Inverting the Hopf map

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March 1, 2015

Abstract

We calculate the η-localization of the motivic stable homotopy ring over $\mathbb{C}$, confirming a conjecture of Guillou and Isaksen. Our approach is via the motivic Adams-Novikov spectral sequence. In fact, work of Hu, Kriz and Ormsby implies that it suffices to compute the $\overline{\alpha}_1$-localization of the classical Adams-Novikov E_2-term, and this is what we do. Guillou and Isaksen also propose a pattern of differentials in the localized motivic classical Adams spectral sequence, which we verify using a method first explored by Novikov.

Dedicated to the memory of Goro Nishida (1943-2014)

1 Introduction

The chromatic approach to stable homotopy theory [8] rests on the fact that non-nilpotent graded endomorphisms of finite complexes can be essentially classified. They always occur in Adams-Novikov filtration zero, and up to taking p^{th} powers every graded BP_*BP-comodule endomorphism survives the Adams-Novikov spectral sequence. At the prime 2, for example, the Hopf map $\eta : S^1 \to S^0$ lies in filtration one, and the celebrated nilpotence theorem of Nishida [17] already guarantees that η is nilpotent; in fact, we know that $\eta^4 = 0$. On the other hand, the element $\overline{\alpha}_1$ that detects η in the Adams-Novikov E_2-term $E_2(S^0; BP)$ is non-nilpotent. This immediate failure of the Adams-Novikov E_2-term to accurately reflect 2-primary stable homotopy has lessened its attractiveness as a tool by which to approach 2-primary stable homotopy theory. One can nevertheless hope to calculate $\overline{\alpha}_1^{-1}E_2(S^0; BP)$ and discover the range in which the localization map is an isomorphism, and this is the main result of our paper.

Some $\overline{\alpha}_1$-free elements have been known for many years. The group $E_2^{1,2n}(S^0; BP)$ is cyclic for $n \geq 1$, generated by elements $\overline{\alpha}_n$ closely related to the image of J ([18, theorem 11.2], [15, corollary 4.23]). In [15] it was shown that for $n \neq 2$, $\overline{\alpha}_1^k \overline{\alpha}_n \neq 0$ for all $k \geq 0$. The Adams-Novikov differentials on these classes are also well-known and due to Novikov [18], [19, pg. 171]. In this paper we show that there are no other $\overline{\alpha}_1$-free generators and that the localization map

$$E_2(S^0; BP) \longrightarrow \overline{\alpha}_1^{-1}E_2(S^0; BP)$$

is an isomorphism above a line of slope 1/5 when we plot the Adams-Novikov spectral sequence in the usual manner. This resolves a question raised by Zahler [22] at the dawn of the chromatic era, in 1972.

Our approach follows Novikov [18] as interpreted in [14]: we filter the BP cobar construction by powers of the kernel of the augmentation $BP_* \to \mathbb{F}_2$. The resulting spectral sequence has the form

$$H^*(P;Q) \implies E_2(S^0;BP)$$

where P is the Hopf subalgebra of squares in the dual Steenrod algebra A and

$$Q = \mathrm{gr}^* BP_* = \mathbb{F}_2[q_0, q_1, \ldots].$$

is the associated graded of BP_*; q_n is the class of the Hazewinkel generator v_n. In this spectral sequence the element $\overline{\alpha}_1$ is represented by the class of $[\xi_1^2]$, which, following the notational conventions in force at an odd prime, we denote by h_0. We proceed by inverting h_0 to obtain a localized algebraic Novikov spectral sequence converging to $\overline{\alpha}_1^{-1} E_2(S^0;BP)$. Our main result is a complete description of this spectral sequence.

The E_1-page of the localized algebraic Novikov spectral sequence is given by $h_0^{-1}H^*(P;Q)$. The computation of this object parallels the well-known fact [12, 13] that if M is a bounded below comodule over A then

$$q_0^{-1}H^*(A;M) = H(M;\mathrm{Sq}^1) \otimes \mathbb{F}_2[q_0^{\pm 1}]$$

where $q_0 \in H^*(A)$ is the class dual to Sq^1. The result we obtain is

$$h_0^{-1}H^*(P;Q) = \mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots].$$

As in [14], the differentials in the localized algebraic Novikov spectral sequence are calculated by exploiting the connection between $E_2(S^0;BP)$ and the theory of formal groups. The facts we rely on are easier to obtain than those used in [14] and date back to [16]. We prove that for $n \geq 2$

$$d_1 q_{n+1} = q_n^2 h_0. \tag{1.1}$$

This calculation allows us to show that the natural map $\mathbb{Z}_{(2)}[\overline{\alpha}_1, \overline{\alpha}_3, \overline{\alpha}_4] \longrightarrow E_2(S^0;BP)$ induces an isomorphism

$$\overline{\alpha}_1^{-1} E_2(S^0;BP) = \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1 \overline{\alpha}_4^2). \tag{1.2}$$

Our way of writing (1.2) requires some explanation. In $E_2(S^0;BP)$ we have $\overline{\alpha}_1\overline{\alpha}_4^2 = 0$ (by Toda's calculation that $\eta\sigma^2 = 0$ [20] if for no other reason). We write "$\overline{\alpha}_1\overline{\alpha}_4^2$" as opposed to "$\overline{\alpha}_4^2$" in (1.2) to remind ourselves that $\overline{\alpha}_1\overline{\alpha}_4^2 = 0$ is the relation, which holds before localization. To see why (1.2) is true we work with the associated graded objects. The images of $\overline{\alpha}_3$ and $\overline{\alpha}_4$ in $\overline{\alpha}_1^{-1}E_2(S^0;BP)$ are detected by $q_1^2 h_0$ and $q_2 h_0$, respectively. This fact, together with the differentials of (1.1), allows us to see that the E_∞-page of the localized algebraic Novikov spectral sequence is $\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2]/(q_2^2)$.

In fact, for $n \geq 0$, the image of $\overline{\alpha}_{2n+1}$ in $\overline{\alpha}_1^{-1}E_2(S^0;BP)$ is detected by $q_1^{2n}h_0$ in the localized algebraic Novikov spectral sequence, and $\overline{\alpha}_{2n+4}$ has image detected by $q_1^{2n}q_2h_0$, as required by the fact that they are $\overline{\alpha}_1$-free elements. It follows, incidentally, that for $m, n \geq 0$, $\overline{\alpha}_{2m}\overline{\alpha}_{2n}$ is $\overline{\alpha}_1$-torsion, and that modulo $\overline{\alpha}_1$-torsion the classes $\overline{\alpha}_{2m+1}\overline{\alpha}_{2n+1}$ and $\overline{\alpha}_{2n+1}\overline{\alpha}_{2m+4}$ depend only on the sum $m+n$.

With these results in hand we can analyze the localized Adams-Novikov spectral sequence. In the unlocalized Adams-Novikov spectral sequence we have $d_3\overline{\alpha}_3 = \overline{\alpha}_1^4$ and this gives the same differential in the localized spectral sequence. Since $\overline{\alpha}_1$ is a unit, this differential terminates the localized spectral sequence.

None of this is particularly interesting from the perspective of the classical homotopy groups of spheres because we already know that $\eta^4 = 0$. However, the advent of motivic homotopy theory has led to interesting related questions. There is a ground field in motivic homotopy theory, which for us will be $\mathbb{C}$. It is known that the motivic analogue of η is non-nilpotent, and one may ask (as Dugger and Isaksen did [5]) to calculate $\eta^{-1}\pi_{*,*}(S^{0,0})$. Hu, Kriz and Ormsby ([9]; see also [5]) study a motivic analogue of the Adams-Novikov spectral sequence and show that its E_2-term is just $E_2(S^0; BP)[\tau]$, where τ detects a certain motivic homotopy class $\theta \in \pi_{0,-1}(S^{0,0})$. The classical Adams-Novikov spectral sequence is recovered by setting $\tau = 1$ and so our calculation allows one to determine the E_2-page and differentials for the localized *motivic* Adams-Novikov spectral sequence. The unique nonzero differential is determined by the motivic analogue of $d_3\overline{\alpha}_3 = \overline{\alpha}_1^4$, namely

$$d_3\overline{\alpha}_3 = \tau\overline{\alpha}_1^4. \tag{1.3}$$

This calculation allows us to show that the map $\mathbb{Z}[\eta, \sigma, \mu_9] \longrightarrow \pi_{*,*}(S^{0,0})$ induces an isomorphism

$$\eta^{-1}\pi_{*,*}(S^{0,0}) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta\sigma^2). \tag{1.4}$$

In this equation $\sigma \in \pi_{7,4}(S^{0,0})$ is a motivic Hopf invariant one element [5] and $\mu_9 \in \pi_{9,5}(S^{0,0})$ is described by the Toda bracket $\langle 8\sigma, 2, \eta\rangle$; θ acts as 0 on $\eta^{-1}\pi_{*,*}(S^{0,0})$ because $\theta\eta^4 = 0$. In $\pi_{*,*}(S^{0,0})$ we have $\eta\sigma^2 = 0$ since Toda's classical result [20] extends to the motivic setting. As before, we have written equation (1.4) in a way that reminds us that this relation holds before localization. To see why (1.4) is true we work with the associated graded objects. σ and μ_9 are detected by $\overline{\alpha}_4$ and $\overline{\alpha}_5$, respectively, in the motivic Adams-Novikov spectral sequence. Thus, their images in $\eta^{-1}\pi_{*,*}(S^{0,0})$ are detected by $\overline{\alpha}_4$ and $\overline{\alpha}_5 = \overline{\alpha}_1^{-1}\overline{\alpha}_3^2$, respectively, in the localized motivic Adams-Novikov spectral sequence. This fact, together with the differential of (1.3), allows us to see that the E_∞-page of the localized Adams-Novikov spectral sequence is $\mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3^2, \overline{\alpha}_4]/(\overline{\alpha}_1\overline{\alpha}_4^2)$. Hu, Kriz, and Ormsby [9] had shown that the elements in (1.4) occur; we prove that there are no more η-free generators.

Our calculation of $\eta^{-1}\pi_{*,*}(S^{0,0})$ verifies a conjecture due to Guillou and Isaksen [7]. They had approached this calculation by means of the motivic classical Adams spectral sequence. Using the motivic May spectral sequence they found that

$$h_0^{-1}E_2(S^{0,0}; (H\mathbb{F}_2)_{\mathrm{mot}}) = \mathbb{F}_2[h_0^{\pm 1}, v_1^4, v_2, v_3, \ldots]. \tag{1.5}$$

Based on extensive computations they conjectured that differentials in the localized motivic Adams spectral sequence are given by

$$d_1 v_{n+1} = v_n^2 h_0, \ n \geq 2. \tag{1.6}$$

In the last part of this paper we recover (1.5) using a localized Cartan-Eilenberg spectral sequence. We then proceed to use the methods of [14] to prove the differentials (1.6), up to higher Cartan-Eilenberg filtration. In fact, the differentials must hold exactly as stated in (1.6), since this is the only way one can obtain the description of $\eta^{-1}\pi_{*,*}(S^{0,0})$ that we prove (see [7]).

Acknowledgements. The first author is particularly grateful to Dan Isaksen and Zhouli Xu for their talks on the motivic Adams spectral sequence and its relationship with the Adams-Novikov spectral sequence. These served as the inspiration to start trying to compute the Adams-Novikov E_2-page. We thank Will Perry, who created charts of the algebraic Novikov E_1-page. The idea for the proof of the main theorem came about while he coded computational software for us. We are indebted to Dan Isaksen for raising these questions about the η-local motivic sphere and for several useful conversations. We thank him and Bert Guillou for keeping us abreast of their work. We also thank Kyle Ormsby and Marc Levine for helpful tutorials on motivic matters.

2 The Adams-Novikov spectral sequence

2.1 The cobar construction

In this paper we follow [15] and work with *right* comodules. Given a Hopf algebroid (A, Γ) and a right Γ-comodule M, the *cobar construction* $\Omega^*(\Gamma; M)$ has $\Omega^s(\Gamma; M) = M \otimes_A \overline{\Gamma} \otimes_A \ldots \otimes_A \overline{\Gamma}$ with s copies of $\overline{\Gamma} = \ker(\epsilon : \Gamma \to A)$, and is equipped with a natural differential [15, (1.10)] of degree 1. $H^*(\Gamma; M)$ denotes the cohomology of this complex. If Γ and M are graded then $H^*(\Gamma; M)$ becomes bigraded; the first index is the cohomological grading and the second is inherited from the gradings on Γ and M.

Recall (e.g. [16, §2]) the Hopf algebroid given to us by the p-typical factor of complex cobordism: (BP_*, BP_*BP). We will work at the prime $p = 2$ throughout this paper.

The Adams-Novikov spectral sequence takes the following form

$$E_2^{s,u} = H^{s,u}(BP_*BP) \overset{s}{\Longrightarrow} \pi_{u-s}(S^0) \otimes \mathbb{Z}_{(2)}, \; d_r : E_r^{s,u} \longrightarrow E_r^{s+r,u+r-1}.$$

Here S^0 denotes the sphere spectrum and $\mathbb{Z}_{(2)}$ denotes the integers localized at 2. Computing the full E_2-page of this spectral sequence is not feasible, but some elements can be written down fairly explicitly.

2.2 A collection of $\overline{\alpha}_1$-free elements

The coefficient ring of BP is a polynomial algebra $\mathbb{Z}_{(2)}[v_1, v_2, \ldots]$. As in [15], we have a short exact sequence of BP_*BP-comodules

$$0 \longrightarrow BP_* \longrightarrow 2^{-1}BP_* \longrightarrow BP_*/2^\infty \longrightarrow 0,$$

which gives rise to a connecting homomorphism

$$\delta : H^{0,*}(BP_*BP; BP_*/2^\infty) \longrightarrow H^{1,*}(BP_*BP; BP_*).$$

Following [15] define $x = v_1^2 - 4v_1^{-1}v_2 \in 2^{-1}v_1^{-1}BP_*$. When $s \geq 2$, the image of $x^s/8s$ in $v_1^{-1}BP_*/2^\infty$ lies in the subgroup $BP_*/2^\infty$. We define

$$\overline{\alpha}_s = \begin{cases} \delta(v_1^s/2) & s \geq 1 \text{ and odd} \\ \delta(v_1^2/4) & s = 2 \\ \delta(x^{s/2}/4s) & s \neq 2 \text{ and even} \end{cases}$$

Proposition 2.2.1 ([15]). *These classes generate $H^{1,*}(BP_*BP)$ and are of the same 2-order as the denominator of the element to which δ was applied.*

The element $\overline{\alpha}_1$ is a permanent cycle detecting $\eta \in \pi_1(S^0)$. In lemma 6.2.2 we will show that for $s \neq 2$, $\overline{\alpha}_s$ is not killed by any power of $\overline{\alpha}_1$. This is [15, corollary 4.23], but our proof is different. We will analyze the images of these elements in $\overline{\alpha}_1^{-1}H^*(BP_*BP)$, an object we will compute explicitly. Our approach shows, in addition, that these classes generate $\overline{\alpha}_1^{-1}H^*(BP_*BP)$ as an $\mathbb{F}_2[\overline{\alpha}_1^{\pm 1}]$-vector space.

3 The algebraic Novikov spectral sequence

One of the best tools for gaining information about the E_2-page of the Adams-Novikov spectral sequence is the algebraic Novikov spectral sequence.

As noted above, the E_2-page of the Adams-Novikov spectral sequence can be calculated using the cobar complex $\Omega^*(BP_*BP)$. The coefficient ring

$$BP_* = \pi_*(BP) = \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$$

admits a filtration by invariant ideals given by powers of $I = \ker(BP_* \to \mathbb{F}_2)$ and this allows us to filter the cobar construction: $F^t\Omega^s(BP_*BP) = I^t\Omega^s(BP_*BP)$. There is a resulting spectral sequence and we need to identify the E_1-page.

3.1 The E_1-page

In this paper we will write A for the dual of the Steenrod algebra. Let P denote the Hopf subalgebra of squares in A. Write

$$\zeta_n = \bar{\xi}_n^2$$

for the square of the conjugate of the Milnor generator ξ_n, so that

$$P = \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$$

with diagonal

$$\Delta\zeta_n = \sum_{i+j=n} \zeta_i \otimes \zeta_j^{2^i}.$$

Write

$$Q = \mathrm{gr}^* BP_*$$

for the graded algebra associated to the I-adic filtration of BP_* and q_n for the class of v_n. Then $Q = \mathbb{F}_2[q_0, q_1, \ldots]$ is naturally a graded algebra in P-comodules. The filtration degree t is the *Novikov weight*. The element q_n has internal degree u given by the degree of v_n, namely $2^{n+1} - 2$. The coaction is determined by

$$q_n \longmapsto \sum_{i+j=n} q_i \otimes \zeta_j^{2^i}.$$

Then [14] on the level of cobar complexes

$$\mathrm{gr}^t\Omega^s(BP_*BP) = \Omega^s(P; Q^t) \tag{3.1.1}$$

and the algebraic Novikov spectral sequence takes the form

$$E_1^{s,t,u} = H^{s,u}(P; Q^t) \overset{t}{\Longrightarrow} H^{s,u}(BP_*BP),\ d_r : E_r^{s,t,u} \longrightarrow E_r^{s+1,t+r,u}. \tag{3.1.2}$$

The following elements will be important for us. We write q_0 for the element in $H^{0,0}(P; Q^1)$ represented by $q_0[\ ]$ and h_n for the element in $H^{1,2^{n+1}}(P; Q^0)$ represented by $[\zeta_1^{2^n}]$.

There are a number of ways to draw $H^*(P; Q)$ in two dimensions and we will display two of them. They are suggested to us by the following fact ([14, 18]): $H^*(P; Q)$ is not only the E_1-page for the algebraic Novikov spectral sequence, but also the E_2-page for the Cartan-Eilenberg spectral sequence (3.1.3). This is the spectral sequence associated to the extension of Hopf algebras

$$P \longrightarrow A \longrightarrow E,$$

where $E = E[\xi_1, \xi_2, \ldots]$; it converges to the E_2-page of the Adams spectral sequence:

$$E_2^{s,t,u} = H^{s,u}(P; Q^t) \overset{s}{\Longrightarrow} H^{s+t,u+t}(A),\ d_r : E_r^{s,t,u} \longrightarrow E_r^{s+r,t-r+1,u+r-1}. \tag{3.1.3}$$

Plotting $(u-s, s)$ has the effect of suppressing the filtration grading in the algebraic Novikov spectral sequence (Novikov weight), while plotting the other gradings in the usual Adams-Novikov spectral sequence format. In low dimensions, drawing $H^*(P; Q)$ in this way results in a picture closely resembling the E_2-page of the Adams-Novikov spectral sequence.

Plotting $(u-s, s+t)$ has the effect of suppressing the filtration grading in the Cartan-Eilenberg spectral sequence, while plotting the other gradings in the usual Adams spectral sequence format. In low dimensions, drawing $H^*(P; Q)$ in this way results in a picture closely resembling the E_2-page of the Adams spectral sequence.

In figure 1 we have plotted $H^{s,u}(P; Q^t)$ in low dimensions. The first chart uses the coordinates $(u-s, s)$ and the second uses the coordinates $(u-s, s+t)$. Vertical black lines indicate multiplication by q_0. The vertical blue arrow indicates a q_0 tower which continues indefinitely. Black lines of slope one indicate multiplication by h_0. The blue arrows of slope one indicate h_0 towers which continue indefinitely. Green arrows denote algebraic Novikov differentials and red arrows denote Cartan-Eilenberg differentials. On the first chart square nodes denote multiple basis elements connected by q_0-multiplication; the number to the upper left indicates how many such basis elements.

3.2 Some permanent cycles

One slow method of finding permanent cycles in the algebraic Novikov spectral sequence is to seek out elements in $H^*(BP_*BP)$ and ask which elements in $H^*(P; Q)$ they represent.

In order to provide the reader with the feeling that some low dimensional computations are possible by hand, we give the following results. These will also be essential for the computation we make later. Before stating the lemma we need to recall that $BP_*BP = BP_*[t_1, t_2, \ldots]$, where the degree of t_n is equal to the degree of v_n.

Lemma 3.2.1. *The following elements are cocycles in the cobar construction $\Omega^*(BP_*BP)$:*

1. $[\ ]$ *and* $[t_1]$;
2. $v_1^2[t_1] + 2v_1[t_1^2] + \frac{4}{3}[t_1^3]$;
3. $v_2[t_1|t_1] + v_1[t_1|t_1^3] - v_1[t_1^2|t_1^2] + v_1[t_1^3|t_1] - 3v_1[t_1|t_2] + 2[t_1|t_1t_2] + 2[t_1^2|t_1^3] - 2[t_1^2|t_2] + 2[t_1t_2|t_1]$.

Proof. Do the calculation by hand or with the aid of a computer: `http://math.mit.edu/~mjandr/bpcobar.jar` □

Corollary 3.2.2. *The following elements are cocycles in the cobar construction $\Omega^*(P; Q)$:*

1. $[\ ]$ *and* $[\zeta_1]$;
2. $q_1^2[\zeta_1] + q_0q_1[\zeta_1^2] + q_0^2[\zeta_1^3]$;

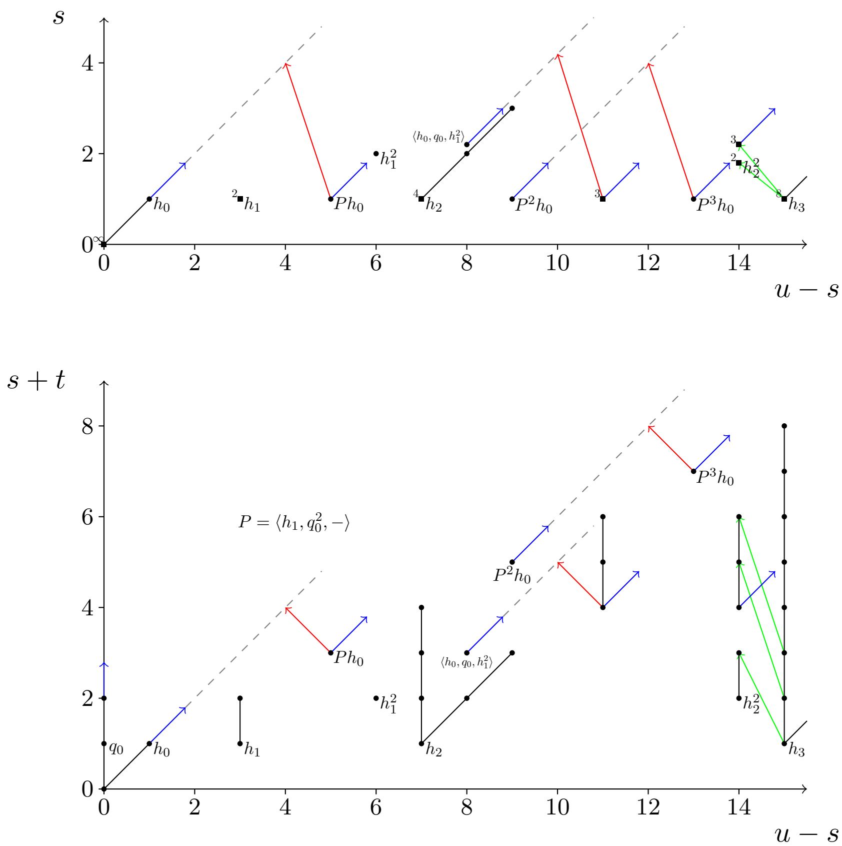

Figure 1: $H^*(P;Q)$ in Novikov and classical Adams projections. Green arrows are algebraic Novikov differentials and red arrows are Cartan-Eilenberg differentials.

3. $q_2[\zeta_1|\zeta_1]+q_1[\zeta_1|\zeta_1^3]+q_1[\zeta_1^2|\zeta_1^2]+q_1[\zeta_1^3|\zeta_1]+q_1[\zeta_1|\zeta_2]+q_0[\zeta_1|\zeta_1\zeta_2]+q_0[\zeta_1^2|\zeta_1^3]+q_0[\zeta_1^2|\zeta_2]+q_0[\zeta_1\zeta_2|\zeta_1]$.

Moreover, these elements define the classes 1, h_0, $\langle h_1, q_0^2, h_0\rangle$ *and* $\langle h_0, q_0, h_1^2\rangle$ *in* $H^*(P;Q)$ *and they are permanent cycles in the algebraic Novikov spectral sequence.*

Proof. Checking the first Massey product is straightforward. The second stops being difficult once one realizes that $d(q_2[\zeta_1] + q_1[\zeta_2 + \zeta_1^3] + q_0[\zeta_1\zeta_2]) = q_0[\zeta_1^2|\zeta_1^2]$. $\square$

3.3 Vanishing Lines

We wish to localize the algebraic Novikov spectral sequence by inverting h_0. In order to identify the resulting E_1-page and check convergence of the spectral sequence we require some basic vanishing lines. The diagrams in figure 1 suggest the following two vanishing lines. Both are proved using the observation that if M is a P-comodule with $M_u = 0$ for $u < m$, then

$$\Omega^{s,u}(P;M) = \left(\overline{P}^{\otimes s} \otimes M\right)_u = 0$$

for $u < m + 2s$ (because $\overline{P}_u = 0$ for $u < 2$).

Lemma 3.3.1. $H^{s,u}(P;Q^t) = 0$ *when* $u - s < s$.

Proof. For any t, $Q^{t,u} = 0$ for $u < 0$. Thus $\Omega^{s,u}(P;Q^t) = 0$ for $u < 2s$. $\square$

Lemma 3.3.2. $H^{s,u}(P;Q^t) = 0$ *when* $0 < u - s < s + t$.

Proof. Suppose from the offset that $0 < u - s$ and define an algebra map $\varphi : Q \longrightarrow P$ by sending q_n to ζ_n (so q_0 maps to 1). If $t < 0$ then $H^{s,u}(P;Q^t) = 0$ so fix a $t \geq 0$. We have a short exact sequence of right P-comodules

$$0 \longrightarrow Q^t \xrightarrow{\varphi|_{Q^t}} P \longrightarrow P/Q^t \longrightarrow 0$$

and the associated long exact sequence gives a surjection $H^{s-1,u}(P;P/Q^t) \longrightarrow H^{s,u}(P;Q^t)$. We see that $(P/Q^t)_u = 0$ if $u < 2(t+1)$, since the first element not in the image of $\varphi|_{Q^t}$ is ζ_1^{t+1}. Thus $H^{s-1,u}(P;P/Q^t)$ is zero provided that $u < 2(s-1) + 2(t+1) = 2s + 2t$. Since $t \geq 0$, $u - s < s + t$ implies $u < 2s + 2t$. $\square$

4 Some localization results

We begin with a well-known localization theorem, dealing with comodules over the dual Steenrod algebra A. Write q_0 for the class of $[\xi_1]$ in $H^{1,1}(A)$. It acts on $H^*(A;M)$ for any A-comodule M. Write E for the quotient Hopf algebra

$$E = A/(\xi_1^2, \xi_2, \xi_3, \ldots)$$

It is the exterior algebra generated by the image of ξ_1. Any A-comodule M becomes an E-comodule and an E-comodule structure on M is equivalent to a degree one differential Sq^1 on M by

$$x \longmapsto x \otimes 1 + \mathrm{Sq}^1 x \otimes \xi_1.$$

Proposition 4.1. *Let M be an A-comodule such that $M_u = 0$ whenever $u < 0$, and consider the following diagram.*

$$\begin{tikzcd} H(A;M) \arrow[r] \arrow[d] & H(E;M) \arrow[d] \\ q_0^{-1}H(A;M) \arrow[r] & q_0^{-1}H(E;M) \end{tikzcd}$$

The top map is surjective in bidegrees (s,u) with $u-s < 2s-2$ and an isomorphism in bidegrees with $u-s < 2s-5$, i.e. above a line of slope $1/2$ in the usual $(u-s,s)$ plot. The bottom map is an isomorphism and the right map is an isomorphism for bidegrees (s,u) with $s > 0$. Moreover,

$$q_0^{-1}H(E;M) = H(M;Sq^1) \otimes \mathbb{F}_2[q_0^{\pm 1}].$$

Proof. The cotensor product $M\square_E A$ is a submodule of $M\otimes A$. Since the coaction map $M \longrightarrow M\otimes A$ is coassociative, it factors through a map $i : M \longrightarrow M\square_E A$. Define L by the following short exact sequence of right A-comodules.

$$0 \longrightarrow M \xrightarrow{i} M\square_E A \longrightarrow L \longrightarrow 0. \tag{4.2}$$

We claim that $H^{s,u}(A;L) = 0$ whenever $u-s < 2s-2$.

If $M = \mathbb{F}_2$, the middle comodule $\mathbb{F}_2\square_E A$ is the homology of the integral Eilenberg-Mac Lane spectrum. It is well known, in that case, that the map i induces an isomorphism in Sq^1-homology. (One way to see this is to think about the dual: left multiplication by Sq^1 gives a bijection between the Cartan-Serre basis elements for $H^*(H\mathbb{Z})$ with even leading entry and those with odd leading entry, with the exception of the basis element 1 in dimension 0.) Filtering the general comodule M by dimension shows that the same is true in general. We deduce that L is Sq^1-acyclic and so we can apply [1, theorem 2.1] or [2, theorem 1.1] to give the claimed vanishing line for $H^*(A;L)$.

Under the identification $H^*(A;M\square_E A) = H^*(E;M)$, the map induced by applying $H(A;-)$ to i is the top map in the proposition statement and so the first statement in the proposition follows from the cohomology long exact sequence associated to (4.2) and the vanishing line just proved.

Since q_0 acts vertically in $(u-s,s)$ coordinates we find that the bottom map is an isomorphism. The remaining statements follow from the indentification

$$H(E;M) = \frac{\ker(\mathrm{Sq}^1)\otimes\mathbb{F}_2[q_0]}{\mathrm{im}(\mathrm{Sq}^1)\otimes(q_0)}.$$

□

Thus the localization of the Adams E_2-page coincides with the E_2-page of the Bockstein spectral sequence. In fact [12,14] the two spectral sequences coincide from E_2 onwards, giving a qualitative strengthening of Serre's observation that $\pi_*(X)\otimes\mathbb{Q} \cong H_*(X;\mathbb{Q})$.

By doubling degrees we obtain a parallel result for the Hopf subalgebra P of A. Now E will be the quotient Hopf algebra $P/(\zeta_1^2, \zeta_2, \ldots)$. Any P-comodule M becomes an E-comodule and just as we wrtoe Sq^1 above, we may write P^1 for the operator on a right E-comodule corresponding to ζ^1. A P-comodule splits naturally into even and odd parts, which one can handle separately to prove the following result.

Proposition 4.3. *Let M be a P-comodule such that $M_u = 0$ whenever $u < 0$, and consider the following diagram.*

$$\begin{tikzcd} H(P;M) \arrow[r] \arrow[d] & H(E;M) \arrow[d] \\ h_0^{-1}H(P;M) \arrow[r] & h_0^{-1}H(E;M) \end{tikzcd}$$

The top map is surjective in bidegrees (s,u) with $u-s < 5s-4$ and an isomorphism in bidegrees with $u-s < 5s-10$, i.e. above a line of slope $1/5$ in the usual $(u-s,s)$ plot. The bottom map is an isomorphism and the right map is an isomorphism for bidegrees (s,u) with $s > 0$. Moreover,

$$h_0^{-1}H(E;M) = H(M;P^1) \otimes \mathbb{F}_2[h_0^{\pm 1}].$$

As an application, we obtain a calculation of the h_0-localization of the E_1-page of the algebraic Novikov spectral sequence together with the range in which the localization map is an isomorphism.

Corollary 4.4. *For any t, the localization map $H^*(P;Q^t) \longrightarrow h_0^{-1}H^*(P;Q^t)$ is injective in bidegrees (s,u) with $u-s < 5s-4$ and an isomorphism in bidegrees with $u-s < 5s-4$. Moreover,*

$$h_0^{-1}H(P;Q) = \mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots].$$

Proof. It is enough to note that $P^1 q_1 = q_0$, $\ker P^1 = \mathbb{F}_2[q_0, q_1^2, q_2, q_3, \ldots]$ and $\operatorname{im} P^1 = (q_0)$. □

5 The localized algebraic Novikov spectral sequence

The algebraic Novikov spectral sequence is multiplicative since it is obtained by filtering the DG algebra $\Omega^*(BP_*BP)$ by powers of a differential ideal. The class $h_0 \in E_1^{1,0,2}$ is a permanent cycle and so multiplication by h_0 defines a map of spectral sequences. Since forming homology commutes with filtered colimits, we may localize the spectral sequence by inverting the class h_0 page by page.

In forming this localization we may lose convergence; this is the issue at stake in the "telescope conjecture" of chromatic homotopy theory. Here we are lucky, however. Convergence is preserved because, as was the case in [14], the operator we are inverting acts parallel to a vanishing line. This vanishing line is visible in the lower diagram in figure 1 and is the content of proposition 3.3.2.

The vanishing line has the following two implications.

1. Suppose given an element $x \in H^*(BP_*BP)$. The vanishing line ensures that there is an $n \in \mathbb{N}$ such that the algebraic Novikov filtration (possibly ∞) of $\overline{\alpha}_1^n x$ and $\overline{\alpha}_1^{n+m} x$ coincide for all $m \geq 0$. Thus, if x is $\overline{\alpha}_1$-free,

   $$F = \{\text{algebraic Novikov filtration of } \overline{\alpha}_1^m x : m \geq 0\}$$

   is bounded above and the image of x in $\overline{\alpha}_1^{-1}H^*(BP_*BP)$ is detected by the localized algebraic Novikov spectral sequence in filtration $(\max F)$.

2. Suppose x is a fixed h_0-free element in the E_1-page of the algebraic Novikov spectral sequence. The vanishing line shows that the length of a non-trivial differential on $h_0^m x$ is bounded above by an integer depending only on x. Thus, if each $h_0^m x$ supports a nontrivial differential, then there exists an $n \in \mathbb{N}$, together with a non-trivial differential $d_r h_0^n x = y$ such that, for $m \geq 0$, each differential $d_r h_0^{n+m} x = h_0^m y$ is non-trivial, and in this case the image of x in the localized algebraic Novikov spectral sequence supports a non-trivial d_r.

The first fact tells us that we detect everything we are supposed to; the second fact tells us that we do not detect more than we are supposed to. A more thorough account of similar convergence issues may be found in [3].

Coupled with the isomorphism range in corollary 4.4, the natural map of spectral sequences from the algebraic Novikov spectral sequence to its localized counterpart implies an isomorphism range for the $\overline{\alpha}_1$-localization of the Adams-Novikov E_2-page for the sphere. Recall that the class $\overline{\alpha}_1 \in H^{1,2}(BP_*BP)$ is detected by $h_0 \in H^{1,2}(P;Q^0)$.

Proposition 5.1. *The localization map*

$$H^*(BP_*BP) \longrightarrow \overline{\alpha}_1^{-1}H^*(BP_*BP)$$

is surjective in bidegrees (s,u) *for which* $u-s<5s-4$ *and an isomorphism in bidegrees for which* $u-s<5s-10$*, i.e. above a line of slope* $1/5$ *in the usual* $(u-s,s)$ *plot.*

Proof. We use the natural map of spectral sequences from the algebraic Novikov spectral sequence to its localized counterpart. Both spectral sequences converge but, for our argument, we must show that the filtration is finite in each bidegree (s,u) under consideration (those for which $u-s<5s-4$).

For the unlocalized spectral sequence there are three cases. If $u-s<0$, then $H^{s,u}(BP_*BP)=0$; if $u-s=0$, then $s>\frac{4}{5}$ and $H^{s,u}(BP_*BP)=0$; if $u-s>0$, then lemma 3.3.2 implies that $E_1^{s,t,u}$ is potentially nonzero only when $0 \le t \le u-2s$. In the localized case lemma 3.3.2 extends, since multiplication by h_0 acts along the vanishing line, to tell us that $E_1^{s,t,u}$ is potentially nonzero only when $0 \le t \le u-2s$. In any case, the filtration is finite.

From these observations it is enough to show that, at the E_∞-page, the map of spectral sequences is surjective in bidegrees (s,u) for which $u-s<5s-4$ and an isomorphism in bidegrees for which $u-s<5s-10$. We know this to be true at the E_1-page by proposition 4.3; suppose it is true for the E_r-page. A d_r-differential in the algebraic Novikov spectral sequence has (s,u) bidegree $(1,0)$. Because $u-s<5s-4$ if and only if $u-(s+1)<5(s+1)-10$, it has source in the surjective region if and only if it has target in the isomorphism region. Thus, we can deduce the result for the E_{r+1}-page using the following simple observation: if a map of cochain complexes is a surjection in degree n and an isomorphism in higher degrees, then the same is true of the map induced in cohomology. Using the vanishing line of lemma 3.3.2 and the computation of the localized algebraic Novikov spectral sequence in the next section we see that for a fixed (s,u) bigrading we have $E_r = E_\infty$ for some large r and so the proof is complete by induction. □

6 Computing the localized algebraic Novikov spectral sequence

6.1 Differentials in the localized algebraic Novikov spectral sequence

The main result of this section is the following proposition, which completely describes the localized algebraic Novikov spectral sequence.

Proposition 6.1.1. *In the localized algebraic Novikov spectral sequence* 1*,* h_0*,* q_1^2 *and* q_2 *are permanent cycles, while* $d_1 q_{n+1} = q_n^2 h_0$ *for* $n \ge 2$*.*

Proof. The images in $\Omega^*(E[\zeta_1];Q)$ of the elements in corollary 3.2.2 are $[\ ]$, $[\zeta_1]$, $q_1^2[\zeta_1]$, $q_2[\zeta_1|\zeta_1]$, respectively. So there are permanent cycles in the algebraic Novikov spectral sequence mapping to

1, h_0, $q_1^2h_0$, and $q_2h_0^2$ in the localized algebraic Novikov spectral sequence. We are left to prove the differential, so suppose that $n \geq 2$ and write E for $E[\zeta_1]$.

By proposition 4.3, there exists an $N \geq 0$ such that $q_{n+1}h_0^N$ is in the image of $H^*(P;Q) \to H^*(E;Q)$. Pick an X in $H^{N,*}(P;Q^1)$ mapping to $q_{n+1}h_0^N$. To complete the proof of the proposition it is enough to calculate d_1X in the unlocalized algebraic Novikov spectral and check that its image under $H^*(P;Q) \to H^*(E;Q)$ is $q_n^2h_0^{N+1}$.

Since $\Omega^*(P;Q) \to \Omega^*(E;Q)$ is surjective, we can find a cocycle in $\Omega^*(P;Q)$ representing X, which maps to $q_{n+1}[\zeta_1]^N$ in $\Omega^*(E;Q)$. We find that all elements of the monomial basis for $\Omega^*(P;Q)$ that include a tensor factor containing some monomial in P other than ζ_1 map to zero in $\Omega^*(E;Q)$. This means that when we write our cocycle in this monomial basis it must contain the term $q_{n+1}[\zeta_1]^N$. We write the cocyle representing X as $q_{n+1}[\zeta_1]^N + x$, where x is a linear combination of other basis elements.

By (3.1.1) we have a surjection

$$I\Omega^*(BP_*BP) \longrightarrow \Omega^*(P;Q^1).$$

To be specific, we choose the set-theoretic splitting that in each term of a linear combination of monomial basis elements, replaces each ζ_i by t_i and each q_i by v_i. (Remember that $v_0 = 2$; this map is not linear.) With this choice of splitting, $v_{n+1}[t_1]^N + y$ is selected to map to our cocycle representing X, where y is a linear combination of terms, each of which involves, as a tensor factor, some monomial in the t_i's other than the monomial t_1, and such that each nonzero coefficient is v_i for some i.

Since $q_{n+1}[\zeta_1]^N + x \in \Omega^*(P;Q)$ is a cocycle, $d(v_{n+1}[t_1]^N + y) \in I^2\Omega^*(BP_*BP)$. Mapping to $\mathrm{gr}^2\Omega^*(BP_*BP) = \Omega^*(P;Q^2)$ gives an element representing $d_1X \in H^*(P;Q)$. As explained at the start of the proof, we wish to understand the image of this element in $H^*(E;Q)$.

To do this we will consider the BP_*-basis of the cobar construction given by placing a monomial in the t_i's in each tensor factor. Any element of $I^2\Omega^*(BP_*BP)$ is uniquely a linear combination of these elements with coefficients in I^2. Of these terms, only those of the form $\alpha[t_1]^j$ with $\alpha \notin I^3$ map nontrivially to $\Omega^*(E;Q^2)$. $d(v_{n+1}[t_1]^N)$ and dy are linear combinations of these basis elements with coefficients in BP_*. Since $q_{n+1}[\zeta_1]^N$ is not a cocycle, neither set of coefficients by themselves need to lie in I^2, though their sums do. First, we look at the contribution from $d(v_{n+1}[t_1]^N)$.

Lemma 6.1.2. *For $n \geq 1$, the coefficient of $[t_1]^{N+1}$ in $d(v_{n+1}[t_1]^N)$ is v_n^2 mod I^3.*

Proof. Because t_1 is primitive it is enough to investigate the coefficient of t_1 in $\eta_R v_{n+1}$. Since the elements $\eta_R v_{n+1}$ and $v_{n+1} = \eta_L v_{n+1}$ have the same augmentation we have

$$\eta_R(v_{n+1}) \equiv v_{n+1} + ct_1 \mod (t_1^2, t_2, t_3, \ldots)$$

for some $c \in BP_{2(2^{n+1}-2)}$. The only monomial in the v_i's of this degree that is not in I^3 is v_n^2. Moreover, $2v_n^2t_1 \in I^3$ so that

$$\eta_R(v_{n+1}) = v_{n+1} + bv_n^2t_1 \mod I^3 + (t_1^2, t_2, t_3, \ldots)$$

where $b = 0$ or 1. Since [16, 5.1]

$$\eta_R(v_{n+1}) \equiv v_{n+1} + v_nt_1^{2^n} - v_n^2t_1 \mod (2, v_1, \ldots, v_{n-1})$$

we must have $b = 1$. $\square$

Mapping $v_n^2[t_1]^{N+1} \in I^2\Omega^*(BP_*BP)$ to $\mathrm{gr}^2\Omega^*(BP_*BP) = \Omega^*(P;Q^2)$ gives $q_n^2[\zeta_1]^{N+1}$. Mapping further to $\Omega^*(E;Q)$, gives a cocycle representing $q_n^2h_0^{N+1}$. In order to complete the proof it suffices to show that the coefficient of $[t_1]^{N+1}$ in dy is zero.

Recall that y is a linear combination of terms, each of which involves, as a tensor factor, some monomial in the t_i's other than the monomial t_1. The differential in the cobar complex makes use of the right unit and the diagonal map in BP_*BP. When evaluating dy, the right unit is used on the coefficients of the terms in y. This cannot lead to a BP_* multiple of $[t_1]^{N+1}$ arising. Thus, we just need to consider terms coming from the diagonal map. The following simple lemma is crucial.

Lemma 6.1.3. *The only monomials in the t_i's that contain a BP_*-multiple of $t_1 \otimes t_1$ in their diagonal are t_1^2 and t_2.*

Proof. Recall that we have an inclusion $BP_* \subset H_*(BP) = \mathbb{Z}_{(2)}[m_1, m_2, \ldots]$ given by the Hurewicz homomorphism. Thus, we can compute in $H_*(BP \wedge BP) = H_*(BP)[t_1, t_2, \ldots]$ and there we have (see [16]) an inductive formula for the diagonal of t_n.

$$\Delta t_n = \sum_{i+j+k=n} m_i t_j^{2^i} \otimes t_k^{2^{i+j}} - \sum_{i=1}^{n} m_i (\Delta t_{n-i})^{2^i}$$

The first sum does not contain a term $t_1 \otimes t_1$. Moreover, the only terms in Δt_{n-i} with a 1 on one side or the other are $t_{n-i} \otimes 1$ and $1 \otimes t_{n-i}$. Thus, in the expression of $(\Delta t_{n-i})^{2^i}$, the only way one can achieve $t_1 \otimes t_1$ is with $i = 1$ and $n - i = 1$ so that $n = 2$.

The diagonal is multiplicative and so one can achieve $t_1 \otimes t_1$ in the diagonal of a monomial only in the cases t_2 and t_1^2. □

Now consider a tensor product of monomials, a basis element in the cobar construction. The differential is computed by applying the reduced diagonal to each factor and taking the alternating sum. One receives a term, which is a BP_*-multiple of $[t_1]^{N+1}$ only by starting with a tensor product of monomials in which all but one term is t_1, and the remaining term is either t_1^2 or t_2.

We should call attention to a subtlety here. When the reduced diagonal is applied to a monomial, the result is a BP_*-linear combination of monomials. Given a basis element of the cobar complex, to express the value of the differential on it as a BP_*-linear combination of tensor products of such monomials, one needs to pull coefficients outside the tensor products. This operation is nontrivial since the tensor products, while formed over BP_*, use the left and right actions on the right and left factors, respectively. In particular, $t \otimes vt' = \eta_R(v)t \otimes t'$. $\eta_R(v)$ will itself be a linear combination of monomials in the t_i's (where we now include 1 as t_0) so if the expression involves more than $[t_1]$'s before this manoeuvre, it will continue to involve more than $[t_1]$'s afterwards as well.

Now recall that y has internal dimension $2(2^{n+1} - 1) + 2N$. The internal dimensions of

$$[t_1]^{N-i}[t_1^2][t_1]^{i-1} \text{ and } [t_1]^{N-i}[t_2][t_1]^{i-1}$$

are $2(N + 1)$ and $2(N + 2)$, respectively and so the coefficients of these basis elements in y must have internal dimensions $2(2^{n+1} - 2)$ and $2(2^{n+1} - 3)$, respectively. But recall that the coefficient of each term appearing in y is a v_i. The first dimension does not occur as the dimension of a v_i, and the second occurs only for $n = 1$. Since we assumed $n \geq 2$, this completes the proof.

We note that this $n = 1$ issue was already apparent in lemma 3.2.1, where the third cocycle contains the terms $v_2[t_1|t_1]$ and $-3v_1[t_1|t_2]$. These provide two cancelling $v_1^2[t_1|t_1|t_1]$ terms. Of course, this is how we saw $q_2h_0^2$ was a permanent cycle. □

We see immediately from the proposition that the E_2-page of the localized algebraic Novikov spectral sequence consists of permanent cycles and so we obtain the following corollary.

Corollary 6.1.4. *The E_∞-page of the localized algebraic Novikov sepctral sequence is*

$$\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2]/(q_2).$$

6.2 What happens to $\overline{\alpha}_s$?

We now return to the elements $\overline{\alpha}_s$ of section 2.2. In order to say what happens to them under the localization map it is convenient to consider the mod 2 Moore spectrum analogues of our results for S^0, which are of interest in their own right. We write $S/2$ for the mod 2 Moore spectrum. In complete analogy with what we have done above we can prove the following proposition.

Proposition 6.2.1. *There is a spectral sequence*

$$E_1^{s,t,u} = H^{s,u}(P; [Q/(q_0)]^t) \overset{t}{\Longrightarrow} H^{s,u}(BP_*BP; BP_*/2), \; d_r : E_r^{s,t,u} \longrightarrow E_r^{s+1,t+r,u},$$

an "algebraic Novikov spectral sequence," for computing the E_2-page of the Adams-Novikov spectral sequence for $\pi_*(S/2)$. We may invert h_0 to obtain a convergent "localized algebraic Novikov spectral sequence." The E_1-page is given by

$$h_0^{-1}H^*(P; Q/(q_0)) = \mathbb{F}_2[h_0^{\pm 1}, q_1, q_2, \ldots].$$

q_1 and q_2 are permanent cycles in this spectral sequence and we have $d_1 q_{n+1} = q_n^2 h_0$ for $n \geq 2$. The map $S^0 \longrightarrow S/2$ induces a map between the localized algebraic Novikov spectral sequences. At the E_1-page, the map is given by the inclusion

$$\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots] \longrightarrow \mathbb{F}_2[h_0^{\pm 1}, q_1, q_2, \ldots].$$

At the E_∞-page, it is given by the inclusion $\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2]/(q_2) \longrightarrow \mathbb{F}_2[h_0^{\pm 1}, q_1, q_2]/(q_2)$.

Lemma 6.2.2. *For $s \neq 2$, $\overline{\alpha}_s$ is $\overline{\alpha}_1$-free and its image in $\overline{\alpha}_1^{-1}H^*(BP_*BP)$ is detected by $q_1^{s-1}h_0$ when s is odd and $q_1^{s-4}q_2h_0$ when s is even.*

Proof. When s is odd $\overline{\alpha}_s$ has a cocycle representative with leading term $sv_1^{s-1}[t_1]$. All other terms have the same filtration and involve higher powers of t_1. Thus, $\overline{\alpha}_s$ is detected by $q_1^{s-1}h_0$.

Suppose $s \neq 2$ is even. Because the map induced by $S^0 \to S/2$ between our localized algebraic Novikov spectral sequences is injective at each page, it suffices to check the result for the image of $\overline{\alpha}_s$ in $H^*(BP_*BP; BP_*/2)$. For $s > 4$ one can find (see [19, 4.4.35], for instance) an explicit cocycle representative for this element

$$v_1^{s-4}v_2[t_1] + v_1^{s-3}[t_2] + v_1^{s-3}[t_1^3].$$

This element is detected by $q_1^{s-4}q_2h_0$ in the localized algebraic Novikov spectral sequence.

For $s = 4$ one finds, by direct computation, an explicit cocycle respresentative for $\overline{\alpha}_4$.

$$[t_1^4] + v_2[t_1] + v_1[t_2] + v_1[t_1^3] + v_1^2[t_1^2].$$

Upon multiplying by $\overline{\alpha}_1^3$ we have leading term $[t_1^4|t_1|t_1|t_1]$. It is classical that $h_0^3h_2 = 0$ in $H^*(P)$. Find $y \in \Omega^3 P$ with $dy = [\zeta_1^4|\zeta_1|\zeta_1|\zeta_1]$. Then obtain $y' \in \Omega^3(BP_*BP)$ by replacing ζ's by t's. Using lemma 6.1.3 we see that dy' cannot contain $v_2[t_1|t_1|t_1|t_1]$. Thus, picking off the elements of filtration 1 with only single powers of t_1's appearing in

$$\left([t_1^4] + v_2[t_1] + v_1[t_2] + v_1[t_1^3] + v_1^2[t_1^2]\right) \cdot [t_1|t_1|t_1] + dy'$$

gives $v_2[t_1|t_1|t_1|t_1]$. We deduce that $\overline{\alpha}_1^3\overline{\alpha}_4$ is detected by $q_2h_0^4$ in the localized algebraic Novikov spectral sequence. □

We can now obtain an explicit description of the localized Adams-Novikov E_2-page, in terms of elements which exist before localizing.

Corollary 6.2.3. $\overline{\alpha}_1^{-1}H^*(BP_*BP) = \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1\overline{\alpha}_4^2)$.

Proof. Consider the natural map $\mathbb{Z}_{(2)}[\overline{\alpha}_1, \overline{\alpha}_3, \overline{\alpha}_4] \longrightarrow H^*(BP_*BP)$. Because $\eta\sigma^2 = 0$ [20] in $\pi_*(S^0)$, we have $\overline{\alpha}_1\overline{\alpha}_4^2 = 0$ in $H^*(BP_*BP)$; we also have $2\overline{\alpha}_1 = 0$. Thus, the map above factors through a map

$$\mathbb{Z}_{(2)}[\overline{\alpha}_1, \overline{\alpha}_3, \overline{\alpha}_4]/(2\overline{\alpha}_1, \overline{\alpha}_1\overline{\alpha}_4^2) \longrightarrow H^*(BP_*BP).$$

Inverting $\overline{\alpha}_1$ gives a map $f : \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1\overline{\alpha}_4^2) \longrightarrow \overline{\alpha}_1^{-1}H^*(BP_*BP)$.

We have shown that $\overline{\alpha}_3$ and $\overline{\alpha}_4$ have images in $\overline{\alpha}_1^{-1}H^*(BP_*BP)$ detected by $q_1^2h_0$ and q_2h_0, respectively. The E_∞-page of the localized algebraic Novikov spectral sequence is $\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2]/(q_2)$; for each bidegree (s, u) there is only one t such that $E_\infty^{s,t,u} \neq 0$ and so the filtration is locally finite. These facts, together with convergence of the localized algebraic Novikov spectral sequence, allow one to check that f is injective and surjective. □

7 The localized motivic Adams-Novikov spectral sequence

We briefly recall the theorems of Voevodsky [9, 21] concerning mod 2 motivic homology (over an algebraically closed field of characteristic 0) and the motivic Steenrod algebra. The coefficient ring of motivic homology is $\mathbb{M}_2 = \mathbb{F}_2[\tau]$. Motivic homology is bigraded by dimension and *weight* and $|\tau| = (0, -1)$. The motivic dual Steenrod algebra is the Hopf algebra over $\mathbb{M}_2$ given by

$$A_{\text{Mot}} = \mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_n^2 = \tau\xi_{n+1}),$$

$$|\tau_n| = (2^{n+1} - 1, 2^n - 1), \ |\xi_n| = (2^{n+1} - 2, 2^n - 1)$$

The diagonal is given precisely by the analogue, with $p = 2$, of Milnor's formula for the diagonal at an odd prime.

Various other grading conventions are available. For example, from a topological perspective it is more natural to grade using the dimension and "Chow" (or "Novikov") degree [6]; it satisfies $\mathit{dimension} = \mathit{Chow} + 2 \cdot \mathit{weight}$, so $\mathit{Chow}\ \tau = 2$, $\mathit{Chow}\ \tau_n = 1$ and $\mathit{Chow}\ \xi_n = 0$. We will use the standard grading.

The simplicity of this picture was extended by Hu, Kriz, and Ormsby, to the motivic analogue of the Brown-Peterson spectrum. They constructed the motivic Adams-Novikov spectral sequence, at the prime 2 [9, (36)] (see also [5]). They showed that the motivic analogue of the Hopf algebroid (BP_*, BP_*BP) is simply the classical one tensored with $\mathbb{Z}[\tau]$. It follows that both the E_2-page of the motivic Adams-Novikov spectral sequence and the corresponding algebraic Novikov spectral sequence are obtained by adjoining τ. Thus, our work above has the following consequence.

Corollary 7.1. *Over an algebraically closed field of charateristic zero, the motivic Adams-Novikov E_2-page localizes to give* $\overline{\alpha}_1^{-1} E_2(S^{0,0}; BP_{\mathrm{Mot}}) = \mathbb{F}_2[\tau, \overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1 \overline{\alpha}_4^2)$.

The motivic Adams-Novikov spectral sequence takes the form

$$E_2^{s,u,w} = H^*(BP_*BP)[\tau]^{s,u,w} \overset{s}{\Longrightarrow} \pi_{u-s,w}(S^{0,0}),\ d_r : E_r^{s,u,w} \longrightarrow E_r^{s+r,u+r-1,w}.$$

If $x \in H^{s,u}(BP_*BP)$ is nonzero then u is even and $\tau^n x$ defines an element of $H^*(BP_*BP)[\tau]^{s,u,u/2-n}$. We can recover the classical Adams-Novikov spectral sequence by forgetting the weight and setting $\tau = 1$.

The motivic Adams-Novikov spectral sequence converges 2-locally [5, 9]. Moreover, powers of $\overline{\alpha}_1$ constitute the vanishing line in this spectral sequence. Since $2\eta = 0$ the telescope $\eta^{-1}S^{0,0}$ is killed by 2 and so the localized spectral sequence converges. We obtain the following theorem.

Theorem 7.2. *The homotopy of the η-localized motivic sphere spectrum $\eta^{-1}S^{0,0}$ is*

$$\eta^{-1}\pi_{*,*}(S^{0,0}) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta\sigma^2),$$

where $\eta \in \pi_{1,1}(S^{0,0})$ and $\sigma \in \pi_{7,4}(S^{0,0})$ are the motivic Hopf invariant one elements described in [6] and $\mu_9 \in \pi_{9,5}(S^{0,0})$ is the unique class in the Toda bracket $\langle 8\sigma, 2, \eta\rangle$. The element $\theta \in \pi_{0,-1}(S^{0,0})$ detected by τ acts trivially on $\eta^{-1}\pi_{*,*}(S^{0,0})$.

Proof. Classically, η, σ, μ_9 and $\eta\mu_9 = \mu_{10}$ are detected by $\overline{\alpha}_1$, $\overline{\alpha}_4$, $\overline{\alpha}_5$ and $\overline{\alpha}_1\overline{\alpha}_5 = \overline{\alpha}_3^2$, respectively, in the Adams-Novikov spectral sequence and we have a differential $d_3\overline{\alpha}_3 = \overline{\alpha}_1^4$.

We have shown that the E_2-page of the localized motivic Adams-Novikov spectral sequence is given by $\mathbb{F}_2[\tau, \overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1\overline{\alpha}_4^2)$ and the classical differential just referenced gives $d_3\overline{\alpha}_3 = \tau\overline{\alpha}_1^4$ in this context. Moreover, $\overline{\alpha}_3^2$ and $\overline{\alpha}_4$ are permanent cycles because they are classically. The E_∞-page is

$$\mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3^2, \overline{\alpha}_4]/(\overline{\alpha}_1\overline{\alpha}_4^2) = \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_4, \overline{\alpha}_5]/(\overline{\alpha}_1\overline{\alpha}_4^2);$$

for each stem d and weight w there is only one Adams-Novikov filtration s such that $E_\infty^{s,d+s,w} \neq 0$.

Consider the natural map $\mathbb{Z}[\eta, \sigma, \mu_9] \to \pi_{*,*}(S^{0,0})$. $\eta\sigma^2 = 0$ in $\pi_{*,*}(S^0)$, since this holds classically [20] and there are no "exotic" classes in the 15-stem with weight 9 [10]. We also have $2\eta = 0$. So the map above factors through a map $\mathbb{Z}[\eta, \sigma, \mu_9]/(2\eta, \eta\sigma^2) \longrightarrow \pi_{*,*}(S^{0,0})$ and inverting η gives a map $\mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta\sigma^2) \longrightarrow \eta^{-1}\pi_{*,*}(S^{0,0})$. Motivically, we still have that η, σ and μ_9 are detected by $\overline{\alpha}_1$, $\overline{\alpha}_4$ and $\overline{\alpha}_5$, respectively, and so using the convergence of the localized motivic Adams-Novikov spectral sequence together with the facts above, we see that this map is an isomorphism.

The proof is completed by observing that $\theta\eta^4 = 0$ [10]. $\square$

8 A comparison of spectral sequences

8.1 The diagram

In this final section, we complete the calculation of a square of spectral sequences, a localized version of the following square.

$$\begin{tikzcd}
H^*(P;Q)[\tau] \arrow[r, Rightarrow, "\text{CESS}"] \arrow[d, Rightarrow, "\text{alg-Nov-SS}[\tau]"'] & E_2(S^{0,0}; (H\mathbb{F}_2)_{\mathrm{Mot}}) \arrow[d, Rightarrow, "\text{MASS}"] \\
E_2(S^{0,0}; BP_{\mathrm{Mot}}) \arrow[r, Rightarrow, "\text{MANSS}"] & \pi_{*,*}(S^{0,0})
\end{tikzcd}$$

The right spectral sequence is the motivic Adams spectral sequence as studied in [5,9]. The bottom spectral sequence is the motivic Adams-Novikov spectral sequence described above, which was first studied in [9]. The left spectral sequence is the *motivic algebraic Novikov spectral sequence*, obtained by filtering $\pi_*(BP_{\mathrm{Mot}}) = BP_*[\tau]$ by powers of the kernel of the augmentation $\pi_*(BP_{\mathrm{Mot}}) \longrightarrow \mathbb{F}_2[\tau]$. By the results of Hu, Kriz, and Ormsby [9] this is simply the algebraic Novikov spectral sequence described in section 3 extended by τ. The grading of $H^*(P;Q)[\tau]$ follows that of $H^*(BP_*BP)[\tau]$. If $x \in H^{s,u}(P;Q^t)$ is nonzero then u is even and $\tau^n x$ defines an element of $H^*(P;Q)[\tau]^{s,t,u,u/2-n}$. The top spectral sequence is the Cartan-Eilenberg spectral sequence associated to the extension of Hopf algebras

$$\mathbb{M}_2 \otimes P \longrightarrow A_{\mathrm{Mot}} \longrightarrow \mathbb{M}_2 \otimes E. \tag{8.1.1}$$

This motivic Cartan-Eilenberg spectral sequence is indexed just as in (3.1.3), but with the additional weight grading that is preserved by differentials. The vanishing lines of 3.3.1 and 3.3.2 ensure that we can localize all the spectral sequences to obtain a square of convergent spectral sequences. The behavior of these spectral sequences is summarized in the following diagram.

$$\begin{tikzcd}
\mathbb{F}_2[\tau, h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots] \arrow[rr, Rightarrow, "d_3 q_1^2 = \tau h_0^3"] \arrow[d, Rightarrow, "d_1 q_{n+1} = q_n^2 h_0,\ n \geq 2"'] & & \mathbb{F}_2[h_0^{\pm 1}, v_1^4, v_2, v_3, \ldots] \arrow[d, Rightarrow, "d_2 v_{n+1} \equiv v_n^2 h_0,\ n \geq 2"] \\
\mathbb{F}_2[\tau, \overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1 \overline{\alpha}_4^2) \arrow[rr, Rightarrow, "d_3 \overline{\alpha}_3 = \tau \overline{\alpha}_1^4"] & & \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta \sigma^2)
\end{tikzcd}$$

We have calculated the left spectral sequence and the bottom one in the earlier sections of this paper. Guillou and Isaksen calculated the E_2-page of the localized motivic Adams spectral sequence in [7]. In the next section, we will give a different proof of their result by calculating the top spectral sequence. In the final section, we will use the techniques of [14] to determine the differentials in the localized motivic Adams spectral sequence, verifying another conjecture of Guillou and Isaksen [7].

8.2 The localized Cartan-Eilenberg spectral sequence

The extension of Hopf algebras (8.1.1) gives rise to a Cartan-Eilenberg spectral sequence, which we may localize by inverting $h_0 \in H(P;Q)[\tau]^{1,0,2,1}$.

Lemma 8.2.1. *In the localized Cartan-Eilenberg spectral sequence we have $d_3 q_1^2 = \tau h_0^3$. The classes q_1^4 and q_n for $n \geq 2$ are permanent cycles and so*

$$E_\infty = \mathbb{F}_2[h_0^{\pm 1}, q_1^4, q_2, q_3, \ldots].$$

Proof. The differential $d_3 q_1^2 = \tau h_0^3$ follows from the unlocalized differential $d_3\langle h_1, q_0^2, h_0\rangle = \tau h_0^4$ and this is forced on us by our limited knowledge of $H(A_{\mathrm{Mot}})$. Degree considerations show that q_1^4, and q_n for $n \geq 2$, are permanent cycles. □

We can now prove the following result of Guillou and Isaksen [7]. We note that they follow the conventions at $p = 2$, which hold classically and denote by h_1 the class which we call h_0.

Corollary 8.2.2. $h_0^{-1} E_2(S^{0,0}; (H\mathbb{F}_2)_{\mathrm{Mot}}) = \mathbb{F}_2[h_0^{\pm 1}, v_1^4, v_2, v_3, \ldots]$ *for some classes $v_1^4, v_2, v_3, \ldots$*

Proof. We choose a representative for q_1^4, which we call v_1^4, and for $n \geq 2$ we choose representatives for q_n, which we call v_n. Since the associated graded algebra is free on the classes of these generators, the result follows. □

8.3 Comparing Adams spectral sequences

In this section we will complete the calculation of the localized motivic Adams spectral sequence. By finding representatives, one sees that in the localized motivic Adams spectral sequence for the η-local sphere spectrum the elements v_1^4 and v_2 are permanent cycles. For the other generators, we have the following proposition, which follows from the techniques of [14].

Proposition 8.3.1. *For $n \geq 2$, we have $d_2 v_{n+1} = v_n^2 h_0$ up to higher Cartan-Eilenberg filtration.*

We will give an improvement, due to the first author, of the statement and the proof of the comparison result of [14] (which, in turn, followed ideas from [18]). The second author is eager to use this opportunity to clarify the proof given in [14], and to fill a gap: lemma 6.7 is not correct as stated there. What follows is a correct statement that serves the purpose in [14], and which will be used in the proof presented here as well. This lemma relates to the comparison of two boundary maps, and its importance cannot be overstated. It deals with the following situation. Let $A \longrightarrow B \longrightarrow C$ and $X \longrightarrow Y \longrightarrow Z$ be cofiber sequences. Smash them together to form the followign commutative diagram of cofiber sequences.

$$\begin{tikzcd}
A \wedge X \arrow[r] \arrow[d] & A \wedge Y \arrow[r] \arrow[d] & A \wedge Z \arrow[d] \\
B \wedge X \arrow[r] \arrow[d] & B \wedge Y \arrow[r] \arrow[d] & B \wedge Z \arrow[d] \\
C \wedge X \arrow[r] & C \wedge Y \arrow[r] & C \wedge Z
\end{tikzcd}$$

Let b be an element of $\pi_n(B\wedge Y)$ that maps to 0 in $\pi_n(C\wedge Z)$. Then there is an element $a \in \pi_n(A\wedge Z)$ mapping to the image of b in $\pi_n(B \wedge Z)$, and an element $c \in \pi_n(C \wedge X)$ mapping to the image of b in $\pi_n(C \wedge Y)$.

Lemma 8.3.2 (May [11])**.** *The elements a and c can be chosen so that they have the same image (up to a conventional sign) in $\pi_{n-1}(A \wedge X)$ under the boundary maps associated to the cofiber sequences along the top and the left edge of the diagram.*

This statement is a small part of an elaborate structure enriching the displayed 3×3 diagram. This structure is described in detail and proved by May in [11]. In the founding days of the theory of triangulated categories, Verdier [4] showed that a 2×2 diagram can always be extended to a 3×3 diagram of cofiber sequences. An analysis of his proof reveals that it actually produces precisely the structure verified by May for the specific case in which the 3×3 diagram occurs by smashing together two cofiber sequences.

For clarity, we will work in the non-motivic context, and in the specific case of BP and $H\mathbb{F}_2$, and the sphere spectrum. We will then indicate the general setting under which the result holds and this will prove the proposition just stated. Write H for the mod 2 Eilenberg Mac Lane spectrum.

So we have the following square of spectral sequences. The initial two are the algebraic Novikov spectral sequence (3.1.2) and the Cartan-Eilenberg spectral sequence (3.1.3).

$$\begin{tikzcd}
H^{s,u}(P;Q^t) \arrow[r, Rightarrow, "\text{CESS}"] \arrow[d, Rightarrow, "\text{alg-Nov-SS}"'] & E_2^{s+t,u+t}(S^0;H) \arrow[d, Rightarrow, "\text{ASS}"] \\
E_2^{s,u}(S^0;BP) \arrow[r, Rightarrow, "\text{ANSS}"] & \pi_{u-s}(S^0)
\end{tikzcd}$$

Theorem 8.3.3. *Suppose $x \in F^s_{CE}E_2^{s+t,u+t}(S^0;H)$. Then the Cartan-Eilenberg filtration of $d_2^{ASS}x$ is higher:*

$$d_2^{ASS}x \in F^{s+1}_{CE}E_2^{s+t+2,u+t+1}(S^0;H).$$

Moreover, if x is detected in the Cartan-Eilenberg spectral sequence by $a \in H^{s,u}(P;Q^t)$ then $d_2^{ASS}x$ is detected by $d_1^{alg\text{-}Nov}a \in H^{s+1,u}(P;Q^{t+1})$.

Proof. The proof hinges on geometric constructions of the two algebraically defined spectral sequences. Both arise from the canonical BP-resolution of S^0 and so we recall how this resolution is constructed. From the unit map of the ring spectrum BP we can construct a cofiber sequence

$$S^0 \longrightarrow BP \longrightarrow \overline{BP}. \tag{8.3.4}$$

Smashing this cofiber sequence with various smash-powers of $\overline{BP}$ gives the canonical BP-resolution of S^0.

$$\begin{tikzcd} S^0 \arrow[d] & \overline{BP} \arrow[l, "+"'] \arrow[d] & \cdots \arrow[l, "+"'] & \overline{BP}^{\wedge s} \arrow[l, "+"'] \arrow[d] & \overline{BP}^{\wedge(s+1)} \arrow[l, "+"'] \arrow[d] & \cdots \arrow[l, "+"'] \\ BP^{[0]} & BP^{[1]} & & BP^{[s]} & BP^{[s+1]} & \end{tikzcd} \tag{8.3.5}$$

Here $BP^{[s]} = BP \wedge \overline{BP}^{\wedge s}$ and the marked arrows indicate that they map from a desuspension.

First, we set up the Cartan-Eilenberg spectral sequence. Smashing (8.3.4) with a spectrum X and applying mod 2 homology gives a short exact sequence

$$0 \longrightarrow H_*(X) \longrightarrow H_*(BP \wedge X) \longrightarrow H_*(\overline{BP} \wedge X) \longrightarrow 0$$

and thus a long exact sequence

$$\cdots \longrightarrow E_2^{t,u}(X;H) \longrightarrow E_2^{t,u}(BP \wedge X;H) \longrightarrow E_2^{t,u}(\overline{BP} \wedge X;H) \xrightarrow{\delta} E_2^{t+1,u}(X;H) \longrightarrow \cdots.$$

This means that applying $E_2(-;H)$ to (8.3.5) gives an exact couple and hence a spectral sequence. We index the spectral sequence so that

$$E_1^{s,t,u} = E_2^{t,u+t}(BP^{[s]};H) \overset{s}{\Longrightarrow} E_2^{s+t,u+t}(S^0;H).$$

We see that $E_2^{t,u+t}(BP^{[s]};H)$ can be identified with $(Q^t \otimes \overline{P}^{\otimes s})_u$ so that our E_1-term is isomorphic, as a complex, to $\Omega^*(P;Q)$; so $E_2^{s,t,u} = H^{s,u}(P;Q^t)$. This spectral sequence is, in fact, the Cartan-Eilenberg spectral sequence of (3.1.3). The Cartan-Eilenberg filtration of $E_2(S^0;H)$ is given by

$$F^s_{\mathrm{CE}}E_2^{s+t,u+t}(S^0;H) = \mathrm{im}\left(\delta^s : E_2^{t,u+t}(\overline{BP}^{\wedge s};H) \longrightarrow E_2^{s+t,u+t}(S^0;H)\right). \tag{8.3.6}$$

If $x = \delta^s z$ then x is detected in the Cartan-Eilenberg E_1-page by the image of z under the map $E_2^{t,u+t}(\overline{BP}^{\wedge s};H) \longrightarrow E_2^{t,u+t}(BP^{[s]};H)$.

We turn to the first part of the theorem statement: that Adams d_2-differentials increase Cartan-Eilenberg filtration. For this we need to recall a construction of the Adams spectral sequence for a spectrum X. We also need to recall the geometric boundary theorem to give a geometric description of δ, the connecting homomorphism appearing in (8.3.6).

Again we have a cofiber sequence $S^0 \longrightarrow H \longrightarrow \overline{H}$ and we define, similarly to before but with a transposition, $H^{[t]} = \overline{H}^{\wedge t} \wedge H$. The cofiber sequences

$$\overline{H}^{\wedge(t+1)} \wedge X \longrightarrow\!\!\!\!\!\!\!\!\!+\quad \overline{H}^{\wedge t} \wedge X \longrightarrow H^{[t]} \wedge X$$

link together as in (8.3.5) and the Adams spectral sequence for X is obtained by applying $\pi_*(-)$. We have $E_1^{t,u} = \pi_u(H^{[t]} \wedge X)$ and $E_2^{t,u} = H^{t,u}(A; H_*(X))$. We note that $H_*(X)$ is a *left* A-comodule here.

The map of ring spectra $BP \longrightarrow H$ descends to a map $\overline{BP} \longrightarrow \overline{H}$. This provides us with maps

$$\overline{H}^{\wedge t} \wedge \overline{BP} \wedge X \longrightarrow \overline{H}^{\wedge(t+1)} \wedge X \tag{8.3.7}$$

for any spectrum X. Moreover, since

$$\begin{tikzcd}
\Sigma^{-1}\overline{H} \wedge \overline{BP} \arrow[r] \arrow[d] & S^0 \wedge \overline{H} \arrow[d, "="] \\
\overline{H} \wedge S^0 \arrow[r, "="] & \overline{H}
\end{tikzcd}$$

commutes, the maps of (8.3.7) descend to maps $H^{[t]} \wedge \overline{BP} \wedge X \longrightarrow H^{[t+1]} \wedge X$. The geometric boundary theorem (or a short calculation) says that $\pi_u(H^{[t]} \wedge \overline{BP} \wedge X) \longrightarrow \pi_u(H^{[t+1]} \wedge X)$ induces

$$\delta : E_2^{t,u}(\overline{BP} \wedge X; H) \longrightarrow E_2^{t+1,u}(X; H).$$

Suppose that $x \in F^s_{\mathrm{CE}} E_2^{s+t,u+t}(S^0; H)$ so that $x = \delta^s z$ for some $z \in E_2^{t,u+t}(\overline{BP}^{\wedge s}; H)$. Since δ is a map of spectral sequences we see that $d_2^{\mathrm{ASS}} x = \delta^s(d_2^{\mathrm{ASS}} z)$. Thus, in order to show that

$$d_2^{\mathrm{ASS}} x \in F^{s+1}_{\mathrm{CE}} E_2^{s+t+2,u+t+1}(S^0; H)$$

it is enough to find an element mapping to $d_2^{\mathrm{ASS}} z$ under

$$\delta : E_2^{t+1,u+t+1}(\overline{BP}^{\wedge(s+1)}; H) \longrightarrow E_2^{t+2,u+t+1}(\overline{BP}^{\wedge s}; H).$$

z is represented by an element $z' \in \pi_{u+t}(H^{[t]} \wedge \overline{BP}^{\wedge s})$ and we claim that $\partial z' \in \pi_{u+t}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge s})$ lifts (see (8.3.10)) to

$$y_0 \in \pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)}). \tag{8.3.8}$$

Well, z' maps to an element $a' \in \pi_{u+t}(H^{[t]} \wedge BP^{[s]})$, which represents a class in $E_2^{t,u+t}(BP^{[s]}; H)$. Since the Adams spectral sequence for $BP^{[s]}$ degenerates at the E_2-page, a' lifts to

$$y_1 \in \pi_{u+t}(\overline{H}^{\wedge t} \wedge BP^{[s]}). \tag{8.3.9}$$

By considering the following diagram in which the bottom row and right column is exact we obtain y_0.

$$\begin{tikzcd}
\pi_{u+t}(\overline{H}^{\wedge t} \wedge \overline{BP}^{\wedge(s+1)}) & & \pi_{u+t}(\overline{H}^{\wedge t} \wedge BP^{[s]}) \arrow[ll, "\partial_h"'] \arrow[d] \\
& \pi_{u+t}(H^{[t]} \wedge \overline{BP}^{\wedge s}) \arrow[r] \arrow[d, "\partial"] & \pi_{u+t}(H^{[t]} \wedge BP^{[s]}) \arrow[d] \\
\pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)}) \arrow[r] \arrow[uu, "\partial_v"] & \pi_{u+t}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge s}) \arrow[r] & \pi_{u+t}(\overline{H}^{\wedge(t+1)} \wedge BP^{[s]})
\end{tikzcd} \tag{8.3.10}$$

To compute a representative for $d_2^{\mathrm{ASS}} z$ we chase z' through the zig-zag in the exact couple defining the Adams spectral sequence, displayed in the following diagram (the wavey arrow indicates that we do not have function, but a correspondence).

$$\begin{tikzcd}
\pi_{u+t}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge s}) & \pi_{u+t+1}(\overline{H}^{\wedge(t+2)} \wedge \overline{BP}^{\wedge s}) \arrow[l] \arrow[d] & \pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)}) \arrow[l] \arrow[d] \\
\pi_{u+t}(H^{[t]} \wedge \overline{BP}^{\wedge s}) \arrow[u, "\partial"] \arrow[r, squiggly, "d_2"] & \pi_{u+t+1}(H^{[t+2]} \wedge \overline{BP}^{\wedge s}) & \pi_{u+t+1}(H^{[t+1]} \wedge \overline{BP}^{\wedge(s+1)}) \arrow[l]
\end{tikzcd}$$

Since $\partial z'$ lifts to $y_0 \in \pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)})$ we find that $d_2^{\mathrm{ASS}} z = \delta \overline{y}_0$, where

$$\overline{y}_0 \in E_2^{t+1,u+t+1}(\overline{BP}^{\wedge(s+1)}; H) \tag{8.3.11}$$

is represented by the image of y_0 under the map

$$\pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)}) \longrightarrow \pi_{u+t+1}(H^{[t+1]} \wedge \overline{BP}^{\wedge(s+1)}).$$

This completes the proof of the first part of the theorem.

Now we interpret the algebraic Novikov spectral sequence geometrically by recalling that the Adams-Novikov spectral sequence is obtained from (8.3.5) by applying $\pi_*(-)$, so that we have an E_1-page given by $E_1^{s,u} = \pi_u(BP^{[s]})$. Moreover, we have an identification $\pi_u(BP^{[s]}) = \Omega^s(BP_*BP)_u$ so that our E_1-term is isomorphic, as a complex, to the cobar complex $\Omega^*(BP_*BP)$. The algebraic Novikov spectral sequence is constructed by filtering $\Omega^*(BP_*BP)$ by powers of the kernel of the augmentation $BP_* \to \mathbb{F}_2$. This kernel consists of the elements in BP_* of Adams filtration greater than or equal to 1; the algebraic Novikov spectral sequence is obtained by filtering $\pi_*(BP^{[*]})$ using the Adams filtration and

$$E_0^{s,t,u} = F^t_{\mathrm{Adams}} \pi_u(BP^{[s]}) / F^{t+1}_{\mathrm{Adams}} \pi_u(BP^{[s]}).$$

Let us turn to the calculation of d_1 in the algebraic Novikov spectral sequence. It is captured by the bottom line of the following diagram.

$$\begin{tikzcd}
 & \pi_{u+t}(\overline{H}^{\wedge t} \wedge \overline{BP}^{\wedge(s+1)}) \arrow[d] & \pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)}) \arrow[l, "\partial_v"'] \arrow[d] \\
\pi_{u+t}(\overline{H}^{\wedge t} \wedge BP^{[s]}) \arrow[ur, "\partial_h"] \arrow[r] \arrow[d] & \pi_{u+t}(\overline{H}^{\wedge t} \wedge BP^{[s+1]}) \arrow[d] & \pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge BP^{[s+1]}) \arrow[l] \arrow[d] \\
F^t_{\mathrm{Adams}} \pi_u(BP^{[s]}) \arrow[r, "d"] & F^t_{\mathrm{Adams}} \pi_u(BP^{[s+1]}) & F^{t+1}_{\mathrm{Adams}} \pi_u(BP^{[s+1]}) \arrow[l, hook']
\end{tikzcd} \tag{8.3.12}$$

Start with $a \in E_1^{s,t,u}$(alg-Nov). This element is represented by some $\tilde{a} \in F^t_{\mathrm{Adams}} \pi_u(BP^{[s]})$. Because $\tilde{a}$ represents an element of E_1(alg-Nov), $d\tilde{a}$ lies in higher Adams filtration $d\tilde{a} \in F^{t+1}_{\mathrm{Adams}} \pi_u(BP^{[s+1]})$. $d\tilde{a}$ represents $d_1^{\text{alg-Nov}} a \in E_1^{s+1,t+1,u}$(alg-Nov). The various filtrations are witnessed by the middle row of (8.3.12).

Since the Adams spectral sequence for $BP^{[s]}$ is degenerate we have the following commutative diagram (the wavy arrow indicates that there is no function, but that elements of $E_2^{t,u+t}(BP^{[s]};H)$ are represented by elements in $\pi_{u+t}(H^{[t]} \wedge BP^{[s]})$).

$$\begin{tikzcd}
& \pi_{u+t}(\overline{H}^{\wedge t} \wedge BP^{[s]}) \arrow[dl] \arrow[d] \arrow[r, two heads] & F^t_{\text{Adams}}\pi_u(BP^{[s]}) \arrow[d] \\
\pi_{u+t}(H^{[t]} \wedge BP^{[s]}) \arrow[r, squiggly] & E_2^{t,u+t}(BP^{[s]};H) \arrow[r, "\cong"] & F^t_{\text{Adams}}\pi_u(BP^{[s]})/F^{t+1}_{\text{Adams}}\pi_u(BP^{[s]})
\end{tikzcd} \tag{8.3.13}$$

We turn to the final statement of the theorem.

Suppose that $x \in F^s_{\text{CE}}E_2(S^0;H)$ is detected by $a \in H^{s,u}(P;Q^t)$ in the Cartan-Eilenberg spectral sequence. Referring back to the elements in the first part of the proof, this means that we can assume that z maps to an element $\hat{a} \in E_2^{t,u+t}(BP^{[s]};H) = (Q^t \otimes \overline{P}^{\otimes s})_u$ representing a in the cobar construction. Moreover, choosing z' determines a', which represents $\hat{a}$, and choosing y_1 (8.3.9) leads to a preferred choice of $\tilde{a}$. These elements fit into the diagram of (8.3.13).

$$\begin{tikzcd}
& y_1 \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dl, mapsto] & \tilde{a} \arrow[d, mapsto] \\
a' \arrow[r, squiggly, mapsto] & \hat{a} \arrow[r, mapsto, "\cong"] & \hat{a}
\end{tikzcd}$$

To compute $d_1^{\text{alg-Nov}}a$, we refer to (8.3.12). We find that it is enough to lift $\partial_h y_1$ along ∂_v. However, we already have the element y_0 of (8.3.8) and we would like for $\partial_v y_0$ to be equal to $\partial_h y_1$. We assume this for now. In the following diagram y_0 is mapped down to $d\tilde{a}$ and right to $\overline{y}_0$ (8.3.11). (Again, wavey arrows indicate where representing elements live.)

$$\begin{tikzcd}
\pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)}) \arrow[r] \arrow[d] & \pi_{u+t+1}(H^{[t+1]} \wedge \overline{BP}^{\wedge(s+1)}) \arrow[r, squiggly] \arrow[d] & E_2^{t+1,u+t+1}(\overline{BP}^{\wedge(s+1)};H) \arrow[d] \\
\pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge BP^{[s+1]}) \arrow[r] \arrow[d] & \pi_{u+t+1}(H^{[t+1]} \wedge BP^{[s+1]}) \arrow[r, squiggly] & E_2^{t+1,u+t+1}(BP^{[s+1]};H) \\
F^{t+1}_{\text{Adams}}\pi_u(BP^{[s+1]}) & &
\end{tikzcd}$$

Thus, using (8.3.13) with s and t replaced by $(s+1)$ and $(t+1)$, respectively, we see that $d_1^{\text{alg-Nov}}a$ is represented by the image of $\overline{y}_0$ in $E_2^{t+1,u+t+1}(BP^{[s+1]};H)$. Recalling that $d_2^{\text{ASS}}x = \delta^s(d_2^{\text{ASS}}z) = \delta^{s+1}\overline{y}_0$, we conclude that $d_2^{\text{ASS}}x$ is detected by $d_1^{\text{alg-Nov}}a$ in the Cartan-Eilenberg spectral sequence.

We are left to show the compatibility

$$\partial_v y_0 = \partial_h y_1.$$

This follows directly from May's lemma 8.3.2 applied to the following cofiber sequences.

$$\begin{tikzcd}
\overline{H}^{\wedge t} \arrow[r] & H^{[t]} \arrow[r] & \overline{H}^{\wedge(t+1)} \\
\Sigma^{-1}\overline{BP}^{\wedge(s+1)} \arrow[r] & \overline{BP}^{\wedge s} \arrow[r] & BP^{[s]}
\end{tikzcd}$$

We start with $z' \in \pi_{u+t}(H^{[t]} \wedge \overline{BP}^{\wedge s})$; see (8.3.10). $\square$

This proof works in much greater generality. As in [14], the square of spectral sequences can be set up for any map of ring spectra $A \longrightarrow B$ and any spectrum X for which the B-Adams spectral sequence

$$E_2(A \wedge \overline{A}^{\wedge s} \wedge X; B) \implies \pi_*(A \wedge \overline{A}^{\wedge s} \wedge X)$$

converges and collapses at the E_2-page for all s. The proof holds whenever $B^1(\overline{B} \wedge \overline{A}) = 0$.

In particular, the proof works in the motivic context. Thus, the differentials constructed in proposition 6.1.1 produce the following differentials in the motivic Adams spectral sequence:

$$d_2 v_{n+1} \equiv v_n^2 h_0, \; n \geq 2$$

modulo terms of higher Cartan-Eilenberg filtration. We have

$$h_0 \in E_2^{1,2,1}(S^{0,0}; (H\mathbb{F}_2)_{\mathrm{mot}}), \; v_n \in E_2^{1,2^{n+1}-1,2^n-1}(S^{0,0}; (H\mathbb{F}_2)_{\mathrm{mot}}).$$

Recalling that $Chow = t - 2w = 1$ we see $Chow(h_0) = 0$ and $Chow(v_n) = 1$. Thus $v_n^2 h_0$ has Chow degree 2 and Adams filtration 3 and any other such element must be of the form $v_i v_j h_0$. We find that

$$v_n^2 h_0 \in E_2^{3,2^{n+2},2^{n+1}-1}(S^{0,0}; (H\mathbb{F}_2)_{\mathrm{mot}})$$

is the only element in its trigrading so, in fact, our calculation has no indeterminancy. This is our last theorem.

Theorem 8.3.14. *In the η-localized motivic Adams spectral sequence,*

$$d_2 v_{n+1} = v_n^2 h_0, \; n \geq 2.$$

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