# Constructing the Squares

Note Title 2/12/2009

We'll give a geometric construction. An algebraic one follows by thinking in terms of chains instead.

<u>Notation</u> Let $X$ be a pointed space, $*$ the base point, let $\pi$ be a subgroup of $\Sigma_n$, & let $K_n = K(R,n)$, $R$ a fixed field.

Since $X$ is pointed, $X^n$ is filtered:

$$\underset{F_0}{\underset{\parallel}{*^n}} \subseteq \underset{F_1 = \overset{n}{\underset{i=1}{\bigvee}} X}{\underset{\parallel}{\{(x_1,\dots,x_n) \mid \text{at most } 1\ x_i \neq *\}}} \subseteq \cdots \subseteq \underset{F_{n-1} = \text{"fat wedge"}}{\underset{\parallel}{\{(x_1,\dots,x_n) \mid \text{at most } n-1\ x_i \neq *\}}} \subseteq X^n$$

The group $\pi$ acts on $X^n$:

$$\sigma(x_1,\dots,x_n) = (x_{\sigma(1)},\dots,x_{\sigma(n)})$$

& the group action respects the filtration (Say $F_i \longrightarrow F_{i+1}$ is $\pi$-equivariant).

$\Rightarrow$ cofibers get a $\pi$-action.

Most important is

<u>Def</u> The n-fold smash power of $X$ is

$$X^{(n)} = X^n / F_{n-1}.$$

This is canonically a $\pi$-space. However, the $\pi$-action isn't free ($\pi(x,\dots,x) = (x,\dots,x)$).

Now we need an equivariant construction.

<u>Def</u> If $X$ is a $G$-space, then the <u>Borel construction</u> is the orbit space

$$X_{hG} := EG \underset{G}{\times} X = (EG \times X)_G$$

Here $EG$ is a free, contractible $G$-space & $G$ acts on $EG \times X$ diagonally: $g(e,x) = (g(e), g(x))$.

**Prop 1** If $G$ acts freely on $X$, then $X_{hG} = X_G$.
If $G$ acts trivially on $X$, then $X_{hG} = BG \times X$.

Idea is that if $G$-acts freely, then $EG \times X \to X$ is $G$-equivariantly a homotopy equivalence.

Can realize $EG \times_G X$ as a bundle over $BG$:

① $X \longrightarrow *$ is a $G$-map $\Rightarrow$ get a map

$$EG \times_G X \longrightarrow EG \times_G * = BG.$$

this is the map

② $$\begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG \end{tikzcd}$$ is a fibration. We form $EG \times_G X$ by replacing the fiber $G$ with $X$ using the $G$-action.

$$\Rightarrow \quad \begin{tikzcd} X \arrow[r] & EG \times_G X \arrow[d] \\ & BG \end{tikzcd}$$

We can apply all of this to $X^{(n)}$ & $\pi$.

$\Rightarrow$ Have bundles over $B\pi$ w/ total space

$$E\pi \times_\pi F_{n-1} \quad \& \quad E\pi \times_\pi X^n.$$

Since $F_{n-1} \subseteq X^n$ equivariantly,

$$E\pi \times_\pi F_{n-1} \subseteq E\pi \times_\pi X^n \quad \text{as a subbundle.}$$

**Def** The $\pi$-extended power of $X$ is

$$D_\pi X = \left(E\pi \times_\pi X^n\right) \Big/ E\pi \times_\pi F_{n-1}.$$

We can rewrite the right-hand side as

$$E\pi_+ \wedge_\pi X^{(n)}.$$

The $\pi$-extended power construction is the source of all Steenrod & power operations.

We need to understand the cohomology, especially in the universal case of $K_n$.

**Thm 1** If $\bar{H}^r(X) = 0$ for $r < q$, then
$$\bar{H}^s(D_\pi X) = 0 \quad \text{for } s < nq.$$
Moreover, $\bar{H}^{nq}(D_\pi X) = (\bar{H}(X)^{\otimes n})^\pi$. ← invariants.

Can give a few proofs. For the first part, let's assume $X$ is s.c.

$\Rightarrow$ the $q$-skeleton of $X$ is a wedge of $q$-spheres

$\Rightarrow$ the $nq$-skeleton of $X^{(n)}$ is a wedge of $(nq)$-spheres.

$\Rightarrow$ the $nq$-skeleton of $D_\pi X$ is a wedge of $(nq)$-spheres.

To better understand $\bar{H}^{nq}(D_\pi X)$, we use the relative Serre SS:

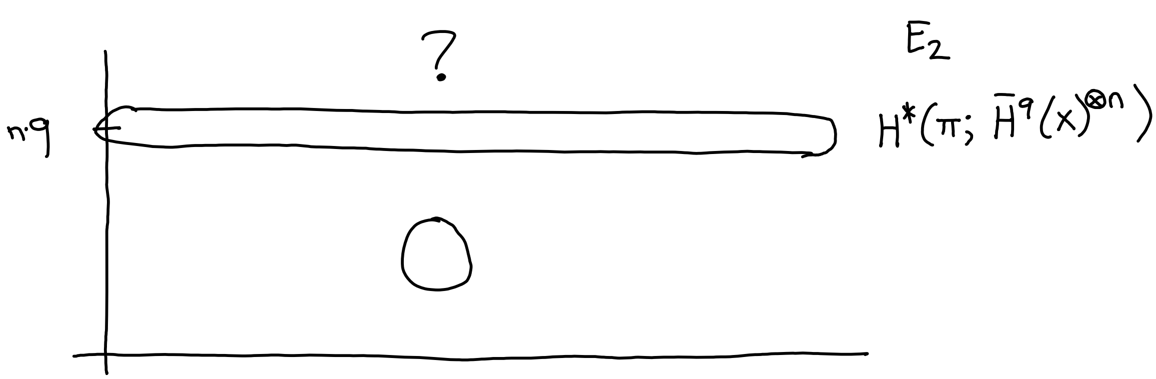
**Thm 2** If
$$\begin{tikzcd} F' \arrow[r, phantom, "\subseteq"] \arrow[d] & F \arrow[d] \\ E' \arrow[r, phantom, "\subseteq"] \arrow[dr] & E \arrow[d] \\ & B \end{tikzcd}$$
are a bundle & subbundle over a fixed base $B$, then there is a 1st quadrant SS with
$$E_2^{p,q} = H^p(B; \mathcal{H}^q(F, F'; R)) \Rightarrow H^{p+q}(E, E'; R).$$

In our case, $E' = E\pi \times_\pi F_{n-1}$, $E = E\pi \times_\pi X^n$, so
$$H^{p+q}(E, E'; R) = \bar{H}^{p+q}(D_\pi X; R).$$

& $B = B\pi$ has $\pi_1 = \pi$. This acts non-trivially on $H^*(F, F')$.

Now $H^*(F, F'; R) = \bar{H}^*(F/F'; R) = \bar{H}^*(X^{(n)})$

$= \bar{H}^*(X)^{\otimes n}$ & this is all $\pi$-equivariant. So we know that
$$H^*(F, F'; R) = \begin{cases} 0 & * < nq \\ \bar{H}^q(X)^{\otimes n} & * = nq \end{cases}$$



So for $s < n\cdot q$, $E_2^{r,s} = 0 \Rightarrow$

① everything in $E_2^{0,n\cdot q}$ is a perm cycle

② $\bar{H}^*(D_\pi X) = \begin{cases} 0 & * < n\cdot q \\ H^0(\pi; \bar{H}^q(x)^{\otimes n}) & * = n\cdot q \end{cases}$

Since $H^0(G; M) = M^G$ (by def), we conclude

$$\bar{H}^*(D_\pi X) = \begin{cases} 0 & * < n\cdot q \\ (\bar{H}^q(x)^{\otimes n})^\pi & * = n\cdot q \end{cases}.$$

$\boxed{\checkmark}$

We apply this to $K_q$.

**Prop 2** $H^*(K_q; R) = \begin{cases} 0 & * < q \\ R & * = q \end{cases}$

**Cor 1** $\bar{H}^{nq}(D_\pi K_q; R) = R$, generated by a class

$P_\pi \iota_q$ s.t.

$$\begin{array}{ccc} P_\pi \iota_q & \longmapsto & \iota_q^{\otimes n} \\ \cap & & \cap \\ \bar{H}^{nq}(D_\pi K_q) & \longrightarrow & \bar{H}^{nq}(K_q^{(n)}) \end{array}$$

So in fact we have a map

$$D_\pi K_q \xrightarrow{P_\pi \iota_q} K_{nq}$$

**Def** Let $u \in H^q(x)$. Then the Total Steenrod power on $u$ is the composite

$$D_\pi X \xrightarrow{D_\pi u} D_\pi K_q \xrightarrow{P_\pi} K_{qn}$$

To get Steenrod ops in the usual form, we pull back along the diagonal: $X \xrightarrow{\Delta} X^{(n)}$ is $\pi$-equivariant $\Rightarrow$

$$\begin{array}{ccc} E\pi_+ \wedge_\pi X & \xrightarrow{\Delta} & D_\pi X \\ \| & & \\ B\pi_+ \wedge X & & \end{array}$$

Now $\bar{H}^*(B\pi_+ \wedge X) = H^*(B\pi) \otimes \bar{H}^*(x)$, so

The composite $B\pi_+ \wedge X \xrightarrow{\Delta} D_\pi X \xrightarrow{P_\pi u} K_{qn}$

is a sum $\sum b_i \otimes x_i$, $b_i \in H^*(B\pi)$, $x_i \in \bar{H}^*(x)$

Let's restrict attention to $n=2$, $\pi = \Sigma_2$, $R = \mathbb{F}_2$

Then $B\pi = \mathbb{RP}^\infty$, so $H^*(B\pi; \mathbb{F}_2) = \mathbb{F}_2[x]$, $|x| = 1$

Def $\quad \Delta^*(P_2 u) = \sum x^{q-i} \otimes Sq^i(u)$

This canonically defines elements $Sq^i(u) \in \tilde{H}^{p+i}(X; \mathbb{F}_2)$.

Prop 3 $Sq^q u = u^2$

Pf Consider the "inclusion of the fiber" $K_q \longrightarrow E\pi_+ \wedge_\pi K_q$

Then we have a square

$$\begin{tikzcd}
E\pi_+ \wedge_\pi K_q \arrow[r, "\Delta"] & D_2 K_q \\
K_q \arrow[u] \arrow[r, "\Delta"] & K_q^{(2)} \arrow[u]
\end{tikzcd}$$

& a map $D_2 K_q \longrightarrow K_{2q}$

we look at what happens to $P_2 \iota_q$ under these maps:

$$\begin{tikzcd}
\sum_i x^{q-i} \otimes Sq^i \iota_q \arrow[ddd, mapsto] & & & P_2 \iota_q \arrow[lll, mapsto, "\text{Def of } Sq"'] \arrow[ddd, mapsto, "\text{Thm 1}"] \arrow[dl, phantom, "\in"] \\
& H^{2q}(B\pi_+ \wedge K_q) \arrow[dd] & H^{2q}(D_2 K_q) \arrow[l] \arrow[dd] & \\
& & & \\
Sq^q \iota_q & H^{2q}(K_q) & H^{2q}(K_q^{(2)}) \arrow[l] \arrow[r, phantom, "\ni"] & \iota_q^{\otimes 2} \arrow[dll, mapsto, "\text{def of } \cup"] \\
& \iota_q^2 \arrow[u, phantom, "\in"] & &
\end{tikzcd}$$

So $Sq^q \iota_q = \iota_q^2$. Naturality gives the result.