Lecture 14 - ko Homology

Note Title 3/27/2009

To get the feel for the Adams SS, we'll look at a simple case: ko-homology.

Def Let ko be the (-1)-connected cover of the real K-theory spectrum.

Recall that $KO^0(X)$ = Grothendieck group of isom classes of real vector bundles on X & $KO^{-n}(X) = KO(\Sigma^n X)$.

Thm 1 (Bott Periodicity) $KO^{-8}(X) \cong KO(X)$.

This can be restated in terms of the spaces in KO, the spectrum representing $KO^*(-)$.

$KO_{n+8} = KO_n$.

n	-1	0	1	2	3		7	8
KO_n	0	$BO\times\mathbb{Z}$	?	?	?	$\cdots$	0	$BO\times\mathbb{Z}$

$\varinjlim_n O(n)$

All of the form BSp, $BSpin$, O/U, etc.

Can do all of this for KU = complex K-theory:

n	0	1	2	3	
KU_n	$\cdots$ $BU\times\mathbb{Z}$	U	$BU\times\mathbb{Z}$	U	$\cdots$

$(KU^{-2}(X) \cong KU(X))$

We want a connective theory.

Recall: $\pi_k \mathbf{X} = \varinjlim \pi_{n+k} X_n$

So for $\mathbf{X}$ to have no lower homotopy, we must have

$$\pi_{n-k} X_n = 0 \quad \text{for } k>0 \quad \text{(at least finally)}.$$

$\Rightarrow$ have to take higher & higher connective covers.

n	-1	0	1	2	3		n
ko_n	$\cdots$ $*$	$BO\times\mathbb{Z}$	$KO_1\langle 1\rangle$	$KO_2\langle 2\rangle$	$KO_3\langle 3\rangle$	$\cdots$	$KO_n\langle n\rangle$

Why would we look at this.

① $ko_0 X = KO_0 X$ if X is connected.

② ASS for $\pi_* ko \wedge X = ko_* X$ is easy!

Def Let $A(n)$ be the subalg of A gen by $Sq^0, Sq^1, Sq^2, \ldots, Sq^{2^n}$.

Let $E(n)$ " " " " " " " $Sq^0, Q_0, Q_1, \ldots, Q_n$,

where Q_i is the Milnor primitive $Q_i = [Sq^{2^i}, Q_{i-1}]$, $Q_0 = Sq^1$.

Both $A(n)$ and $E(n)$ are sub-Hopf algebras of A $\Rightarrow$ have an action of each on A on the right and on $\mathbb{F}_2$.

Thm 2 •) $H^*(ko) = A \otimes_{A(1)} \mathbb{F}_2$

as A-modules.

•) $H^*(ku) = A \otimes_{E(1)} \mathbb{F}_2$

If B is a sub-Hopf alg of A, then sometimes write $A//B$ for $A \otimes_B \mathbb{F}_2$.

read "mod-mod"

Cor 1: For any X, $H^*(ko \wedge X) = A \otimes_{A(1)} H^*(X)$ as A-modules.

This means that $H^*(ko \wedge X)$ is an induced A-module, so the A-module structure is completely determined by the $A(1)$-module structure of $H^*(X)$.

Thm 3 (Change-of-rings / Frobenius reciprocity)

$$\mathrm{Ext}_A^{s,t}(A \otimes_{A(1)} H^*(X), \mathbb{F}_2) \cong \mathrm{Ext}_{A(1)}^{s,t}(H^*(X), \mathbb{F}_2).$$

Big Reason to expect this.

If R is a subring of S, and M is an R-module, then

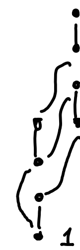
$$\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M, N)$$

So Thm 3 is the "derived" version of this.

Cor 2 The Adams E_2-term for $ko_* X$ is

$$\mathrm{Ext}_{A(1)}^{s,t}(H^*(X), \mathbb{F}_2).$$

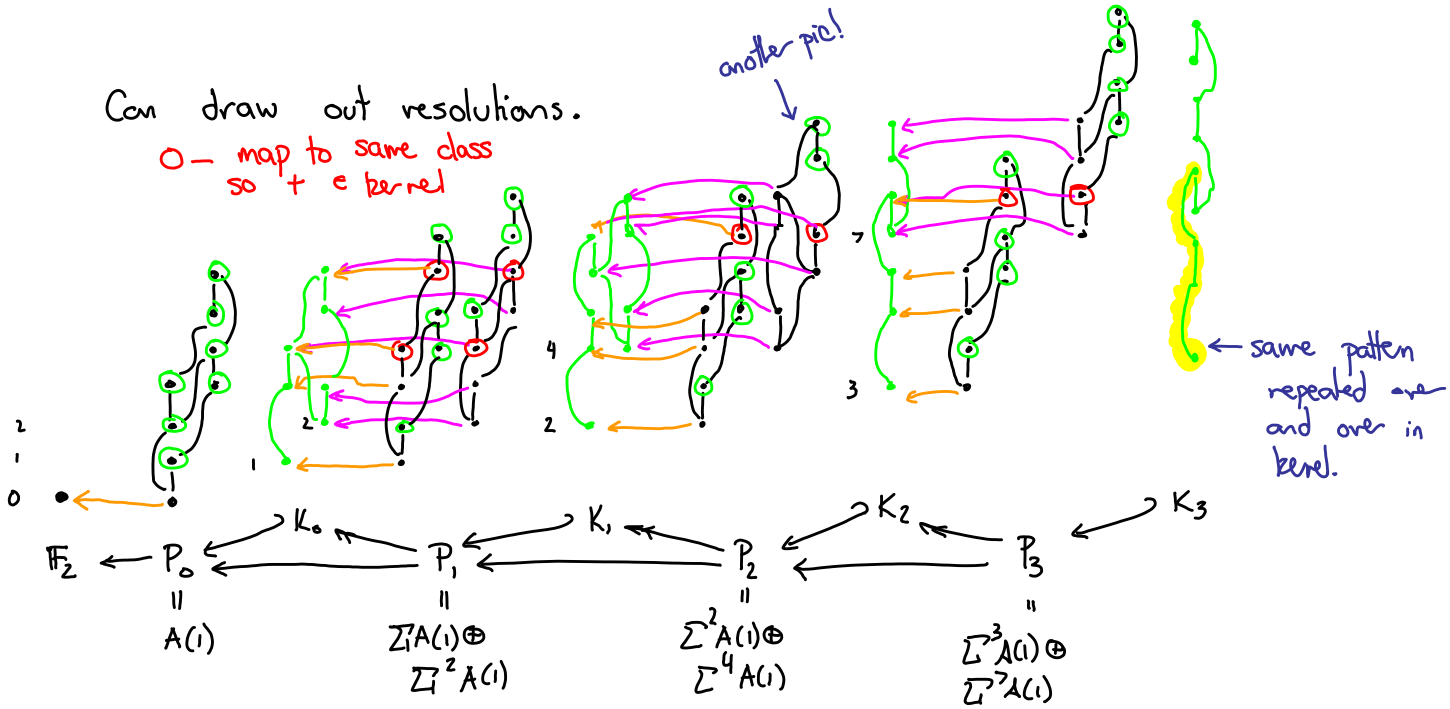
$A(1)$ is an 8-dim $\mathbb{F}_2$-algebra:



$= Sq^2 \cdot (-)$

$= Sq^1(-)$

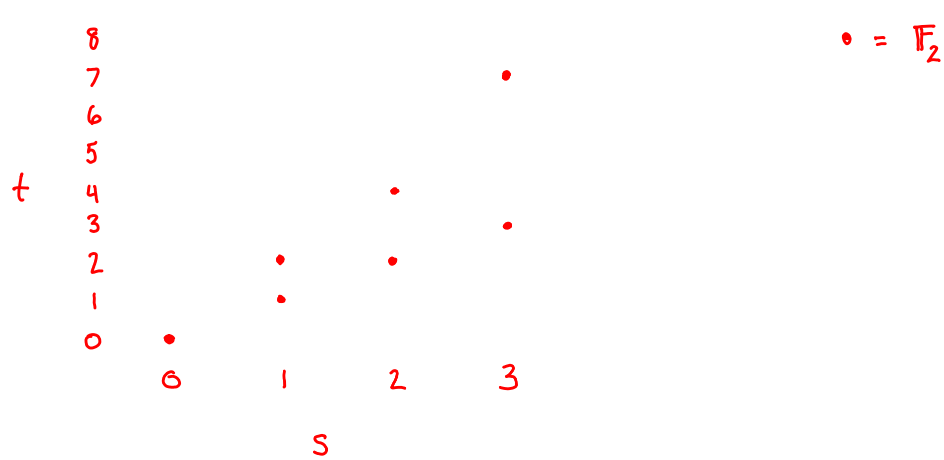
Can draw out resolutions.



So we can compute $\mathrm{Ext}^{s,t}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ by Hom ing this resolution into $\mathbb{F}_2$ & applying homology.

Prop 1 $\mathrm{Hom}^r_{A(1)}(\Sigma^t A(1), \mathbb{F}_2) = \begin{cases} 0 & r \neq t \\ \mathbb{F}_2 & r = t \end{cases}$

Cor 3 $\mathrm{Hom}^t_{A(1)}(P_s, \mathbb{F}_2) =$



Def A resolution is *minimal* if $\ker(P_s \to P_{s-1}) \subseteq I(A(1)) \cdot P_s$

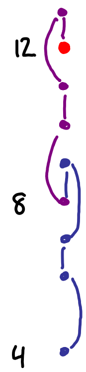
(alt: $\mathrm{Im}(P_s \to P_{s-1}) \subseteq I(A(1)) \cdot P_{s-1}$).

Thm 4 If $P_\bullet$ is a minimal resolution, then

$$\mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = \mathrm{Hom}^t_{A(1)}(P_s, \mathbb{F}_2).$$

Our resolution is by construction minimal, so we have computed Ext.

Look more closely at K_3:



Each piece is something understandable:

$\bullet = \Sigma^{12} \mathbb{F}_2$

$\bullet$ & $\bullet$ are modules of the form $A(1) \otimes_{A(0)} \mathbb{F}_2$, so have easy to describe Ext groups.

$\Rightarrow$ we see another copy of $\mathrm{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ repeated periodically from the $\bullet$ in dim 12. We'll return to this in lecture 17.

Since $A(1)$ is a nice Hopf algebra, $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ is a bigraded commutative algebra.

Thm 5 $\mathrm{Ext}^{s,t}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, x_4, v_1^4] / h_0h_1,\ h_1x_4,\ h_1^3,\ x_4^2 = h_0^2 v_1^4.$

$h_0 \in \mathrm{Ext}^{1,1}$

$h_1 \in \mathrm{Ext}^{1,2}$ & $v_1^4 \in \mathrm{Ext}^{4,12}$

$x_4 \in \mathrm{Ext}^{3,7}$

rep by the red dot in above pic.