Massey Products, the Steenrod Alg, & Kudo Transgression

Note Title 2/20/2009

We'll recast the Kudo transgression thm as a consequence of a lemma for higher products.

Definition: Let $(M_\bullet, d)$ be an associative dga, and let $a \in M_m$, $b \in M_n$, $c \in M_k$ be cycles s.t. $[a\cdot b] = [b \cdot c] = 0$ in $H^*(M)$.

Then the Massey product of $a, b, \&\ c$ is

$$\langle a,b,c\rangle = \left\{ [e\cdot c - \bar{a}\cdot f] \;\middle|\; \begin{array}{l} e \xmapsto{d} a\cdot b \\ f \mapsto b\cdot c \end{array} \;\&\; \bar{a} = (-1)^{|a|} a \right\} \subseteq H^{m+n+k-1}(M).$$

There is a lot packed into the statement.

① $\langle a,b,c\rangle$ is a subset of $H^{n+m+k-1}(M)$, not an element.

More on this later.

② the elements $e\cdot c - \bar{a}\cdot f$ are cycles.

This is easy to check:

$$d(e\cdot c - \bar{a}\cdot f) = d(e\cdot c) - d(\bar{a}\cdot f) =$$

$$\Big(d(e)\cdot c + (-1)^{|e|}\cancel{e\cdot d(c)}\Big) - \Big(\cancel{d(\bar{a})}\cdot f + (-1)^{|a|}\bar{a}\cdot d(f)\Big)$$

0 ← a & c are cycles → 0

$\| \leftarrow$ def of e $\qquad\qquad \| \leftarrow$ def of f $\qquad\qquad$ associativity of M

$$(a\cdot b)\cdot c \;-\; a\cdot(b\cdot c) \;=\; 0.$$

Often indicate our choice of "null-homotopies" by placing them over what they bound:

$$\langle a \overset{e}{,} b \overset{f}{,} c\rangle.$$

When thusly adorned, we are referring to an element of $\langle a,b,c\rangle$.

Prop 1 $\langle a,b,c\rangle$ is a coset of $[a]\cdot H^{n+k-1}(M) + H^{m+n-1}(M)\cdot[c]$.

Pf We need to show $\langle a \overset{e}{,} b \overset{f}{,} c\rangle - \langle a \overset{e'}{,} b \overset{f'}{,} c\rangle \in a\cdot H^{n+k-1} + H^{m+n-1}\cdot c$.

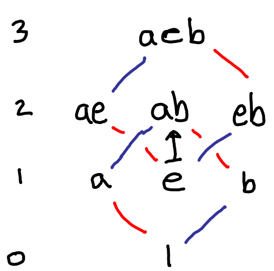
Since $e, e' \xmapsto{d} a\cdot b$ & $f, f' \xmapsto{d} b\cdot c$, $e-e'$ & $f-f'$ are cycles in M_{m+n-1} & M_{n+k-1} resp. Thus by def, we have

$$\langle a, \overset{e}{b}, \overset{f}{c}\rangle - \langle a, \overset{e'}{b}, \overset{f'}{c}\rangle = (e\cdot c - \bar{a}\cdot f) - (e'\cdot c - \bar{a}\cdot f')$$
$$= (e-e')\cdot c - \bar{a}\cdot(f-f')$$
$$\in \underbrace{Z(M_{m+n-1})}_{\text{"cycles"}}\cdot c + a\cdot \underbrace{Z(M_{n+k-1})}_{\text{"cycles"}}. \quad \boxtimes$$

Def The group $a\cdot H^{n+k-1} + H^{m+n-1}\cdot c$ is the <u>indeterminacy</u> of $\langle a,b,c\rangle$.

If the indeterminacy is zero, we identify $\langle a,b,c\rangle$, the coset, with the unique element therein.

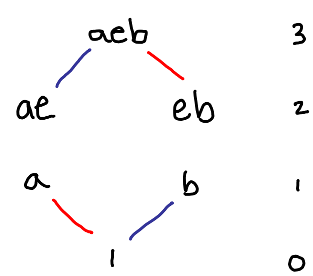
Ex: Let $M_\bullet = E(a,b,e)$ with deg $\to$ 1, 1, 1, $d(e) = ab$. Then can depict $M_\bullet$ by



$\backslash = a\cdot(\ )$

$/ = (\ )\cdot b$

so H^*:



This is again a Poincaré duality alg.

Then $ae = \langle a,a,b\rangle$ while $eb = \langle a,b,b\rangle$.

1st $aH^1 + H^1 b = 0$ (no red or blue lines off 1 line) $\Rightarrow$ indeterminacy $= 0$.

Then $b^2 = a^2 = 0$ in $M_\bullet$, so 0 is a good null-homotopy. $d(e) = ab$, so e is a good null-homotopy:

$$\langle a, \overset{0}{a}, \overset{e}{b}\rangle = -\bar{a}\cdot e = a\cdot e$$
$$\langle a, \overset{e}{b}, \overset{0}{b}\rangle = e\cdot b.$$

Thus far, this is basically naming.

Prop 2 If $[a]\cdot[b] = [b]\cdot[c] = [c]\cdot[d] = 0$, then
$$a\langle b,c,d\rangle = \langle a,b,c\rangle d$$
as subsets of $H_*(M)$.

This will let us deduce several results about multiplicative relations on later pages of spectral sequences.

In fact, this is a very strong result, since we have control over indeterminacy too!

If $0 \in \langle a,b,c\rangle$ and $\langle b,c,d\rangle$, then can form larger products:

$$\langle a \overset{e}{,} b \overset{f}{,} c \overset{g}{,} d\rangle = hd \pm eg \pm ai. \quad (h \text{ over } a,b,c;\ i \text{ over } b,c,d)$$

We can obviously continue this, but the indeterminacy becomes a nightmare.

If $a \in M_\bullet$ has $[a]\cdot[a]=0$, then can consider Massey Powers

$$a^{\langle n\rangle} = \langle \underbrace{a,\dots,a}_{n}\rangle.$$

Since each slot is a itself, we can symmetrically choose null-homotopies, making the set of choices slightly smaller.

Can now tie all of this to the Steenrod algebra:

New unstable axiom:

If $|x|=2i$, then $P^i(x)=x^p$ ← old one

If $|x|=2i+1$, then $\beta P^i(x) \in x^{\langle p\rangle}$ ← new part

In practice, $x^{\langle p\rangle}$, when chosen symmetrically, will be small enough to identify it w/ $\beta P^i(x)$.

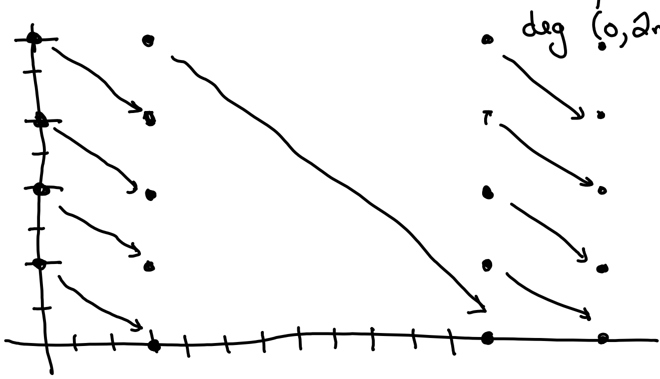
<u>Lemma 1</u> In the Serre SS, working over a field of char p, if x is a polynomial gen (possibly truncated $> p-1$), and if

$$d(x)=y, \quad \text{then} \quad d(yx^{p-1}) = y^{\langle p\rangle}.$$

As always, the equality is an inclusion if there is indeterminacy.

<u>Ex</u>: Universal example: $E_2 = \mathbb{F}_p[x]/x^p \otimes E(y) \otimes \mathbb{F}_p[y^{\langle p\rangle}]$

x: deg $(0,2n)$ $\quad y$: deg $(2n+1,0)$ $\quad y^{\langle p\rangle}$: deg $(2pn+2,0)$



$p=5$

This has $d_{2n+1}(x)=y$ and

$$d_{2n(p-1)+1}(x^{p-1}y) = y^{\langle p\rangle}.$$

Combining this with the result about Steenrod ops gives the "Kudo Transgression thm"

Thm 1 If x is a transgressive element hitting y, then $x^{p-1}y$ hits $\beta P^{\frac{|x|}{2}} y$.

Can use this w/ Borel's thm to get the cohom of odd primary EM spaces.

Def Let $I = (\epsilon_0, s_1, \epsilon_1, s_2, \dots, s_m, \epsilon_m)$ be a seq of ints w/ $\epsilon_i = 0, 1$.

I is admissible if $s_i = p \cdot s_{i+1} + \epsilon_i$. The excess is

$$\epsilon_0 + \sum_i (2s_i - 2ps_{i+1} - \epsilon_i).$$

The ϵ_0 part is a little odd, and it makes what follows messier.

Thm 2 (Cartan) For p an odd prime, $H^*(K(\mathbb{Z}/p, n); \mathbb{F}_p)$

$=$ free commutative alg on classes $P^I(\iota_n)$ for $e(I) < n$

(or $e(I) = n$ on an odd class).

Why the other part? We really put in all classes and their p^{th} powers of any form. The last classes are p^{th} Massey powers, given by something of excess n, so we need to include them at this stage.