Math 31B
Integration and Infinite Series

Practice Final

Instructions: You have 180 minutes to complete this exam. There are ??? questions, worth a total of ??? points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

Student ID number: ______________________

Discussion: ___________________________

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	0	
9	0	
Total:	0	

Problem 1.

(a) Calculate the following indefinite integral:

$$\int (\ln x)^2 \, dx.$$

Solution: Do integration by parts with $u = (\ln x)^2$, $dv = dx$, to get

$$\begin{aligned} \int (\ln x)^2 \, dx &= x(\ln x)^2 - 2\int \ln x \, dx \\ &= x(\ln x)^2 - 2(x \ln x - x) + c = x((\ln x)^2 - 2\ln x + 2) + c. \end{aligned}$$

(b) Calculate the following indefinite integral:

$$\int \frac{\ln(\arcsin x)}{(\arcsin x) \cdot \sqrt{1-x^2}} \, dx.$$

Solution: Let $u = \ln(\arcsin x)$. Then $du = \frac{1}{(\arcsin x) \cdot \sqrt{1-x^2}} \, dx$.
Carrying through the u-sub gives $\frac{(\ln(\arcsin x))^2}{2} + c$.

(c) Calculate the following indefinite integral:

$$\int \frac{xe^x}{(x+1)^2} \, dx.$$

Solution: Do integration by parts with $u = xe^x$ and $dv = \frac{1}{(x+1)^2} \, dx$, to get

$$\int \frac{xe^x}{(x+1)^2} \, dx = -\frac{xe^x}{x+1} + \int e^x \, dx = e^x - \frac{xe^x}{x+1} + c = \frac{e^x}{x+1} + c.$$

(d) Use integration by parts, together with a trigonometric identity (the only one you *have* to know) to calculate the following definite integral:

$$\int_0^\pi \cos^2 x \; dx.$$

Solution: Take $u = \cos x$ and $dv = \cos x$. Then

$$\int_0^\pi \cos^2 x \; dx = [\cos x \sin x]_0^\pi + \int_0^\pi \sin x^2 \; dx = \int_0^\pi 1 - \cos^2 x \; dx,$$

so

$$\int_0^\pi \cos^2 x \; dx = \int_0^\pi \frac{1}{2} \; dx = \frac{\pi}{2}.$$

Problem 2.

(a) Calculate $\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}}$.

Solution: $\lim_{n\to\infty} \ln(n^{\frac{1}{\sqrt{n}}}) = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0.$

So $\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}} = e^0 = 1$.

(b) Calculate $\lim_{n\to\infty} \frac{(\ln n)^2}{n}$.

Solution: $\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{\frac{2\ln n}{n}}{1} = \lim_{n\to\infty} \frac{2\ln n}{n} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1} = 0.$

Problem 3.

Calculate

$$\int_0^\infty x^5 e^{-x}\ dx.$$

Reminder: many points for this question would be for adhering to definitions correctly, and performing IBP on *proper* integrals.

Solution: You can use integration by part to show that for $n \geq 1$,

$$\int_0^\infty x^n e^{-x}\ dx = n \int_0^\infty x^{n-1} e^{-x}\ dx.$$

Direct calculation gives

$$\int_0^\infty x^0 e^{-x}\ dx = \int_0^\infty e^{-x}\ dx = 1 = 0!$$

and so

$$\int_0^\infty x^n e^{-x}\ dx = n!$$

The answer to the question is $5! = 120$.

Problem 4.

(a) Use the direct comparison theorem to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n + e^{-n}}{n\sqrt{4n-1}}.$$

Solution: Compare with $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ to see it diverges.

(b) Use the integral test to determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} ne^{-n}.$$

Solution: Let $f(x) = xe^{-x}$.
As long as $x \geq 1$, we have $f'(x) = (1-x)e^{-x} \leq 0$, so that $f(x)$ is decreasing.
As remarked in question 3, $\int_0^\infty f(x)\ dx = 1$. Thus, $\int_1^\infty xe^{-x}\ dx$ converges.
The integral test says the series converges too.

Problem 5.

(a) Prove the convergence or divergence of the improper integral

$$\int_{-\infty}^{\infty} 1 \, dx.$$

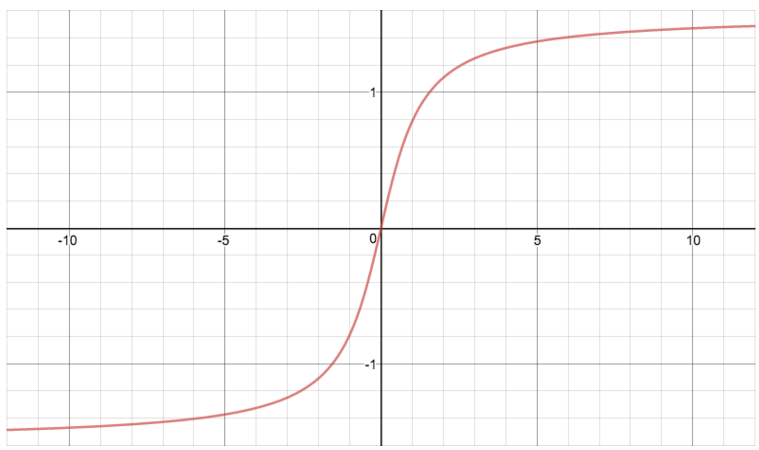
Most of the points in this question would be for adhering to definitions correctly.

> **Solution:** If this integral is not obviously divergent to you then you are missing some key intuition. You should be able to verify this result from the definition.

(b) Prove the convergence or divergence of the improper integral

$$\int_{1}^{\infty} \frac{\arctan(e^x)}{1+x^2} \, dx.$$

Standard tests can be used but less familiar improper integrals (if you need them) should be dealt with as though you've never seen them before (this means that you should prove everything about them).



Graph of $y = \arctan(x)$.

> **Solution:** Compare with $\int_1^\infty \frac{\pi}{2x^2} \, dx$ to see it converges.

(c) Use the comparison test for integrals to prove whether the following integral converges or diverges:

$$\int_0^{\pi/4} \frac{\cos x}{x^2} \, dx.$$

> **Solution:** Compare with $\int_0^{\pi/4} \frac{1}{x^2\sqrt{2}} \, dx$ to see it diverges.

Problem 6.

(a) Calculate the improper integral $\int_{\frac{3}{5}}^{\frac{6}{5}} \frac{1}{\sqrt{36-25x^2}}\,dx$.

Solution: Using the u-sub $u = \frac{5}{6}x$, we get

$$\begin{aligned}
\int_{\frac{3}{5}}^{\frac{6}{5}} \frac{1}{\sqrt{36-25x^2}}\,dx &= \lim_{S\to\frac{6}{5}-} \int_{\frac{3}{5}}^{S} \frac{1}{\sqrt{36-25x^2}}\,dx \\
&= \lim_{S\to\frac{6}{5}-} \int_{\frac{1}{2}}^{\frac{5S}{6}} \frac{\frac{6}{5}}{\sqrt{36-36u^2}}\,du \\
&= \lim_{S\to\frac{6}{5}-} \int_{\frac{1}{2}}^{\frac{5S}{6}} \frac{\frac{1}{5}}{\sqrt{1-u^2}}\,du \\
&= \lim_{S\to\frac{6}{5}-} \frac{1}{5}\arcsin\left(\frac{5S}{6}\right) - \frac{1}{5}\arcsin\left(\frac{1}{2}\right) \\
&= \frac{1}{5}\arcsin(1) - \frac{1}{5}\arcsin\left(\frac{1}{2}\right) \\
&= \frac{\pi}{15}.
\end{aligned}$$

(b) Use u-substitution(s) to verify whether the following improper integral converges or diverges. If it converges, calculate what it converges to.

$$\int_{\frac{1}{e}}^{1} \frac{2\ln(-\ln x)}{x\ln x}\,dx$$

Solution: Using the u-sub $u = \ln(-\ln x)$, we get

$$\begin{aligned}
\int_{\frac{1}{e}}^{1} \frac{2\ln(-\ln x)}{x\ln x}\,dx &= \lim_{S\to 1-} \int_{\frac{1}{e}}^{S} \frac{2\ln(-\ln x)}{x\ln x}\,dx \\
&= \lim_{S\to 1-} \int_{0}^{\ln(-\ln S)} 2u\,du = \lim_{S\to 1-} (\ln(-\ln S))^2 = \infty
\end{aligned}$$

so the integral diverges.

Reminder: almost half the points for these question would be for adhering to definitions correctly, and performing u-subs on *proper* integrals.

Problem 7.

Note that $\int_{\frac{\pi}{2}}^{\infty} |\frac{\cos x}{x^2}|\, dx$ converges by comparison with $\int_{\frac{\pi}{2}}^{\infty} \frac{1}{x^2}\, dx$. The concept of "absolute convergence" makes sense for integrals too(!) so $\int_{\frac{\pi}{2}}^{\infty} \frac{\cos x}{x^2}\, dx$ converges.

Using this information prove the convergence or divergence of the following integral:

$$\int_0^{\infty} \frac{\sin x}{x}\, dx.$$

A strategy is as follows:

- Break the integral into one from 0 to $\frac{\pi}{2}$, and one from $\frac{\pi}{2}$ to ∞.
- Deal with the one from $\frac{\pi}{2}$ to ∞ by integrating by part. You'll need to use the fact given above, together with the fact that $\lim_{R\to\infty} \frac{\cos R}{R} = 0$ (which follows from the squeeze theorem).

Solution: First, note that for $0 < x \leq \frac{\pi}{2}$, we have $0 \leq \frac{\sin x}{x} \leq 1$.

The comparison test says that $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx$ converges.

Integrating by parts with $u = \frac{1}{x}$ and $dv = \sin x\, dx$ gives

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\infty} \frac{\sin x}{x}\, dx = \lim_{R\to\infty} \int_{\frac{\pi}{2}}^{R} \frac{\sin x}{x}\, dx &= \lim_{R\to\infty} \left[\left[\frac{-\cos x}{x} \right]_{\frac{\pi}{2}}^{R} - \int_{\frac{\pi}{2}}^{R} \frac{\cos x}{x^2}\, dx \right] \\ &= -\lim_{R\to\infty} \frac{\cos R}{R} - \int_{\frac{\pi}{2}}^{\infty} \frac{\cos x}{x^2}\, dx \\ &= -\int_{\frac{\pi}{2}}^{\infty} \frac{\cos x}{x^2}\, dx.\end{aligned}$$

Thus, $\int_{\frac{\pi}{2}}^{\infty} \frac{\sin x}{x}\, dx$ converges too. We conclude $\int_0^{\infty} \frac{\sin x}{x}\, dx$ converges.

(The related question of whether $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges or not is much more difficult. In particular, the integral test does NOT apply.)

Problem 8.

To prepare for series I recommend picking examples from the homework and trying to solve them using every possible test just like I did in the final lecture.

Problem 9.

For Taylor polynomials, try the midterm 2 question again without looking at the solutions. Make sure you understand the solutions and why I could do it so quickly. Also, try to get to a point where the homework questions feel easier. Practice some pattern spotting regarding derivatives.