Lecture 17 - More Ext Techniques & A differential

Note Title 4/4/2009

Begin with two additional ways to compute Ext.

① Long Exact Sequence

② Truncated resolutions

<u>Notation</u> $\mathrm{Ext}(M)$ will denote $\mathrm{Ext}_A(M, \mathbb{F}_2)$ or $\mathrm{Ext}_{A(1)}(M, \mathbb{F}_2)$

If M fits into a SES

$$0 \to M' \to M \to M'' \to 0,$$

the have a LES on Ext:

$$\cdots \to \mathrm{Ext}^{s,t}(M'') \to \mathrm{Ext}^{s,t}(M) \to \mathrm{Ext}^{s,t}(M') \xrightarrow{\delta} \mathrm{Ext}^{s+1,t}(M'') \to \cdots$$

So we have a SES which gives $\mathrm{Ext}^{s,t}(M)$:

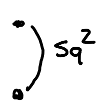
$$0 \to \mathrm{coker}(\delta) \to \mathrm{Ext}^{s,t}(M) \to \ker(\delta) \to 0,$$

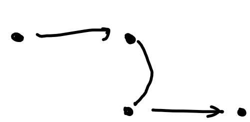
and if we can determine δ, $\mathrm{Ext}(M')$, $\mathrm{Ext}(M'')$, we'll be close to understanding $\mathrm{Ext}(M)$.

<u>Remark</u> This vastly generalizes. If $F_\bullet M$ is a filtration of M, then there is a spectral sequence w/ E_1-term $\mathrm{Ext}(Gr_\bullet(M))$ and where the d_r-differentials take $\mathrm{Ext}^{s,t}(Gr_k(M)) \to \mathrm{Ext}^{s+1,t}(Gr_{k-r} M)$.

<u>Prop 1</u> The long exact sequence in Ext is a long exact sequence of modules over $\mathrm{Ext}(\mathbb{F}_2)$.

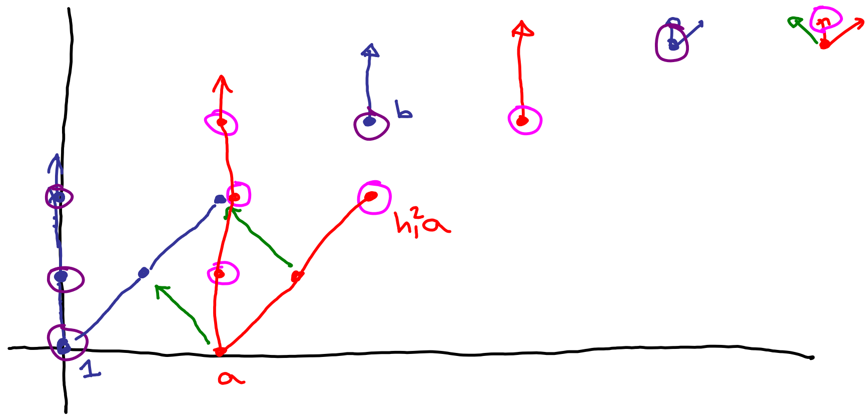
This actually makes computation strikingly easy.

<u>Ex</u>: $M = C(\eta) =$ 

Sits in a SES 

$$\Sigma^2 \mathbb{F}_2 \to C(\eta) \to \mathbb{F}_2$$

So $\delta: \mathrm{Ext}^{s,t}(\Sigma^2 \mathbb{F}_2) \to \mathrm{Ext}^{s+1,t}(\mathbb{F}_2)$.



$\bullet = \mathrm{Ext}(\mathbb{F}_2)$

$\bullet = \mathrm{Ext}(\Sigma^2 \mathbb{F}_2)$

take resolution for $\mathbb{F}_2$ and shift 2 units over.

$\nwarrow = \delta$

$\circ$ = coker

$\circ = \ker(\delta)$

We find δ by using the following fact:

Prop 2 If $0 \to M' \to M \to M'' \to 0$, then

$$\delta : \mathrm{Hom}(M', M') \to \mathrm{Ext}^1(M'', M')$$

sends $\mathrm{Id}_{M'}$ to the class of the extension.

In our case, we have an element $\mathrm{Id} \in \mathrm{Hom}^2(\Sigma^2 \mathbb{F}_2, \mathbb{F}_2)$, and this goes to h_1, the extension of M.

These are extensions as $\mathrm{Ext}(\mathbb{F}_2)$-modules! Elements in $\ker(\delta)$ are in a quotient module of $\mathrm{Ext}(M)$, not in $\mathrm{Ext}(M)$ itself. In fact, $h_0 \cdot (h_1^2 a) = b$.

① Is not useful for proving that some Ext group is periodic. For that, we need another result.

② Truncated Resolutions.

Given a module M, let P_s be the s^{th} stage of a projective resolution and K_s the kernel of $P_s \to P_{s-1}$ $(= \mathrm{Im}(P_{s+1} \to P_s))$.

Prop 3 If $P_\bullet$ is a minimal resolution, then

$$\mathrm{Ext}^r(M) = \begin{cases} \mathrm{Hom}(P_r, \mathbb{F}_2) & r \leq s \\ \mathrm{Ext}^{r-s-1}(K_s, \mathbb{F}_2) & r \geq s+1 \end{cases}$$

In some sense this is obvious: a minimal resolution of K_s continues the resolution of M.

Remark if $P_\bullet$ is not minimal, then we get a SS w/ $E_1 = \mathrm{Ext}(\mathrm{res})$.

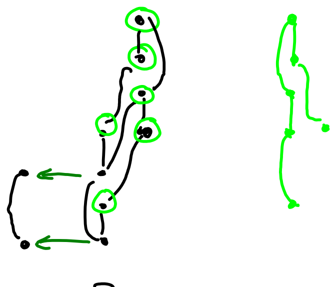
This even works if none of the stages is projective:

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \quad \Rightarrow \quad SS \text{ w/ } E_1 = \bigoplus \mathrm{Ext}(M_i) \Rightarrow \mathrm{Ext}(M).$$

This is the hypercohomology SS.

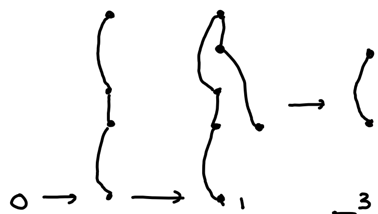
This works very well when coupled w/ ①.

Ex $M = C(\eta)$



P_0 K_0

Now K_0 sits in a SES:

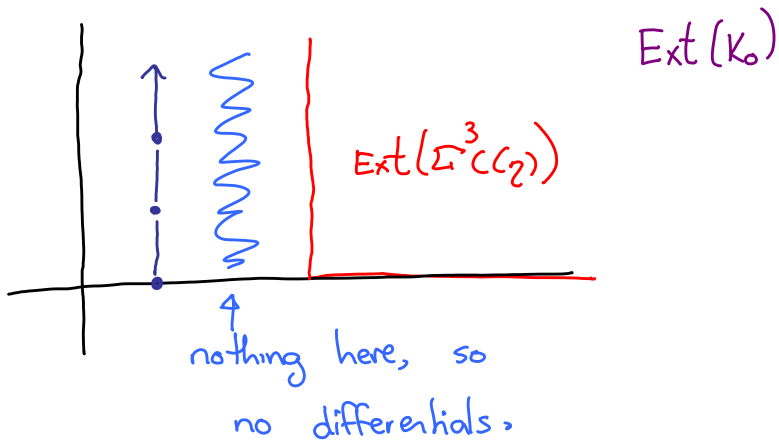


$$0 \to \Sigma H\mathbb{Z} \to K_0 \to \Sigma^3 C(\eta) \to 0$$

Use ① to compute $\mathrm{Ext}(K_0)$:

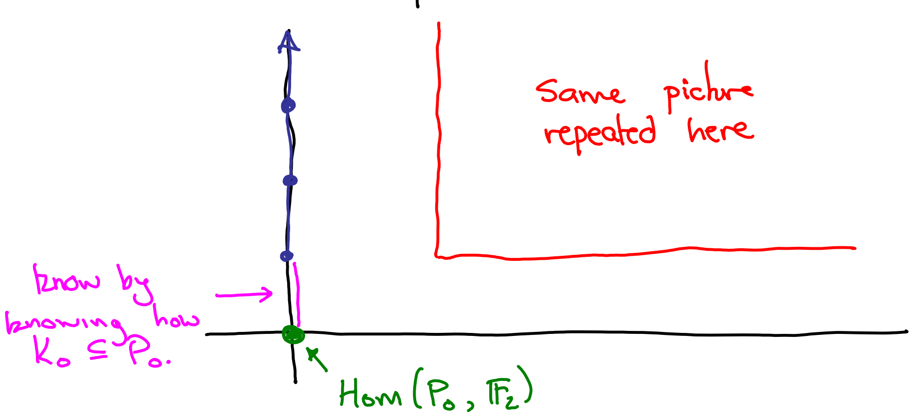
$\mathrm{Ext}(\Sigma H\mathbb{Z}) = \bullet$

$\mathrm{Ext}(\Sigma^3 C(\eta)) = ?$



This picture goes into that for $\mathrm{Ext}(C(\eta))$ in filtration 1

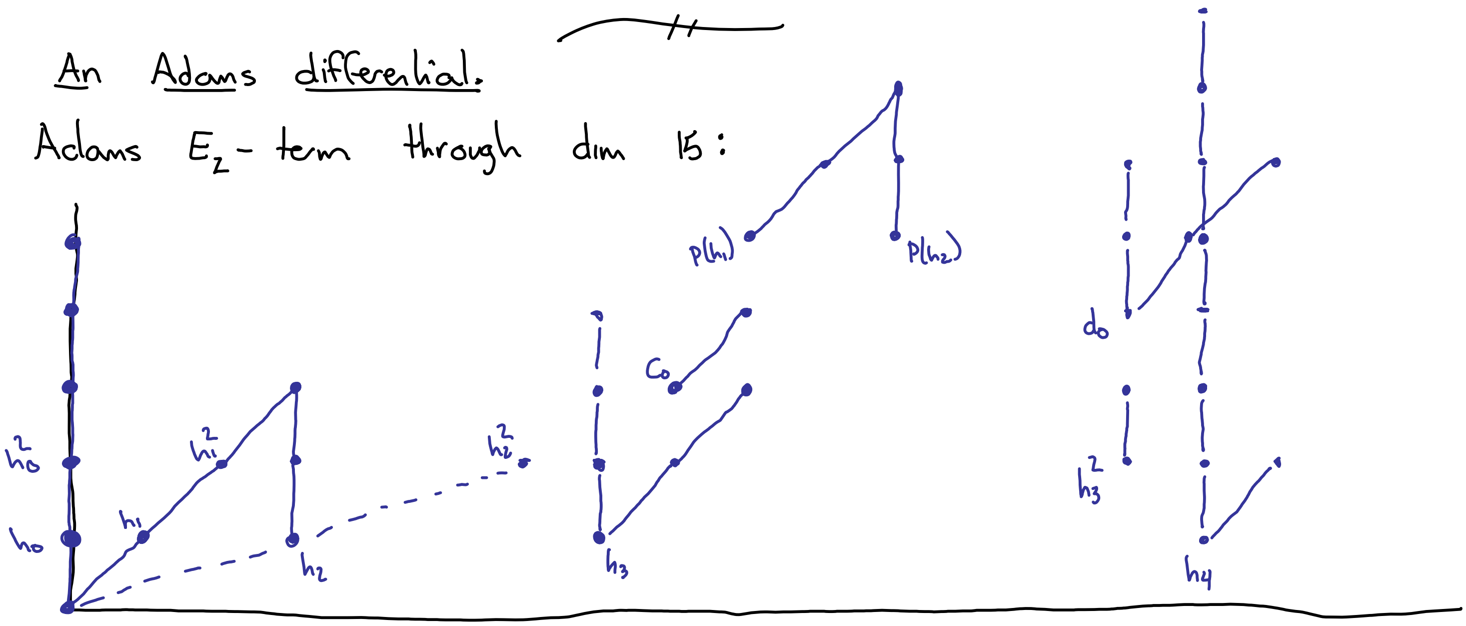
$\Rightarrow$ shift up 1 and 1 to the left



So we see that the picture is periodic w/ period $(2,1)$.

An Adams differential.

Adams E_2-term through dim 15:



Thm 1 The Adams SS is a SS of algebras:

$$d_r(a \cdot b) = d_r(a) b + a \cdot d_r(b),$$

converging to $\pi_* S^0$ as a ring.

Can use this to show that there are no possible diffs through dim 14.

Lem 1 ① h_0 is a perm cycle.

② h_1 is a perm cycle.

Proof A d_r-differential changes $t-s$ by -1 and s by r. So ① is immediate. For ②, assume $d_r(h_1) = h_0^{r+1}$. Then

$$0 = d_r(0) = d_r(h_0 \cdot h_1) = h_0 \cdot d_r(h_1) + h_1 d_r(h_0) = h_0^{r+2}.$$

Since h_0 is a polynomial gen, no power is zero. ☒

Lem 2 h_2, h_3, c_0, Ph_1, Ph_2, d_0 all perm cycles.

Pf In all cases, the targets of possible diffs are zero for degree reasons. ☒

Cor 1 All classes of dim ≤ 14 are perm cycles.

Pf They are all products of perm cycles. ☒

Now we can get a differential.

Prop 4: $h_0 \cdot (-)$ detects mult by 2.

Pf $\pi_0 S^0 = \mathbb{Z}$ & filt is by powers of 2. Alt: $H^*(H\mathbb{Z}) = A \otimes_{A(0)} \mathbb{F}_2$. ☒

The class h_3 survives to a homotopy class in $\pi_7 S^0$, σ.
$\pi_* S^0$ is a graded commutative ring, so we know

$$2\cdot\sigma^2 = 0, \quad \text{since } \sigma \text{ is odd.}$$

However, $h_0 \cdot h_3^2$ detects this, so it must also be zero by E_∞.

<u>Thm 2</u> There is a d_2-differential

$$d_2(h_4) = h_0 h_3^2.$$

<u>Pf</u> This is the only way to kill $h_0 h_3^2$. ☒

<u>Cor 2</u> There is no map $S^{15} \xrightarrow{f} S^0$ s.t. $Sq^{16} \neq 0$ in $H^*(C(f))$.

<u>Pf</u> h_4 would detect such a map. ☒

<u>Thm 3</u> There is a d_3-differential

$$d_3(h_0 h_4) = h_0 d_0.$$

This completes our analysis through dim 15.