Math 31B: Sequences and Series

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1 Sequences

1.1 What is one?

A sequence is a list which goes on forever. Here's an example.

$$31,\ 30,\ 31,\ 31,\ 28,\ 31,\ 30,\ 31,\ 30,\ 31,\ 31,\ 30,\ 31,\ 30,\ 31,\ \ldots$$

This sequence lists the number of days in each month starting in October 2017. There are some things we can demonstrate with this sequence.

- There's not a particular nice formula for this sequence and that doesn't matter.
- We often write a_n for the n-th term of a sequence. In this case,

  $$a_1 = 31,\ a_2 = 30,\ a_3 = 31,\ a_4 = 31,\ a_5 = 28,\ \ldots.$$

- We often write (a_n) or $(a_n)_{n=1}^{\infty}$ for a sequence, so in this case $(a_n)_{n=1}^{\infty}$ stands for

  $$31,\ 30,\ 31,\ 31,\ 28,\ 31,\ 30,\ 31,\ 30,\ 31,\ 31,\ 30,\ 31,\ 30,\ 31,\ \ldots$$

Here are some other examples of sequences:

- $1,\ 2,\ 3,\ 4,\ 5,\ \ldots$
- $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots$
- $2,\ 4,\ 8,\ 16,\ 32,\ \ldots$
- $\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots$

The above sequences have nice formulas for their n-th term. We have

$$a_n = n,\ a_n = \frac{1}{n},\ a_n = 2^n,\ a_n = \frac{1}{2^n},$$

respectively.

1.2 What does convergence mean?

If the sequence has a nice formula for its n-th term then one way you can figure out its limit (if it exists) is by typing in the formula into a calculator and then plugging in a massive positive integer for n.

In the examples above we have the following.

1. Plugging in a massive postive integer into $a_n = n$ gives back the same huge positive integer. The sequence *diverges to* ∞.

2. Plugging in a big enough postive integer into the formula $a_n = \frac{1}{n}$ will force a rubbish calculator to return 0. The sequence *converges to* 0.

3. Plugging in a massive positive integer into $a_n = 2^n$ will return an even bigger huge positive integer. The sequence *diverges to* ∞.

4. Plugging in a big enough positive integer into the formula $a_n = \frac{1}{2^n}$ will force a rubbish calculator to return 0. The sequence *converges to* 0.

There is a formal definition of what it means for a sequence (a_n) to *converge* to a number L. We can visualize a sequence $(a_n)_{n=1}^{\infty}$ on a graph by putting a dot at the point (n, a_n) for $n = 1, 2, 3, \ldots$ Without using mathematical symbols, the definition says, "if some annoying person (Cauchy) puts their arms either side of the line $y = L$, then you can specify how far off to the right someone else (Weierstrass) has to walk until all the subsequent points of the sequence lie between Cauchy's arms." When this definition is satisfied we write

$$\lim_{n\to\infty} a_n = L.$$

This definition (due to Monsieur Cauchy) is clever: although we say "a_n tends to L as n tends to ∞," the formal definition does not depend on any hand-waving about ∞. This is good because ∞ is not a real number!

Writing the definition just mentioned out in symbols and learning how to use it is the best way to understand the convergence of sequences. However, many students (including myself, 12 years ago) take a long to get to grips with the formal definition. There is not much time, and so I will not expect you to come to terms with the formal definition, but you might still find it useful to think about.

Similarly, there is formal definition of what it means for a sequence (a_n) to *diverge to* ∞. When this definition is satisfied we write

$$\lim_{n\to\infty} a_n = \infty.$$

Even more similarly, we can make sense of $\lim_{n\to\infty} a_n = -\infty$, too.

If (a_n) is a sequence and none of the above conditions hold, we say (a_n) *diverges* and that $\lim_{n\to\infty} a_n$ *does not exist.*

1.3 The function case

Something you may be more familiar with is the limit of a *function* $f(x)$ as x goes to ∞, $\lim_{x\to\infty} f(x)$. This can help you!

Sequences via functions. Suppose $a_n = f(n)$ for some function $f(x)$ and that $\lim_{x\to\infty} f(x) = L$. Then $\lim_{n\to\infty} a_n = L$.

The point of this theorem is that a sequence only has values for each positive integer: it is a list. A function takes on even more values: it can make sense at $\sqrt{2}$, e, π, $\frac{10^{10}}{3}$. This means that the condition $\lim_{x\to\infty} f(x) = L$ is a stronger one than $\lim_{n\to\infty} f(n) = L$. However, once we have a function, methods of calculus (e.g. L'Hôpital's rule) might be applicable, whereas, before they were not.

For example, if you want to calculate $\lim_{n\to\infty}(1+\frac{1}{n})^n$, then it is enough to calculate $\lim_{x\to\infty}(1+\frac{1}{x})^x$. We have seen, using L'Hôpital's rule, that $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$, and so $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

1.4 Your friends

When we differentiate, we rarely have to go near the definition of the derivative. When we differentiate, our friends are x^n, $\cos x$, $\sin x$, e^x, $\ln x$, $\arcsin x$, and $\arctan x$. Once we know how to differentiate our friends, and know some rules about differentiation, we can differentiate almost any function we want to. It's like all our friends showed up at some product rule, quotient rule, chain rule party, got on really swell, and had a load of babies - isn't $e^x \cos(2x)$ cute?! Now they're our friends too.

The same is true for sequences. We remember the limits of our sequence friends, and most other limits will follow from some rules about convergent sequences. Here are your two best sequence friends.

1. The sequence with n-th term $a_n = \frac{1}{n}$ converges to 0. That is,
$$\lim_{n\to\infty} \frac{1}{n} = 0.$$
2. If r is a number with $-1 < r < 1$, then the sequence with n-th term $a_n = r^n$ converges to 0. That is,
$$\lim_{n\to\infty} r^n = 0.$$
If $|r| > 1$ then the sequence with n-th term $a_n = r^n$ diverges.

1.5 Rules for sequences

Here are the rules your sequence friends use to make babies.

Suppose (a_n) and (b_n) are covergent sequences, that (c_n) is a divergent sequence, that k is a real number, and $f(x)$ is a continuous function defined at all a_n and $\lim_{n\to\infty} a_n$.

1. $\lim_{n\to\infty} k = k$;
2. $\lim_{n\to\infty}(ka_n) = k \cdot (\lim_{n\to\infty} a_n)$;
3. $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$;
4. $(a_n + c_n)$ diverges;
5. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$;
6. $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, as long as $\lim_{n\to\infty} b_n \neq 0$;
7. $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.

As an example, we can use the rules to verify that
$$\lim_{n\to\infty} \frac{\sqrt{4n^2 + 2n + 1}}{\sqrt{9n^2 + 3n + 227}} = \frac{2}{3}.$$
First, we note that
$$\frac{\sqrt{4n^2 + 2n + 1}}{\sqrt{9n^2 + 3n + 227}} = \frac{\sqrt{4 + 2 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n}}}{\sqrt{9 + 3 \cdot \frac{1}{n} + 227 \cdot \frac{1}{n} \cdot \frac{1}{n}}}$$

Next, we calculate

$$\begin{aligned}
\lim_{n\to\infty}\left[9+3\cdot\frac{1}{n}+227\cdot\frac{1}{n}\cdot\frac{1}{n}\right] &= \lim_{n\to\infty} 9+\lim_{n\to\infty}\left[3\cdot\frac{1}{n}\right]+\lim_{n\to\infty}\left[227\cdot\frac{1}{n}\cdot\frac{1}{n}\right]\\
&= \lim_{n\to\infty} 9+3\left[\lim_{n\to\infty}\frac{1}{n}\right]+227\lim_{n\to\infty}\left[\frac{1}{n}\cdot\frac{1}{n}\right]\\
&= \lim_{n\to\infty} 9+3\left[\lim_{n\to\infty}\frac{1}{n}\right]+227\left[\lim_{n\to\infty}\frac{1}{n}\right]\left[\lim_{n\to\infty}\frac{1}{n}\right]\\
&= 9+3\cdot 0+227\cdot 0\cdot 0 = 9.
\end{aligned}$$

The first equality uses 3; the second uses 2; the third uses 5; the final equality uses 1 and the fact that $\lim_{n\to\infty}\frac{1}{n}=0$.

Since $\sqrt{x}$ is continuous, 7 tells us that

$$\lim_{n\to\infty}\sqrt{9+3\cdot\frac{1}{n}+227\cdot\frac{1}{n}\cdot\frac{1}{n}} = \sqrt{\lim_{n\to\infty}\left[9+3\cdot\frac{1}{n}+227\cdot\frac{1}{n}\cdot\frac{1}{n}\right]} = \sqrt{9} = 3.$$

Similarly, we can verify that $\lim_{n\to\infty}\sqrt{4+2\cdot\frac{1}{n}+\frac{1}{n}\cdot\frac{1}{n}} = 2$. Finally,

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sqrt{4n^2+2n+1}}{\sqrt{9n^2+3n+227}} &= \lim_{n\to\infty}\frac{\sqrt{4+2\cdot\frac{1}{n}+\frac{1}{n}\cdot\frac{1}{n}}}{\sqrt{9+3\cdot\frac{1}{n}+227\cdot\frac{1}{n}\cdot\frac{1}{n}}}\\
&= \frac{\lim_{n\to\infty}\sqrt{4+2\cdot\frac{1}{n}+\frac{1}{n}\cdot\frac{1}{n}}}{\lim_{n\to\infty}\sqrt{9+3\cdot\frac{1}{n}+227\cdot\frac{1}{n}\cdot\frac{1}{n}}} = \frac{2}{3}.
\end{aligned}$$

The middle equality follows from 6, which is okay to use because $3 \neq 0$.

I would never expect you to do this in so much detail on the exam, but I do think it is beneficial for you to see where everything is coming from. The point is that all we used was knowledge of our friend $(a_n) = (\frac{1}{n})$. Everything else followed from the rules.

Even if you're not amazing at saying exactly what rules you're using, you MUST be able to see that

$$\lim_{n\to\infty}\frac{\sqrt{4n^2+2n+1}}{\sqrt{9n^2+3n+227}} = \frac{2}{3},$$

and I think the best way of doing this is writing

$$\frac{\sqrt{4n^2+2n+1}}{\sqrt{9n^2+3n+227}} = \frac{\sqrt{4+2\cdot\frac{1}{n}+\frac{1}{n}\cdot\frac{1}{n}}}{\sqrt{9+3\cdot\frac{1}{n}+227\cdot\frac{1}{n}\cdot\frac{1}{n}}}.$$

In calculating such a limit, this is the standard technique to show that the highest degree terms in the numerator and denominator are all that matter.

1.6 Ignoring terms at the beginning of a sequence

Since calculating a limit requires understanding what happens late on in the sequence, the limit won't change if we delete or change some terms at the beginning. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, the sequences

$$20,\ 20,\ 20,\ 20,\ 20,\ 20,\ 20,\ \frac{1}{8},\ \frac{1}{9},\ \frac{1}{10},\ \frac{1}{11},\ \frac{1}{12},\ \frac{1}{13},\ \frac{1}{14},\ \ldots$$
$$\frac{1}{100},\ \frac{1}{101},\ \frac{1}{102},\ \frac{1}{103},\ \frac{1}{104},\ \frac{1}{105},\ \frac{1}{106},\ \frac{1}{107},\ \frac{1}{108},\ \frac{1}{109},\ \frac{1}{110},\ \ldots$$

also converge to 0.

1.7 The squeeze theorem

The squeeze theorem is a useful result for calculating limits. It is a gateway theorem before we get hooked on the tests for the convergence and divergence of series because the type of thinking used to apply such theorems is similar.

Squeeze theorem. Suppose (LOWER_n), $(\mathrm{SQUEEZED}_n)$ and (UPPER_n) are sequences with

$$\mathrm{LOWER}_n \le \mathrm{SQUEEZED}_n \le \mathrm{UPPER}_n$$

for each n. If $\lim_{n\to\infty} \mathrm{LOWER}_n = L = \lim_{n\to\infty} \mathrm{UPPER}_n$, then

$$\lim_{n\to\infty} \mathrm{SQUEEZED}_n = L.$$

Example. $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ since $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$, and $\lim_{n\to\infty} \pm\frac{1}{n} = 0$; we can take

$$\mathrm{LOWER}_n = -\frac{1}{n},\ \mathrm{SQUEEZED}_n = \frac{\sin n}{n},\ \text{and } \mathrm{UPPER}_n = \frac{1}{n}$$

in the squeeze theorem.

Example. $\lim_{n\to\infty} \frac{8^n}{n!} = 0$ because $0 \le \frac{8^n}{n!} \le \frac{8^8}{8!} \cdot (\frac{8}{9})^{n-8}$; you could also use the inequality $0 \le \frac{8^n}{n!} \le \frac{8^8}{8!} \cdot \frac{8}{n}$.

We'll expand on the second example after the discussion.

HELP! Students always struggle with finding the lower and upper sequences in the squeeze theorem. Here are some pointers to help you.

The first thing to note is the purpose of the squeeze theorem. It is used to formalize intuition you have about why a sequence converges to a limit. In the previous example, $(\frac{8^n}{n!})$, I'd have said, "I think that $n!$ grows faster than exponents, so my guess is 0." This is the first part of using the squeeze theorem.

- Make a sensible guess about what the sequence in question converges to. For the rest of the discussion let's call that guess "L."

As soon as you have made a sensible guess, L, for the limit of the sequence in question, this imposes conditions on what your lower and upper sequences can be. If the following bullet points are not fulfilled then you have gone badly wrong.

- You'd better be able to calculate what (LOWER_n) and (UPPER_n) converge to easily. For this to be true, they need to be "friends" or at least things closely related to friends. For example, the limits of $(\frac{100}{n})$, $((\frac{5}{6})^n)$, $(1 + \frac{1}{n})$, (0) are 0, 0, 1, and 0, respectively.

- In light of the previous bullet point, (LOWER_n) and (UPPER_n) should probably look a little dissimilar to (SQUEEZED_n): if (SQUEEZED_n) looks like a friend, then you don't need the squeeze theorem; if it doesn't look like a friend but (UPPER_n) looks similar to it, then the limit of (UPPER_n) is too difficult to calculate.

- (LOWER_n) and (UPPER_n) must have the SAME limit, and that limit better be your guess L; otherwise, you're not squeezing! For example, suppose $\lim_{n\to\infty} \text{LOWER}_n = 0$ and $\lim_{n\to\infty} \text{UPPER}_n = 1$. If we take 0 to be BROAD2160E and 1 to be the hill, this is like asking "can your friends both be huggging you if one is in my lecture, and the other is up the hill?" They'd need bloody long arms!

With all of these points in mind, the most difficult part is the following.

- Make sure that the inequality

  $$\text{LOWER}_n \leq \text{SQUEEZED}_n \leq \text{UPPER}_n$$

  holds for each n.

Let's return to the example $(\frac{8^n}{n!})$. Going through the bullet points...

- This sequence should converge to 0 since I think $n!$ grows faster than exponents.
- (LOWER_n) and (UPPER_n) must be "friends."
- (LOWER_n) and (UPPER_n) should not involve complicated things like $n!$, (though fixed numbers expressed in factorials are okay).
- We need $\lim_{n\to\infty} \text{LOWER}_n = 0 = \lim_{n\to\infty} \text{UPPER}_n$.
- We need $\text{LOWER}_n \le \text{SQUEEZED}_n \le \text{UPPER}_n$.

It is hopefully clear that $\text{LOWER}_n = 0$ is a fine choice. (0) has the easiest of all limits to calculate, it is 0, and it is trivially true that $0 \le \text{SQUEEZED}_n$ because the sequence under consideration consists of positive terms.

To figure out (UPPER_n) requires more thought about the sequence under consideration. Imagine that you bet your mother's house on the fact that this sequence converges and you start writing down the terms in the sequence one after another.

$$\frac{8}{1}, \quad \frac{8}{1}\cdot\frac{8}{2}, \quad \frac{8}{1}\cdot\frac{8}{2}\cdot\frac{8}{3}, \quad \frac{8}{1}\cdot\frac{8}{2}\cdot\frac{8}{3}\cdot\frac{8}{4}, \quad \frac{8}{1}\cdot\frac{8}{2}\cdot\frac{8}{3}\cdot\frac{8}{4}\cdot\frac{8}{5}, \quad \cdots$$

OH NOOOOOO! We keep multiplying by a number bigger than 1. This is a disaster. What am I gonna tell her? Why wasn't I better at math? Why'd a make a stupid bet? "Never bet," she told me, "unless it's selling all your pounds before the EU referendum." Fingers crossed. The 9-th term...

$$\left[\frac{8}{1}\cdot\frac{8}{2}\cdot\frac{8}{3}\cdot\frac{8}{4}\cdot\frac{8}{5}\cdot\frac{8}{6}\cdot\frac{8}{7}\cdot\frac{8}{8}\right]\cdot\frac{8}{9}.$$

HOLY CRAP! We just multiplied by a number less than 1 to get from the 8-th term to the 9-th term! Maybe it's gonna be alright? Maybe I should go to Vegas after class?! Buy her a condo on Wilshire Blvd?

$$\frac{8^n}{n!} = \left[\frac{8}{1}\cdot\frac{8}{2}\cdot\frac{8}{3}\cdot\frac{8}{4}\cdot\frac{8}{5}\cdot\frac{8}{6}\cdot\frac{8}{7}\cdot\frac{8}{8}\right]\cdot\left[\frac{8}{9}\cdot\frac{8}{10}\cdot\frac{8}{11}\cdots\frac{8}{n-1}\cdot\frac{8}{n}\right]$$

OH WOW! It just keeps getting better! I keep multiplying by numbers less than 1, in fact, numbers less than or equal to $\frac{8}{9}$. This is great! Maybe I'll make sure my first child is born on August 9th? Oof. I gotta calm myself. That all got a little much.

So when I write out $\frac{8^n}{n!}$, there are eight crappy terms at the beginning, which almost gave me a heart attack, and then, the rest are lovely and less than or equal to $\frac{8}{9}$. If there are n terms in total, and eight crappy terms, then there are $n-8$ lovely terms. We just said:

$$\begin{aligned}\frac{8^n}{n!} &= \left[\frac{8}{1}\cdot\frac{8}{2}\cdots\frac{8}{7}\cdot\frac{8}{8}\right]\cdot\left[\frac{8}{9}\cdot\frac{8}{10}\cdot\frac{8}{11}\cdots\frac{8}{n-1}\cdot\frac{8}{n}\right] \\ &= \frac{8^8}{8!}\cdot\left[\frac{8}{9}\cdot\frac{8}{10}\cdot\frac{8}{11}\cdots\frac{8}{n-1}\cdot\frac{8}{n}\right] \leq \frac{8^8}{8!}\cdot\left[\frac{8}{9}\right]^{n-8}\end{aligned}$$

Can we take $\mathrm{UPPER}_n = \frac{8^8}{8!}\cdot(\frac{8}{9})^{n-8}$? Well, it is a friend: we can see this by writing it as

$$\frac{\frac{8^8}{8!}}{(\frac{8}{9})^8}\cdot\left[\frac{8}{9}\right]^n.$$

It doesn't involved any $n!$; sure, there's an 8!, but this is not depending on n, it's a FIXED number. Since $\frac{8}{9} < 1$, the sequence converges to 0. We just proved the requisite inequality. DONE!

If we'd have enjoyed the thrill of betting and not celebrated $\frac{8}{9}$ so much, we might have let $\mathrm{UPPER}_n = \frac{8^8}{8!}\cdot\frac{8}{n}$. This works since

$$\frac{8^n}{n!} = \left[\frac{8}{1}\cdot\frac{8}{2}\cdots\frac{8}{7}\cdot\frac{8}{8}\right]\cdot\left[\frac{8}{9}\cdot\frac{8}{10}\cdot\frac{8}{11}\cdots\frac{8}{n-1}\right]\cdot\frac{8}{n} \leq \frac{8^8}{8!}\cdot 1\cdot\frac{8}{n}.$$

1.8 Bounded, monotone sequences (non-examinable)

Here's a couple of useful results for guaranteeing that a sequence converges.

The driving-a-car-at-a-wall theorem. Suppose (a_n) is a sequence and that there is an M such that $a_n \leq a_{n+1} \leq M$ for all n. Then (a_n) converges to some L which is less than or equal to M. That is, "a sequence, which is increasing and bounded above, converges."

The reversing-a-car-at-a-wall theorem. Suppose (a_n) is a sequence and that there is an m such that $m \leq a_{n+1} \leq a_n$ for all n. Then (a_n) converges to some L which is bigger than or equal to m. That is, "a sequence, which is decreasing and bounded below, converges."

If one cares about the development of the real numbers, then these results are actually incredibly important. We wouldn't know e existed without such theorems. In Math 31B, we just need to know these results are out there.

1.9 Recursively defined sequences (non-examinable)

Let $a_1 = 1$ and for $n > 1$, $a_n = \sqrt{2a_{n-1}}$. This means

$$a_1 = 1,\ a_2 = \sqrt{2},\ a_3 = \sqrt{2\sqrt{2}},\ a_4 = \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots$$

To get a feel for this you could press "1," then "=" on your calculator, then enter "$\sqrt{2 \cdot \mathrm{ANS}}$," and press "=" over and over again. What does your answer end up at? It should be 2.

To prove this mathematically there are two steps.

1. Pretend there is an answer, and figure out what it must be using limit laws.

2. Demonstrate there *is* an answer.

You've actually done this sort of thing before. I know you'd tell me that the solutions to $ax^2 + bx + c = 0$ are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. To write this formula, you assumed there were solutions. After that, you must consider whether they make sense: if $b^2 - 4ac < 0$, then there are no real roots, and it turned out the calcuation was a bit illegal (unless you know complex numbers).

Here's the "pretend there is an answer, and figure out what it must be using limit laws" part. This is the most important for exam purposes.

Suppose that (a_n) converges to some number L. Since $\lim_{n\to\infty} a_n = L$, we have $\lim_{n\to\infty} a_{n+1} = L$; this is because the sequence (b_n) with $b_n = a_{n+1}$ is the same sequence with the first term missing. Since $a_{n+1} = \sqrt{2a_n}$, taking limits gives

$$L = \sqrt{2L},\ \text{so that}\ L^2 = 2L,\ \text{and}\ L(L-2) = 0.$$

We can see that $a_n \geq 1$ for all n and so L cannot be zero. Thus, $L = 2$, i.e. $\lim_{n\to\infty} a_n = 2$.

To verify that such an L actual exists, we use the driving-a-car-at-a-wall theorem. We will show that $a_n \leq a_{n+1} \leq 2$ for all n. Driving-a-car-at-a-wall will then say that (a_n) converges to some $L \leq 2$. It is important that you know that you can use such a theorem. But I will never ask you to show that a sequence is increasing and bounded above; I would always tell you such information.

$a_1 = 1 \leq \sqrt{2} = a_2$ and, if $n > 1$ and $a_{n-1} \leq a_n$, then

$$a_n = \sqrt{2a_{n-1}} \leq \sqrt{2a_n} = a_{n+1}.$$

Also, $a_1 \leq 2$ and, if $n > 1$ and $a_{n-1} \leq 2$, then

$$a_n = \sqrt{2a_{n-1}} \leq \sqrt{2 \cdot 2} = 2.$$

From this (and induction) we learn that $a_n \leq a_{n+1} \leq 2$ for all n. The keen student would ask how I guessed 2 was an upper bound. The way I have presented the problem answers that question: I knew what the sequence had to converge to before I even showed it converges.

2 Series

2.1 What is one?

A series is an infinite sum. The difference with a sequence is that the commas are replaced by addition signs. For example,

$$1 + 2 + 3 + 4 + 5 + \dots$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

$$2 + 4 + 8 + 16 + 32, \ \dots$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

2.2 Σ-notation

If p, q are integers, $p \geq q$, and $a_p, a_{p+1}, \dots, a_{q-1}, a_q$ are real numbers, we write $\sum_{n=p}^{q} a_n$ for

$$a_p + a_{p+1} + \dots + a_{q-1} + a_q.$$

Here, n is the "indexing number." You can think of this just as you think of the x in $\int_a^b f(x)\, dx$. p and q tell you where to start and end the summation. You can think of these like the limits in a definite integral. Just like with integration, and anything in math, we can use different letters. Sometimes it is useful to index by k instead. I will try to stick with n unless I have to use another letter.

We can use Σ-notation for series too. For the above series, we write

$$\sum_{n=1}^{\infty} n, \quad \sum_{n=1}^{\infty} \frac{1}{n}, \quad \sum_{n=1}^{\infty} 2^n, \quad \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

2.3 What does this mean? What does convergence mean?

A common strategy in math is to make sense of something new using something old; we're too unimaginative/lazy to come up with new ideas (*jk!*). Since we got so good at sequences it'd be great if we could reduce a series to a sequence. We can.

We'd like to make sense of

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

Let

$$s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \dots + a_N.$$

Then $(s_N)_{N=1}^{\infty}$ is called the *sequence of partial sums*, and we can talk about its convergence or divergence. This is *exactly the same* as talking about the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$. In one equation:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} s_N.$$

Later on, you will see the similarity to improper integrals. In that case, we will make sense of an improper integral as a limit of proper integrals. Now, we're making sense of an infinite sum as a limit of finite sums. Considering the series $\sum_{n=1}^{\infty} a_n$ is *exactly the same* as considering the sequence

$$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ a_1 + a_2 + a_3 + a_4,\ \dots$$

As an example of the content of the previous discussion, let's turn to our old sequence friend, the one with $a_n = \frac{1}{n}$. We love this sequence and will never forget how grateful we are that it converges to 0.

Its cousin series, $\sum_{n=1}^{\infty} \frac{1}{n}$, has a different personality. We love it too, but for different reasons. This series is so special that it has a name: *the harmonic series*. Let's show it diverges.

$$\begin{aligned}
s_1 &= 1 = \frac{2}{2} \\
s_2 &= s_1 + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2} \\
s_4 &= s_2 + \left(\frac{1}{3} + \frac{1}{4}\right) \geq \frac{3}{2} + \frac{1}{2} = \frac{4}{2} \\
s_8 &= s_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \geq \frac{4}{2} + \frac{1}{2} = \frac{5}{2} \\
s_{16} &= s_8 + \left(\frac{1}{9} + \frac{1}{10} + \dots + \frac{1}{16}\right) \geq \frac{5}{2} + \frac{1}{2} = \frac{6}{2} \\
s_{32} &= s_{16} + \left(\frac{1}{17} + \frac{1}{18} + \dots + \frac{1}{32}\right) \geq \frac{6}{2} + \frac{1}{2} = \frac{7}{2}
\end{aligned}$$

We find that $s_{2^n} \geq \frac{n+2}{2}$ so that the sequence $(s_n)_{n=1}^{\infty}$ diverges. By definition, this means the harmonic series diverges.

This is important. Most people do not expect this result. It is tempting to think that the series will converge since the terms get smaller; far along in the summation, you are adding on a very small amount. The calculation above shows, that although the terms start becoming very small, by grouping them together cleverly, in sums which are a power of 2 long, they are still significant.

If you are happier with the improper integral stuff, think about the fact that $\int_1^\infty \frac{1}{x}\,dx$ diverges. It is the same deal: the curve gets real close to the x-axis, the area available under it is less and less, but there's still enough that there is an infinite area overall, since $\lim_{S\to\infty} \ln S = \infty$.

2.4 Your friends

The series we know and love are the p-series and the geometric series.

By a *p-series* we mean one of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

This **converges if** $p > 1$ **and diverges if** $p \leq 1$**.** We can check this by using the integral test which we'll talk about later. When a p-series converges, we rarely know what to (see philosphical discussion later). Ramanujan would have known by now.

By a *geometric series* we mean one of the form (notice the sum starts at 0)

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \dots$$

The point is that to get from one term to the next you multiply by r, so to spot one you see if the ratio between successive terms is a constant. Suppose $c \neq 0$, since otherwise it is not very interesting. The series **converges if** $-1 < r < 1$ **and diverges if** $|r| \geq 1$**.** When it converges, it converges to

$$\frac{c}{1-r}.$$

[This is because, for $r \neq 1$,

$$s_N = c + cr + \dots + cr^N = c(1 + r + \dots + r^N) = \frac{c(1-r^{N+1})}{1-r}.]$$

2.5 Rules for series

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, $\sum_{n=1}^{\infty} c_n$ is a divergent series, and that k is a real number.

1. $\sum_{n=1}^{\infty} ka_n$ converges to $k\sum_{n=1}^{\infty} a_n$; if $k \neq 0$, $\sum_{n=1}^{\infty} kc_n$ diverges.
2. $\sum_{n=1}^{\infty}(a_n + b_n)$ converges to $\left[\sum_{n=1}^{\infty} a_n\right] + \left[\sum_{n=1}^{\infty} b_n\right]$.
3. $\sum_{n=1}^{\infty}(a_n + c_n)$ diverges.

Notice, that there are far fewer rules. For instance, there is not one for $\sum_{n=1}^{\infty} a_n b_n$. This is because, even in the finite case, it is not always true that $(a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_2 b_2$; only a Freshman dream would allow such a thing.

An example of these rules (and recognizing geometric series) is as follows.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{8 + 2^{n+5}}{5^n} &= \sum_{n=1}^{\infty} \left[\frac{8}{5^n} + \frac{2^{n+5}}{5^n}\right] \\
&= \left[\sum_{n=1}^{\infty} \frac{8}{5^n}\right] + \left[\sum_{n=1}^{\infty} \frac{2^{n+5}}{5^n}\right] \\
&= \left[8\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n\right] + \left[2^5 \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n\right] = \frac{8 \cdot \frac{1}{5}}{1 - \frac{1}{5}} + \frac{2^5 \cdot (\frac{2}{5})}{1 - \frac{2}{5}}
\end{aligned}$$

$\sum_{n=1}^{\infty} \frac{2}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converge (by the p-test) and the rules say that

$$\sum_{n=1}^{\infty} \left[\frac{2}{n^2} + \frac{1}{n^3}\right] = \sum_{n=1}^{\infty} \frac{2}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

We can also use the rules to show $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3}$ diverges. First, notice

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n^3} = \sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^3} = \sum_{n=1}^{\infty} \left[\frac{1}{n} + \left(\frac{2}{n^2} + \frac{1}{n^3}\right)\right].$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \left[\frac{2}{n^2} + \frac{1}{n^3}\right]$ converges part 3. applies.

It is true that $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ and $\sum_{n=1}^{\infty} -\frac{1}{2n+1}$ diverge (later on, you will be able to prove this using the limit comparison test). However, it is NOT true that $\sum_{n=1}^{\infty} \left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$ diverges; it converges to 1 (see the homework). The rules say *nothing* about this situation and they cannot because of the following silly series: $\sum_{n=1}^{\infty}(1 + 1)$ and $\sum_{n=1}^{\infty}(1 - 1)$.

3 A philosophical discussion

This section is completely non-examinable but some of the things mentioned are important to understand. Without thinking about some of these issues, I don't think you can really understand any of the questions I'm asking you about series!

3.1 Why are series more difficult than sequences?

We reduced the question of talking about the convergence/divergence of a series $\sum_{n=1}^{\infty} a_n$ to that of talking about the sequence of partial sums (s_N), that is, the sequence (s_N) with

$$s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \ldots + a_N.$$

So, you ask, aren't series just a special case of sequences? And, if so, why do we bother talking about them?

Well, yes, series are just a special type of series. But, here's the issue... A non-series-type sequence often has a nice and simple formula attached to it; in this case, it is often possible to calculate the limit of the sequence. On the other hand, when we use sums to define the terms of a sequence, our formula often becomes complicated so that it is difficult or even impossible to calculate what the limit might be.

For example, what do you think

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \ldots,$$

converge to?

A sensible guess for the first was made by Euler using an infinite *product*

$$\sin x = x \cdot \prod_{n=1}^{\infty} \left[1 - \frac{x^2}{n^2 \pi^2} \right].$$

I say "guess," but, at the time (Euler lived from 1707 to 1783), his argument was accepted as a proof. His answer was $\frac{\pi^2}{6}$ and this turned out to be correct. You can also let $x = \frac{\pi}{2}$ in his product formula to see that

$$\frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdot \frac{8 \cdot 8}{7 \cdot 9} \cdot \frac{10 \cdot 10}{9 \cdot 11} \cdot \frac{12 \cdot 12}{11 \cdot 13} \cdot \frac{14 \cdot 14}{13 \cdot 15} \cdots = \frac{\pi}{2}.$$

This is Wallis' product for π (1655). On wikipedia, there is a proof of this result using integration by part and the squeeze theorem: you've made it to 17th century mathematics!

The second number is called Apéry's constant. In 1978, it was proved that this number is irrational, meaning not a fraction. It took until then to know this!

3.2 Convergent things we cannot calculate

For both integral and series, we talk about convergence and divergence. In many instances, we will be able to show that an integral or series converges but we will not be able to calculate the value which it converges to.

Why on earth it would be useful to know an integral or series converges if we cannot calculate its value exactly? Here's a question: "do you even know what π is?" Answer: no, but you know arbitrarily good approximations to it. Once it is known that an integral or series converges, you can calculate approximations to it and these can be useful.

Here are two crazily important examples:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}}\,dt, \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

The integral defines the normal distribution with mean 0 and variance 1; normal distributions describe many things in real life. The integral is impossible to evaluate exactly. However, it is important to know it converges. Approximations to it are very useful and can be found in any statistics book. [We will see, later, that it converges by comparing with the integrand with function $e^{1-|t|}$.]

The series defines the cosine function. Mathematicians can prove all the familiar formulae involving cosine using this definition. The pitfall of the formula is that it doesn't allow us to know the values of cos exactly. But, this never phased you before; you were happy with your calculator giving you a decimal to 10 decimal places. Now, you can always use Taylor's Error Bound to calculate values of cosine as accurately as you need. [We will see, later, that the series defining cosine converges using the ratio test.]

In lecture, I will note that decimals only make sense because of series!!

My favorite application of infinite series is called Fourier series, which tells you about the amplitude of the harmonics in a periodic signal: part of the reason musical instruments sound different to one another.

Having said all this, I think I now feel less bad about telling you all the tests for convergence/divergence of integrals and series.

4 Tests for convergence/divergence of a series

4.1 The n-th term test

In the previous section, we remarked that most people do not expect for the harmonic series to diverge: the terms get smaller and smaller; we're adding on less and less. It is important to remember the harmonic series.

The terms getting small is not enough for a series to converge.

On the other hand, you would not expect

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + \dots$$

to converge. The terms do not get small; we are always adding on a significant amount; we cannot expect the series to converge. To see this even more explicitly, let's consider the sequence of partial sums.

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + 1 = 2 \\ s_3 &= 1 + 1 + 1 = 3 \\ s_4 &= 1 + 1 + 1 + 1 = 4 \\ s_5 &= 1 + 1 + 1 + 1 + 1 = 5 \end{aligned}$$

We have $s_n = n$, and so (s_n) is very definitely divergent.

The n-th term test. If $\lim_{n\to\infty} |a_n| \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

4.2 Ignoring terms at the beginning of a series when checking convergence or divergence

The convergence or divergence of a series is dictated by the behaviour of the later terms. Consider the following series.

$$0 + 0 + 0 + \dots + 0 + 0 + 0 + \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \frac{1}{1004} + \frac{1}{1005} + \dots$$
$$1000 + 1000 + \dots + 1000 + 1000 + \frac{1}{2^{100}} + \frac{1}{2^{101}} + \frac{1}{2^{102}} + \frac{1}{2^{103}} + \frac{1}{2^{104}} + \dots$$

The first is the harmonic series, with its first 1000 terms replaced by 0. This does not miraculously save the harmonic series from diverging. This is because (I'm very naughty for writing this down: infinity is *not* a number!)

$$\text{“}\infty - \sum_{n=1}^{1000} \frac{1}{n} = \infty - \text{some finite number} = \infty.\text{”}$$

The second is the geometric series, with $r = \frac{1}{2}$, with its first 100 terms replaced by 1000. This does not suddenly make it diverge. It just changes its value by the number

$$100,000 - \sum_{n=0}^{99} \frac{1}{2^n}.$$

When someone asks you about the convergence of a series, it is useful to know that you can ignore the first $1,000,000,000$ terms if it is convenient. For this reason, in later theorems, you only have to check hypothesis "for sufficently large n" even if it says "for all n."

Ignoring the first 100 values in a convergent sequence, **will change the value** which it converges to, however. So if you're asked for a value, don't randomly ignore the first 100 terms; you'd be silly to do so.

4.3 Integral test (inaccessible until after 8.7)

Normally, the last test I try is the integral test. However, this test is useful for proving when our friends, the p-series, converge, something we have not done yet. Also, there is a nice picture associated to the theorem. The picture demonstrates a comparison between two areas, and understanding this idea will help in understanding later ideas. I drew it in lectures.

The integral test. Suppose $\sum_{n=1}^{\infty} a_n$ is a series and that $a_n = f(n)$ where

1. $f(x)$ is defined for $x \geq 1$ and continuous;
2. $f(x) \geq 0$;
3. $f(x)$ is decreasing, that is, for $x \geq y$ we have $f(x) \leq f(y)$.

Then

1. $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges.
2. $\sum_{n=1}^{\infty} a_n$ diverges if and only if $\int_1^{\infty} f(x)\, dx$ diverges.

Example.

Let $f(x) = \frac{1}{x^p}$, where $p > 0$. Then $f(x)$ satisfies the hypothesis of the integral test. Thus,

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $\int_1^{\infty} \frac{1}{x^p}\, dx$ converges.
2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if and only if $\int_1^{\infty} \frac{1}{x^p}\, dx$ diverges.

This means $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$ (this includes $p \leq 0$, although we are not using the integral test for this).

Important. The integral test can be used to show a series converges, but it does not say much about what it converges to. For example,

$$\int_1^{\infty} \frac{1}{x^2}\, dx = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

If one is careful, and understands the proof of the integral test, all it says about $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is that $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\, dx = 1 + 1 = 2.$

Additional. In light of the remarks made in section 4.2, one can relax all the assumptions in the integral test, demanding only that they are true eventually. How does one express this mathematically? We mean that there is an $M \geq 0$ such that:

- $a_n = f(n)$ when $n \geq M$;
- $f(x)$ is defined and continuous for $x \geq M$;
- $f(x) \geq 0$ for $x \geq M$;
- $f(x)$ is decreasing when $x \geq M$.

In this case,

1. $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_M^{\infty} f(x)\, dx$ converges.
2. $\sum_{n=1}^{\infty} a_n$ diverges if and only if $\int_M^{\infty} f(x)\, dx$ diverges.

Honest thoughts. I like the integral test because it has an intuitive proof that highlights many of the ideas which are going on in this stuff. However, this course is not dedicated to proofs. For this reason, I feel the integral test is verging on useless for us. After using it to say when p-series converge, one almost never needs it; other tests are more practical. Did I waste your time by setting questions on it? No! Thankfully, the saving grace is that to use the integral test one has to practice improper integrals, u-subs with them, integration by part with them, and L'Hôpital's rule, so it is good for review. Also, it gets you to grips with checking hypotheses. In the integral test, the first few are normally very easy, but the decreasing condition might require a little work. Since I don't care much for the integral test, I won't grill you on this on the exam.

Example. Here's one example where I would choose the integral test over other tests... $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges. This is because

$$\begin{aligned}
\int_e^{\infty} \frac{1}{x \ln x}\, dx &= \lim_{S\to\infty} \int_e^{S} \frac{1}{x \ln x}\, dx \\
&= \lim_{S\to\infty} \int_1^{\ln S} \frac{1}{u}\, du \\
&= \lim_{S\to\infty} \Big[\ln |u|\Big]_1^{\ln S} \\
&= \lim_{S\to\infty} \Big[\ln(\ln S) - \ln(1)\Big] = \lim_{S\to\infty} \ln(\ln S) = \infty.
\end{aligned}$$

and, letting $f(x) = \frac{1}{x \ln x}$, we have

1. $f(x)$ is defined for $x \geq e$ and continuous;
2. $f(x) \geq 0$ for $x \geq e$;
3. $f'(x) = -\frac{1+\ln x}{(x \ln x)^2} \leq 0$ for $x \geq e$, so that $f(x)$ is decreasing for $x \geq e$.

   (You could also deduce this from the fact that x and $\ln x$ are increasing *and* positive for $x > 1$, but the derivative is a tool that will pretty much always work for you.)

Example. Here's one other example where I would choose the integral test over other tests... $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges. This is because

$$\begin{aligned}
\int_e^{\infty} \frac{1}{x(\ln x)^2}\, dx &= \lim_{S\to\infty} \int_e^{S} \frac{1}{x(\ln x)^2}\, dx \\
&= \lim_{S\to\infty} \int_1^{\ln S} \frac{1}{u^2}\, du \\
&= \lim_{S\to\infty} \Big[-\frac{1}{u}\Big]_1^{\ln S} \\
&= \lim_{S\to\infty} \Big[1 - \frac{1}{\ln S}\Big] = 1,
\end{aligned}$$

and, letting $f(x) = \frac{1}{x(\ln x)^2}$, we have

1. $f(x)$ is defined for $x \geq e$ and continuous;
2. $f(x) \geq 0$ for $x \geq e$;
3. $f'(x) = -\frac{2+\ln x}{x^2(\ln x)^3} \leq 0$ for $x \geq e$, so that $f(x)$ is decreasing for $x \geq e$.

4.4 Direct comparison test

The direct comparison test is my favorite test. It is the most theoretically useful by a long way; it is, arguably, the most practically useful. However, it is also the one that students find the most difficult.

The direct comparison test. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series and that $a_n \geq b_n \geq 0$ for all n.

1. If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} b_n$ converges.
2. If $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

It might be helpful to remember this more descriptive statement of the theorem.

Suppose $\sum_{n=1}^{\infty} \mathrm{BIG}_n$ and $\sum_{n=1}^{\infty} \mathrm{SMALL}_n$ are two series and that $\mathrm{BIG}_n \geq \mathrm{SMALL}_n \geq 0$ for all n.

1. If $\sum_{n=1}^{\infty} \mathrm{BIG}_n$ converges then $\sum_{n=1}^{\infty} \mathrm{SMALL}_n$ converges.
2. If $\sum_{n=1}^{\infty} \mathrm{SMALL}_n$ diverges then $\sum_{n=1}^{\infty} \mathrm{BIG}_n$ diverges.

The theorem says, if you are less than or equal to a finite number then you're finite; if you're bigger than or equal to infinity, then you're infinity.

The idea of the theorem is similar to the squeeze theorem. We compare things which are too complicated for us to argue about directly, with familiar things, our friends. For this reason, you MUST know your friends super well. We definitely don't want one of those embarassing moments when you come to introduce a friend and you can't even remember their name, their major, and think they like Taylor Swift when, in fact, they just love Meshuggah.

Example 1.

How do we use this theorem? Well, suppose we want to prove the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3n+2}}.$$

Where can we start? We go to our friends: the p-series and the geometric series. Does it look like either? It definitely looks more like a p-series, but it sort of looks like lots of them and that is confusing. A couple of Freshman dreams down the line, you might say

$$\frac{1}{\sqrt{n^3+3n+2}} = \frac{1}{\sqrt{n^3}} + \frac{1}{\sqrt{3n}} + \frac{1}{\sqrt{2}} \qquad \text{(NOOOOOO!!!!)}.$$

These are, indeed, Freshman dreams, and you should check that you **do not do this**. However, they do highlight what the difficulty is. We have a sum underneath a square root, at the bottom of a fraction. Square roots play badly with sums, and sums play badly with fractions if they are on the bottom. The only thing we can really do is to try and ignore terms or pair them up together in some way. Let's try ignoring them and see if we get anywhere.

Do we know about the convergence of any of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{3n}} = \frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} 1?$$

YES! They are all our friends; they are all p-series. The first converges since $\frac{3}{2} > 1$. The others diverge since $0, \frac{1}{2} \leq 1$.

Now we need to figure out what effect ignoring the various terms had. In each case, we forgot about a positive quantity, under the square root of the bottom of a fraction. In each case, this makes the bottom of the fraction smaller, and so the whole thing bigger. That is,

$$\frac{1}{\sqrt{n^3+3n+2}} \leq \frac{1}{\sqrt{n^3}},\ \frac{1}{\sqrt{3n}},\ \frac{1}{\sqrt{2}}.$$

We now see that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3n+2}}$ is smaller than a convergent thing, so we can apply the direct comparison test to say it converges.

Here's what you write. Let

$$\text{SMALL}_n = \frac{1}{\sqrt{n^3+3n+2}} \quad \text{and} \quad \text{BIG}_n = \frac{1}{\sqrt{n^3}}, \quad \text{so} \quad \text{BIG}_n \geq \text{SMALL}_n \geq 0.$$

Because $\frac{3}{2} > 1$, the p-series test tells us $\sum_{n=1}^{\infty} \text{BIG}_n$ converges. The direct comparison theorem tells us $\sum_{n=1}^{\infty} \text{SMALL}_n$ converges, i.e. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3n+2}}$ converges.

Example 2.

Suppose we want to prove the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4+6n+3}}.$$

Do we know about the convergence of any of

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{6n}} = \frac{1}{\sqrt[5]{6}} \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{3}} = \frac{1}{\sqrt[5]{3}} \sum_{n=1}^{\infty} 1?$$

YES! They are all our friends; they are all p-series. They all diverge since $\frac{4}{5}, \frac{1}{5}, 0 \leq 1$.

This suggests that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4+6n+3}}$ diverges. To use the comparison test we better have

$$\mathrm{BIG}_n = \frac{1}{\sqrt[5]{n^4+6n+3}}.$$

We need SMALL_n. To make $\frac{1}{\sqrt[5]{n^4+6n+3}}$ smaller we need to make the things under the square root bigger. We'd like it if they paired up nicely, so that no freshman dreams are required. Here's how.

$$\frac{1}{\sqrt[5]{n^4+6n+3}} \geq \frac{1}{\sqrt[5]{n^4+6n^4+3n^4}} = \frac{1}{\sqrt[5]{10}} \frac{1}{\sqrt[5]{n^4}}$$

We let

$$\mathrm{SMALL}_n = \frac{1}{\sqrt[5]{10}} \frac{1}{\sqrt[5]{n^4}}.$$

$\sum_{n=1}^{\infty} \mathrm{SMALL}_n$ diverges by the p-test since $\frac{4}{5} \leq 1$. The direct comparison test tells us that $\sum_{n=1}^{\infty} \mathrm{BIG}_n$ diverges, i.e. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4+6n+3}}$ diverges.

Notice that the term we compared to was the one "closest" to making $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4+6n+3}}$ converge.

4.5 Limit comparison test

In the last example of the direct comparison theorem, wouldn't it have been great if, after realizing that $\frac{1}{\sqrt[5]{n^4}}$ was the important term, we did not have to do as much messing around? The limit comparison theorem makes this whole process far easier.

Limit comparison theorem. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series consisting of positive terms and that $\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0, \infty$.

1. $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
2. $\sum_{n=1}^{\infty} a_n$ diverges if and only if $\sum_{n=1}^{\infty} b_n$ diverges.

Example.

Let $a_n = \frac{1}{\sqrt[5]{n^4+6n+3}}$ and $b_n = \frac{1}{\sqrt[5]{n^4}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[5]{n^4}}{\sqrt[5]{n^4+6n+3}} = \lim_{n\to\infty} \frac{1}{\sqrt[5]{1+\frac{6}{n^3}+\frac{3}{n^4}}}$$
$$= \frac{1}{\sqrt[5]{1+0+0}} = 1 \neq 0, \infty.$$

Thus, the limit comparison theorem says that $\sum_{n=1}^{\infty} a_n$ diverges if $\sum_{n=1}^{\infty} b_n$ diverges, i.e. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4+6n+3}}$ diverges if $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4}}$ diverges.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4}}$ diverges by the p-test, since $\frac{4}{5} \leq 1$, so

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4+6n+3}} \text{ diverges.}$$

There are also versions of the limit comparison test which account for the cases when $L = 0$ or $L = \infty$. The first time I taught this class, I found that students would often misuse these versions, and so, the second time I taught the class, I didn't encourage their use as much. Since I found that some students still wanted to use them, I've compromised. Questions on the final will not *require* these versions - another test will always be applicable - but if you like them, you are free to use them, and they might help you. Look up the statements in the textbook. Here's an example of when $L = 0$.

Example.

Let $a_n = \frac{10 \ln n}{n^2}$ and $b_n = \frac{1}{n^{\frac{3}{2}}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{10 \ln n}{\sqrt{n}} = \lim_{x\to\infty} \frac{10 \ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\left(\frac{10}{x}\right)}{\left(\frac{1}{2\sqrt{x}}\right)} = \lim_{x\to\infty} \frac{20}{\sqrt{x}} = 0.$$

This limit calculation says that for large values of n, a_n is much smaller than b_n. (WolframAlpha shows that $a_n < b_n$ as long as $n > 8100$.)

The p-test says $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges and the $L = 0$ version of the limit comparison test allows us to conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{10 \ln n}{n^2}$$

converges.

4.6 Ratio and root test

All of our tests so far have relied on comparison with one of our friends. Wouldn't it be great if we had a test that didn't require us being so social?

Ratio test. Suppose $\sum_{n=1}^{\infty} a_n$ is a series consisting of non-zero terms, and that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).
2. If $L > 1$ (including $L = \infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, then we learn nothing.

Root test. Suppose $\sum_{n=1}^{\infty} a_n$ is a series, and that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).
2. If $L > 1$ (including $L = \infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, then we learn nothing.

The ratio test is particularly good if you see factorials. $n!$ factorial grows quicker than pretty much anything. The only way to beat it is with things like, n^n, double exponents, or double factorials. So, if you see a factorial on the top of a fraction in the n-th term, then the series is likely to diverge, and if you see it on the bottom of a fraction in the n-th term, the series is likely to converge. The ratio test is then a good check, but, with the ratio test, unlike the comparison tests, you do not even need an educated guess.

Example 1.

Consider the series $\sum_{n=0}^{\infty} \frac{1000^n}{n!}$. The $n!$ on the bottom of the fraction making up the n-th term makes us think it will converge, but even if we didn't think this, we can use the ratio test with $a_n = \frac{1000^n}{n!}$ and we get

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left[\frac{1000^{n+1}}{(n+1)!}\right]}{\left[\frac{1000^n}{n!}\right]} = \lim_{n\to\infty} \frac{1000}{n+1} = 0 < 1.$$

Thus, $\sum_{n=0}^{\infty} \frac{1000^n}{n!}$ converges. In fact, it converges to e^{1000}.

Example 2.

Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$. Using the ratio test becomes a little complicated: we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left[\frac{e^{n+1}}{(n+1)^{n+1}}\right]}{\left[\frac{e^n}{n^n}\right]} = \frac{e \cdot n^n}{(n+1) \cdot (n+1)^n} = \frac{1}{n+1} \cdot \frac{e}{(1+\frac{1}{n})^n}$$

which converges to $0 \cdot \frac{e}{e} = 0 < 1$ and so the series converges.

The root test is far easier.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{e^n}{n^n}} = \lim_{n\to\infty} \frac{e}{n} = e \cdot \lim_{n\to\infty} \frac{1}{n} = e \cdot 0 = 0 < 1$$

so the series converges.

Note. If the ratio or root test is inconclusive because the calculation gives $L = 1$, the other will not help you. Don't waste time trying both.

4.7 Alternating series test

If we forget all but one thing about series, then what we remember is that the harmonic series diverges. Funnily enough, by inserting some minus signs, we get a convergent series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots \ = \ln 2.$$

Alternating series test. Suppose $\sum_{n=1}^{\infty} b_n$ is a series where $b_n = (-1)^{n-1} a_n$ and

1. $a_{n+1} \leq a_n$ for all n;
2. $\lim_{n\to\infty} a_n = 0$.

Then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

Example.

The alternating harmonic series satisfies the hypothesis for the alternating series test, since $\frac{1}{n+1} \leq \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ converges.}$$

Example.

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^3+1}$ converges. Let $a_n = \frac{n^2}{n^3+1}$. One can see that $a_2 \leq a_1$. Also, if $f(x) = \frac{x^2}{x^3+1}$, then $f'(x) = -\frac{x(x^3-2)}{(x^3+1)^2} \leq 0$ when $x \geq \sqrt[3]{2}$ so that $a_{n+1} \leq a_n$ for $n \geq 2$. Finally, $0 \leq a_n \leq \frac{1}{n}$, so the squeeze theorem tells us that $\lim_{n\to\infty} a_n = 0$. We can now apply the alternating series test.

4.8 Absolutely and conditionally convergent series

The alternating harmonic series is strange. By changing the order of the summation we can get any answer we like. For example,

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \dots = \frac{3}{2}\ln 2.$$

The reason for this is that the harmonic series diverges, i.e. once we change all the negative signs to be positive it diverges.

Definition. If $\sum_{n=1}^{\infty} a_n$ but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is said to be *conditionally convergent*.

Conditionally convergent series always have the bizarre aforementioned property. One could view this as an interesting but bad property. If we demand that $\sum_{n=1}^{\infty} |a_n|$ converges, then the order of summation never matters.

Definition. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent*.

Absolutely convergent series are convergent.

Example.

Consider the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$. We stare at this and think that, other than the $\cos n$, it looks a lot like our friend $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges since $2 > 1$. $\cos n$ varies between positive and negative, but it is always between -1 and 1. By letting

$$\text{BIG}_n = \frac{1}{n^2} \quad \text{and} \quad \text{SMALL}_n = \frac{|\cos n|}{n^2}$$

we have $\text{BIG}_n \geq \text{SMALL}_n \geq 0$, and $\sum_{n=1}^{\infty} \text{BIG}_n$ converges. The direct comparison test tells us that $\sum_{n=1}^{\infty} \text{SMALL}_n$ converges,

$$\text{i.e.} \quad \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \quad \text{converges.}$$

This tells us $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent, and thus convergent.

4.9 The hierachy of tests

I don't generally have an order of tests which I apply to figure out whether a series converges or diverges. With enough practice and experience, you almost always know straightaway. But... if I had to give a suggested order of things to try on $\sum_{n=1}^{\infty} a_n$, it'd be this... This is a fail-safe method; not the most efficient.

1. The n-th term test: if $\lim_{n\to\infty} |a_n| \neq 0$ the series diverges.

   This is the best initial check to do. What if you can see $\lim_{n\to\infty} |a_n| \neq 0$ but find that limit difficult to explain? Well, you can say that you think $\lim_{n\to\infty} |a_n| \neq 0$ so that you think $\sum_{n=1}^{\infty} a_n$ diverges. Then use another test. At least you know what your answer should be!

   If $\lim_{n\to\infty} |a_n| = 0$, move along, and try something else.

2. Look out for friends. If it's a p-series or a geometric series you have been handed a gift. You understand the convergence and divergence of p-series, and, in the case of geometric series, when they converge, you even know what they converge to.

   Also, look out for things which are just sums or multiples of such things, and apply series rules. For example, $\sum_{n=1}^{\infty}[\frac{2}{n^2} + \frac{100}{n^3}]$ converges, and $\sum_{n=0}^{\infty}[\frac{6^n - 2^n}{3^n}]$ diverges.

   What if you can kind of see a friend? Hands rub together... Comparison tests might apply!!

3. If you have negative terms, first, check to see if the alternating series test applies. If it doesn't, apply absolute values to every term in the hope that the series is absolutely convergent; then do subsequent tests on $\sum_{n=1}^{\infty} |a_n|$.

4. Try the direct comparison test. If you get really good at these then this will pretty much always work, since the proofs of the ratio test, the root test, and limit comparison test, all depend on the direct comparison test. In spirit, this is more like number one on my non-existent list.

5. Perhaps, while trying to use the comparison test, you see that it is one of those ones where the limit comparison test is a little easier. If so, use the limit comparison test instead. Beware of the $L = 0, \infty$ cases, if you use them. I avoided them in lecture since I did not want to cause additional confusion. If you're confident, then feel free to use them.

6. If there are factorials or powers of n in sight, the ratio test might work. If there are n-th powers, the root test might work. These could also work in other situations too. Sometimes a factorial might make you head to the ratio test straightaway, one way in which you can see this list is a little too inflexible.

7. Use the integral test, an almost last resort.

   A good exercise is to check that for $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, the integral test is the only option, i.e. check why all the other tests don't help for these series.

8. Sometimes you might have to rewrite the series a little before you can apply a test to it. For instance, how would you prove

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \dots$$

   converges? Use parentheses cleverly!

   Sometimes using more than one test one after another can work too.

4.10 Examples

1. $\sum_{n=1}^{\infty} (-1)^{n-1} 3n^2 e^{-n^3}$

   This converges for pretty much every reason! e^{-n^3} decays really quickly so... However, some tests are easier than others. I'll make a couple of observations before beginning.

   Let $f(x) = 3x^2 e^{-x^3}$. Then $f'(x) = 3xe^{-x^3}(2 - 3x^3)$ and so $f'(x) \leq 0$ when $x \geq 1$. This means $f(x)$ is decreasing when $x \geq 1$. Also,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2}{e^{x^3}} = \lim_{x\to\infty} \frac{6x}{3x^2 e^{x^3}} = \lim_{x\to\infty} \frac{2}{xe^{x^3}} = 0,$$

   where the second equality used L'Hôpital's rule.

   Let $a_n = f(n) = 3n^2 e^{-n^3}$. We care about $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$.

   (a) By using the calculations above, we have

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = 0.$$

   So $\lim_{n\to\infty} (-1)^{n-1} a_n = 0$ and $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ passes the n-th term test. This required L'Hôpital's rule; since we expected it to pass, I wouldn't have bothered doing such a careful calculation.

(b) We don't see any friends.

(c) By using the calculations above, we have

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = 0,$$

and $a_{n+1} \leq a_n$ for all n. The alternating series test tells us that $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = \sum_{n=1}^{\infty}(-1)^{n-1}3n^2e^{-n^3}$ converges.

This required some calculus and L'Hôpital's rule: a bit annoying.

We will show $\sum_{n=1}^{\infty} a_n$ converges so that $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is absolutely convergent.

(d) The direct comparison test is applicable. We can let $\mathrm{SMALL}_n = a_n$ and $\mathrm{LARGE}_n = 3e^{-n}$. Proving $\mathrm{SMALL}_n \leq \mathrm{LARGE}_n$ is a bit annoying.

(e) The limit comparison is possible, though my choice of $b_n = e^{-n}$ results in $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and calculating this requires L'Hôpital: a bit annoying.

(f)

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{3(n+1)^2e^{-(n+1)^3}}{3n^2e^{-n^3}} \\
&= \lim_{n\to\infty} \left[\frac{n+1}{n}\right]^2 \lim_{n\to\infty} e^{-3n^2-3n-1} = 1 \cdot 0 = 0 < 1
\end{aligned}$$

so the ratio test says the series converges relatively painlessly.

The root test is applicable but requires knowing $\lim_{n\to\infty} \sqrt[n]{n} = 1$, which requires L'Hôpital: a bit annoying.

(g) The integral test is applicable since

$$\int_0^{\infty} f(x)\ dx = \lim_{S\to\infty} [-e^{-x^3}]_0^S = \lim_{S\to\infty} [1 - e^{-S^3}] = 1$$

and we proved that $f(x)$ is decreasing for $x \geq 1$; it's a bit annoying that we had to do this.

In summary, the ratio test was the best; in fact, I would have gone for it straight away since powers play well with the ratio test.

2. $\sum_{n=1} \frac{e^{\frac{n}{1000}}+n}{n^2+2}$.

Since $e^{\frac{n}{1000}}$ grows faster than everything in sight I would expect this to diverge. It also diverges for pretty much every reason, but some are easier to check than others.

Let $a_n = \frac{e^{\frac{n}{1000}}+n}{n^2+2}$.

(a) Using L'Hôpital's rule twice gives

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{e^{\frac{x}{1000}}+x}{x^2+2} = \lim_{x\to\infty} \frac{\frac{e^{\frac{x}{1000}}}{1000}+1}{2x} = \lim_{x\to\infty} \frac{\frac{e^{\frac{x}{1000}}}{1{,}000{,}000}}{2} = \infty$$

so it fails the n-th term test.

(b) We vaguely see our friend $\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$. However, we ignored $e^{\frac{n}{1000}}$ to see this friend which is a little risky. We will see later that the inequality "goes the right way."

(c) All terms are positive.

(d) $\frac{e^{\frac{n}{1000}}+n}{n^2+2} \geq \frac{e^{\frac{n}{1000}}+n}{n^2+2n^2} = \frac{e^{\frac{n}{1000}}+n}{3n^2} \geq \frac{n}{3n^2} = \frac{1}{3n}$. So the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{3n}$ shows it diverges.

(e) Limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ works. Letting $b_n = \frac{1}{n}$, we have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{\frac{n}{1000}}+n}{n+\frac{2}{n}} = \infty$.

Limit comparison with a friend will always result in the ∞ case so this is a little annoying.

(f) $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n^2+2}{(n+1)^2+2} \lim_{n\to\infty} \frac{e^{\frac{n+1}{1000}}+(n+1)}{e^{\frac{n}{1000}}+n} = e^{\frac{1}{1000}} > 1$ so the ratio test tells us it diverges.

The root test is annoying.

(g) The integral test is terrible.

I think the direct comparison was easiest. The n-th term test wasn't so bad either.

5 Direct Comparison for Integrals

Suppose that BIG(x) and SMALL(x) are continuous on $a < x < b$ and that for these values we have

$$\text{BIG}(x) \geq \text{SMALL}(x) \geq 0.$$

1. If $\int_a^b \text{BIG}(x)\ dx$ converges, then $\int_a^b \text{SMALL}(x)\ dx$ converges.
2. If $\int_a^b \text{SMALL}(x)\ dx$ diverges, then $\int_a^b \text{BIG}(x)\ dx$ diverges.

These are separate statements so you will use one of 1 or 2.
The assumption that the functions are positive is very important.

Example 1: $\int_0^\infty e^{-x^2} dx$ is convergent.

Let SMALL(x) $= e^{-x^2}$ and note that SMALL(x) ≥ 0. We must choose BIG(x) so that

$$\text{BIG}(x) \geq e^{-x^2} \text{ when } x > 0, \text{ and } \int_0^\infty \text{BIG}(x)\ dx \text{ converges.}$$

Someone in lecture wisely suggested letting BIG(x) $= e^{-x}$. Is it true that $e^{-x} \geq e^{-x^2}$ when $x > 0$? Sadly not :(Why? Suppose $x > 0$. We find that

$$e^{-x} \geq e^{-x^2} \iff -x \geq -x^2 \iff x^2 \geq x \iff x \geq 1.$$

(The first equivalence uses the fact that e^t is an increasing function of t; the last equivalence makes use of the fact that $x > 0$.)

Do we give up? Never! We just avoid our problems. Notice that

$$\int_0^\infty e^{-x^2}\ dx = \int_0^1 e^{-x^2}\ dx + \int_1^\infty e^{-x^2}\ dx.$$

The first integral in the sum is fine because:

- e^{-x^2} is a continuous function;
- the limits of the integral are finite.

We're left with the second integral, and the argument that we just tried to make *does* work now. We have BIG(x) $\geq$ SMALL(x) ≥ 0 when $x \geq 1$, and

$$\int_1^\infty \text{BIG}(x)\ dx = \lim_{R\to\infty} \int_1^R e^{-x}\ dx = \lim_{R\to\infty} (e^{-1} - e^{-R})$$

converges to $\frac{1}{e}$. So the comparison theorem says that $\int_1^\infty \mathrm{SMALL}(x)\; dx = \int_1^\infty e^{-x^2}\; dx$ converges. We have now completed the proof.

An alternative proof, avoiding dividing the integral up, would have been to take $\mathrm{BIG}(x) = e^{\frac{1}{4}-x}$:

$$e^{\frac{1}{4}-x} \geq e^{-x^2} \iff \frac{1}{4} - x \geq -x^2$$
$$\iff x^2 - x + \frac{1}{4} \geq 0 \iff \left(x - \frac{1}{2}\right)^2 \geq 0.$$

Example 2: $\int_2^\infty \frac{1}{\ln x} dx$ diverges.

Let $\mathrm{BIG}(x) = \frac{1}{\ln x}$ and $\mathrm{SMALL}(x) = \frac{1}{x}$.

$\int_2^\infty \mathrm{SMALL}(x)\; dx$ diverges by the p-test. To apply the comparison test, we just need to show that $\mathrm{BIG}(x) \geq \mathrm{SMALL}(x) \geq 0$ when $x > 2$.

If $x \geq 1$, then

$$\frac{d}{dx}(x) = 1 \geq \frac{1}{x} = \frac{d}{dx}(\ln x).$$

Since $1 \geq \ln(1)$, this shows that for $x > 2$, $x \geq \ln x$, and so

$$\frac{1}{\ln x} \geq \frac{1}{x} \text{ when } x > 2.$$

(By using the prime number theorem (a very difficult theorem) this example gives a very roundabout proof that there are infinitely many prime numbers.)

6 More 8.7 help.

Here are the type of inequalities I use for solving questions 61-75 in section 8.7.

1. (a) If $f(x) > 0$, $g(x) \geq 0$, and $a \geq 0$, then
$$\frac{1}{(f(x)+g(x))^a} \leq \frac{1}{(f(x))^a}.$$
   (b) If $f(x) > 0$, $g(x) \geq k > 0$, then
$$\frac{1}{(f(x)g(x))^a} \leq \frac{1}{(kf(x))^a}.$$
2. (a) If $f(x) > g(x) \geq 0$, and $a \geq 0$, then
$$\frac{1}{(f(x)-g(x))^a} \geq \frac{1}{(f(x))^a}.$$
   (b) If $nf(x) \geq g(x) > 0$ and $a \geq 0$, then
$$\frac{1}{(f(x)+g(x))^a} \geq \frac{1}{((n+1)f(x))^a}.$$
   (c) If $f(x) > 0$, $k \geq g(x) > 0$, and $a \geq 0$, then
$$\frac{1}{(f(x)g(x))^a} \geq \frac{1}{(kf(x))^a}.$$
   This is used in 75 with $f(x) = x$, $g(x) = e^x + x$, $k = e + 1$, $a = 1$, on the interval $0 < x \leq 1$.
3. (a) $e^x \geq 1$ for $x \geq 0$.
   (b) If $f(x) \geq 0$, $g(x) \geq 0$, then
$$e^{-(f(x)+g(x))} \leq e^{-f(x)}.$$
4. (a) $|\cos x|, |\sin x| \leq 1$.
   (b) $|\sin x| \leq |x|$; this is useful when x is near 0.
5. If $p(x) = Ax^n + q(x)$ where $q(x)$ is a polynomial of degree less than n, then it is possible to find a positive constant C with the property that
$$\frac{1}{p(x)} \leq \frac{C}{x^n}$$
   for all x. One can do something similar with non-integer powers.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$