Lecture 13 - Adams Spectral Sequence

Note Title 3/18/2009

Primary tool for computing stable homotopy classes of maps

<u>Def</u> $\{X,Y\}_t = \varinjlim [\Sigma^{n+t} X, \Sigma^n Y]$

$= \pi_t F(X,Y)$

↑ function spectrum

<u>Thm 1</u> There is a spectral sequence of the form

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(H^*(Y), H^*(X))$$
$$\Downarrow$$
$$\pi_{t-s} F(X,Y)_2^\wedge.$$

There is a lot packed into this. First, the Ext referenced is the Ext from before. The t-part refers to the internal grading of the modules. This is not a spectral sequence like the previous ones (it's more 2^{nd} or 4^{th} quadrant than 1^{st}).

Let's look at where this comes from.

We'll normally call this just $[X,Y]$, rather than the cumbersome $\{\ \}$.

Given an element $f \in [\Sigma^t X, Y]$, we get a homomorphism of $\mathcal{A}$-modules:

$$H^*(Y) \longrightarrow H^*(\Sigma^t X) = \Sigma^t H^*(X)$$

We've seen this for spaces. Cartan formula & naturality show that for all spaces X $\quad \tilde{H}^*(\Sigma X) \cong \Sigma \tilde{H}^*(X)$. → passes through to spectra.

(Quick recollection: If M is a graded module, then

$(\Sigma M)_n = M_{n-1}$)

So given a stable map $\Sigma^k X \xrightarrow{f} Y$, we get

$$f^* \in \mathrm{Hom}_A(H^* Y, H^* \Sigma^k X).$$

If f is null-homotopic, then $f^* = 0$.

On the other hand, if $f^* = 0$, then it is not nec true that f is null.

If f is null, then $C(f) = Y \vee \Sigma^{k+1} X$. This is equivalent to f being null. In this case, $H^*C(f) = H^*(Y) \oplus H^*(\Sigma^{k+1}X)$, as A-modules.

If $f^* = 0$, then the LES in cohomology becomes a SES of graded A-modules

$$0 \to H^*(\Sigma^{k+1}X) \to H^*(C(f)) \to H^*(Y) \to 0$$

This gives us an extension of $H^*(Y)$ by $H^*(\Sigma^{k+1}X)$ in the category of A-modules.

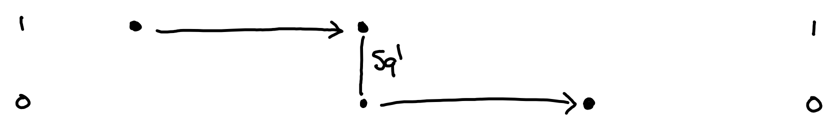
Thus we get an element of $\mathrm{Ext}^1_A(H^*(Y), H^*(\Sigma^{k+1}X))$.

This is another definition of Ext^1: equivalence classes of extensions (hence the name).

Ex: $\cdot 2 : S^0 \to S^0$ in $H^*(-;\mathbb{F}_2)$, $(\cdot 2)^* = 0$

The extension: $S^0 \to M(\mathbb{Z}/2) \to S^1$

$$0 \to H^*(S^1) \to H^*(M(\mathbb{Z}/2)) \to H^*(S^0) \to 0$$



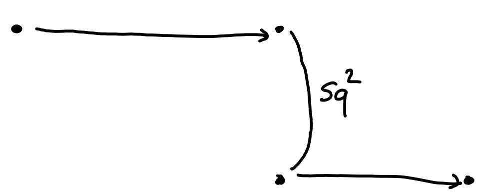
$H^*(M(\mathbb{Z}/2)) \not\cong H^*(S^0) \oplus H^*(S^1)$ as A-modules (there is no splitting $H^*(S^0) \to H^*(M(\mathbb{Z}/2))$ as A-modules)

$\Rightarrow \cdot 2$ is not null.

We have "detected" this map as an extension of A-modules.

Ex: $\eta : S^1 \to S^0$

$$H^*S^2 \to H^*C(\eta) \to H^*S^0$$

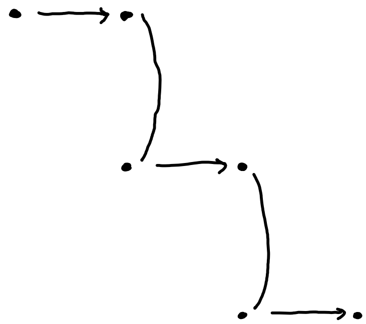


$H^*C(\eta) \not\cong H^*S^0 \oplus H^*S^2$ as A-modules, so the class η is not-null.

Can repeat this process (though it's slightly more complicated):

Ex:

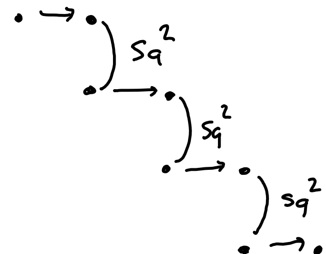
$\eta^2 : S^2 \to S^0$



$\in \mathrm{Ext}^2(H^*S^0, H^*S^4)$

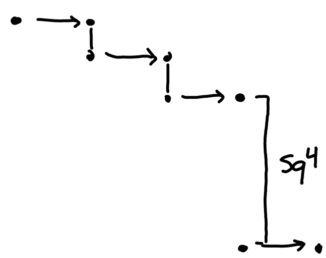
Can be hard to use actual extensions when s gets large:

Ex:



$\in \mathrm{Ext}^3(\mathbb{F}_2, \Sigma^6\mathbb{F}_2)$

$\cong$



$\in \mathrm{Ext}^3(\mathbb{F}_2, \Sigma^6\mathbb{F}_2)$

Now we can construct the spectral sequence.

Def A generalized EM spectrum is a wedge of EM spectra.

Prop 1 If $M = H\mathbb{F}_2 \wedge X$, then $[Y, M]_* \cong \mathrm{Hom}_A^*(H^*(M), H^*(Y))$ (Y finite).

Pf: By Spanier-Whitehead duality, it suffices to show this for $Y = S^0$. Here
$\pi_*(M) = H_*X \cong (H^*X)^* = \mathrm{Hom}(H^*X, \mathbb{F}_2) = \mathrm{Hom}_A(A \otimes H^*X, \mathbb{F}_2) = \mathrm{Hom}_A(H^*M, \mathbb{F}_2)$. ▨

We will resolve a general space by generalized EM spaces, allowing us to identify the E_1-term.

Let $S^0 \longrightarrow H\mathbb{F}_2$ be the map giving the non-zero element of $\overline{H}^*S^0 = \mathbb{F}_2$. (This is actually the unit map for the E_∞-ring structure on $H\mathbb{F}_2$!)

Let $\overline{H\mathbb{F}_2}$ be the fiber of $S^0 \to H\mathbb{F}_2$

Then for any X, smashing with $\overline{H\mathbb{F}_2} \to S^0 \to H\mathbb{F}_2$ gives a fiber sequence

$$\begin{tikzcd} \overline{H\mathbb{F}_2}\wedge X \arrow[d] & \\ X \arrow[r] & X\wedge H\mathbb{F}_2 \end{tikzcd}$$

This lets us build a tower of fibrations for any Y:

$$\begin{tikzcd} \vdots \arrow[d] & \\ Y\wedge(\overline{H\mathbb{F}_2})^{\wedge 2} \arrow[r] \arrow[d] & {} \\ Y\wedge\overline{H\mathbb{F}_2} \arrow[r] \arrow[d] & (Y\wedge\overline{H\mathbb{F}_2})\wedge H\mathbb{F}_2 \\ Y \arrow[r] & Y\wedge H\mathbb{F}_2 \end{tikzcd}$$

Let $Y_n = Y\wedge(\overline{H\mathbb{F}_2})^{\wedge n}$ and Y_{n+1} sits in a fiber sequence

$$\begin{tikzcd} Y_n\wedge\overline{H\mathbb{F}_2} \arrow[r] \arrow[d, equal] & Y_n \arrow[r] & Y_n\wedge H\mathbb{F}_2 \\ Y_{n+1} & & \end{tikzcd}$$

This is the <u>canonical</u> <u>Adams</u> <u>resolution</u>.

We can convert this to a sequence: all GEM spaces

$$\begin{tikzcd} Y \arrow[r] & Y\wedge H\mathbb{F}_2 \arrow[r] \arrow[d] & Y\wedge(\Sigma\overline{H\mathbb{F}_2})\wedge H\mathbb{F}_2 \arrow[r] \arrow[d] & Y\wedge(\Sigma\overline{H\mathbb{F}_2})^{\wedge 2}\wedge H\mathbb{F}_2 \arrow[r] & \cdots \\ & \Sigma Y\wedge\overline{H\mathbb{F}_2} \arrow[ur] & Y\wedge(\Sigma\overline{H\mathbb{F}_2})^{\wedge 2} \arrow[ur] & & \end{tikzcd}$$

$- \wedge (S^0 \to H\mathbb{F}_2)$

$\Sigma\overline{H\mathbb{F}_2}$ is the cofiber of $S^0 \to H\mathbb{F}_2$.

If we apply $[X,-]$, then we get a SS w/ $E_1 = [X, Y\wedge(\Sigma\overline{H\mathbb{F}_2})^{\wedge n}\wedge H\mathbb{F}_2]$

by Prop 1, $[X, Y\wedge(\Sigma\overline{H\mathbb{F}_2})^{\wedge n}\wedge H\mathbb{F}_2] = \mathrm{Hom}_A(A\otimes H^*(\Sigma\overline{H\mathbb{F}_2})^{\otimes n}\otimes H^*Y, H^*X)$

<u>Prop 2</u> $H^*(\Sigma\overline{H\mathbb{F}_2}) = I(A)$, the ideal of positive degree elements.

The map $S^0 \to H\mathbb{F}_2$ gives the augmentation $A \to \mathbb{F}_2$.

Then $A\otimes H^*(\Sigma\overline{H\mathbb{F}_2})^{\otimes n}\otimes H^*Y = A\otimes I(A)^{\otimes n}\otimes H^*Y$.

<u>Prop 3</u> ① $A\otimes I(A)^{\otimes n}\otimes H^*Y$ is the n^{th} stage of the can. projective resolution of H^*Y as an A-module.

② $H^*(Y \to H\mathbb{F}_2\wedge Y \to \cdots)$ realizes this projective resolution.

As a corollary, the homotopy classes $[X, Y\wedge \cdots \wedge H\mathbb{F}_2]$ are $\mathrm{Hom}_A(P_n, H^*X)$, where P_n is the n^{th} stage in the projective resolution. The maps in spectra realizes the d_1-differential in the exact couple from the tower of fibrations & it is $\mathrm{Hom}_A(-, H^*(X))$ of the maps in the projective resolution.

<u>Cor</u> $E_2 = \mathrm{Ext}_A(H^*Y, H^*X)$.