines the lower parallelogram,

$$\begin{tikzcd}
& x & \bullet \arrow[l, mapsto] \arrow[d, mapsto] \\
\tilde{z}' \arrow[ur, mapsto] & \tilde{x} \arrow[u, mapsto] \arrow[ur, mapsto] \arrow[d, mapsto] & (d_2^H\tilde{z})' \\
& d_2\tilde{z}' \arrow[ur, mapsto] &
\end{tikzcd}$$

where $(d_2^H\tilde{z})'$ is a representative for $d_2^H\tilde{z}$ in the H-ASS for X_s. Using the map of SSs determined by the first huge diagram we drew we see that

$$\begin{tikzcd}
\pi_{u-t-1}(X_0 \wedge J_{t+2}) & \pi_{u-t-1}(X_s \wedge J_{t-s+2}) \arrow[l] & \pi_{u-t-1}(X_{s+1} \wedge J_{t-s+1}) \arrow[l] \arrow[d] \\
& & \pi_{u-t-1}(I_{s+1} \wedge J_{t-s+1})
\end{tikzcd}$$

$$\begin{tikzcd}
(d_2^H z)' & (d_2^H\tilde{z})' \arrow[l, mapsto] & d_2\tilde{z}' \arrow[l, mapsto] \arrow[d, mapsto] \\
& & (d_2^{\mathrm{MAY}}a)''
\end{tikzcd}$$

and so

$$\begin{tikzcd}
R_H^{t+2}\pi_{u+1}(X_0) & R_H^{t-s+2}\pi_{u-s+1}(X_s) \arrow[l] & R_H^{t-s+1}\pi_{u-s}(X_{s+1}) \arrow[l] \arrow[d] \\
 & & R_H^{t-s+1}\pi_{u-s}(I_{s+1})
\end{tikzcd}$$

$$\begin{tikzcd}
d_2^H z & d_2^H \tilde{z} \arrow[l, mapsto] & \bullet \arrow[l, mapsto] \arrow[d, mapsto] \\
 & & (d_2^{\mathrm{MAY}} a)'
\end{tikzcd}$$

which says that $d_2^H z \in F_{\mathrm{MAH}}^{s+1} R_H^{t+2}\pi_{u+1}(X)$ and is detected by $d_2^{\mathrm{MAY}} a = [(d_2^{\mathrm{MAY}} a)']$. □

2 $p = 2$

In this section we examine whether our methods are applicable when $p = 2$.

2.1 $v_1 \in \mathbf{Cotor}_A(H_*(\mathbf{End}(S/2 \wedge S/\eta)))$

Let A denote the dual of the Steenrod algebra at the prime 2, so as an algebra $A = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$. Recall $H_*(S/2 \wedge S/\eta) = \mathbb{F}_2\langle 1, \xi_1, \xi_1^2, \xi_1^3\rangle$ so that $H_*(\mathrm{End}(S/2 \wedge S/\eta)) = \mathrm{End}(H_*(S/2 \wedge S/\eta))$ consists of 4×4 matrices. The degree of the matrix $E_{i,j}$ is $i-j$ and using the identification $\mathrm{End}(S/2\wedge S/\eta) = (S/2 \wedge S/\eta) \wedge \Sigma^{-3}(S/2 \wedge S/\eta)$ together with the Kunneth formula E_{ij} corresponds to $\xi_1^i \otimes \sigma^{-3}\xi_1^{3-j}$, where we index the matrix using $\{0, 1, 2, 3\}^2$. Consider the three matrices

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that I and K are A-comodule primitives and that $J \longmapsto \xi_1 \otimes I + 1 \otimes J + \xi_1^4 \otimes K$. Let $x \in \Omega(A; H_*(\mathrm{End}(S/2 \wedge S/\eta)))$ be the element

$$[\xi_2|I] + [\xi_1^2|J] + [\overline{\xi_2^2}|K].$$

x is a cocycle representing an element we call v_1. We claim v_1 generates a polynomial algebra $\mathbb{F}_2[v_1]$ in $\mathrm{Cotor}_A(H_*(\mathrm{End}(H_*(S/2 \wedge S/\eta))))$. Thus $\mathrm{Cotor}_A(H_*(S/2 \wedge S/\eta))$ is a module over $\mathbb{F}_2[v_1]$.

2.2 $v_1^{-1}\mathbf{Cotor}_A(H_*(S/2 \wedge S/\eta))$

Let $B = A/(\xi_1^4, \xi_2^4, \ldots)$ and $C = B/(\xi_1) = A/(\xi_1, \xi_2^4, \xi_3^4 \ldots)$. The map $A \longrightarrow B$ induces a map

$$\mathrm{Cotor}_A(H_*(S/2 \wedge S/\eta)) \longrightarrow \mathrm{Cotor}_B(H_*(S/2 \wedge S/\eta)).$$

The square

$$\begin{tikzcd}
H_*(S/2 \wedge S/\eta) \arrow[r] \arrow[d] & B \otimes H_*(S/2 \wedge S/\eta) \arrow[d] \\
\mathbb{F}_2 \arrow[r] & C \otimes \mathbb{F}_2
\end{tikzcd}$$

induces a map

$$\mathrm{Cotor}_B(H_*(S/2 \wedge S/\eta)) \longrightarrow \mathrm{Cotor}_C(\mathbb{F}_2),$$

(which is an isomorphism). Thus we obtain a map (which we call φ)

$$\mathrm{Cotor}_A(H_*(S/2 \wedge S/\eta)) \longrightarrow \mathrm{Cotor}_C(\mathbb{F}_2) = \mathbb{F}_2[h_{j,0}, h_{j,1} : j \geq 2] \longrightarrow \mathbb{F}_2[h_{2,0}] \otimes \mathbb{F}_2[h_{j,1} : j \geq 2]$$

mapping $v_1 \cdot 1$ to $h_{2,0}$. [In fact, one can see that $v_1^n \cdot 1$ is mapped to $h_{2,0}^n$, proving that v_1 generates a polynomial algebra.] We claim that after inverting v_1 we obtain an isomorphism:

$$v_1^{-1}\mathrm{Cotor}_A(H_*(S/2 \wedge S/\eta)) = \mathbb{F}_2[v_1, v_1^{-1}] \otimes \mathbb{F}_2[h_{j,1} : j \geq 2].$$

2.3 The h_1-BSS

Let X be $H_*(S/2) = \mathbb{F}_2\langle 1, \xi\rangle$ and Y be $H_*(S/2 \wedge S/\eta) = \mathbb{F}_2\langle 1, \xi, \xi^2, \xi^3\rangle$. Applying $H_*(-)$ to the Puppe sequence

$$S/2 \wedge S^1 \xrightarrow{1\wedge\eta} S/2 \wedge S^0 \longrightarrow S/2 \wedge S/\eta \longrightarrow S/2 \wedge S^2$$

gives the SES

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & \Sigma^2 X \arrow[r] & 0 \\
 & 1, \xi_1 \arrow[r, mapsto] & 1, \xi_1 & & \\
 & & \xi_1^2, \xi_1^3 \arrow[r, mapsto] & 1, \xi &
\end{tikzcd}$$

Applying $\mathrm{Cotor}_A(-)$ gives the h_1-BSS. To check the connecting homomorphism is $(-)\cdot h_1$ we draw

$$\begin{tikzcd}
\Omega(A;X) \arrow[r] & \Omega(A;Y) \arrow[r] & \Omega(A;X) \\
 & \tilde{x} \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
x \cdot [\xi_1^2|1] \arrow[r, mapsto] & x \cdot [\xi_1^2|1] & 0
\end{tikzcd}$$

where $\tilde{x}$ is defined by multiplying elements in the comodule variable of x by ξ_1^2. Notice that we are using the right action of $\Omega(A;k)$ on $\Omega(A;X)$ and $\Omega(A;Y)$.

2.4 Structure of the h_1-BSS with respect to v_1^4-mulitplication

We have $v_1 \in \mathrm{Cotor}_A(H_*(\mathrm{End}(S/2\wedge S/\eta))$ and we have a map $\mathrm{End}(S/2) \longrightarrow \mathrm{End}(S/2\wedge S/\eta)$ given by smashing with the identity $S/\eta \longrightarrow S/\eta$. Formally this is adjoint to the map

$$\mathrm{End}(S/2, S/2) \wedge (S/2 \wedge S/\eta) = (\mathrm{End}(S/2, S/2) \wedge S/2) \wedge S/\eta \xrightarrow{\mathrm{ev}\wedge 1} S/2 \wedge S/\eta.$$

We claim that v_1^4 lifts to $\mathrm{Cotor}_A(H_*(\mathrm{End}(S/2)))$ and that this is a permanent cycle in the ASS detecting an element $A \in \pi_8(\mathrm{End}(S/2))$. Thus

$$\begin{tikzcd}
\Sigma^8 S/2 \wedge S^0 \arrow[r] \arrow[d, "A\wedge 1"] & \Sigma^8 S/2 \wedge S/\eta \arrow[r] \arrow[d, "A\wedge 1"] & \Sigma^8 S/2 \wedge S^2 \arrow[d, "A\wedge 1"] \\
S/2 \wedge S^0 \arrow[r] & S/2 \wedge S/\eta \arrow[r] & S/2 \wedge S^2
\end{tikzcd}$$

shows that $v_1^4 \cdot (-)$ commutes with the maps

$$\mathrm{Cotor}_A(X) \longrightarrow \mathrm{Cotor}_A(Y) \longrightarrow \mathrm{Cotor}_A(\Sigma^2 X).$$

Since

$$\begin{tikzcd}
\Sigma^8 S/2 \wedge S^1 \arrow[r, "1\wedge\eta"] \arrow[d, "A\wedge 1"] & \Sigma^8 S/2 \wedge S^0 \arrow[d, "A\wedge 1"] \\
S/2 \wedge S^1 \arrow[r, "1\wedge\eta"] & S/2 \wedge S^0
\end{tikzcd}$$

commutes $v_1^4 \cdot (-)$ commutes with $(-) \cdot h_1$. This is also evident since they are left and right actions, respectively. We deduce that $d_r x = y \implies d_r v_1^4 \cdot x = v_1^4 \cdot y$.

2.5 Some differentials in the h_1-BSS

We note that $v_1 \cdot 1 \in \mathrm{Cotor}_A^{1,3}(Y)$ is represented by $[\xi_2|1] + [\xi_1^2|\xi_1]$ and this element lifts to $\widetilde{v_1 \cdot 1} \in \mathrm{Cotor}_A^{1,3}(X)$. We compute v_1^2:

$$\Big([\xi_2|\xi_2|I] + [\xi_2|\xi_1^2|J] + [\xi_2|\overline{\xi_2^2}|K]\Big) + \Big([\xi_1^2|\xi_1\xi_2|I] + [\xi_1^2|\xi_2|J] + [\xi_1^2|\xi_1^4\xi_2|K]\Big) +$$

$$\Big([\xi_1^2|\xi_1^3|J] + [\xi_1^2|\xi_1^2|J^2] + [\xi_1^2|\xi_1^6|KJ]\Big) + \Big([\xi_1^2|\xi_1\overline{\xi_2^2}|K] + [\xi_1^2|\overline{\xi_2^2}|JK]\Big) + \Big([\overline{\xi_2^2}|\xi_2|K] + [\overline{\xi_2^2}|\xi_1^2|KJ]\Big).$$

i.e.

$$\Big([\xi_2|\xi_2|I] + [\xi_1^2|\xi_1\xi_2|I]\Big) + \Big([\xi_2|\xi_1^2|J] + [\xi_1^2|\xi_2|J] + [\xi_1^2|\xi_1^3|J]\Big) + [\xi_1^2|\xi_1^2|J^2] +$$

$$\Big([\xi_2|\overline{\xi_2^2}|K] + [\overline{\xi_2^2}|\xi_2|K] + [\xi_1^2|\xi_1^4\xi_2 + \xi_1\overline{\xi_2^2}|K]\Big) + [\xi_1^2|\overline{\xi_2^2}|JK] + \Big([\xi_1^2|\xi_1^6|KJ] + [\overline{\xi_2^2}|\xi_1^2|KJ]\Big).$$

Thus $v_1^2 \cdot 1 \in \mathrm{Cotor}_A^{2,6}(Y)$ is represented by

$$[\xi_2|\xi_2|1] + [\xi_1^2|\xi_1\xi_2|1] + [\xi_2|\xi_1^2|\xi_1] + [\xi_1^2|\xi_2|\xi_1] + [\xi_1^2|\xi_1^3|\xi_1] + [\xi_1^2|\xi_1^2|\xi_1^2].$$

The image in $\mathrm{Cotor}_A^{2,4}(X)$ is $1 \cdot h_1^2$; we see that $d_3 v_1^2 \cdot 1 = 1$.

We compute v_1^3, only recording terms involving the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}:$$

$$[\xi_2|\xi_1^2|\xi_1^2|J^2] + \Big([\xi_1^2|\xi_1^3|\xi_1^2|J^2] + [\xi_1^2|\xi_1^2|\xi_1^3|J^2]\Big) +$$

$$\Big([\xi_1^2|\xi_2|\xi_1^2|J^2] + [\xi_1^2|\xi_1^2|\xi_2|J^2] + [\xi_1^2|\xi_1^2|\xi_1^3|J^2]\Big) + [\xi_1^2|\xi_1^2|\xi_1^2|J^3].$$

Thus we see that the image of $v_1^3 \cdot 1 \in \mathrm{Cotor}_A^{3,9}(Y)$ in $\mathrm{Cotor}_A^{3,7}(X)$ is represented by

$$[\xi_2|\xi_1^2|\xi_1^2|1] + [\xi_1^2|\xi_1^3|\xi_1^2|1] + \Big([\xi_1^2|\xi_2|\xi_1^2|1] + [\xi_1^2|\xi_1^2|\xi_2|1]\Big) + [\xi_1^2|\xi_1^2|\xi_1^2|\xi_1] =$$

$$\Big(([\xi_2|1] + [\xi_1^2|\xi_1]) \cdot [\xi_1^2|\xi_1^2|1]\Big) + \Big([\xi_1^2|\xi_1^2|\xi_1^3|1] + [\xi_1^2|\xi_2|\xi_1^2|1] + [\xi_1^2|\xi_1^2|\xi_2|1]\Big)$$

The first term represents $\widetilde{(v_1 \cdot 1)} \cdot h_1^2$ and the second term lifts to an element $x \in \mathrm{Cotor}_A^{3,7}(\mathbb{F}_2)$. x/h_1 is represented by $[\xi_1^2|\xi_1^3] + [\xi_2|\xi_1^2] + [\xi_1^2|\xi_2]$. Adding $d[\xi_1\xi_2]$ gives $[\xi_1^4|\xi_1]$, which represents h_0h_2. Thus $x = h_0h_1h_2 = 0$ and the image of $v_1^3 \cdot 1$ in $\mathrm{Cotor}_A^{3,7}(X)$ is

$$\widetilde{(v_1 \cdot 1)} \cdot h_1^2.$$

We conclude that $d_3(v_1^3 \cdot 1) = v_1 \cdot 1$.

We have an element in $y \in \mathrm{Cotor}^{2,6}(Y)$ represented by $[\xi_2^2|1] + [\xi_1^4|\xi_1^2]$ and we see that $\varphi y = h_{2,1}$. We can describe $v_1^3 \cdot y$ if we multiply. Anyway, by looking at the tables we see that we could prove

$$d_3 v_1^3 \cdot y = v_1 \cdot y \quad \text{and} \quad d_3 v_1^4 \cdot y = v_1^2 \cdot y.$$

In a localised h_1-BSS, after inverting v_1^4, this would give

$$d_3 v_1^3 h_{2,1} = v_1 h_{2,1} \quad \text{and} \quad d_3 v_1^4 h_{2,1} = v_1^2 h_{2,1}.$$

2.6 Some formulae

For reference we note the coactions

$$J^2 \longmapsto 1 \otimes J^2 + \xi_1^2 \otimes I + \xi_1^4 \otimes (JK + KJ),$$

$$JK \longmapsto 1 \otimes JK + \xi_1 \otimes K, \quad KJ \longmapsto 1 \otimes KJ + \xi_1 \otimes K.$$

and that $v_1 \cdot y$ is represented by

$$\Big[[\xi_2|I] + [\xi_1^2|J]\Big] \cdot \Big[[\xi_2^2|1] + [\xi_1^4|\xi_1^2]\Big] =$$

$$[\xi_2|\xi_2^2|1] + [\xi_2|\xi_1^4|\xi_1^2] + [\xi_1^2|\xi_1\xi_2^2|1] + [\xi_1^2|\xi_2^2|\xi_1] + [\xi_1^2|\xi_1^5|\xi_1^2] + [\xi_1^2|\xi_1^4|\xi_1^3].$$

In $\Omega(A; H_*(\mathrm{End}(S/2)))$ we have

$$d\left[\left[\xi_1|\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] + \left[\overline{\xi}_2|\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]\right] = [\xi_1|\xi_1|I]$$

which shows algebraically that multpilication by 4 is zero on $S/2$.

2.7 $\pi_*(v_1^{-1}S/2 \wedge S/\eta)$

We claim that $v_1 \in \mathrm{Cotor}_A(H_*(\mathrm{End}(S/2 \wedge S/\eta)))$ is a permanent cycle in the ASS detecting a map $v_1 : S/2 \wedge S/\eta \longrightarrow \Sigma^{-2}S/2 \wedge S/\eta$. We form the mapping telescope $v_1^{-1}S/2 \wedge S/\eta$. Because $\mathrm{Cotor}_A(Y)$ has a vanishing line of slope parallel to that given by v_1 we can calculate the homotopy using a localised ASS:

$$v_1^{-1}\mathrm{Cotor}_A(H_*(S/2 \wedge S/\eta)) \implies \pi_*(v_1^{-1}S/2 \wedge S/\eta).$$

We saw that $v_1^{-1}\mathrm{Cotor}_A(H_*(S/2 \wedge S/\eta)) = \mathbb{F}_2[v_1, v_1^{-1}] \otimes \mathbb{F}_2[h_{j,1} : j \geq 2]$. We claim that $d_2 v_1 = d_2 h_{2,1} = 0$, that $d_2 h_{j,1} = v_1 h_{j-1,1}^2$ for $j > 2$ and that the spectral sequence is multiplicative so that

$$\pi_*(v_1^{-1}S/2 \wedge S/\eta) = \mathbb{F}_2[v_1, v_1^{-1}] \otimes E[h_{2,1}].$$

To justify the multiplicativity maybe we can prove that $v_1^{-1}S/2 \wedge S/\eta$ is a ring. Alternatively, we might use the map which Mahowald and Davis talk of: $Y \wedge Y \longrightarrow Y'$.

2.8 $\pi_*(A^{-1}S/2)$

We have a cofiber sequences

$$S/2 \wedge S^1 \xrightarrow{1\wedge\eta} S/2 \wedge S^0 \longrightarrow S/2 \wedge S/\eta$$

$$A^{-1}S/2 \wedge S^1 \xrightarrow{1\wedge\eta} A^{-1}S/2 \wedge S^0 \longrightarrow A^{-1}S/2 \wedge S/\eta$$

and of course, $A^{-1}S/2 \wedge S/\eta = v_1^{-1}S/2 \wedge S/\eta$. Thus we have Bockstein SSs

$$\pi_*(S/2 \wedge S/\eta) \implies \pi_*(S/2), \qquad \pi_*(v_1^{-1}S/2 \wedge S/\eta) \implies \pi_*(A^{-1}S/2).$$

Recall that we have classes $1, \widetilde{v_1 \cdot 1}, \widetilde{v_1 \cdot y}, \widetilde{v_1^2 \cdot y} \in \mathrm{Cotor}_A(X)$, which are nonzero under $(-) \cdot h_1^2$. We immediately see from the charts that all the elements just considered are permanent cycles in the ASS and are not boundaries; the non-h_1-powers detect elements mapping to $1, v_1, v_1 h_{2,1}$ and $v_1^2 h_{2,1}$ in $\pi_*(v_1^{-1}S/2 \wedge S/\eta)$. In $\pi_*(S^0)$, $\eta^3 = 4\nu$ and so because multiplication by 4 is zero on $S/2$ we have $(-) \cdot \eta^3 = 0$ on $\pi_*(S/2)$. Thus $1, v_1 \cdot 1, v_1 \cdot y, v_1^2 \cdot y$ are targets of d_3's but not d_2's and in the localised SS the $1, v_1, v_1 h_{2,1}, v_1^2 h_{2,1}$ are targets of d_3's. Looking at the charts we deduce the first and fourth of the following differentials.

$$d_3 v_1^2 \cdot 1 = 1, \quad d_3 v_1^3 \cdot 1 = v_1 \cdot 1, \quad d_3 v_1^3 \cdot y = v_1 \cdot y, \quad d_3 v_1^4 \cdot y = v_1^2 \cdot y.$$

Since $(\widetilde{v_1 \cdot 1}) \cdot h_1^2$ lies in highest possible filtration we deduce the second differential from the corresponding differential in the algebraic setting; however, the third differential might be false. Mapping into the localised SS gives the following differentials except, perhaps, the third.

$$d_3 v_1^2 = 1, \quad d_3 v_1^3 = v_1, \quad d_3 v_1^3 h_{2,1} = v_1 h_{2,1}, \quad d_3 v_1^4 h_{2,1} = v_1^2 h_{2,1}.$$

We know $v_1 h_{2,1}$ is the target of a d_3 but perhaps $d_3 0 = v_1 h_{2,1}$. We need to discount

$$d_1 v_1^2 h_{2,1} = v_1 h_{2,1}, \quad d_2 v_1^5 = v_1 h_{2,1}.$$

Well, $d_3 v_1^4 h_{2,1} = v_1^2 h_{2,1} \implies d_1 v_1^2 h_{2,1} = 0$ and $d_3 v_1^3 = v_1 \implies d_2 v_1 = 0 \implies d_2 v_1^5 = 0$.

So, in fact, all the differentials above in the localised SS are true. These differentials are nontrivial, since after multiplication by powers of v_1^4 the sources and targets run through a basis of $\pi_*(v_1^{-1} S/2 \wedge S/\eta)$. We conclude that the associated graded of $\pi_*(A^{-1}S/2)$ with the 2-adic filtration is:

$$\mathbb{F}_2[A, A^{-1}] \otimes \mathbb{F}_2[\eta]/(\eta^3) \otimes \mathbb{F}_2\langle 1, v_1, v_1 h_{2,1}, v_1^2 h_{2,1} \rangle.$$

We'd like to recover the fact that $2v_1 = \eta^2$ and $2v_1^2 h_{2,1} = v_1 h_{2,1}$. One checks directly that

$$(\widetilde{v_1 \cdot 1}) \cdot h_0 = 1 \cdot h_1^2 : \ \Big[[\xi_2|1] + [\xi_1^2|\xi_1] \Big] \cdot [\xi_1|1] = [\xi_2|\xi_1|1] + [\xi_1^2|\xi_1|\xi_1] + [\xi_1^2|\xi_1^2|1] = d[\xi_2|\xi_1] + [\xi_1^2|\xi_1^2|1].$$

Similarly, we can check the other relation. Thus,

$$\pi_*(A^{-1}S/2) = \frac{\mathbb{Z}/4[A, A^{-1}] \otimes \mathbb{Z}/4[\eta]/(\eta^3) \otimes \mathbb{Z}/4\langle 1, v_1, v_1 h_{2,1}, v_1^2 h_{2,1} \rangle}{(2 \cdot 1 = 0,\ 2v_1 = \eta^2,\ 2v_1 h_{2,1} = 0,\ 2v_1^2 h_{2,1} = v_1 \eta^2 h_{2,1},\ 2\eta = 0)}.$$

Bibliography

[1] J. F. Adams, *The Kahn-Priddy theorem.*

[2] J. F. Adams, *On the groups $J(X)$-IV.*

[3] M. Behrens, M. Hill, M.J. Hopkins, M. Mahowald, *On the existence of a v_2^{32}-self map on $M(1,4)$ at the prime* 2.

[4] J. M. Boardman, *Conditionally Convergent Spectral Sequences.*

[5] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ Ring Spectra and their Applications.*

[6] M. C. Crabb and K. Knapp, *Adams periodicity in stable homotopy.*

[7] D. M. Davis and M. Mahowald, *The image of the stable J-homomorphism.*

[8] D. M. Davis, M. Mahowald and H. Miller, *Mapping telescopes and K_*-localization.*

[9] A. D. Elmendorf, I. Kriz, M. A. Mandell and J. P. May, *Rings, modules, and algebras in stable homotopy theory.*

[10] M. J. Hopkins, J. H. Smith, *Nilpotence and Stable Homotopy Theory II.*

[11] M. Mahowald, *The order of the image of the J-homomorphism.*

[12] M. Mahowald, D. C. Ravenel and P. Shick, *The triple loop space approach to the telescope conjecture.*

[13] M. Mahowald and H. Sadofsky, *v_n telescopes and the Adams spectral sequence.*

[14] J. P. May, *A general algebraic approach to Steenrod operations.*

[15] H. R. Miller, *A localization theorem in homological algebra.*

[16] H. R. Miller, *Ext above the b_{10}^n line*, `http://www-math.mit.edu/~hrm/papers/loc-ass.pdf`

[17] H. R. Miller, *On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space.*

[18] H. R. Miller and V. P. Snaith, *On the K-theory of the Kahn-Priddy map.*

[19] C. Nassau, `http://www.nullhomotopie.de/charts/z3ext0.pdf`

[20] D. C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres.*

[21] D. C. Ravenel, *Localization with Respect to Certain Periodic Homology Theories.*

[22] D. C. Ravenel, *The non-existence of odd primary Arf invariant elements in stable homotopy.*

[23] D. C. Ravenel, *The Segal Conjecture for Cyclic Groups and its Consequences.*

[24] E. Riehl, *Categorical Homotopy Theory.*