Warm-Up Computations with the Serre SS

Note Title 2/2/2009

Basic recollections:

<u>Def</u> A map $E \to B$ is a Serre fibration if it satisfies homotopy lifting for cell complexes. Ie if

$$\begin{tikzcd} K \arrow[r] \arrow[d] & E \arrow[d] \\ K \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

is commutative, then the dashed arrow exists.

<u>Def</u> The Fiber of $E \xrightarrow{\pi} B$ is $\pi^{-1}(b)$, $b \in B$ the basepoint.

If B is path connected, then homotopy lifting implies that the fibers over any point is $\simeq$ to F.

More is true:

<u>Prop 1</u>: $\pi_1(B)$ acts on F (in the homotopy category).

This is analogous to the statement that $\pi_1(B)$ acts on the universal cover by Deck transformations.

<u>Cor 1</u>: $\pi_1(B)$ acts on $\pi_n F$, $H_n F$, etc

<u>Def</u>: $F \to E \to B$ is <u>simple</u> if $\pi_1 B$ acts trivially

The vast majority of cases we consider will be simple.

One of the big features of a fibration is the long exact sequence in homotopy:

$$\longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

We will sometimes use this to start SS comps. We should think of the Serre SS as "dual" to this.

<u>Thm 1</u> Let $F \to E \to B$ be a simple fibration. Then there is a first quadrant, cohomological SS of algebras

$$E_2^{p,q} = H^p(B; H^q(F; R)) \Longrightarrow H^{p+q}(E; R).$$

The notation "$\Rightarrow H^{p+q} \ldots$" tells which groups form the associated graded for which degrees. In this case

$$Gr(H^n(E)) = \bigoplus_{p+q=n} E_\infty^{p,q}.$$

Rather than prove this theorem, we will show how it is used in two essential cases:

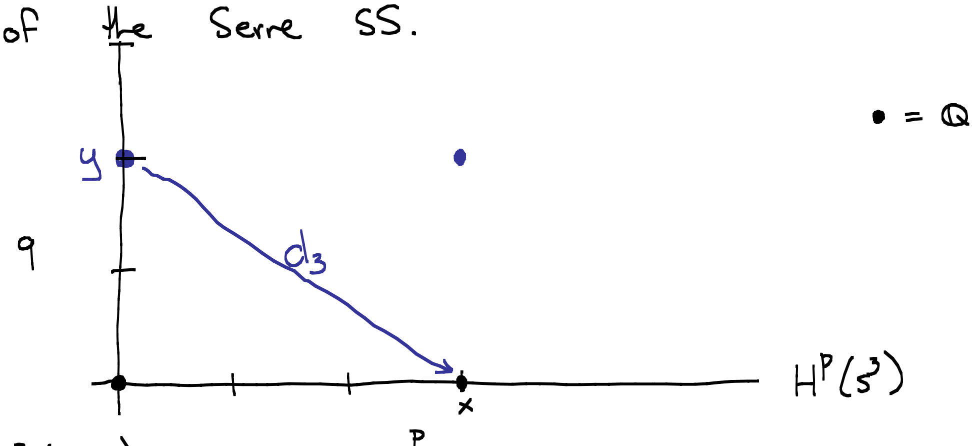
① $\Omega X \longrightarrow PX = Map_*(I, X)$; ② $H \longrightarrow G$, G, H Lie, H closed
$$\begin{array}{ccc} & PX & \\ & \downarrow & \\ & X & \end{array} \qquad\qquad \begin{array}{c} G \\ \downarrow \\ G/H \end{array}$$

<u>Remark</u> If $F \to E \to B$ is not simple, then we must consider the action of $\pi_1(B)$ on $H^q(F; R)$. This gives "cohomology with twisted / local coefs" and $E_2 = H^p(B; \mathcal{H}^q(F; R))$.

In case ①, the total space is contractible! We imagine just pulling all paths to the base point. So the Serre E_2-term looks like
$$E_2^{p,q} = H^p(X; H^q(\Omega X; R))$$
while $E_\infty^{p,q} = 0$ unless $p = q = 0$. So $H^q(\Omega X; R)$ must cancel out the $H^p(X)$ stuff.

<u>Example 1</u> : $H^*(\Omega S^3; \mathbb{Q})$

We know $H^0(\Omega S^3; \mathbb{Q}) = \mathbb{Q}$ since $\pi_1 S^3 = 0$. So we know the 0^{th} row of the Serre SS.



Since $H^3(PS^3) = 0$, the class x must be the target of a diff.

Only option is a d_3 from $(0,2)$. So we must have
$E_2^{0,2} = \mathbb{Q} = H^0(S^3; H^2(\Omega S^3; \mathbb{Q}))$
$\Rightarrow H^2(\Omega S^3; \mathbb{Q}) = \mathbb{Q}$. (Also easy to see H^1 must be 0).

Now there is a new class, $x \cdot y$ in $H^3(S^3; H^2(\Omega S^3; \mathbb{Q}))$.
Since $E_\infty^{3,2} = 0$, this must be the target of a d_3-differential.
Continuing with an induction argument gives

Cor 2 $H^*(\Omega S^3; \mathbb{Q}) = \begin{cases} \mathbb{Q} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$

We can also work out the ring structure. Since $H^*(S^3; \mathbb{Z})$ is free, Künneth $\Rightarrow$

$$H^p(S^3; H^q(\Omega S^3; R)) = H^p(S^3; R) \otimes_R H^q(\Omega S^3; R).$$

(For a more general X, need that one of these is projective)

This is actually a splitting as algebras. Let $e_i \in H^{2i}(\Omega S^3; \mathbb{Q})$, $e_0 = 1$
be the class s.t.

$$1 \otimes e_i \in H^0(S^3) \otimes H^{2i}(\Omega S^3) \quad \text{hits} \quad x \otimes e_{i-1} \in H^3(S^3) \otimes H^{2i-2}(\Omega S^3).$$

under the aforementioned d_3. In fact, d_3 establishes an isomorphism $H^0(S^3) \otimes H^{2i}(\Omega S^3) \xrightarrow{\simeq} H^3(S^3) \otimes H^{2i-2}(\Omega S^3)$
for $i > 0$.

We can then determine the ring structure on $H^*(\Omega S^3)$. Since this is concentrated in even degrees, everything commutes.

Prop 2 $e_1^k = k!\, e_k$

Proof This is by induction on k. The base case is too obvious, so let's look at $k=2$. The class e_1^2 is in $H^4(\Omega S^3)$, so we look at $d_3(1 \otimes e_1^2) = d_3((1 \otimes e_1)^2)$. By the Leibnitz rule:

$$d_3((1 \otimes e_1)^2) = 2 \cdot (1 \otimes e_1)\, d_3(1 \otimes e_1) \qquad (|1 \otimes e_1| \text{ is even})$$
$$= 2 \cdot x \otimes e_1$$

Since d_3 is an iso $H^0(S^3)\otimes H^4(\Omega S^3) \to H^3(S^3)\otimes H^2(\Omega S^3)$,

we learn that $1\otimes e_1^2 = d_3^{-1}(2\cdot x\otimes e_1) = 1\otimes 2e_2$, so

$e_1^2 = 2e_2$.

The same argument shows that

$$d_3((1\otimes e_1^k)) = k\cdot x\otimes e_1^{k-1}, \quad \text{so by induction}$$
$$= k!\ x\otimes e_{k-1}. \qquad \boxtimes$$

Nothing here depended on us working over $\mathbb{Q}$ or that it was S^3, rather than S^{2k+1}.

<u>Def</u> The divided powers algebra over R on a class x in degree k is the R-algebra generated by classes

$$x_i, \quad |x_i| = k\cdot i, \quad \text{subject to}$$
$$x_i\cdot x_j = \binom{i+j}{i} x_{i+j}$$

This algebra will be denoted $\Gamma_R(x)$. It is the Hopf alg dual to $R[x]$.

<u>Exercise</u>

1. Show that $H^*(\Omega S^{2n+1};\mathbb{Z}) = \Gamma_{\mathbb{Z}}(x)$, $|x| = 2n$
2. Show that $H^*(\Omega S^{2n};\mathbb{Z}) = E(x)\otimes\Gamma(y)$, $|x| = 2n-1$, $|y| = 4n-2$.

We can also compute the cohom of the total space. Our big example comes from the unitary groups.

<u>Prop 3</u> $U(n)/U(n-1) = S^{2n-1}$

<u>Pf</u> $U(n)$ acts transitively on S^{2n-1} = unit sphere in $\mathbb{C}^n$.

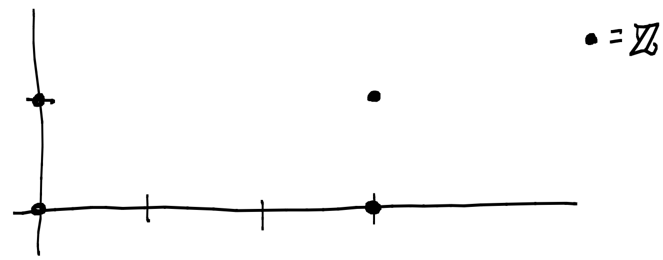
If we pick a vector, then the stabilizer is those unitary transformations of the orthogonal $\mathbb{C}^{n-1}$. $\boxtimes$

So we have a fibration $U(n-1)\longrightarrow U(n)\longrightarrow S^{2n-1}$.

Additionally, $U(1) = S^1$, so we can use induction.

$n=2$: $\quad U(1) \hookrightarrow U(2) \longrightarrow S^3$ gives

$E_2^{p,q} = H^p(S^3; H^q(U(1);\mathbb{Z})) =$



For degree reasons, this collapses: $E_2 = E_\infty$, so

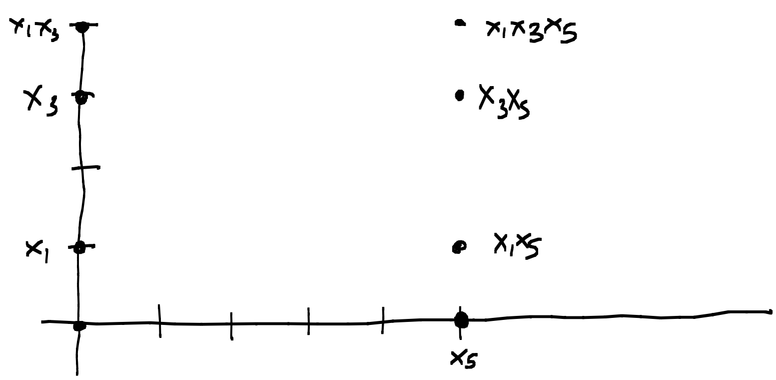
$$H^*(U(2)) = \begin{cases} \mathbb{Z} & 0 \\ \mathbb{Z} & 1 \\ 0 & 2 \\ \mathbb{Z} & 3 \\ \mathbb{Z} & 4 \end{cases}$$

Moreover, since this is a SS of algebras,

$H^*(U(2)) = E(x_1, x_3) \qquad |x_i| = i$

$n=3$ $\quad U(2) \longrightarrow U(3) \longrightarrow S^5$

$E_2^{p,q} = H^p(S^5; H^q(U(2))) = E(x_1, x_3, x_5)$:



For degree reasons, x_1, x_3, x_5 all perm cycles. The generate E_2, so all classes are perm cycles.

<u>Exercise</u> Complete the induction argument to show

$$H^*(U(n)) = E(x_1, \dots, x_{2n-1})$$

Same argument gives $H^*(Sp(n)) = E(x_3, x_7, \dots, x_{4n-1})$

using $Sp(n-1) \longrightarrow Sp(n) \longrightarrow S^{4n-1}$, $\quad Sp(1) = SU(2) = S^3$.

$SO(n)$ is more subtle: $H^*(SO(n); \mathbb{Z}/2) \neq E(x_1, x_2, \dots)$

(try $SO(3) = \mathbb{RP}^3$).