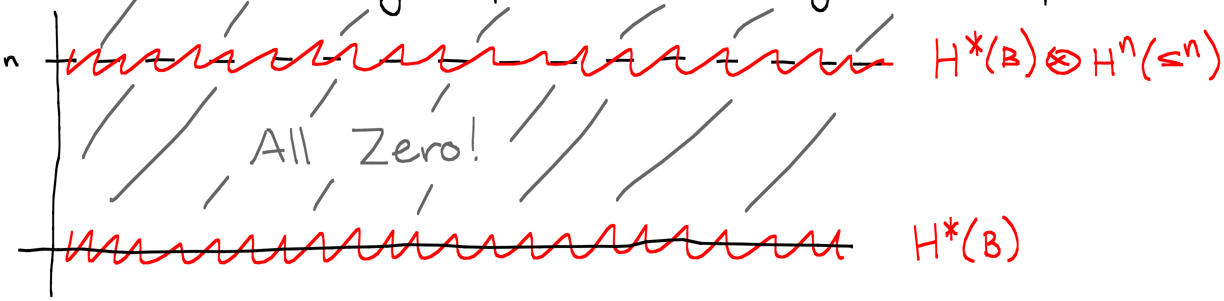
Gysin Sequence & Hochschild-Serre Spectral Sequence

Note Title 2/5/2009

Look today at the Serre SS for the case $F = S^n$

So if $S^n \to E \to B$ is a (simple) spherical fibration, then the Serre SS is especially simple: it is a long exact sequence



As an algebra, $E_2 = H^*(B) \otimes H^*(S^n) = H^*(B) \otimes E(x_n)$

Since $H^*(B)$ sits on the zero line, everything in $H^*(B)$ is a permanent cycle. $\Rightarrow$ differentials are completely determined by those on x_n.

For degree reasons, only possibility is a d_{n+1}.

So $d_{n+1}(x) = e \in H^{n+1}(B)$, and if $b \in H^*(B)$,

$$d_{n+1}(x \cdot b) = d_{n+1}(x) \cdot b + \cancel{(-1)^{|x|} x \, d_{n+1}(b)}$$
$$= e \cdot b.$$

Thus
$$H^{*,q}(E_{n+1}, d_{n+1}) = \begin{cases} \ker\left(H^*(B) \xrightarrow{\cdot e} H^*(B)\right) & \text{in } q = n \\ 0 & q \neq 0, n \\ \operatorname{coker}\left(H^*(B) \xrightarrow{\cdot e} H^*(B)\right) & \text{in } q = 0 \end{cases}$$

For degree reasons, this is also E_∞.

We'll return to reassembling this into $H^*(E)$ in a minute.

Def The class e is called the Euler class of the spherical fibration.

This notation comes from vector bundles & characteristic classes.

If $V \to B$ is a vector bundle w/ a metric (say B compact enough) then we have an associated sphere bundle $S(V) \to B$ whose fiber over b is the unit sphere of V_b.

Then the class e for the Serre SS is the Euler class for the vector bundle V. (If V is the tangent bundle to a manifold M, the e = (euler characteristic) · Poincaré duality class)

<u>Ex</u>: $H^*(V_2(\mathbb{R}^{n+1}))$.

<u>Def</u> $V_2(\mathbb{R}^{n+1})$ = space of orthogonal pairs of unit vectors in $\mathbb{R}^{n+1}$

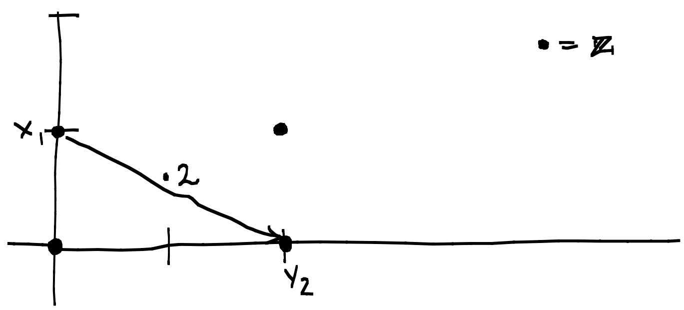
So we can identify $V_2(\mathbb{R}^{n+1})$ with the unit sphere bundle to $T(S^n)$: $T(S^n) = \{(\bar{x}, \bar{v}) \in (\mathbb{R}^{n+1})^2 \mid \begin{smallmatrix} \bar{x}\cdot\bar{x} = 1 \\ \bar{x}\cdot\bar{v} = 0 \end{smallmatrix} \}$, $\quad T(S^n) \to S^n$, $(\bar{x}, \bar{v}) \mapsto \bar{x}$

so the unit sphere bundle is $S(T) = \{(\bar{x}, \bar{v}) \mid \begin{smallmatrix} \bar{x}\cdot\bar{x} = \bar{v}\cdot\bar{v} = 1 \\ \bar{x}\cdot\bar{v} = 0 \end{smallmatrix}\}$

has fiber S^{n-1} over S^n.

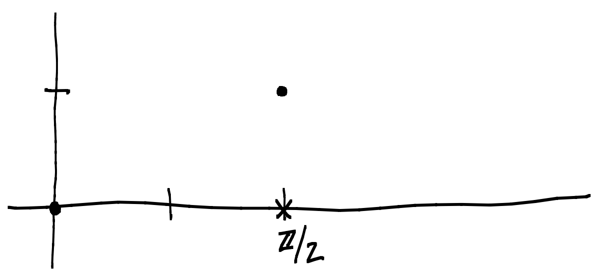
Euler Class: $(1+(-1)^n)[S^n]$.

<u>n = 2</u>:



$d(x_1) = 2y_2$

E_3:



$$\Rightarrow H^*(V_2(\mathbb{R}^3)) = \begin{cases} \mathbb{Z} & 3 \\ \mathbb{Z}/2 & 2 \\ 0 & 1 \\ \mathbb{Z} & 0 \end{cases}$$

We should expect this: Given $\bar{x}, \bar{v}$ s.t. $\bar{x}\cdot\bar{v} = 0$, $\|\bar{x}\| = \|\bar{v}\| = 1$, there is a unique $\bar{w}$ s.t. $\begin{bmatrix} \bar{x} & \bar{v} & \bar{w} \end{bmatrix} \in SO(3)$.

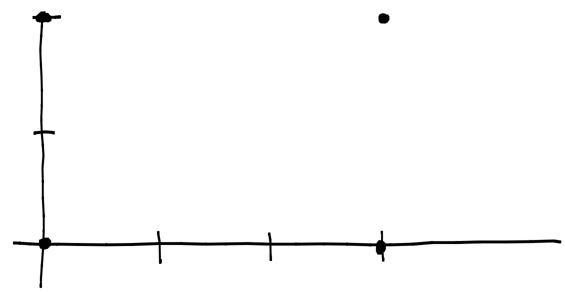
So $V_2(\mathbb{R}^3) = SO(3) = \mathbb{RP}^3$

this is done by looking at the conjugation action of the unit quaternions on the imaginary ones.

$\underline{n=3}$: $\quad S^2 \to V_2(\mathbb{R}^4)$
$\qquad\qquad\qquad\quad \downarrow$
$\qquad\qquad\qquad\quad S^3$

$e = 0$, so

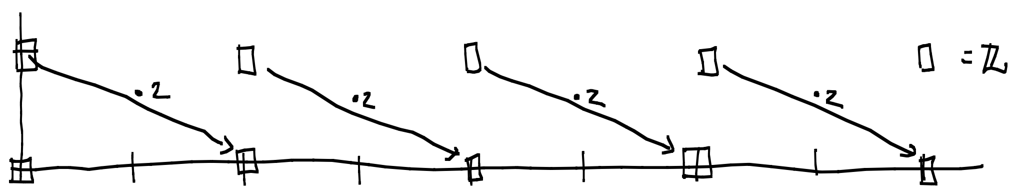
$E_2 = E_\infty$!



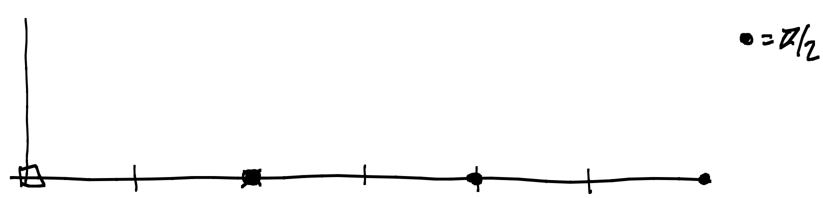
$$H^*(V_2(\mathbb{R}^4)) = \begin{cases} \mathbb{Z} & 5 \\ 0 & 4 \\ \mathbb{Z} & 3 \\ \mathbb{Z} & 2 \\ 0 & 1 \\ \mathbb{Z} & 0 \end{cases}$$

In general, $$H^*(V_2(\mathbb{R}^{n+1})) = \begin{cases} E(x_{n-1}, x_n) & n \text{ odd} \\ \left\{\begin{matrix} \mathbb{Z} & 2n-1 \\ \vdots & \\ 0 & \\ \mathbb{Z}/2 & n \\ \vdots & \\ 0 & \\ \mathbb{Z} & 0 \end{matrix}\right\} & n \text{ even} \end{cases}$$

$\underline{\text{Ex}}$ Let $S^1 \to E \to \mathbb{C}P^\infty$ be the spherical fib w/ euler class $2x \in H^2(\mathbb{C}P^\infty)$ (this is the sphere bundle associated to $L^{\otimes 2} \to \mathbb{C}P^\infty$, L the canonical line bundle & $E \simeq \mathbb{R}P^\infty$)



So E_3 :
$\quad \|$
$\quad E_\infty$



We can now return to the case of rebuilding the cohomology. First look at effects of the maps in the fibration.

$$F \xrightarrow{\iota} E \xrightarrow{\pi} B$$

Prop 1 ① $\pi^*(H^*(B))$ is the subring of $H^*(E)$ given by $E_\infty^{*,0}$

② The image of $\iota^*(H^*(E))$ in $H^*(F)$ is given by $E_\infty^{0,*}$

We can see this by comparing with the SSS for the fibrations $F \to F \to *$ and $* \to B \to B$

A more careful analysis of the filtration gives the following.

Prop 2 $H^r(E)$ has a filtration F_i s.t.

$$F_0 = E_\infty^{r,0} \quad \& \quad F_i / F_{i-1} = E_\infty^{r-i,i}.$$

So
$$\begin{tikzcd}
E_\infty^{r,0} \arrow[r, phantom, "\subseteq"] & F_1 \arrow[d] \arrow[r, phantom, "\subseteq"] & F_2 \arrow[d] \arrow[r, phantom, "\subseteq \cdots \subseteq"] & F_r \arrow[d] \\
& E_\infty^{r-1,1} & E_\infty^{r-2,2} & E_\infty^{0,r}
\end{tikzcd}$$

Now we piece together the Serre SS for $S^n \to E \to B$ w/ euler class e.

Already saw that we have a SES
$$0 \to E_\infty^{k,n} \hookrightarrow \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ H^k(B)}}{H^k(B) \otimes H^n(S^n)} \to H^{k+n+1}(B) \to E_\infty^{k+n+1,0} \to 0$$

The previous props, together with sparceness, show that there is a SES
$$0 \to E_\infty^{k,0} \to H^k(E) \to E_\infty^{k-n,n} \to 0$$

Splicing these all together gives a long exact seq.

Thm 1 If $S^n \to E \xrightarrow{\pi} B$ is a simple fibration, then
$$\cdots \longrightarrow H^{k+n}(E) \to H^k(B) \xrightarrow{\cup e} H^{k+n+1}(B) \xrightarrow{\pi^*} H^{k+n+1}(E) \to \cdots$$
is exact, where e is the euler class.

There is a big case where we <u>don't</u> want simple fibrations: group cohomology.

Def Let $B: \mathcal{G}rps \to \mathcal{T}op$ be the classifying space functor.

If G is an abelian group, can choose BG to be one as well. Moreover, $\pi_{i+1} BG = \pi_i G$ & $G \simeq \Omega BG$.

Prop 3 If $1 \to N \to G \to H \to 1$ is a SES, then

$$BN \to BG \to BH \text{ is a fibration.}$$

Remark In fact, $G \to H \to BN$, etc are also fibrations.

This is very often not simple.

Def $H^*(G) = H^*(BG)$.

We can incorporate non-trivial G-modules by looking at bundles over BG & homology w/ twisted coefficients.

The Hochschild-Serre SS is the SS associated to the fibration $BN \longrightarrow BG \to BH$:

$$H^p(H; \mathcal{H}^q(N; M)) \Longrightarrow H^{p+q}(G; M)$$

M a G-module, $\mathcal{H}^q(N; M)$ an H-module by twisting.

Ex: $H^*(\Sigma_3; \mathbb{Z}_{(3)})$.

Need some facts.

① $\mathbb{Z}/3 \hookrightarrow \Sigma_3 \twoheadrightarrow \mathbb{Z}/2$ is exact

② $\mathbb{Z}/2$ acts on $\mathbb{Z}/3$ by inversion.

③ If p is a unit in R, then $H^{*>0}(\mathbb{Z}/p; R) = 0$

In our case, ③ is satisfied. So the HSSS takes the form

$$E_2^{p,q} \quad H^p(\mathbb{Z}/2; H^q(\mathbb{Z}/3; \mathbb{Z}_{(3)})) \Longrightarrow H^{p+q}(\Sigma_3; \mathbb{Z}_{(3)})$$

&, if $p>0$, then $E_2^{p,q} = 0$.

For $p=0$, $H^*(\mathbb{Z}/3; \mathbb{Z}_{(3)}) = \mathbb{Z}_{(3)}[x_2]/3x_2$ & $\mathbb{Z}/2$ acts by $x_2 \longmapsto -x_2$. So $H^0(\mathbb{Z}/2; \mathbb{Z}_{(3)}[x_2]/3x_2) = \mathbb{Z}_{(3)}[x_2^2]/3x_2^2$, and this is also E_∞!