Cohomology of Eilenberg-MacLane Spaces

Note Title 2/12/2009

We need a few more tools.

Def The $d_{n+1}: E_{n+1}^{0,n} \longrightarrow E_{n+1}^{n+1,0}$ is called the transgression, τ.

If $f \in E_2^{0,n}$ is a d_i-cycle for $i < n+1$ &

$d_{n+1}(f) \neq 0$, say f is transgressive.

This is the last possible differential from $E^{0,n}$ & it is related to actual geometric concepts:

Let δ be the connecting map

$$H^n(E) \longrightarrow H^n(F) \xrightarrow{\delta} H^{n+1}(E,F) \to H^{n+1}(E)$$

There is a map of pairs $(E,F) \xrightarrow{\pi} (B,*)$,

So have

$$\begin{array}{ccc} H^n(F) \xrightarrow{\delta} & H^{n+1}(E,F) & \\ & \uparrow \pi^* & \\ & H^{n+1}(B,*) & \cong \tilde{H}^{n+1}(B) \end{array}$$

Prop 1 The transgression is $\pi^{*-1} \circ \delta$.

This is not totally well defined, so there is a lot wrapped-up in this statement.

One restatement: $\delta(f) = \pi^*(b) \longleftrightarrow d_{n+1}(f) = b$.

(so really get that $d_{n+1}(f)$ hits a coset $= \mathrm{im}(b)$ in $H^*(E)$).

Cor 1 If f is transgressive, then so is $Sq^i f$ for all i.

"Pf": $\delta(Sq^i f) = Sq^i \delta(f) = Sq^i \pi^*(b) = \pi^*(Sq^i b)$.

So in fact, $\tau(Sq^i f) = Sq^i \tau(f)$.

This is the first important piece. Part II is a thm of Borel.

Def $\{x_1, \dots\}$ is a simple system of generators for $H^*(X)$ if the simple products $x_{i_1} \cdots x_{i_n}$ form a basis.

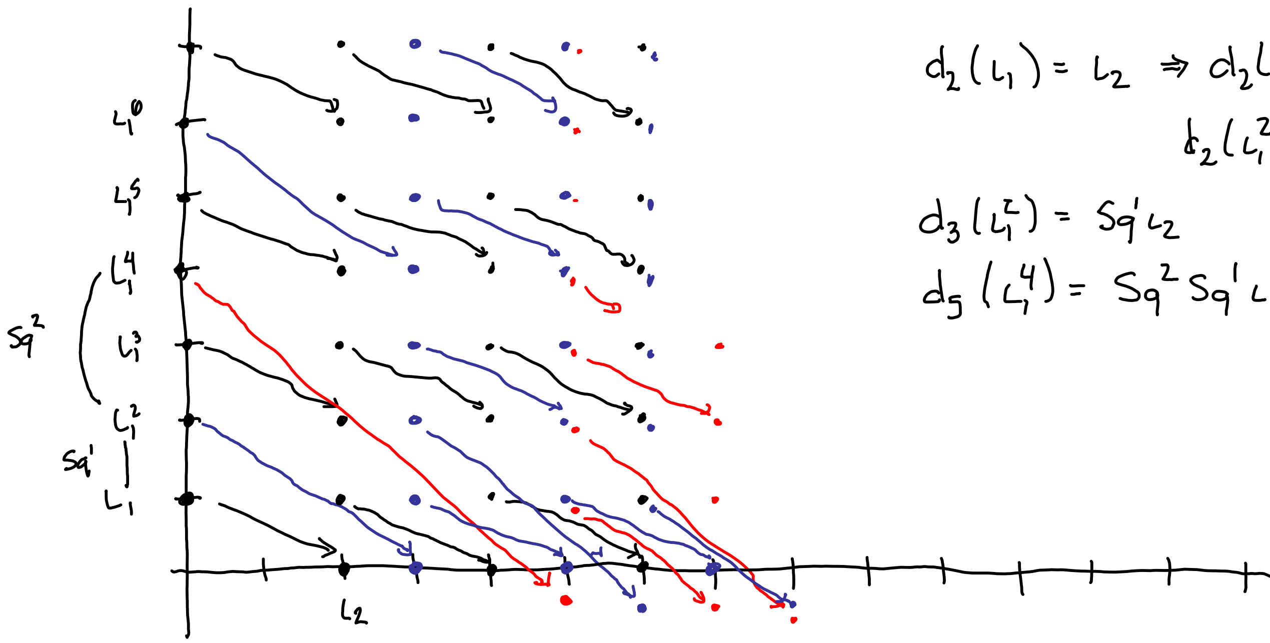
Also ask for finite type.

Ex $\mathbb{F}_2[x]$ has $\{x, x^2, x^4, x^8, \ldots\}$ as a simple sys of gen.

If $\{x_1, \ldots\}$ is a simple sys for A & $\{y_1, \ldots\}$ is for B, then $\{x_1, \ldots, y_1, \ldots\}$ is a simple sys for $A \otimes B$.

Thm 1 (Borel) Let $F \to E \to B$ be a fib w/ $E \simeq *$. Then if $H^*(F)$ has a simple system of transgressive generators, then $H^*(B)$ is polynomial on the transgressions.

Let's see if this is plausible. $\mathbb{RP}^\infty \to * \to K(\mathbb{Z}/2, 2)$



$d_2(\iota_1) = \iota_2 \Rightarrow d_2(\iota_1^{2k+1}) = \iota_1^{2k} \cdot \iota_2$

$d_2(\iota_1^{2k}) = 0$

$d_3(\iota_1^2) = Sq^1 \iota_2$

$d_5(\iota_1^4) = Sq^2 Sq^1 \iota_2$

Thm 2 (Serre) $H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) = \mathbb{F}_2[Sq^I \iota_n]$, $e(I) < n$.

Ex $n=1$: $e(I) = 0 \Rightarrow I = 0$, so $H^*(K(\mathbb{Z}/2, 1)) = \mathbb{F}_2[\iota_1]$

$n=2$: $e(I) = 1 \Rightarrow I = (2^k, 2^{k-1}, \ldots, 2, 1)$

$$H^*(K(\mathbb{Z}/2, 2)) = \mathbb{F}_2[\iota_2, Sq^1 \iota_2, Sq^2 Sq^1 \iota_2, \ldots]$$

Proof: First, a few observations:

1. $e(I) > n$, then $Sq^I \iota_n = 0$
2. $e(I) = n$, then $Sq^I \iota_n = (Sq^J \iota_n)^{2^k}$

some subsequence J w/ $e(J) < n$.

So the proof is by induction on n. $n=1$ is done

Induction hypothesis: $H^*(K(\mathbb{Z}/2,n);\mathbb{F}_2) = \mathbb{F}_2[Sq^I\iota_n]$, $e(I)<n$.

This has a simple system of generators:

$$\{(Sq^I\iota_n)^{2^k}\}$$

This can be rewritten as

$$\{\iota_n, Sq^I\iota_n \mid e(I)\leq n\}$$

Now ι_n is transgressive $(\iota_n \overset{\tau}{\longmapsto} \iota_{n+1})$

$\Rightarrow$ $Sq^I\iota_n$ is transgressive.

$\Rightarrow$ $H^*(K(\mathbb{Z}/2,n))$ has a simple sys of transg. elements.

$\Rightarrow$ $H^*(K(\mathbb{Z}/2,n+1)) = \mathbb{F}_2[\iota_{n+1}, Sq^I\iota_{n+1}]$, $e(I)<n+1$. $\boxtimes$

<u>Cor</u> <u>2</u> The admissible sequences form a basis for $\mathcal{A}$.

<u>Pf</u> The map of $\mathcal{A}$-modules

$$\begin{array}{ccc} 1 & \longmapsto & \iota_n \\ \mathcal{A} & \longrightarrow & H^*(K(\mathbb{Z}/2,n)) \end{array}$$

is injective on classes of $e\geq n$ & sends them to lin. ind elements. $\boxtimes$

Same arguments show $H^*(K(\mathbb{Z},n))$ & $H^*(K(\mathbb{Z}/2^k,n))$

<u>Thm</u> <u>3</u> (Serre) $H^*(K(\mathbb{Z},n)) = \mathbb{F}_2[Sq^I\iota_n]$, $e(I)<n$ & Sq^I does not end in Sq^1.

The Sq^1 condition comes from $K(\mathbb{Z},1)$ or $K(\mathbb{Z},2)$. in both, $Sq^1=0$.

So for $\mathbb{C}P^\infty$: S. System: $\{\iota_2, Sq^2\iota_2, Sq^4Sq^2\iota_2, \dots\}$

for $K(\mathbb{Z},3)$: $\{\iota_3, Sq^2\iota_3, Sq^4Sq^2\iota_3, \dots$

$Sq^3\iota_3, Sq^5Sq^2\iota_3, Sq^9Sq^4Sq^2\iota_3, \dots\}$

& we never see a sequence ending w/ Sq^1.

Thm 4 (Serre) $H^*(K(\mathbb{Z}/2^k, n)) = \mathbb{F}_2[Sq^I \iota_n]$ $\quad e(I) < n$

} if Sq^I ends in Sq^1, replace that with the mod 2^k Bockstein β_k.

Here β_k is the connecting hom for $- \otimes (\mathbb{Z}/2 \to \mathbb{Z}/2^{k+1} \to \mathbb{Z}/2^k)$.